

University of Ghana



**On the Geometric view of Pentagram Integrals of Polygons
Inscribed in Non-Degenerate Conics.**

BY

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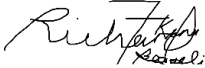
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DECLARATION

This thesis was written in the Department of Mathematics, University of Ghana, Legon from August 2017 to July 2018 in partial fulfilment of the requirements for the award of Master of Philosophy degree in Mathematics under the supervision of Dr. Richard Kena Boadi and Dr. Ralph Twum of the Kwame Nkrumah University of Science and Technology and the University of Ghana respectively.

I hereby declare that except where due acknowledgement is made, this work has never been presented wholly or in part for the award of a degree at the University of Ghana or any other University.

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ABSTRACT

The Pentagon map is a well notable integrable system that is defined on the moduli space of polygons. In 2005, Richard Evan Schwartz introduced certain polynomials called **pentagram integrals** (Monodromy invariants) of the pentagram map and defined certain associated integrals, the analogous **first integrals**. Schwartz further studied in 2011 with S. Tabachnikov on how these integrals behave on inscribed polygons.

They discovered that the integrals are equal for every given weight of polygons inscribed in non-degenerate conics. However, the proof of their outcome was combinatorial which appeared to be more involving hence there was a need for quite a simple proof.

Anton Izosimov in 2016 gave quite a simple conceptual geometric proof of these invariants of polygons inscribed in non-degenerate conics.

In this thesis, we seek to analyse the geometry of these invariants by reviewing Anton's work. Our core analyses is that for any polygon inscribed in a non-degenerate conic, the analogous monodromy should satisfy a certain self-duality relation.

DEDICATION

I humbly dedicate this thesis to God for giving me heavenly knowledge, wisdom, strength and directions to produce this project. All pages of this work are dedicated to my parents, Mr. Isaac Opoku Gyasi and Mrs Janet Opoku Gyasi for their support and all my loved ones.

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Contents

Declaration	i
Abstract	iii
Dedication	iv
Acknowledgements	v
1 Introduction	2
2 Preliminaries	7
2.1 Projective Space	7
2.1.2 Real Projective Line	8
2.1.5 Real Projective Plane	8
2.2 Projective Transformations	8
2.3 Cross Ratio	11
2.4 Twisted Polygon	18
2.5 Corner Invariant (The Invariant Function)	18
2.6 Projective Duality	20
2.7 Dual Polygon	21
2.8 Self-dual Polygons	22

2.8.1	Example	22
2.9	Non-degenerate Conic	23
2.10	Inscribed Polygon	24
2.11	Involution	24
3	Pentagram Map and Monodromy Invariants	26
3.1	Formula for the Pentagram Map	26
3.2	Construction of Monodromy Invariants	30
4	Monodromy Invariants of the Spectral Curve and Corner Invariants in the Monodromy Matrix	33
4.1	Monodromy Invariants and the Spectral Curve	33
4.2	Monodromy Matrix and the Corner Invariant	38
5	Inscribed Polygons and their Monodromy Invariants	48
5.1	Proof of Theorem 1.1	53
6	Conclusion and Further Discussion	54
6.1	Conclusion	54
6.2	Further Discussion - Degenerate Conics	55
A	Other Useful Concepts	57
A.1	DEFINITIONS	57
A.2	Integrable System	59
A.2.1	Zero locus of set of Polynomial	59
	References	61

List of Figures

1.1	A Pentagonam	3
1.2	Pentagramma Mirificum	3
1.3	Construction of the pentagram	4
1.4	Extending the sides of a pentagon in the euclidean plane: The red dash lines represent the extension of the two parallel sides which do not meet in the Euclidean plane	4
2.1	Cross ratio (definition)	11
2.2	$(A, B : C, D)$ is well defined	13
2.3	corner invariant; definition.	19
2.4	corner invariant; example.	20
2.5	Construction of corner invariants at different vertices on a twisted m-gon.	21
2.6	Projective duality	21
2.7	Duality of the pentagon	22
2.8	Self Dual polygon	23
2.9	Non-degenerate conics shown in red	24
2.10	A polygon inscribed in a non-degenerate conic (ellipse)	25
2.11	Involution of the Pentagonam map	25
3.1	Projective duals	28
3.2	Duality maps	29

3.3	Labelling schemes of the Pentagram Map	30
5.1	Lemma 4.2.2, proof.	50
6.1	Degenerate Conics	55
A.1	Convex hull	58

NOTATIONS

$\mathbb{RP}^2 \equiv \mathbb{P}^2(\mathbb{R})$	Real projective plane
$\mathbb{RP}^1 \equiv \mathbb{P}^1(\mathbb{R})$	Real projective line
$GL(n, \mathbb{R})$	General linear group of dimension n over the real numbers
X_k, Y_k	Monodromy invariants
$\mathbb{PGL}_n$	Projective general linear group of dimension n
M	Pentagram map
T	Monodromy
x_i, y_i	Corner invariants
S^2	Sphere
(p_i)	Twisted polygon
(P_i)	Lifts of the points (p_i)
$\acute{T}$	Monodromy of the lifts (P_i)
$\mathcal{V}_m$	Space of twisted m -gons modulo projective transformation
q_i	Cross ratio of inscribed points projected onto a projective line

Chapter 1

Introduction

The pentagram is intuitively defined as a five star-like figure, (see Figure 1.1) mostly used by the ancient Greeks and now by most religions as a symbol representing an object they worshiped. For instance Christians used it to represent the five wounds of Jesus Christ on the cross. It is said that every man or woman is a pentagram, with reference to Leonardo da Vinci's relation of Mathematics and Arts. One novelty of the pentagram is that, it has connections with the number phi ($\phi \approx 1.168$) which has several applications in real life. The pentagram, in its real life applications, is also used in drawing maps (Google maps) with reference to the pentagram that was found carved into the surface of the earth in Kazakhstan.

In the theory of mathematics, the pentagram was first introduced by Gauss when he studied the geometry of a right-angle triangle (A) inside a sphere S^2 in 1876. He used the word **Pentagramma Mirificum** to describe the construction of the pentagram as shown in Figure 1.2.

Studies showed that the pentagram can be constructed by either extending the edges of a pentagon or by joining the corresponding diagonals of the vertices of a pentagon as shown in Figure 1.3.

One can locate a pentagon either inside or outside (by joining the vertices of the pentagram together) the pentagram. Several pentagons can be constructed from any given initial pentagon. The map that transforms one pentagon P into another pentagon P' is called **Pentagram map** as shown in Figure 1.3. M. Glick studied the pentagram map for only pentagons in 1945 [7]. However, in 1992, Richard Evan Schwartz introduced the pentagram map and defined it generally for all polygons [11].

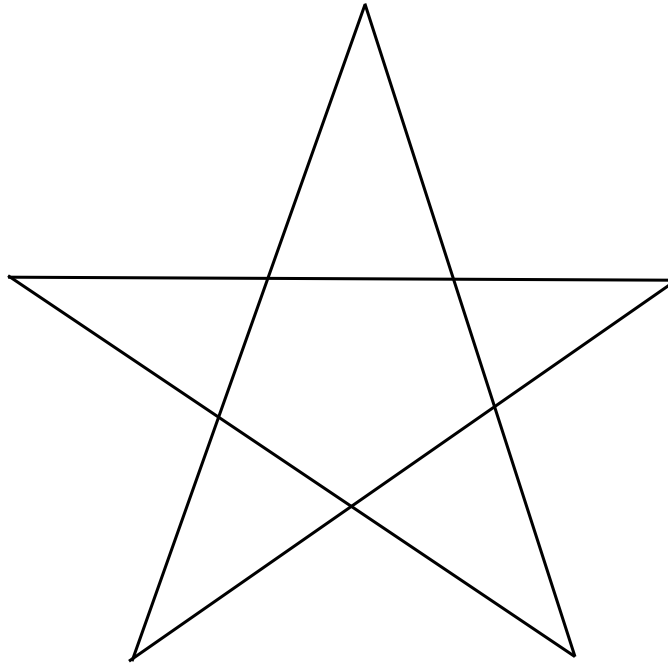


Figure 1.1: A Pentagram

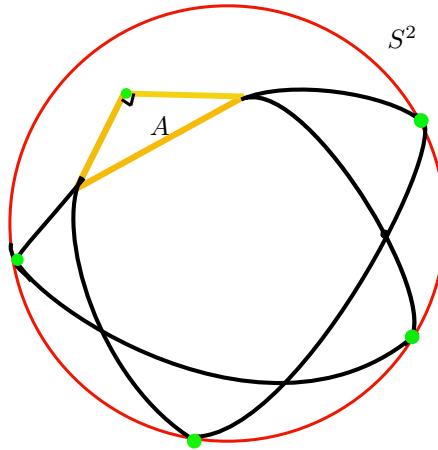


Figure 1.2: Pentagramma Mirificum

Hence, the pentagram map in general terms is defined as the map that transforms one polygon into another polygon. Because we obtain an equivalent polygon based on the above constructions, we say that any two polygons P and P' under the pentagram map are projectively equivalent.

The construction of the Pentagram map is similar to the constructions shown in Figure 1.3 in the usual Euclidean plane. However, the pentagram map acting on certain polygons cannot be constructed by the approach in Figure 1.3(a) since they

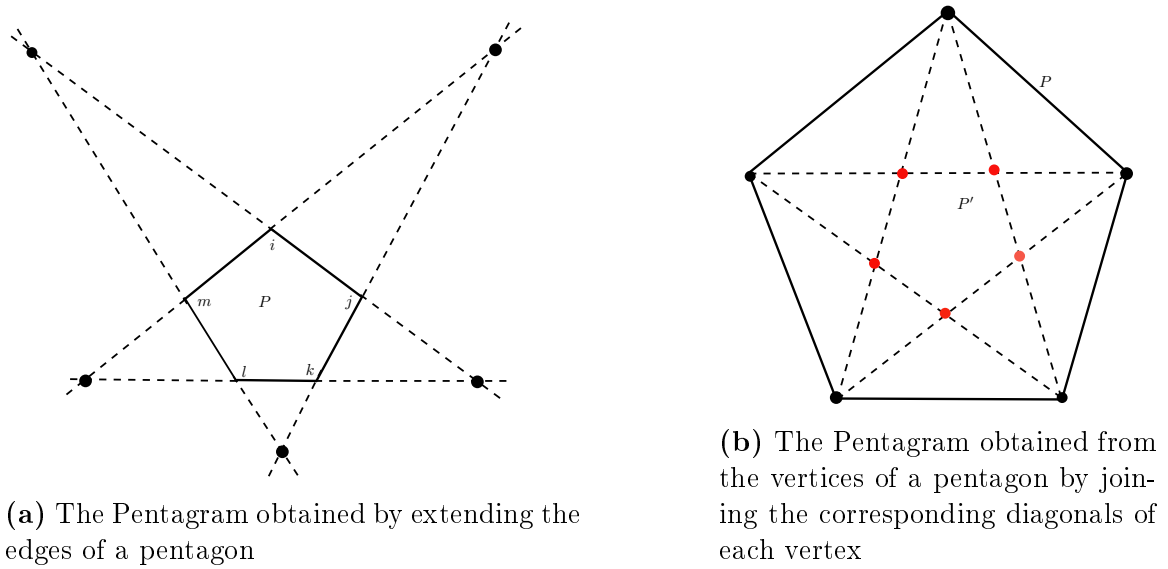


Figure 1.3: Construction of the pentagram

may require the extension of parallel lines as shown in Figure 1.4

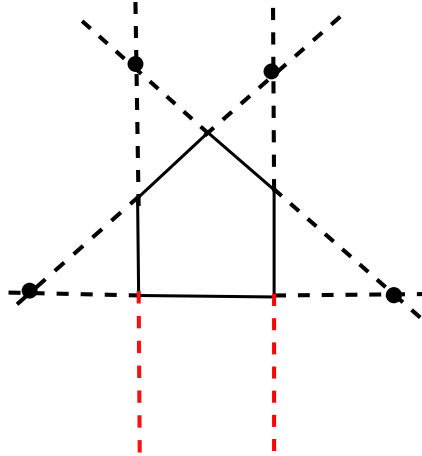


Figure 1.4: Extending the sides of a pentagon in the euclidean plane: The red dash lines represent the extension of the two parallel sides which do not meet in the Euclidean plane

Now, since parallel lines do not meet in the Euclidean plane, we require a much more convenient space for the efficient construction of the Pentagram map under the constructions of both figures in Figure 1.3. Hence, the **projective plane** defined over any field is mostly convenient for our constructions since parallel lines meet at infinity in this space.

Schwartz wrote a number of papers such as [11], [15], [12] for general polygons and also in recent papers [10], [17], [3]. It is discussed that the map commutes with projective transformations and induces a generally defined map on the space of cyclically n -gons

modulo projective transformations [18]. A bigger space called the space of *twisted polygons* was introduced in [15]. In this space, the invariant polynomials,

$$O_1, \dots, O_{\lfloor \frac{n}{2} \rfloor}, n, E_1, \dots, E_{\lfloor \frac{n}{2} \rfloor}, n : \mathcal{P}_n \rightarrow \mathbb{R},$$

under the pentagram map were produced for every n -gon (see Chapter 3). These invariant polynomials are called **Monodromy Invariants**, also known as **Pentagram Integrals** in [16].

Several works have been done in recent papers on the Pentagram map and its Integrals. In [15], it was shown that the map is a completely integrable system on $\mathcal{P}_n$, that is obtained from the construction of an invariant poisson bracket on $\mathcal{P}_n$ together with monodromy invariants. Additionally, [9] mainly considered the integrability of the map on the space of cyclically ordered polygons (main space of interest in this paper). The first result of complete integrability of the map was introduced in [13].

[19] discusses the algebraic-geometric integrability for any monodromy either in twisted or closed polygons. Here, the notion of spectral data which consists of Riemann surface called the spectral curve was introduced. Due to this, it was shown that the pentagram map can be written as a zero-curvature equation with a spectral parameter with corresponding spectral curve and the dynamics of the Jacobian.

This approach of integrability was more preferable to that in [10] since it works on both closed and twisted polygons. Additionally, it is equally fit in both continuous and discrete cases, which can be used to integrate the continuous limit of the map (the Boussinesq equation).

The Lax representation was the major ingredient used to prove this algebraic integrability since it organises certain invariant functions in the form of a spectral curve.

The Lax representation was used over Arnold integrability since it is useful in both continuous and discrete cases and also works for both twisted and closed polygons.

A discrete analogue of the monodromy matrix is a monodromy operator. In relation to [19], we shall represent our monodromy operator as scaled monodromy matrix.

In 2011, Schwartz and S. Tabachnikov had a further research on how the monodromy invariants behave when polygons are inscribed in non-degenerate conics [18]. Their studies resulted in the theorem below;

Theorem 1.0.1. For every n -gon P inscribed in a non-degenerate conic, $O_k = E_k$,

$$\forall k = 1, \dots, \lfloor \frac{n}{2} \rfloor, n.$$

Explicit formulas were used to prove Theorem(1.0.1) combinatorially. However, their proof seemed to be more involving and complicated.

In 2016, Anton Izosimov gave a quite simple geometric proof of Theorem (1.0.1). The focus of his study was that, for every given polygon that is inscribed in a non-degenerate conic, its analogous monodromy should satisfy a particular self-duality relation.

The core of this work is to discuss in details the geometry of the above theorem by reviewing Anton's work. The structure of this thesis is shown as follows; Chapter 2 discusses some relevant definitions and their associated examples. In Chapter 3, the work is discussed in two (2) sections. Whereas the first section discusses the formula for the pentagram map in coordinates, the second section discusses how the monodromy invariants are constructed (in this work, O_k and E_k are represented by X_k and Y_k respectively). Similarly, Chapter 4 discusses the occurrence of monodromy invariants as coefficients of the spectral curve in the first section. The second section is dedicated for the relationship between the monodromy matrix and corner invariants. Chapter 5 discusses monodromy invariants of inscribed polygons using the self-duality relation coupled with idea of stereographic projection of a point on a conic onto the projective line. The results obtained is finally used to prove Theorem(1.0.1). Chapter 6 is devoted to conclusion and Appendix for further discussion.

Chapter 2

Preliminaries

In this chapter, we define some relevant concepts with their useful examples.

2.1 Projective Space

Definition 2.1.1. Let X be a vector space. The projective space $\mathbb{P}(X)$ is defined as the set of all 1-dimensional vector subspaces of X . Thus, it is the set of lines through the origin of the vector space X . The projective space $\mathbb{P}^n(\mathbb{X})$ can be defined as

$$\mathbb{P}^n(X) = \mathbb{P}^n(\mathbb{R}) = \mathbb{R}^n \cup \mathbb{P}^{n-1}(\mathbb{R}), \quad \text{for } X = \mathbb{R}$$

$\mathbb{P}^n(\mathbb{R})$ is called the *real projective space*. If $\mathbb{R}$ is replaced by $\mathbb{C}$, then the projective space is called the *complex projective space*.

Suppose X is an $(n+1)$ dimensional vector space, then the projective space $P(X)$ has dimension n . The 1-dimensional and 2-dimensional projective spaces are termed as the *Projective line* and the *Projective plane* respectively. Elements (points) in the projective space are called *homogeneous coordinates*.

2.1.2 Real Projective Line

Definition 2.1.3. The *real projective line* is the set of lines in $\mathbb{R}^2$ which pass through the origin and is obtained by the extension of the usual line when a point at infinity is added. Thus

$$\mathbb{P}^1(\mathbb{R}) = \mathbb{R} \cup \text{point} = \mathbb{R} \cup \{\infty\}.$$

Example 2.1.4. [8] Let $n = 1$. Since a 0-dimensional projective space is a single point, we obtain the projective line $\mathbb{P}^1(\mathbb{R}) = \mathbb{R} \cup \mathbb{P}^0$, where $\mathbb{P}^0$ is a point in the projective line. Usually, we call this extra point ‘the point at ∞ ’. When $X = \mathbb{C}$, then the projective line becomes the *extended complex plane* $(\mathbb{C} \cup \{\infty\})$.

2.1.5 Real Projective Plane

Definition 2.1.6. The *real projective plane*, $\mathbb{P}^2(\mathbb{R}) = \mathbb{R}^3 \setminus \sim$ is the set of lines through the origin of $\mathbb{R}^3$. Similar to the projective line where points meet at infinity, a plane in projective geometry contains a single line at infinity at which the edges of the plane meet. Note that points of $\mathbb{P}^2(\mathbb{R})$ consists of points of $\mathbb{R}^2$ and points in $\mathbb{P}^1(\mathbb{R})$ whiles lines of $\mathbb{P}^2(\mathbb{R})$ consist of: lines in $\mathbb{R}^2$ with point at infinity and points $\mathbb{P}^1(\mathbb{R})$ at infinity. For any set of lines and points in a projective plane, there exists some incidence relations that describes the lines and points in the space. Nevertheless, there exists a unique projective line that passes through any two distinct points in a projective plane [4].

2.2 Projective Transformations

Points in $\mathbb{P}^n(\mathbb{R})$ are just lines which pass through the origin of $\mathbb{R}^{n+1}$, we therefore need a group of transformations which map such lines in $\mathbb{R}^{n+1}$ onto lines which pass through the origin of $\mathbb{R}^{n+1}$. The appropriate transformations for such purposes are the invertible linear transformations which map $\mathbb{R}^{n+1}$ onto itself. These transformations map points $x \in \mathbb{R}^{n+1}$ onto points $Nx \in \mathbb{R}^{n+1}$. Therefore the projective transformation

with an associated matrix N maps points x of $\mathbb{P}^n(\mathbb{R})$ onto points of $\mathbb{P}^n(\mathbb{R})$.

Definition 2.2.1. Let $N \in GL(n+1, \mathbb{R})$ be a non-singular matrix. Then the function $T : \mathbb{P}^n(\mathbb{R}) \rightarrow \mathbb{P}^n(\mathbb{R})$ defined by the map $x \mapsto Nx$, for $x \in \mathbb{R}^{n+1}$ which maps a line $[x]$ into another line $[Nx]$ is called a *projective transformation* of $\mathbb{P}^n(\mathbb{R})$.

For every $\alpha \in K$, where K is a field, αT and T define the same projective transformation. Thus,

$$[\alpha T(x)] = \alpha[T(x)] = [T(x)]$$

which are of the same equivalence class.

Example 2.2.2. Let us consider the Möbius transformation of the extended complex plane. These are projective transformations of the complex projective line $\mathbb{P}^1(\mathbb{C})$ onto itself. Points in $\mathbb{P}^1(\mathbb{C})$ are described by homogeneous coordinates. Let us consider the homogeneous coordinates $(z_1, z_2) \sim (z'_1, z'_2)$ such that $z_1 = \lambda z'_1$ and $z_2 = \lambda z'_2$, $0 \neq \lambda \in \mathbb{C}$. Then the linear transformation with coordinates $[z_1, z_2]$ is

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \mapsto \begin{pmatrix} \bar{z}_1 \\ \bar{z}_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \quad (2.1)$$

having its corresponding invertible linear transformation $T : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ as $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Now, if $z_2 \neq 0$, then $z = \frac{z_1}{z_2}$ which give the coordinates $[z, 1]$. Following from Equation (2.1), we obtain

$$\bar{z}_1 = az_1 + bz_2 \quad \text{and} \quad \bar{z}_2 = cz_1 + dz_2.$$

Then

$$\frac{\bar{z}_1}{\bar{z}_2} = \bar{z} = \frac{az_1 + bz_2}{cz_1 + dz_2} = \frac{az + b}{cz + d}, \quad z = \frac{z_1}{z_2}.$$

is called a *Möbius transformation* [8].

Example 2.2.3. The projective transformation of the real projective plane, $\mathbb{P}^2(\mathbb{R})$.

We have that,

$$\mathbb{P}^2(\mathbb{R}) = \mathbb{R}^2 \cup \mathbb{P}^1(\mathbb{R})$$

having its projective line at infinity is $x_0 = 0$. A linear transformation

$$T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$$

is given by

$$\begin{pmatrix} X'_1 \\ X'_2 \\ X'_3 \end{pmatrix} = \begin{pmatrix} d & b_1 & b_2 \\ c_1 & a_{11} & a_{12} \\ c_2 & a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$$

The components of the vector in $\mathbb{R}^3$ are *homogeneous coordinates* of a point in $P^2(\mathbb{R})$. The coordinates where $X_1 = 0$ represent the points on the line at infinity. All other points have $X_1 \neq 0$ and a special choice of such coordinates can be made. For example

$$\begin{pmatrix} 1 \\ \frac{X_2}{X_1} \\ \frac{X_3}{X_1} \end{pmatrix} = \begin{pmatrix} 1 \\ x \\ y \end{pmatrix}$$

The x and y defined here are affine coordinates. From the above transformation, we can write that

$$\begin{pmatrix} X'_1 \\ X'_2 \\ X'_3 \end{pmatrix} = \begin{pmatrix} d & b_1 & b_2 \\ c_1 & a_{11} & a_{12} \\ c_2 & a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 1 \\ x \\ y \end{pmatrix}$$

and so defining $x' = \frac{X'_2}{X'_1}$ and $y' = \frac{X'_3}{X'_1}$ in a similar way to x and y , we obtain;

$$x' = \frac{c_1 + a_{11}x + a_{12}y}{d + b_1x + b_2y} \quad \text{and} \quad y' = \frac{c_2 + a_{21}x + a_{22}y}{d + b_1x + b_2y}$$

We can write this matrix as

$$v' = \frac{c + Av}{d + b \cdot v}$$

where v is the column 2-vector with components x and y , c is the column 2-vector

with components c_1 and c_2 etc. and

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

2.3 Cross Ratio

Let A, B, C, D be a set of four collinear points. We consider what happens to these points when they are projected from an external point O to another set of four collinear points A', B', C', D' as shown in Figure 2.1. This relation is termed as *cross ratio*. The cross ratio is described as the most important invariant in projective geometry which is preserved by all projective transformations and is said to be the ratio of ratios of the set of four distinct collinear points [8].

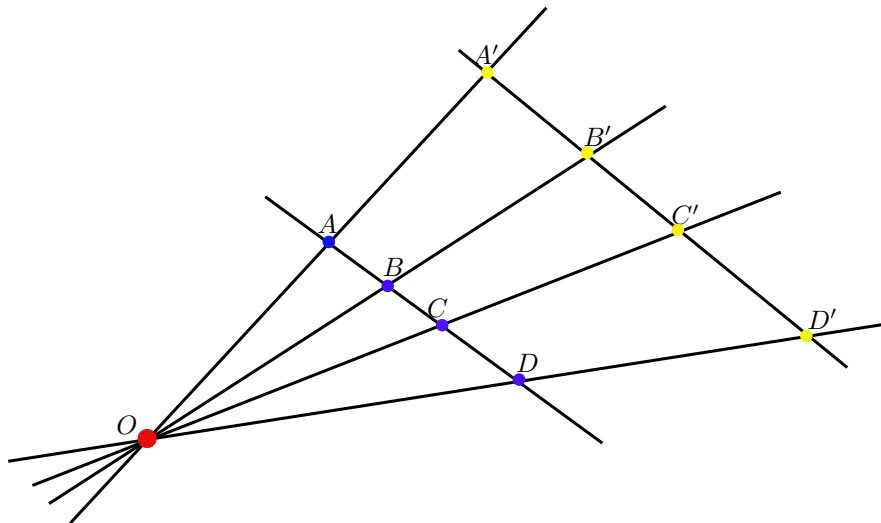


Figure 2.1: Cross ratio (definition)

The cross ratio is defined in terms of homogeneous coordinates in the projective plane. This definition will be used to prove the two relevant theorems below:

- a. The cross ratio is well defined
- b. The cross ratio is projectively invariant.

Definition 2.3.1. Let A, B, C, D be four collinear points with homogeneous coordinates (non-zero vectors) $[a], [b], [c], [d]$ respectively in $\mathbb{P}^2(\mathbb{R})$. Since A, B, C, D are

collinear, one can express c and d as a linear combination of a and b . Thus $\exists \alpha, \beta, \gamma, \delta \in \mathbb{R}$,

$$c = \alpha a + \beta b$$

$$d = \gamma a + \delta b.$$

We define the *cross ratio* of A, B, C, D as the ratio of the ratios $\frac{\beta}{\alpha}$ and $\frac{\delta}{\gamma}$ [8]. Thus

$$[A, B, C, D] = \frac{\beta}{\alpha} \div \frac{\delta}{\gamma} = \frac{\beta\gamma}{\alpha\delta}. \quad (2.2)$$

Example 2.3.2. Let us consider the collinear points $A = [a] = [1, -1, -1]$, $B = [b] = [1, 3, -2]$, $C = [c] = [3, 5, -5]$, $D = [d] = [1, -5, 0]$.

By the definition of cross ratio $[A, B, C, D]$, we first have to express c and d as linear combinations of a and b then we find the real numbers $\forall \alpha, \beta, \gamma, \delta \in \mathbb{R}$, associated to each of them to compute the ratio. Thus for $c = \alpha a + \beta b$, we have

$$(3, 5, -5) = \alpha(1, -1, -1) + \beta(1, 3, -2).$$

Comparing each corresponding coordinate, we have the following system of equations

$$3 = \alpha + \beta \quad (2.3)$$

$$5 = -\alpha + 3\beta \quad (2.4)$$

$$-5 = -\alpha - 2\beta. \quad (2.5)$$

Solving equations, (2.3), (2.4) and (2.5), we have $\alpha = 1$ and $\beta = 2$. Next, given that $d = \gamma a + \delta b$ then,

$$(1, -5, 0) = \gamma(1, -1, -1) + \delta(1, 3, -2).$$

Comparing each corresponding coordinate, we have the following system of equations

$$1 = \gamma + \delta \quad (2.6)$$

$$-5 = -\gamma + 3\delta \quad (2.7)$$

$$0 = -\gamma - 2\delta. \quad (2.8)$$

Solving equations, (2.6), (2.7) and (2.8), we have $\gamma = 2$ and $\delta = -1$. Therefore, from (2.2)

$$[A, B, C, D] = \frac{2}{1} \times \frac{2}{-1} = -4.$$

Theorem 2.3.3. The cross ratio, $[A, B, C, D]$ is well defined (that is, it is independent of the choices of homogeneous coordinates used to represent the collinear points A, B, C, D).

Proof. [8] Suppose $A = [a]$, $B = [b]$, $C = [c]$, $D = [d]$ are homogeneous coordinates of collinear points A, B, C, D . Let

$$c = \alpha a + \beta b$$

$$d = \gamma a + \delta b.$$

Now, suppose $A = [a']$, $B = [b']$, $C = [c']$, $D = [d']$. Then we have by the equivalence relation of homogeneous coordinates that $a = \lambda a'$, $b = \mu b'$, $c = \nu c'$, $d = \xi d'$, for some non-zero $\lambda, \mu, \nu, \xi \in \mathbb{R}$ as shown in the figure 2.2.

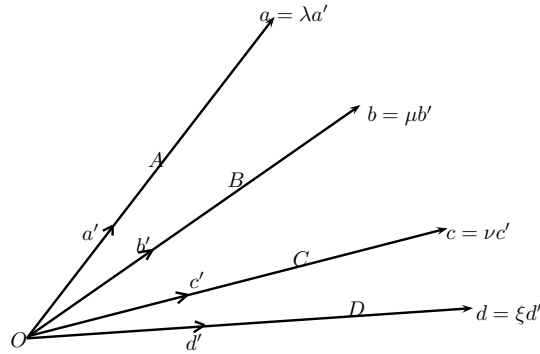


Figure 2.2: $(A, B : C, D)$ is well defined

Substituting the new values into c and d , we have

$$\nu c' = \alpha \lambda a' + \beta \mu b' \implies c' = \frac{\alpha \lambda}{\nu} a' + \frac{\beta \mu}{\nu} b'.$$

and

$$\xi d' = \gamma \lambda a' + \delta \mu b' \implies d' = \frac{\gamma \lambda}{\xi} a' + \frac{\delta \mu}{\xi} b'$$

respectively.

Let, $\frac{\alpha \lambda}{\nu} = \alpha'$, $\frac{\beta \mu}{\nu} = \beta'$, $\frac{\gamma \mu}{\xi} = \gamma'$ and $\frac{\delta \lambda}{\xi} = \delta'$. Then

$$c' = \alpha' a' + \beta' b' \quad \text{and} \quad d' = \gamma' a' + \delta' b'$$

and we have,

$$\begin{aligned} (A, B : C, D) &= \frac{\beta'}{\alpha'} \div \frac{\delta'}{\gamma'} \\ &= \left(\frac{\beta \mu}{\nu} \div \frac{\alpha \lambda}{\nu} \right) \div \left(\frac{\delta \mu}{\xi} \div \frac{\gamma \lambda}{\xi} \right) \\ &= \frac{\beta \mu}{\alpha \lambda} \times \frac{\gamma \lambda}{\delta \mu} \\ &= \frac{\beta \gamma}{\alpha \delta}. \end{aligned}$$

Since the cross ratio of A, B, C, D with coordinates a, b, c, d is the same when coordinates are changed to a', b', c', d' , we say that the cross-ratio is independent of the choice of homogeneous coordinates and hence it is well-defined. $\square$

Example 2.3.4. The cross-ratio for the coordinates $A = [1, 2, 3]$, $B = [2, 2, 4]$, $C = [-3, -5, -8]$, $D = [3, -3, 0]$ is $-\frac{1}{3}$. Now, If the values of the homogeneous coordinates are changed to $A = [1, 2, 3]$, $B = [1, 1, 2]$, $C = [3, 5, 8]$, $D = [1, -1, 0]$, the cross-ratio remains the same because the corresponding homogeneous coordinates are equivalent.

We now want to describe how the cross ratio is preserved by projective transformation.

Theorem 2.3.5. Let τ be a projective transformation having the coordinates A, B, C, D . If $A' = \tau(A)$, $B' = \tau(B)$ and $C' = \tau(C)$, then

$$[A, B, C, D] = [A', B', C', D'].$$

Proof. [8] Recall the definition of projective transformation. If $N \in GL(3, \mathbb{R})$ is an invertible matrix defined by the projective map, then there exist a projective transformation τ such that $[x] \mapsto [Nx]$. Then we have that $A' = [a'] = [Na]$, $B' = [b'] = [Nb]$, $C' = [c'] = [Nc]$ and $D' = [d'] = [Nd]$. Since A, B, C, D are collinear, then we have that

$$c = \alpha a + \beta b \quad \text{and} \quad d = \gamma a + \delta b$$

and $[A, B, C, D] = \frac{\beta \gamma}{\alpha \delta}$. Therefore,

$$Nc = \alpha Na + \beta Nb \quad \text{and} \quad Nd = \gamma Na + \delta Nb$$

implies

$$c' = \alpha a' + \beta b' \quad \text{and} \quad d' = \gamma a' + \delta b'.$$

Hence,

$$[A', B', C', D'] = \frac{\beta \gamma}{\alpha \delta}.$$

Since $[A, B, C, D] = [A', B', C', D']$, we conclude that the cross-ratio is projectively invariant. $\square$

Permutations of the Cross Ratio

For every number of points in a projective space, we can find the number of permutations of its cross ratio. For instance, if there are four points, A, B, C, D , then the number of permutations will be $4! = 24$.

Theorem 2.3.6. Let A, B, C, D be four distinct collinear points in the projective plane, and let $[A, B, C, D] = n$. Then

$$[B, A, C, D] = [A, B, D, C] = \frac{1}{n} \quad \text{and} \quad [A, C, B, D] = [D, B, C, A] = 1 - n.$$

Proof. Suppose $[a], [b], [c], [d]$ are homogeneous coordinates of four collinear points

A, B, C, D such that $A = [a]$, $B = [b]$, $C = [c]$, $D = [d]$. Let

$$c = \alpha a + \beta b \quad \text{and} \quad d = \gamma a + \delta b.$$

Then by the definition of cross-ratio, we let $[A, B, C, D] = \frac{\beta\gamma}{\alpha\delta} = n$. Next, we interchange the positions of A and B then we obtain

$$c = \beta b + \alpha a \quad \text{and} \quad d = \delta b + \gamma a.$$

Therefore,

$$[B, A, C, D] = \frac{\alpha}{\beta} \div \frac{\gamma}{\delta} = \frac{\alpha}{\beta} \times \frac{\delta}{\gamma} = \frac{1}{n}.$$

Again, we interchange the positions of C and D and obtain

$$d = \gamma a + \delta b \quad \text{and} \quad c = \alpha a + \beta b$$

Therefore,

$$[A, B, D, C] = \frac{\delta}{\gamma} \div \frac{\beta}{\alpha} = \frac{\delta}{\beta} \times \frac{\alpha}{\gamma} = \frac{1}{n}.$$

Hence, $[B, A, C, D] = [A, B, D, C] = \frac{1}{n}$.

Now, for $[A, C, B, D]$, we interchange the positions of C and B then we obtain

$$\beta b = -\alpha a + c \implies b = -\frac{\alpha}{\beta}a + \frac{1}{\beta}c$$

and

$$d = \gamma a + \delta b \implies d = \gamma a + \delta \left(-\frac{\alpha}{\beta}a + \frac{1}{\beta}c \right) = \left(\frac{\gamma\beta - \delta\alpha}{\beta} \right) a + \frac{\delta}{\beta}c.$$

Therefore,

$$\begin{aligned}
[A, C, B, D] &= \left(\frac{1}{\beta} \div -\frac{\alpha}{\beta} \right) \div \left(\frac{\delta}{\beta} \div \frac{\gamma\beta - \delta\alpha}{\beta} \right) \\
&= -\frac{1}{\alpha} \div \left(\frac{\delta}{\beta} \times \frac{\beta}{\gamma\beta - \delta\alpha} \right) \\
&= -\frac{(\gamma\beta - \delta\alpha)}{\alpha\delta} \\
&= 1 - \frac{\beta\gamma}{\alpha\delta} \\
&= 1 - n.
\end{aligned}$$

Following from $[A, C, B, D] = 1 - n$, we can have that

$$\begin{aligned}
[A, C, B, D] &= 1 - n \\
&= 1 - [A, B, C, D] \\
&= 1 - \frac{1}{[A, B, D, C]} \\
&= 1 - [B, A, D, C] \\
&= \frac{1}{[B, D, C, A]} \\
&= [D, B, C, A].
\end{aligned}$$

Hence, $[A, C, B, D] = [D, B, C, A] = 1 - n$. □

If the cross ratio of $[A, B, C, D]$ is n , then the following identities are true [6]:

- a. $[A, C, D, B] = (1 - n)^{-1}$.
- b. $[A, D, B, C] = 1 - n^{-1}$.
- c. $[A, D, C, B] = (1 - n^{-1})^{-1}$.

2.4 Twisted Polygon

Definition 2.4.1. A **twisted m -gon** is defined as a map

$$\rho : \mathbb{Z} \rightarrow \mathbb{P}^2(\mathbb{R})$$

and a projective transformation T such that

$$\rho(i + m) = T(\rho(i)), \forall m,$$

where $\mathbb{Z} = \{1, 2, \dots, 2m\}$. Here, $T \in \text{PGL}(3, \mathbb{R})$ is a projective transformation of the plane $\mathbb{P}^2(\mathbb{R})$. T is also called the monodromy of ρ . Suppose $f \in \text{PGL}(3, \mathbb{R})$ such that $f \circ \rho_1 = \rho_2$ and let ρ_1 and ρ_2 be two twisted m -gons then ρ_1 and ρ_2 are said to be equivalent. Note that we require none of these three consecutive points $\rho(i)$, $\rho(i + 1)$, $\rho(i + 2)$ to be collinear (they are in general position). Any two monodromies T_1 and T_2 will satisfy $T_2 = fT_1f^{-1}$. Every twisted polygon is uniquely determined by its m consecutive vertices and monodromy $T \in \text{PGL}(3, \mathbb{R})$. Furthermore, because the dimension of a polygon with bi-infinite vertices is $2m$ and that of the monodromy $T \in \text{PGL}(3, \mathbb{R})$ is 8. We obtain the dimension of the space of twisted m -gons as $2m + 8$.

2.5 Corner Invariant (The Invariant Function)

Let q_1, q_2, q_3, q_4 be four collinear points on a projective line $\mathbb{RP}^1$. We define the cross ratio for corner invariants as

$$[q_1, q_2, q_3, q_4] = \frac{(q_1 - q_3)(q_2 - q_4)}{(q_1 - q_2)(q_3 - q_4)}.$$

Let (z_j) be a twisted m -gon. Consider the point z_j (in red), a vertex of (z_j) and let $z_{j+1}, z_{j+2}, z_{j-1}, z_{j-2}, w_{j-1}, w_{j+1}, \bar{z}_j, \tilde{z}$, represent other points as shown in Figure 2.3.

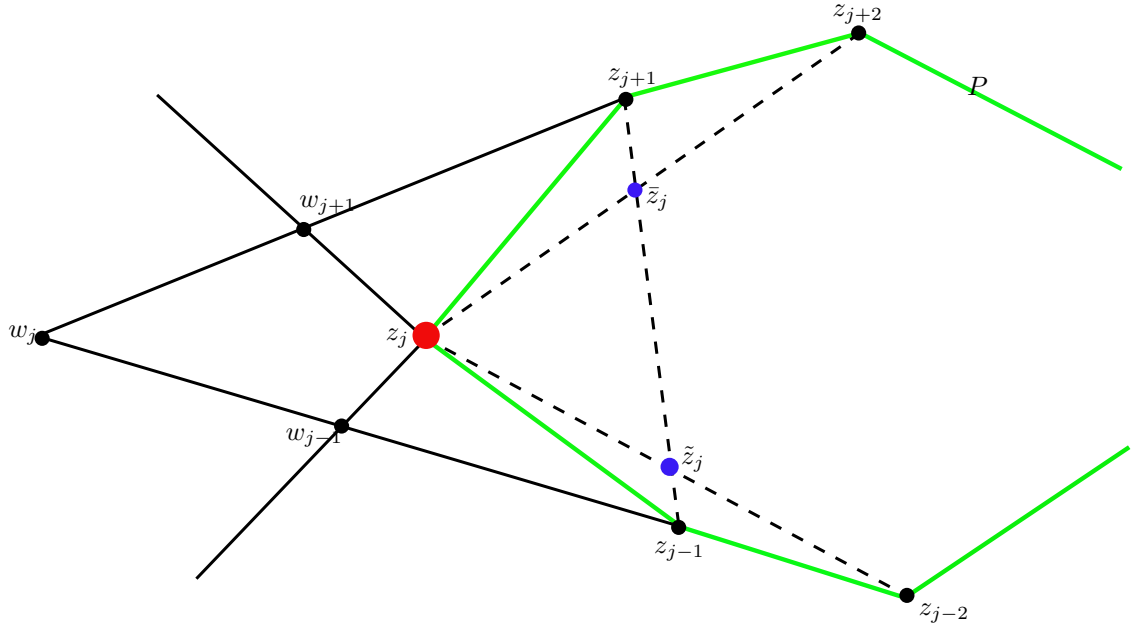


Figure 2.3: corner invariant; definition.

Definition 2.5.1. We define,

$$x_j := [w_j, w_{j+1} : z_{j+1}, z_{j+2}] \quad \text{and} \quad y_j := [w_j, w_{j-1} : z_{j-1}, z_{j-2}]$$

The coordinates x_j, y_j are called **Corner Invariants** of the point z_j .

Example 2.5.2. Let z_j represent the vertices of a polygon. Consider the vertex $j = 6$ as shown in Figure 2.4, then

$$x_6 := [w_6, w_7 : z_1, z_2], \quad y_6 := [w_6, w_5 : z_5, z_4].$$

The corner invariant at $j = 6$ is $([w_6, w_7 : z_1, z_2], [w_6, w_5 : z_5, z_4])$.

Example 2.5.3. This example describes the corner invariants at different vertices (z_2, z_3, z_4) on a given twisted m -gon as shown in Figure 2.5. The corresponding invariants are shown in the table below.

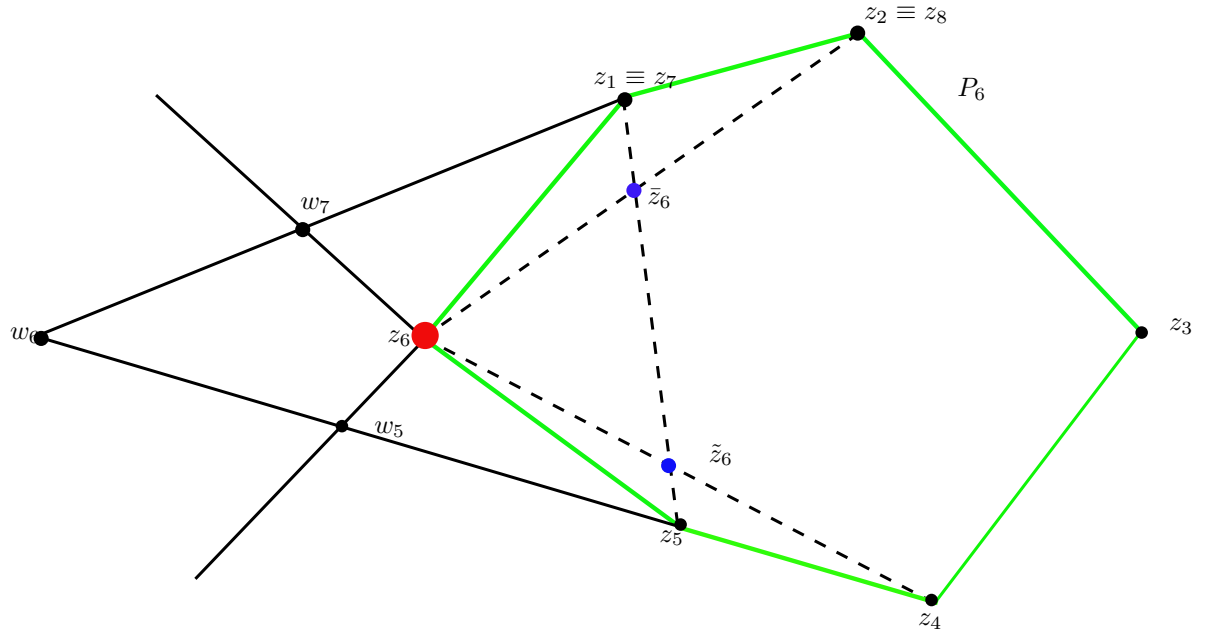


Figure 2.4: corner invariant; example.

z_j	x_j	y_j
z_2	$x_2 := [z_4, z_3 : C, D]$	$y_2 := [z_0, z_1 : E, D]$
z_3	$x_3 := [z_5, z_4 : A, B]$	$y_3 := [z_1, z_2 : C, B]$
z_4	$x_4 := [z_6, z_5 : G, F]$	$y_4 := [z_2, z_3 : A, F]$

2.6 Projective Duality

According to the definition of duality, given any point P in projective plane, there is a correspondence which carries collinear points to concurrent lines [17]. This correspondence is termed, *projective duality*. Let A, B be two distinct points in the real projective plane and C a circle. If two different tangent lines are drawn from each point to meet the circle, then a line is drawn to join the tangential points. Thus, for A , we have $A'A''$ and $B'B''$ for point B . Let $A'A'' \cap B'B'' = p$. It is observed that $A'A'' \rightarrow A$, $B'B'' \rightarrow B$ and $p \rightarrow l$ as shown in Figure 2.6. This gives a relation between points in $\mathbb{RP}^2$ and lines in $(\mathbb{RP}^2)^*$.

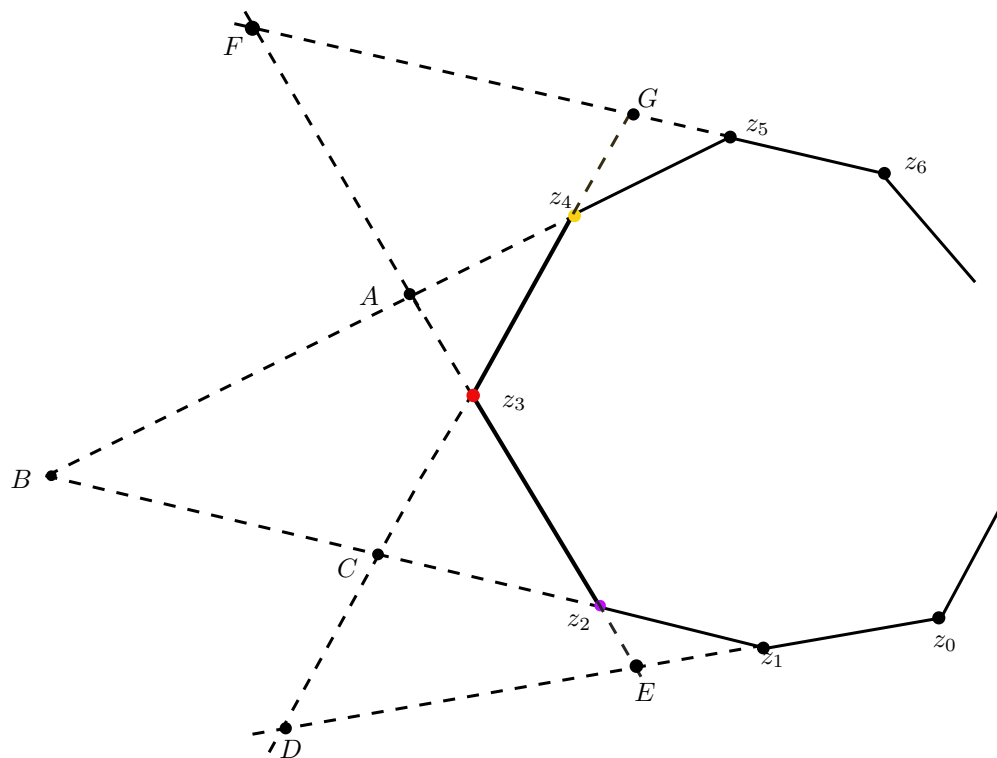


Figure 2.5: Construction of corner invariants at different vertices on a twisted m-gon.

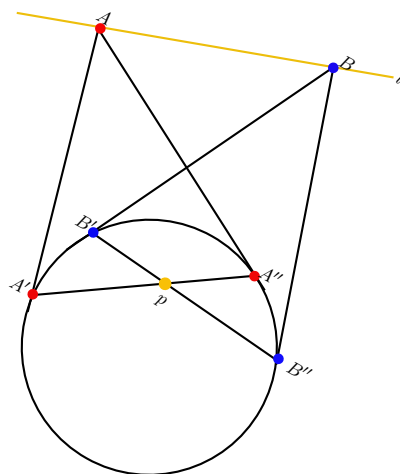


Figure 2.6: Projective duality

2.7 Dual Polygon

We describe the *dual polygon* $P' \in (\mathbb{RP}^2)^*$ as the cyclically ordered collection of n -tuple of points $(p_1, \dots, p_n)$ and lines $(l_1, \dots, l_n)$ which represent the edges and the vertices of the polygon respectively.

Each point l_j of the dual is obtained when any two lines in P meet whilst each line p_n is obtained from the points p_n of the original polygon P . An example is shown in Figure 2.7 .

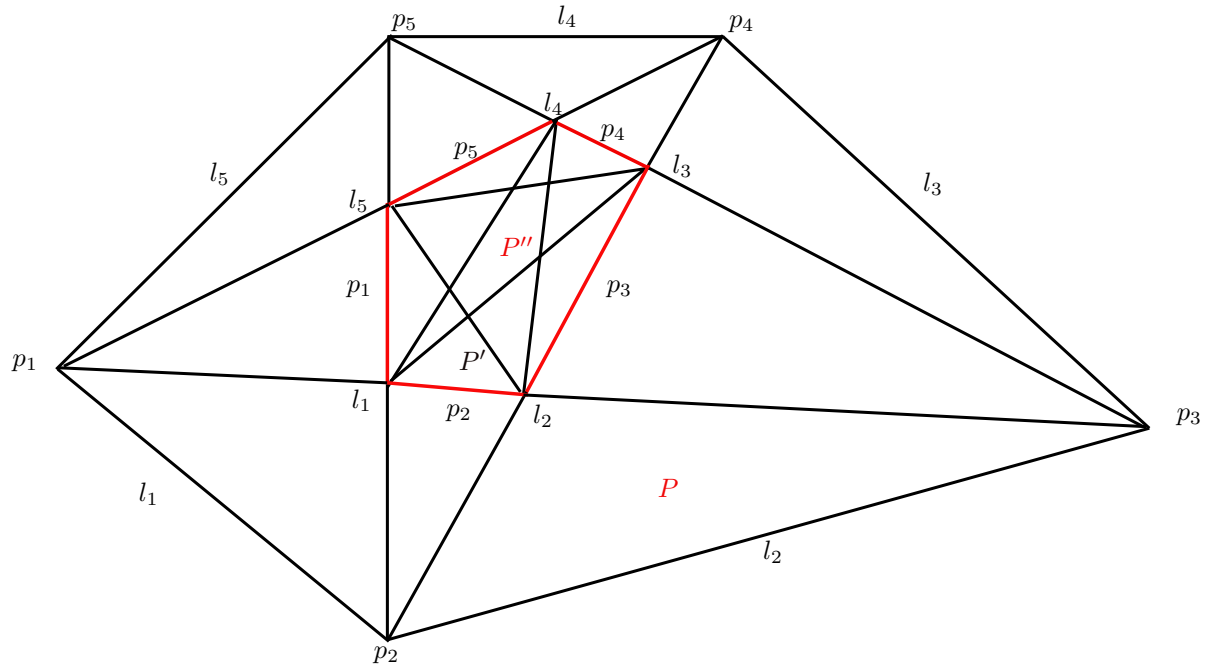


Figure 2.7: Duality of the pentagon

2.8 Self-dual Polygons

Let A' be the dual of an original polygon A . The polygon A is said to be self dual if A is isomorphic to A' , thus, $A \simeq A'$. Intuitively, whenever a polygon is equal to its dual, then it is called a **self dual polygon** [20].

2.8.1 Example

Consider a polygon of sides 6 (P_6) as shown in the Figure 2.8.

Since $P_6 \simeq P'_6$, we call P_6 a self dual polygon.

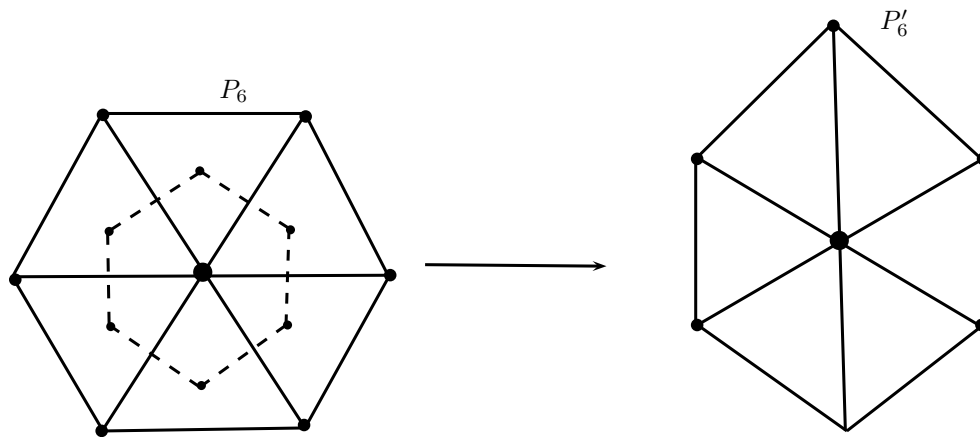


Figure 2.8: Self Dual polygon

2.9 Non-degenerate Conic

A conic is obtained when a plane intersects a cone at a given angle. There are two types of conics: degenerate and non-degenerate conics. When a plane intersects a cone in any angle apart from its vertex, we obtain a **non-degenerate conic**. Non-degenerate conics are the ellipse, parabola and hyperbola as shown in Figure 2.9

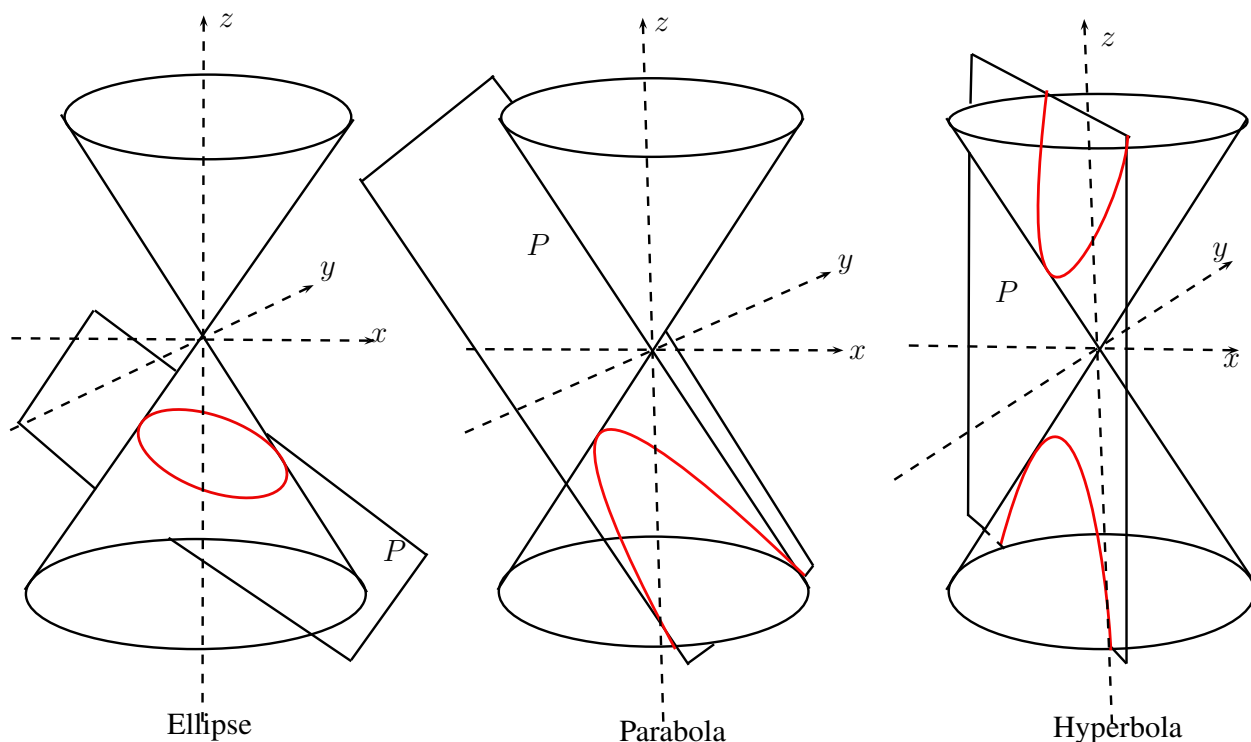


Figure 2.9: Non-degenerate conics shown in red

2.10 Inscribed Polygon

A polygon is said to be inscribed in a non-degenerate conic if all its vertices lie on the conic. Consider an example in Figure 2.10

2.11 Involution

Definition 2.11.1. An involution is a transformation or a function that is equal to its inverse. It is also described as a linear operator T such that $T^2 = I$.

Example 2.11.2. The identity map is an involution. Let H be a set, then the identity function f on H is defined as the function with domain and codomain H which satisfies $f(x) = x, \forall x \in H$. Then $f(f(x)) = x, \forall x \in H$ is an involution.

Example 2.11.3. The squared Pentagon map $T^2(P)$ modulo scaling is an involution

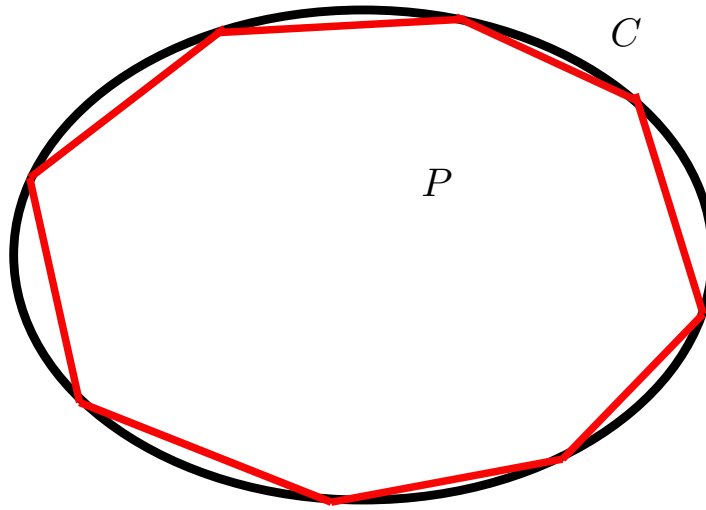


Figure 2.10: A polygon inscribed in a non-degenerate conic (ellipse)

since P and $T^2(P)$ are the same as shown in Figure 2.11

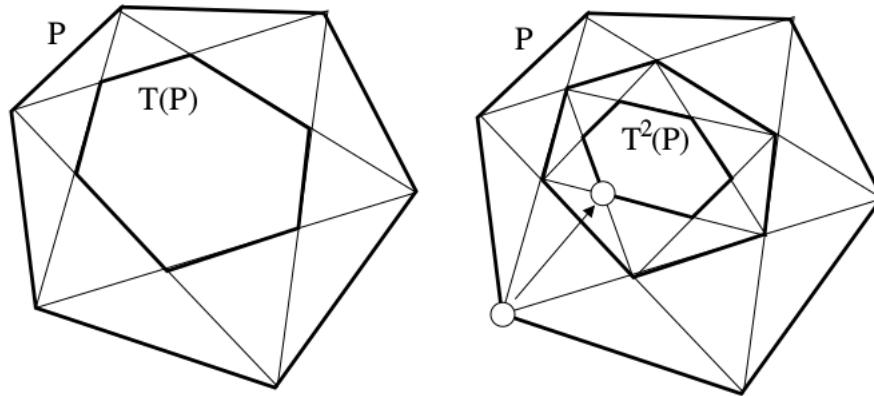


Figure 2.11: Involution of the Pentagram map

Another example of an involution is projective duality. Thus $(P^*)^* = P$ for $P \in \mathbb{RP}^2$ and $P^* \in (\mathbb{RP}^2)^*$ [2]. The composition of two involutions h and g , $(g \circ h)$ is an involution if and only if they commute. Thus $g \circ h = h \circ g$.

Chapter 3

Pentagram Map and Monodromy Invariants

In this chapter, we first describe the formula for the pentagram map using the corner invariants definition and later construct the monodromy invariants from this definition.

3.1 Formula for the Pentagram Map

The goal of this section is to describe the formula for the pentagram map in coordinates. Nevertheless, we shall first describe the pentagram map as the composition of two involutions which are all projective duals.

Recall the definition of twisted polygons. Let $\mathcal{V}_m$ be the space of twisted m -gons modulo projective transformations. The dimension of the space $\mathcal{V}_m$ is $2m$ since the dimension of the space of twisted m -gons is $2m + 8$ [5].

A twisted m -gon depends on $2m$ variables which represents coordinates of vertices $p_i := \rho(i)$, $\forall i = 1, \dots, m$ and consists of 8 parameters of the monodromy matrix T , while the $PSL(3, \mathbb{R})$ equivalence reduces the dimension by 8. The pentagram map T is generically defined on the space $\mathcal{V}_m$. Namely, for every twisted m -gon the vertices of its image are the intersections of pairs of consecutive shortest diagonals: $T(p_i) := (p_{i-1}, p_{i+1}) \cap (p_i, p_{i+2})$. Such intersections are well defined for each generic

point in $\mathcal{V}_m$.

There are two coordinates on the space $\mathcal{V}_m$ called the $(c_i), (d_i)$ coordinates and the corner invariants x_i, y_i . Because the corner invariants are already defined in Definition 2.5.1, we shall now consider the definition of (c_i, d_i) coordinates and state their relationship with the x_i, y_i coordinates. In what follows, we assume that any three points p_i, p_{i+1}, p_{i+2} are in general position (not collinear).

Definition 3.1.1. $((c_i), (d_i)$ Coordinates): Assume m is not divisible by 3 [10]. Then there should exist a unique lift of points $p_i = \rho(i) \in \mathbb{RP}^2$ to vectors $P_i \in \mathbb{R}^3$ such that the determinant of the lifts, $\det(P_i, P_{i+1}, P_{i+2}) = 1, \quad \forall i \in \mathbb{Z}$. We identify a difference equation to the sequence of vectors $P_i \in \mathbb{R}^3$ by setting

$$P_{i+3} = c_i P_{i+2} + d_i P_{i+1} + P_i \quad \forall i \in \mathbb{Z}. \quad (3.1)$$

Because $P_{i+m} = T(P_i) \quad \forall i \in \mathbb{Z}$, we say that the sequences $(c_i), (d_i)$ are m -periodic for any monodromy $T \in \text{SL}(3, \mathbb{R})$ and $c_i, d_i : 0 \leq i \leq m-1$. We shall prove this as a lemma in the next chapter.

The relationship between these two coordinate systems is defined [10] as;

$$x_i = \frac{c_{i-2}}{d_{i-2}d_{i-1}}, \quad y_i = -\frac{d_{i-1}}{c_{i-2}c_{i-1}}. \quad (3.2)$$

In [15], the pentagram map is described as the composition of two involutions, $\alpha(p_i)$ and $\beta(p_i)$ of which each of them is a kind of projective duality. The idea of projective duality in $\mathbb{RP}^2$ is as a result of the fact that the space $(\mathbb{RP}^2)^*$ consists of 1-dimensional subspaces of $\mathbb{RP}^2$.

Lemma 3.1.2. For any given sequence of points $p_i \in \mathbb{RP}^2$. We define $\alpha(p_i) \in (\mathbb{RP}^2)^*$ as the line (p_i, p_{i+1}) and $\beta(p_i) \in (\mathbb{RP}^2)^*$ as the line (p_{i-1}, p_{i+1}) as shown in Figure 3.1. Then $\alpha^2 = \tau$, $\beta^2 = Id$ and $\alpha \circ \beta = M$, where M is the pentagram map and τ is a cyclic permutation; $\tau^*(p_i) = p_{i+1}$.

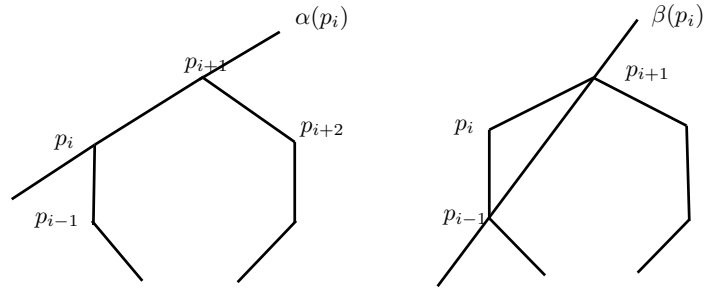


Figure 3.1: Projective duals

Proof. From the definitions of $\alpha(p_i)$ and $\beta(p_i)$ it implies that

$$\begin{aligned}\alpha^2(p_i) &= \alpha(p_i, p_{i+1}) \\ &= \alpha(p_i) \cap \alpha(p_{i+1}) \\ &= (p_i, p_{i+1}) \cap (p_{i+1}, p_{i+2})\end{aligned}$$

and

$$\begin{aligned}\beta^2(p_i) &= \beta(p_{i-1}, p_{i+1}) \\ &= \beta(p_{i-1}) \cap \beta(p_{i+1}) \\ &= (p_{i-2}, p_i) \cap (p_i, p_{i+2})\end{aligned}$$

and hence,

$$\begin{aligned}(\alpha \circ \beta)p_i &= \alpha(\beta(p_i)) \\ &= \alpha(p_{i-1}, p_{i+1}) \\ &= \alpha(p_{i-1}) \cap \alpha(p_{i+1}) \\ &= (p_{i-1}, p_{i+1}) \cap (p_{i+1}, p_{i+2}) \\ &= \beta(p_i) \cap \beta(p_{i+1}) \\ &= M\end{aligned}$$

Because $(\alpha \circ \beta)p_i = M$, we say that the pentagram map is the composition of the two involutions. From this definition, we state that the pentagram map M preserves the monodromy T . Geometry of the proof is shown in Figure 3.2

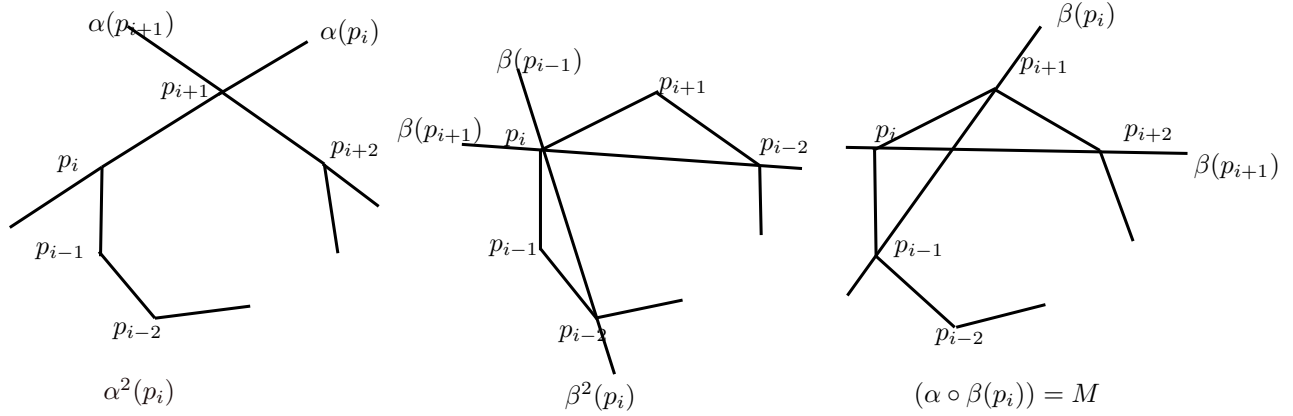


Figure 3.2: Duality maps

□

Definition 3.1.3. Given $\mathcal{V}_m$, we say that the map M that transforms one polygon to the other also descends to a densely defined map

$$M : \mathcal{V}_m \rightarrow \mathcal{V}_m,$$

called the *Pentagram map*

.

To define the Pentagram map in its coordinates, we use the right labelling approach for our convenience, though, we could seemingly use any of the other two labelling schemes in Figure 3.3.

In terms of the corner invariants, we define the map M by

$$x'_i = x_i \frac{1 - x_{i-1}y_{i-1}}{1 - x_{i+1}y_{i+1}} \quad \text{and} \quad y'_i = y_{i+1} \frac{1 - x_{i+2}y_{i+2}}{1 - x_i y_i}. \quad (3.3)$$

The proof of this formula combines results in Lemma 3.1.2 and Equation 3.2 [10].

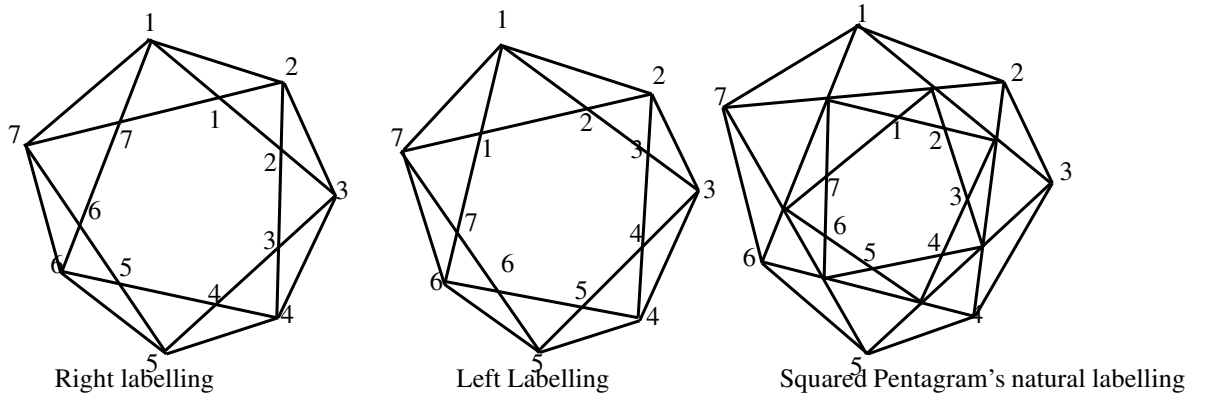


Figure 3.3: Labelling schemes of the Pentagram Map

3.2 Construction of Monodromy Invariants

From **Section 2.5**, we define

$$X_m = \prod_{j=1}^m x_j \quad \text{and} \quad Y_m = \prod_{j=1}^m y_j.$$

Following from Equation (3.3), the functions X_m, Y_m are invariant under the pentagram map.

For any arbitrary equivalence class $V \in \mathcal{V}_m$ of twisted m -gons, the analogous monodromy T defined up to scalar multiplication and conjugation is a square matrix of dimension 3.

We define

$$\varphi_1 = \frac{\text{trace}^3(T^{-1})}{\det(T^{-1})} \quad \text{and} \quad \varphi_2 = \frac{\text{trace}^3(T)}{\det(T)} \quad (3.4)$$

as well-defined invariants under the Pentagram map since the map preserves T . [15] By definition,

$$\varphi_1 = \frac{1}{X_m^2 Y_m} \left(\sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} X_k \right)^3 \quad \text{and} \quad \varphi_2 = \frac{1}{Y_m^2 X_m} \left(\sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} Y_k \right)^3, \quad (3.5)$$

where,

$$\text{trace}(T^{-1}) = \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} X_k, \quad \det(T^{-1}) = X_m^2 Y_m \quad \text{and} \quad \text{trace}(T) = \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} Y_k, \quad \det(T) = Y_m^2 X_m$$

Because the pentagram map M on every m -gon preserves the monodromy T , it implies that the map $M : \mathcal{V}_m \rightarrow \mathcal{V}_m$, defined on equivalence classes of polygons preserves φ_1 and φ_2 [5]. Hence we also define

$$\tilde{\varphi}_1 := X_m^2 Y_m \varphi_1 \quad \text{and} \quad \tilde{\varphi}_2 := Y_m^2 X_m \varphi_2$$

as invariant polynomials in the corner invariants x_i, y_i (Proof is shown in [15]). From Equation 3.4, we will have that

$$\tilde{\varphi}_1 = \left(\sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} X_k \right)^3 \quad \text{and} \quad \tilde{\varphi}_2 = \left(\sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} Y_k \right)^3.$$

X_0 and Y_0 are just constant functions $X_0 = 0 = Y_0$ so we obtain

$$\tilde{\varphi}_1 = \left(1 + \sum_{k=1}^{\lfloor \frac{m}{2} \rfloor} X_k \right)^3 \quad \text{and} \quad \tilde{\varphi}_2 = \left(1 + \sum_{k=1}^{\lfloor \frac{m}{2} \rfloor} Y_k \right)^3$$

The remaining monodromy invariants are described as the weighted homogeneous parts of these polynomials.

The Pentagon map commutes with a rescaling operation

$$R_u(x_1, y_1, x_2, y_2, \dots, x_m, y_m) \mapsto (ux_1, u^{-1}y_1, ux_2, u^{-1}y_2, \dots, ux_m, u^{-1}y_m). \quad (3.6)$$

Note that a polynomial in the corner invariants has weight k if $R_u^*(V) = u^k V$, where R_u^* is the natural action of R_u on the polynomials. For instance, the weight of X_m is m and that of Y_m is $-m$.

By the definition of the rescaling operator, we have

$$R_u^*(\tilde{\varphi}_1) = \left(\sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} X_k u^k \right)^3 \quad \text{and} \quad R_u^*(\tilde{\varphi}_2) = \left(\sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} Y_k u^{-k} \right)^3.$$

Definition 3.2.1. The functions X_k and Y_k which represent the weight k polynomials in each sum where $k = \{1, \dots, \lfloor \frac{1}{2} \rfloor, m\}$ are called the **monodromy invariants**

Chapter 4

Monodromy Invariants of the Spectral Curve and Corner Invariants in the Monodromy Matrix

This chapter describes two major concepts in two sections.

1. In the first section, we describe the relationship between monodromy invariants and the spectral curve.
2. In the second section, we discuss the relationship between the monodromy matrix and the corner invariants. We shall do this by defining the monodromy in terms of corner invariants.

4.1 Monodromy Invariants and the Spectral Curve

This section basically discuss the occurrence of monodromy invariants as coefficients of the spectral curve.

Suppose $V \in \mathcal{V}_m$ is an equivalent class of twisted m -gons. Then the map $R_u : \mathcal{V}_m \rightarrow \mathcal{V}_m$ (Rescaling operation) defines every family $V_u \in \mathcal{V}_m$, where $V_u = R_u(V)$ parametrised by $u \in \mathbb{C}^*$.

Note that each equivalence class has a corresponding conjugacy class of monodromies.

Thus if $V_2 = \lambda V_1, V_3 = \lambda V_2, \dots$ such that the set $V = \{V_1, V_2, V_3, \dots\}, \forall V \in \mathcal{V}_m$, then we could have $T = \{T_1, T_2, T_3, \dots\}$ as the set of each corresponding equivalence class of $\mathcal{V}_m$.

We take any representative of $\{V_1, V_2, V_3, \dots\} \in V$ and lift it to GL_3 . Hence we obtain a set $T(u)$ of 3×3 matrices that is parametrized by $u \in \mathbb{C}^*$. $T(u)$ is called the *scaled monodromy* of the equivalence class V .

We define the spectral function of scaled monodromy matrix $T(u)$ is given as

$$R(u, v) := \frac{\det(\lambda(u)vI - T(u))}{\lambda(u)^3}$$

where $\lambda(u) := (u^m \det T(u)^{\frac{1}{3}})$, I is the identity matrix and $R(u, v)$ is the characteristic polynomial of $T(u)$. The polynomial $R(u, v)$ of $T(u)$ is normalised in a way such that it resists changes if we multiply $T(u)$ by a scalar function of u . Because $T(u)$ is a square matrix, we will have the characteristic polynomial in the form

$$\Delta(\lambda) = \lambda^3 - \text{tr}(T(u))\lambda^2 + Pm(T(u))\lambda - \det(T(u)).$$

Explicitly, we have

$$R(u, v) = v^3 - Y(u^{-1})v^2 + X(u)u^{-m}v - u^{-m}, \quad (4.1)$$

where

$$\text{tr}(T(u)) = Y(u^{-1}), \quad Pm(T(u)) = X(u)u^{-m} \quad \det(T(u)) = u^{-m}$$

and

$$Y(u) := Y_m^{-\frac{2}{3}} X_m^{-\frac{1}{3}} \left(1 + \sum_{k=1}^{\lfloor \frac{m}{2} \rfloor} Y_k u^k \right), \quad X(u) := X_m^{-\frac{2}{3}} Y_m^{-\frac{1}{3}} \left(1 + \sum_{k=1}^{\lfloor \frac{m}{2} \rfloor} X_k u^k \right).$$

Definition 4.1.1. (Spectral Curve) We define the spectral curve as the zero locus of the characteristic equation. Thus, the value of λ such that $\Delta(\lambda) = 0$.

For Example

Let $T(u)$ be a scaled monodromy matrix of the equivalence class of twisted m -gons defined as

$$T(u) = \begin{pmatrix} a(u) & b(u) & c(u) \\ d(u) & e(u) & f(u) \\ g(u) & h(u) & i(u) \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} (u) \quad (4.2)$$

Then we will have that

$$\lambda(u)vI - T(u) = \begin{pmatrix} \lambda(u)v & 0 & 0 \\ 0 & \lambda(u)v & 0 \\ 0 & 0 & \lambda(u)v \end{pmatrix} - \begin{pmatrix} a(u) & b(u) & c(u) \\ d(u) & e(u) & f(u) \\ g(u) & h(u) & i(u) \end{pmatrix} \quad (4.3)$$

Let $\lambda(u) = \lambda$, $a(u) = a$ and all other entries in $T(u)$, then we have that

$$\lambda(u)vI - T(u) = \begin{pmatrix} \lambda v - a & -b & -c \\ -d & \lambda v - e & -f \\ -g & -h & \lambda v - i \end{pmatrix} \quad (4.4)$$

$$\begin{aligned} \det(\lambda(u)vI - T(u)) &= \lambda^3 v^3 + v^2(-\lambda^2 i - e\lambda^2 - a\lambda^2) + v(\lambda ei + a\lambda i + ae\lambda - \lambda hf - bd\lambda - c\lambda g) \\ &\quad + (-aei + ahf + bdi - bgf - cdh + ceg) \end{aligned}$$

Dividing through by $\lambda(u)^3$, we obtain

$$\begin{aligned} \frac{1}{\lambda(u)^3} \det(\lambda(u)vI - T(u)) &= v^3 - \lambda^{-1}v^2(i + e + a) + \lambda^{-2}v(ei + ai + ae - hf - bd - cg) \\ &\quad + \lambda^{-3}(-aei + ahf + bdi - bgf - cdh + ceg) \end{aligned}$$

where $\lambda(u) = (v^m \det T(u))^{\frac{1}{3}}$.

$$\begin{aligned}
 \frac{1}{\lambda(u)^3} \det(\lambda(u)vI - T(u)) &= v^3 - \lambda^{-1}v^2(i + e + a) + \lambda^{-2}v(ei + ai + ae - hf - bd - cg) \\
 &\quad + \lambda^{-3}(-aei + ahf + bdi - bgf - cdh + ceg) \\
 &= v^3 - [\lambda^{-1}(i + e + a)]v^2 \\
 &\quad + [\lambda^{-2}(ei + ai + ae - hf - bd - cg)]v \\
 &\quad + \lambda^{-3}(-aei + ahf + bdi - bgf - cdh + ceg) \\
 &= v^3 - [\lambda^{-1}(u)(i + e + a)(u)]v^2 \\
 &\quad + [\lambda^{-2}(u)(ei + ai + ae - hf - bd - cg)(u)]v \\
 &\quad + \lambda^{-3}(u)(-aei + ahf + bdi - bgf - cdh + ceg)(u) \\
 &= v^3 - [\lambda^{-1}(u)\text{trace}(T(u))]v^2 \\
 &\quad + [\lambda^{-2}(u)Pm(T(u))]v + \lambda^{-3}(u)\det(T(u)) \\
 &= v^3 - \frac{\text{trace}(T(u))}{\lambda(u)}v^2 + \frac{Pm(T(u))}{\lambda^2}v - \frac{\det(T(u))}{\lambda^3(u)}
 \end{aligned}$$

However, since $\lambda(u) = (v^m \det T(u))^{\frac{1}{3}}$, we will have that

$$R(u, v) = v^3 - \frac{\text{trace}(T(u))}{(v^m \det T(u))^{\frac{1}{3}}}v^2 + \frac{Pm(T(u))}{(v^m \det T(u))^{\frac{2}{3}}}v - \frac{\det(T(u))}{(v^m \det T(u))} \quad (4.5)$$

Recall the definition

$$\text{trace}T(u) = \sum_{k=0}^{\frac{m}{2}} Y_m \quad \text{and} \quad \det T(u) = Y_m^2 X_m,$$

$$\begin{aligned}
 R(u, v) &= v^3 - \frac{\sum_{k=0}^{\frac{m}{2}} Y_m}{(u^m Y_m^2 X_m)^{\frac{1}{3}}}v^2 + \frac{Pm(T(u))}{(u^m Y_m^2 X_m)^{\frac{2}{3}}}v - u^{-m} \\
 &= v^3 - (u^{-m})^{\frac{1}{3}} Y_m^{-\frac{2}{3}} X_m^{-\frac{1}{3}} \sum_{k=0}^{\frac{m}{2}} Y_m v^2 + (u^{-m})^{-\frac{2}{3}} Y_m^{\frac{4}{3}} X_m^{-\frac{2}{3}} Pm(T(u))v - u^{-m}
 \end{aligned}$$

Hence,

$$R(u, v) = v^3 - Y(u^{-1})v^2 + X(u)u^{-m} - u^{-m}, \quad (4.6)$$

where,

$$Y(u^{-1}) = (u^{-m})^{\frac{1}{3}} Y_m^{-\frac{2}{3}} X_m^{-\frac{1}{3}} \sum_{k=0}^{\frac{m}{2}} Y_m \quad \text{and} \quad X(u) = (u^{-m})^{-\frac{2}{3}} Y_m^{\frac{4}{3}} X_m^{-\frac{2}{3}} Pm(T(u))$$

Therefore, the spectral curve is the value of v such that

$$v^3 - Y(u^{-1})v^2 + X(u)u^{-m}v - u^{-m} = 0.$$

Corollary 4.1.2. In [5], we have that the spectral curve being invariant under the involution

$$(u, v) \leftrightarrow (u^{-1}, v^{-1})$$

and the monodromy invariants satisfying $Y_k = X_k, \quad \forall k = 1, \dots, [\frac{1}{2}], m$ are equivalent.

Proof. From Equation 4.1, we have $R(u, v) = 0$. Thus,

$$= v^3 - Y(u^{-1})v^2 + X(u)u^{-m}v - u^{-m} = 0. \quad (4.7)$$

Next, we find $-\frac{u^m}{v^3}R(u, v)$ (involution of $R(u, v)$) and obtain

$$u^m - u^m Y(u^{-1})v^{-1} + X(u)v^{-2} - v^{-3} = 0. \quad (4.8)$$

Hence,

$$v^{-3} - X(u)v^{-2} + u^m Y(u^{-1})v^{-1} - u^m = 0. \quad (4.9)$$

Additionally,

$$R(u^{-1}, v^{-1}) = v^{-3} - Y(u)v^{-2} + u^m X(u^{-1})v^{-1} - u^m = 0. \quad (4.10)$$

Hence the spectral curve is invariant under the involution

$$(u, v) \leftrightarrow (u^{-1}, v^{-1}).$$

Next, comparing coefficients in Equations 4.9 and 4.10, we have that $X(u) = Y(u)$ and $X(u^{-1}) = Y(u^{-1})$. Because $X(u) = Y(u)$ we will have that

$$X_m^{-\frac{2}{3}} Y_m^{-\frac{1}{3}} \left(1 + \sum_{k=1}^{\lfloor \frac{m}{2} \rfloor} X_k u^k \right) = Y_m^{-\frac{2}{3}} X_m^{-\frac{1}{3}} \left(1 + \sum_{k=1}^{\lfloor \frac{m}{2} \rfloor} Y_k u^k \right),$$

which implies $X_k = Y_k$. Hence the monodromy invariants satisfy $Y_k = X_k, \quad \forall k = 1, \dots, \lfloor \frac{1}{2} \rfloor, m$.

□

The Newton polygon is used to demonstrate the involution on the spectral curve as defined in chapter 2.

For instance the Newton polygon of the pentagram spectral curve is obtained as a parallelogram when it has 7 sides [5].

4.2 Monodromy Matrix and the Corner Invariant

Here, we describe the monodromy in terms of the corner invariants.

We shall use the field $\mathbb{C}$ instead of $\mathbb{R}$ though this does not cause any difference in our constructions.

We recall the coordinate system of V_m in Chapter 3 as introduced in [10] called the (c_i, d_i) coordinates where $c_i, d_i \in \mathbb{C}, \forall i \in \mathbb{Z}$ such that $c_{i+m} = c_i, d_{i+m} = d_i$ together with the corner invariant coordinates (x_i, y_i) .

In addition, we also recall that these sequence of coordinates are associated by a difference equation in Equation (3.1).

The solution to Equation (3.1) are described as the vectors $P = P_i, \quad \forall P_i \in \mathbb{C}$ that satisfies Equation(3.1). Because m is periodic, it implies that there exists $V \in$

$SL(3, \mathbb{C})$ such that

$$P_{i+m} = V_i \quad (4.11)$$

The lemma below and corollary are proved in order to help us view the relationship between the monodromy invariant and the corner invariant.

Lemma 4.2.1. Given any three non-collinear points p_i, p_{i+1}, p_{i+2} of a twisted polygon $(p_i) \in SL(3, \mathbb{C})$, there exists an associated difference equation of the form:

$$P_{i+3} = c_i P_{i+2} + d_i P_{i+1} + P_i. \quad (4.12)$$

Proof. Suppose $R_i = (P_i, P_{i+1}, P_{i+2})$ and $Q_i = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & d_i \\ 0 & 1 & c_i \end{pmatrix}$ for some sequences $c_i, d_i \in \mathbb{C}$, where $\det (P_i, P_{i+1}, P_{i+2}) = 1$ such that

$$R_{i+1} = R_i Q_i. \quad (4.13)$$

Then we will have that

$$\begin{aligned} (P_{i+1}, P_{i+2}, P_{i+3}) &= (P_i, P_{i+1}, P_{i+2}) \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & d_i \\ 0 & 1 & c_i \end{pmatrix} \\ &= (P_{i+1}, P_{i+2}, P_i + d_i P_{i+1} + c_i P_{i+2}) \end{aligned} \quad (4.14)$$

Comparing the right and left hand side of Equation (4.14), we have that $P_{i+1} = P_{i+1}$, $P_{i+2} = P_{i+2}$ and hence $P_{i+3} = c_i P_{i+2} + d_i P_{i+1} + P_i$. $\square$

In general, let's compute R_{i+m} . If $P_{i+3} = c_i P_{i+2} + d_i P_{i+1} + P_i$, then we can obtain R_{i+n} as follows:

$$R_i = (P_i, P_{i+1}, P_{i+2})$$

$$\begin{aligned}
R_{i+1} &= (P_{i+1}, P_{i+2}, P_{i+3}) \\
R_{i+2} &= (P_{i+2}, P_{i+3}, P_{i+4}) \\
&\vdots \\
R_{i+m} &= (P_{i+m}, P_{i+m+1}, P_{i+m+2}).
\end{aligned} \tag{4.15}$$

Similarly from Equation (4.13) we have that:

$$\begin{aligned}
R_{i+2} &= R_i(Q_i Q_{i+1}) \\
R_{i+3} &= R_i(Q_i Q_{i+1} Q_{i+2}) \\
&\vdots \\
R_{i+m} &= R_i(Q_i Q_{i+1} \cdots Q_{i+m-1})
\end{aligned} \tag{4.16}$$

Lemma 4.2.2. [5] Suppose $(p_i) \in V_m$ for $V_m \in \mathbb{P}^2$ having the monodromy $T \in \mathbb{P}GL_3$ and corner invariants (x_i, y_i) . Then T is conjugate to any of the matrices $T_i := L_i L_{i+1} \cdots L_{i+m-1}$, for any arbitrary $i \in \mathbb{Z}$ and

$$L_j = \begin{pmatrix} 0 & 0 & 1 \\ -x_j y_j & 0 & 1 \\ 0 & -y_j & 1 \end{pmatrix}. \tag{4.17}$$

Proof. We consider the following steps in our proof.

1. Suppose p_i, p_{i+1}, p_{i+2} are non-collinear points in the real projective plane. Taking the lifts P_i, P_{i+1}, P_{i+2} , we write the monodromy of the lifts $\acute{T}_i$ as a conjugate of the product of the matrices Q_i and D_i .
2. Use the corner invariants of the lift to describe the relationship between L_j and Q_i .
3. Write T_i in terms of $\acute{T}_i$, D_i and λ_i .

4. We find λ_{i+m-2} by describing quasi periodicity conditions for λ .
5. We shall use the above result to prove the lemma.

Assume p_i, p_{i+1}, p_{i+2} are in general position. Then we lift the points p_i to vectors $P_i \in \mathbb{C}^3$ in the special linear group (thus, $\det (P_i, P_{i+1}, P_{i+2}) = 1 \quad \forall i \in \mathbb{Z}$). From Lemma (4.2.1) we have that

$$P_{i+3} = c_i P_{i+2} + d_i P_{i+1} + P_i,$$

for some sequences $c_i, d_i \in \mathbb{C}$.

Re-writing in matrix form, we obtain Equation (4.13) and in general we have Equation (4.16).

We consider $\acute{T}$, an arbitrary lift of the monodromy T . Because every twisted polygon (p_i) is of the form $p_{i+1} = T(p_i)$, we can state that

$$\acute{T}P_i = t_i P_{i+n} \quad \text{for some sequence } t_i \in \mathbb{C}. \quad (4.18)$$

For $P_i \in R_i$ and $t_i \in D_i$, we have that

$$\acute{T}R_i = R_{i+m}D_i \quad \text{where } D_i := \text{diag}(t_i, t_{i+1}, t_{i+2}). \quad (4.19)$$

Substituting Equation (4.16) into Equation (4.19), we have that

$$\acute{T}R_i = R_i(Q_i Q_{i+1} \cdots Q_{i+m-1})D_i$$

which implies

$$\acute{T} := (Q_i Q_{i+1} \cdots Q_{i+m-1})D_i$$

Therefore by the comparison of Equations (4.16) and (4.19), we say that $\acute{T}_i$ is conjugate to any of the matrices

$$\hat{T}_i := (Q_i Q_{i+1} \cdots Q_{i+m-1}) D_i \quad \forall i \in \mathbb{Z}. \quad (4.20)$$

Using the definition of corner invariant of the lift P_i of a twisted polygon satisfying Equation (4.12)

$$x_i = \frac{c_{i-2}}{d_{i-2}d_{i-1}}, \quad y_i = -\frac{d_{i-2}}{c_{i-2}c_{i-1}}, \quad (4.21)$$

it is observed that L_i and Q_i are related by means of the *gauge transformation*

$$L_i = \frac{1}{c_{i-1}} \lambda_{i-2}^{-1} Q_{i-2} \lambda_{i-1}, \quad \lambda_i := \text{diag}(1, d_i, c_i). \quad (4.22)$$

Verification

$$Q_{j-2} \lambda_{j-1} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & d_i \\ 0 & 1 & c_i \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & d_{i-1} & 0 \\ 0 & 0 & c_{i-1} \end{pmatrix} = \begin{pmatrix} 0 & 0 & c_{i-1} \\ 1 & 0 & d_i c_{i-1} \\ 0 & d_{i-1} & c_i c_{i-1} \end{pmatrix}.$$

$$\lambda_{i-2}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & d_{i-2} & 0 \\ 0 & 0 & c_{i-2} \end{pmatrix}^{-1} = \frac{1}{d_{i-2}c_{i-2}} \begin{pmatrix} d_{i-2}c_{i-2} & 0 & 0 \\ 0 & c_{i-2} & 0 \\ 0 & 0 & d_{i-2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{d_{i-2}} & 0 \\ 0 & 0 & \frac{1}{c_{i-2}} \end{pmatrix}$$

$$\lambda_{i-2}^{-1} Q_{i-2} \lambda_{i-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{d_{i-2}} & 0 \\ 0 & 0 & \frac{1}{c_{i-2}} \end{pmatrix} \begin{pmatrix} 0 & 0 & c_{i-1} \\ 1 & 0 & d_i c_{i-1} \\ 0 & d_{i-1} & c_i c_{i-1} \end{pmatrix} = \begin{pmatrix} 0 & 0 & c_{i-1} \\ \frac{1}{d_{i-2}} & 0 & \frac{d_i c_{i-1}}{d_{i-2}} \\ 0 & \frac{d_{i-1}}{c_{i-2}} & \frac{c_i c_{i-1}}{c_{i-2}} \end{pmatrix}$$

$$\frac{1}{c_{i-1}} \lambda_{i-2}^{-1} Q_{i-2} \lambda_{i-1} = \frac{1}{c_{i-1}} \begin{pmatrix} 0 & 0 & c_{i-1} \\ \frac{1}{d_{i-2}} & 0 & \frac{d_i c_{i-1}}{d_{i-2}} \\ 0 & \frac{d_{i-1}}{c_{i-2}} & \frac{c_i c_{i-1}}{c_{i-2}} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{c_{i-1}d_{i-1}} & 0 & \frac{d_i}{d_{i-2}} \\ 0 & \frac{d_{i-1}}{c_{i-1}c_{i-2}} & \frac{d_i}{d_{i-2}} \end{pmatrix} \quad (4.23)$$

We observe that Equation (4.23) is equal to L_i since

$$-x_i y_i = \frac{1}{c_{i-1} d_{i-1}} \quad \text{and} \quad -y_i = \frac{d_{i-1}}{c_{i-1} c_{i-2}}.$$

This transformation is called the *gauge transformation*.

From Equation (4.20), we have that

$$(Q_i Q_{i+1} \cdots Q_{i+m-1}) = \acute{T}_i D_i^{-1}. \quad (4.24)$$

Substituting Equation 4.24 into Equation 4.22, we obtain the product of our matrices $T_i := L_i L_{i+1} \cdots L_{i+m-1}$ as

$$T_i = a_i \lambda_{i-2}^{-1} \acute{T}_{i-2} D_{i-2}^{-1} \lambda_{i+m-2} \quad (4.25)$$

In what is left, we find λ_{i+m-2} from Equation (4.25).

From the left, we multiply $\acute{T}$ by Equation (4.13) to obtain

$$\acute{T} R_{i+1} = \acute{T} R_i Q_i. \quad (4.26)$$

Substituting Equation (4.19) into Equation (4.26) we have that

$$R_{i+m+1} D_{i+1} = R_{i+m} D_i Q_i \quad (4.27)$$

We now compare Equation (4.27) with Equation (4.13).

From Equation (4.13), if $R_{i+1} = R_i Q_i$, then $R_{i+m+1} = R_{i+m} Q_{i+m}$ (same as $R_{i+n} R = R_{i+m} Q_{i+m}$) and hence

$$R = Q_{i+m} \quad (4.28)$$

Substituting this into Equation (4.27)(which can be re-written as $R_{i+m} R D_{i+1} =$

$R_{i+m}D_iQ_i$), we have

$$\begin{aligned} R_{i+m}Q_{i+m}D_{i+1} &= R_{i+m}D_iQ_i \\ Q_{i+m}D_{i+1} &= D_iQ_i, \end{aligned}$$

and hence

$$Q_{i+m} = D_iQ_iD_{i+1}^{-1}. \quad (4.29)$$

Because $D_i = \text{diag}(t_i, t_{i+1}, t_{i+2})$, then $D_{i+1} = \text{diag}(t_{i+1}, t_{i+2}, t_{i+3})$ and therefore

$$\begin{aligned} D_{i+1}^{-1} &= \begin{pmatrix} t_{i+1} & 0 & 0 \\ 0 & t_{i+2} & 0 \\ 0 & 0 & t_{i+3} \end{pmatrix}^{-1} \\ &= \frac{1}{t_{i+1}t_{i+2}t_{i+3}} \begin{pmatrix} t_{i+2}t_{i+3} & 0 & 0 \\ 0 & t_{i+1}t_{i+3} & 0 \\ 0 & 0 & t_{i+1}t_{i+2} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{t_{i+1}} & 0 & 0 \\ 0 & \frac{1}{t_{i+2}} & 0 \\ 0 & 0 & \frac{1}{t_{i+3}} \end{pmatrix} \end{aligned}$$

where $\det(D_{i+1}) = t_{i+1}t_{i+2}t_{i+3}$.

Next we have that

$$Q_iD_{i+1}^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & b_i \\ 0 & 1 & a_i \end{pmatrix} \begin{pmatrix} \frac{1}{t_{i+1}} & 0 & 0 \\ 0 & \frac{1}{t_{i+2}} & 0 \\ 0 & 0 & \frac{1}{t_{i+3}} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{1}{t_{i+3}} \\ \frac{1}{t_{i+1}} & 0 & \frac{d_i}{t_{i+3}} \\ 0 & \frac{1}{t_{i+2}} & \frac{c_i}{t_{i+3}} \end{pmatrix},$$

and

$$\begin{aligned}
 D_i Q_i D_{i+1}^{-1} &= \begin{pmatrix} t_{i+1} & 0 & 0 \\ 0 & t_{i+2} & 0 \\ 0 & 0 & t_{i+3} \end{pmatrix} \begin{pmatrix} 0 & 0 & \frac{1}{t_{i+3}} \\ \frac{1}{t_{i+1}} & 0 & \frac{d_i}{t_{i+3}} \\ 0 & \frac{1}{t_{i+2}} & \frac{c_i}{t_{i+3}} \end{pmatrix} \\
 &= \begin{pmatrix} 0 & 0 & \frac{t_i}{t_{i+3}} \\ \frac{t_{i+1}}{t_{i+1}} & 0 & \frac{d_i t_{i+1}}{t_{i+3}} \\ 0 & \frac{t_{i+2}}{t_{i+2}} & \frac{c_i t_{i+2}}{t_{i+3}} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{t_i}{t_{i+3}} \\ 1 & 0 & \frac{d_i t_{i+1}}{t_{i+3}} \\ 0 & 1 & \frac{c_i t_{i+2}}{t_{i+3}} \end{pmatrix}
 \end{aligned}$$

From Equation (4.29), we have that

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & d_{i+m} \\ 0 & 1 & c_{i+m} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{t_i}{t_{i+3}} \\ 1 & 0 & \frac{d_i t_{i+1}}{t_{i+3}} \\ 0 & 1 & \frac{c_i t_{i+2}}{t_{i+3}} \end{pmatrix}. \quad (4.30)$$

By comparison, we have that

$$\frac{t_i}{t_{i+3}} = 1 \quad \text{implies} \quad t_i = t_{i+3},$$

$$d_{i+m} = \frac{d_i t_{i+1}}{t_{i+3}} = d_i \frac{t_{i+1}}{t_i} \quad \text{and} \quad c_{i+m} = \frac{c_i t_{i+2}}{t_{i+3}} = c_i \frac{t_{i+2}}{t_i}.$$

Using the above results, we obtain the quasi periodicity condition for $\lambda_i := \text{diag}(1, d_i, c_i)$ in the equation below.

$$\lambda_{i+n} = \frac{1}{t_i} D_i \lambda_i \quad (4.31)$$

From Equation(4.31), we have that

$$\begin{aligned} \lambda_{i+m} &= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & d_{i+m} \\ 0 & 1 & c_{i+m} \end{pmatrix} = \frac{1}{t_i} \begin{pmatrix} t_i & 0 & 0 \\ 0 & t_{i+1} & 0 \\ 0 & 0 & t_{i+2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & d_i & 0 \\ 0 & 0 & c_i \end{pmatrix} \\ &= \frac{1}{t_i} \begin{pmatrix} t_i & 0 & 0 \\ 0 & d_i t_{i+1} & 0 \\ 0 & 0 & c_i t_{i+2} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & d_i \frac{t_{i+1}}{t_i} & 0 \\ 0 & 0 & c_i \frac{t_{i+2}}{t_i} \end{pmatrix} \end{aligned}$$

From Equation (4.31), we have that

$$\lambda_{i+m-2} = \frac{1}{t_{i-2}} D_{i-2} \lambda_{i-2} \quad (4.32)$$

Now, substituting Equation (4.32) into 4.25, we have that

$$\begin{aligned} T_i &= a_i \lambda_{i-2}^{-1} \acute{T}_{i-2} D_{i-2}^{-1} \frac{1}{t_{i-2}} D_{i-2} \lambda_{i-2} \\ &= \frac{a_i}{t_{i-2}} \lambda_{i-2}^{-1} \acute{T}_{i-2} \lambda_{i-2} \end{aligned} \quad (4.33)$$

Thus, $\acute{T}_{i-2}$ is conjugate to T_i and this completes the proof. $\square$

From the above Lemma, we obtain that for any given equivalence class $V \in \mathcal{V}_m$ with corner invariants (x_i, y_i) , the scaled monodromy is given by any of the matrices $T_i(u) := L_i(u)L_{i+1}(u) \cdots L_{i+n-1}(u)$ with

$$L_j = \begin{pmatrix} 0 & 0 & 1 \\ -x_j y_j & 0 & 1 \\ 0 & -u^{-1} y_j & 1 \end{pmatrix}. \quad (4.34)$$

Chapter 5

Inscribed Polygons and their Monodromy Invariants

The main goal of this chapter is demonstrate that the scaled monodromy satisfies a certain self-duality relation for inscribed polygons. We shall prove two Lemmas and finally use these to prove Theorem 1.0.1.

Suppose $(\cdots, p_{-1}, p_0, p_1, \cdots)$ is an inscribed polygon. Considering the points (p_i) as points on the projective line, we describe their cross ratios and set

$$q_i := 1 - [p_{i-2}, p_{i-1}, p_i, p_{i+1}] \quad (5.1)$$

By a way of Stereographic projection right from a point of the conic, we can identify a non-degenerate conic with $\mathbb{R}P^1$.

An arbitrary different choice of the center of projection gives a projective transformation of $\mathbb{R}P^1$. This implies that the identification of this projection can be described as unique up to projective transformation of the projective line.

Lemma 5.0.1. [18] Suppose

$$x_i = [p_{i-2}, p_{i-1}, p_i, p_{i+2}] \quad \text{and} \quad y_i = [p_{i-2}, p_i, p_{i+1}, p_{i+2}] \quad (5.2)$$

then the map T mapping $(q_i) \rightarrow (x_i, y_i)$ is given by $(p_i) \rightarrow (x_i, y_i)$ such that

$$x_i = \frac{1 - q_i}{q_{i+1}} \quad \text{and} \quad y_i = \frac{1 - q_{i+1}}{q_i} \quad (5.3)$$

Proof. Considering projection from the point p_{i+1} in Figure 5.1, we obtain

$$\begin{aligned} x_i &= [p_{i-2}, p_{i-1}, E, F] \\ &= [(p_{i+1}p_{i-2}), (p_{i+1}p_{i-1}), (p_{i+1}E), (p_{i+1}F)] \\ &= [p_{i-2}, p_{i-1}, p_i, p_{i+2}] \end{aligned}$$

Now, expanding x_i in terms of q_i 's, we obtain

$$\begin{aligned} [p_{i-2}, p_{i-1}, p_i, p_{i+2}] &= \frac{1 - [1 - [p_{i-2}, p_{i-1}, p_i, p_{i+1}]]}{1 - [p_{i-1}, p_i, p_{i+1}, p_{i+2}]} \\ &= \frac{[p_{i-2}, p_{i-1}, p_i, p_{i+1}]}{1 - [p_{i-1}, p_i, p_{i+1}, p_{i+2}]} \end{aligned}$$

Similarly for y , we have that

$$\begin{aligned} [p_{i-2}, p_{i-1}, p_i, p_{i+2}] &= \frac{1 - [1 - [p_{i-1}, p_i, p_{i+1}, p_{i+2}]]}{1 - [p_{i-2}, p_{i-1}, p_i, p_{i+1}]} \\ &= \frac{[p_{i-1}, p_i, p_{i+1}, p_{i+2}]}{1 - [p_{i-2}, p_{i-1}, p_i, p_{i+1}]} \end{aligned}$$

□

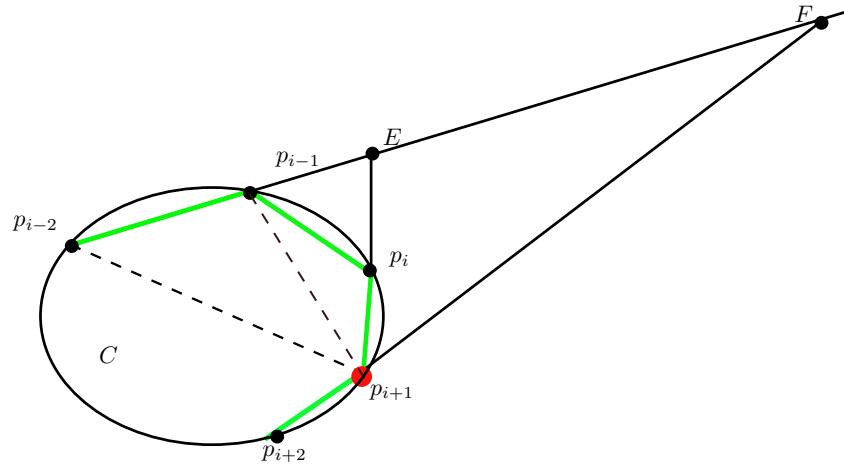


Figure 5.1: Lemma 4.2.2, proof.

Lemma 5.0.2. Suppose $V \in \mathcal{V}_m$ is an equivalence class of polygons inscribed in a non-degenerate conic. Then the analogous scaled monodromy matrix can be chosen to satisfy the self-duality relation

$$T(u) = (T(u^{-1})^{-1})^t.$$

Proof. Considering any polygon (p_i) in the equivalence class of V_m inscribed in a non-degenerate conic C , from Lemma 5.0.1 and results of scaled monodromy associated to Lemma 4.2.2 (thus, Equation (4.34)). We can express the scaled monodromy in

terms of q_i 's where $T_i(u) := L_i(u)L_{i+1}(u) \cdots L_{i+m-1}(u)$ with

$$L_j(u) = \begin{pmatrix} 0 & 0 & 1 \\ (1 - q_j^{-1})(q_{j+1}^{-1} - 1) & 0 & 1 \\ 0 & u^{-1}q_j^{-1}(q_{j+1} - 1) & 1 \end{pmatrix}. \quad (5.4)$$

Verification

From Equation (5.3),

$$\begin{aligned} x_j y_j &= \frac{1 - q_j}{q_{i+j}} \cdot \frac{1 - q_{i+1}}{q_i} \\ &= \frac{1 - q_{i+1} - q_i + q_i q_{i+1}}{q_{i+1} q_i} \\ &= q_{i+1}^{-1} q_i^{-1} - q_i^{-1} - q_{i+1}^{-1} + 1 \\ &= q_i^{-1} (q_{i+1}^{-1} - 1) - 1 (q_{i+1}^{-1} - 1) \\ &= (q_i^{-1} - 1) (q_{i+1}^{-1} - 1), \end{aligned}$$

and hence

$$-x_j y_j = (1 - q_i^{-1})(q_{i+1}^{-1} - 1), \quad y_j = q_j^{-1}(1 - q_{j+1}), \quad -y_j = q_j^{-1}(q_{j+1} - 1).$$

It is observed that $L_j(u)$ and $\hat{L}_j(u)$ are related by gauge transformation:

$$\hat{L}_j(u) := \frac{q_j}{1 - q_j} H_j L_j H_{j+1}^{-1}, \quad H_j = \text{diag}(q_j^{-1} - 1, 1 - q_j, 1), \quad (5.5)$$

where, $H_{j+1} = \text{diag}(q_{j+1}^{-1} - 1, 1 - q_{j+1}, 1)$,

$$H_{j+1}^{-1} = \begin{pmatrix} \frac{1}{q_{j+1}} & 0 & 0 \\ 0 & \frac{1}{q_{j+1}} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad L_j H_{j+1}^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ (1 - q_j^{-1}) & 0 & 1 \\ 0 & -u^{-1}q_j^{-1} & 1 \end{pmatrix} \quad (5.6)$$

and

$$H_j L_j H_{j+1}^{-1} = \begin{pmatrix} 0 & 0 & (q_j^{-1} - 1) \\ (1 - q_j)(1 - q_j^{-1}) & 0 & (1 - q_j) \\ 0 & -u^{-1}q_j^{-1} & 1 \end{pmatrix}.$$

Hence Equation (5.5) becomes

$$L'_j(u) := \frac{q_j}{1 - q_j} H_j L_j H_{j+1}^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ (q_j - 1) & 0 & q_j \\ 0 & u^{-1}(q_j - 1)^{-1} & q_j(1 - q_j)^{-1} \end{pmatrix}$$

Because $\hat{L}_i(u)$ is conjugate to $L_i(u)$, it implies that the matrix $T_i(u)$ is also conjugate to $\hat{T}(u) := \hat{L}_i(u)\hat{L}_{i+1}(u) \cdots \hat{L}_{i+m-1}(u)$.

From the results of the proof of the above Lemma in [5], it is observed that $\hat{L}_j(u)$ and $\hat{T}(u)$ satisfy the relations Equations (20) and (21) in [5]. By the equivalency in $L(u)$ and $T(u)$, since $L(u)$ satisfies the self duality relation, one can conclude that $T(u)$ satisfies the self-duality relation. This completes the proof. $\square$

5.1 Proof of Theorem 1.1

Proof. Following from Lemma 5.0.2, suppose (p_i) is a polygon inscribed in a non-degenerate conic. Then the analogous equivalence class $V \in \mathcal{V}_m$ exhibits a scaled monodromy associated with the self-duality relation.

With reference to the self-duality relation also is the normalised form of the characteristic polynomial satisfying the relation $R(u^{-1}, v^{-1}) = -u^m v^{-3} R(u, v)$.

This relation implies that the spectral curve is invariant under the involution defined earlier. Furthermore by the equivalency in the spectral curve and Lemma 5.0.2, we conclude that $\forall k \ X_k = Y_k$ for every m -gon inscribed in a non-degenerate conic. $\square$

Chapter 6

Conclusion and Further Discussion

This chapter gives the summary of this thesis and also discusses some future works.

6.1 Conclusion

The core of this thesis which discusses the geometry of pentagram integrals of polygons inscribed in non-degenerate conics began with a historical view of the pentagram, its integrals and some useful concepts mainly in projective geometry.

Some concepts such as cross ratio, corner invariant, projective plane etc aid in our construction of both the monodromy invariants and the formula for the pentagram map. The pentagram map was defined in terms of corner invariants. This gave a result that the analogous first integrals for the pentagram map are invariant. It was also observed that the invariants are certain polynomials in the corner invariant.

Further discussions on the relationship between the spectral curve and monodromy invariants were also considered. The spectral curve defined as the zero locus of the characteristic polynomial was introduced for our algebraic-geometric integrability. We were able to define the monodromy invariants as coefficients of the spectral curve. It was also proved that the spectral curve being invariant under an involution and that the corresponding monodromy invariants being equal.

It was further discussed that the corner invariants can be used to define the monodromy matrix and the scaled monodromy matrix. We proved a lemma that for every three points in general position, the monodromy should be conjugate to any of the

some matrices defined in terms of the corner invariants

We finally discussed the behavior of monodromy invariants on polygons inscribed in non-degenerate conics. By the method of stereographic projection, we established a formula between the corner invariants and the cross ratio of points on inscribed polygons. Additionally, we showed that the scaled monodromy matrix were chosen to satisfy a self-duality relation.

Using the results in the non-degenerate conic and the normalisation of the characteristic polynomial of the spectral curve we proved that the invariants are equal when polygons are inscribed in non-degenerate conics.

6.2 Further Discussion - Degenerate Conics

Degenerate conics are obtained when a plane intersect the vertex of a cone. The various types of degenerate conics are discussed below (also see Figure 6.1).

1. If an ellipse degenerates, we obtain a single point
2. If a parabola degenerates, a straight line is obtained.
3. If a hyperbola degenerates, two intersecting lines are obtained

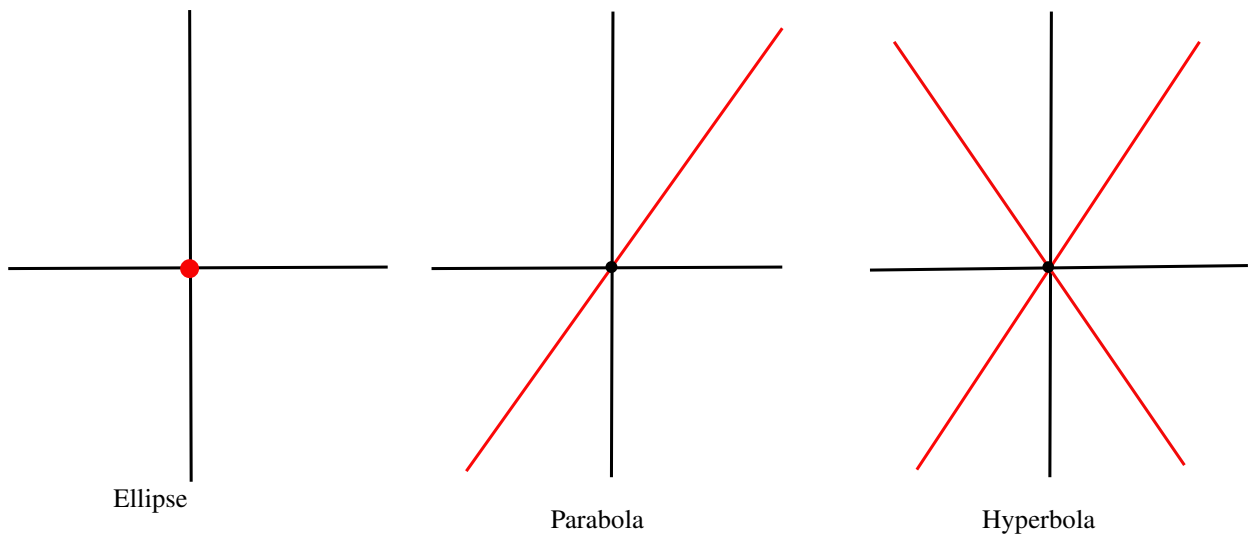


Figure 6.1: Degenerate Conics

1. How does Theorem 1.1 generalise to polygons that are inscribed in degenerate conics?

Appendix A

Other Useful Concepts

This Chapter talks about other concepts that were quite useful to the work.

A.1 DEFINITIONS

Definition A.1.1 (Group Action). Suppose G is a group and X is a set. We denote

$$(g, x) \longmapsto gx,$$

a mapping from $G \times X$ into X is called **action** of G on X if the following two conditions are satisfied:

- a. $ex = x \forall x \in X$
- b. $(gh)x = g(hx), \forall x \in X \text{ and } g, h \in G.$

Definition A.1.2 (Orbits). Assume G is a group acting on a set X . The orbit of an element $x \in X$ is the set

$$Gx = \{gx | g \in G\}.$$

Intuitively, the orbit of an element means taking a single element and acting on it with everything in the group. The set of all orbits is denoted X/G .

Definition A.1.3. Let G be a group and $gxg^{-1} = {}^g x, \forall g, x \in G$. The resulting map

$$G \times G \rightarrow G, \quad (g, x) \mapsto {}^g x$$

is called **Conjugation**. We describe the conjugation as a group action G onto itself.

Definition A.1.4 (Conjugacy Classes). Let G be a group and $x \in G$. The orbit of the element x under conjugation is called the **conjugacy class** of x and is denoted G_x .

Definition A.1.5. (Convex Hull) Let $x_1, x_2, \dots, x_n$ be the set of points in the plane, $\mathbb{RP}^2$. The smallest polygon that contains all points in the plane (including all its interior points) as shown in Figure A.1

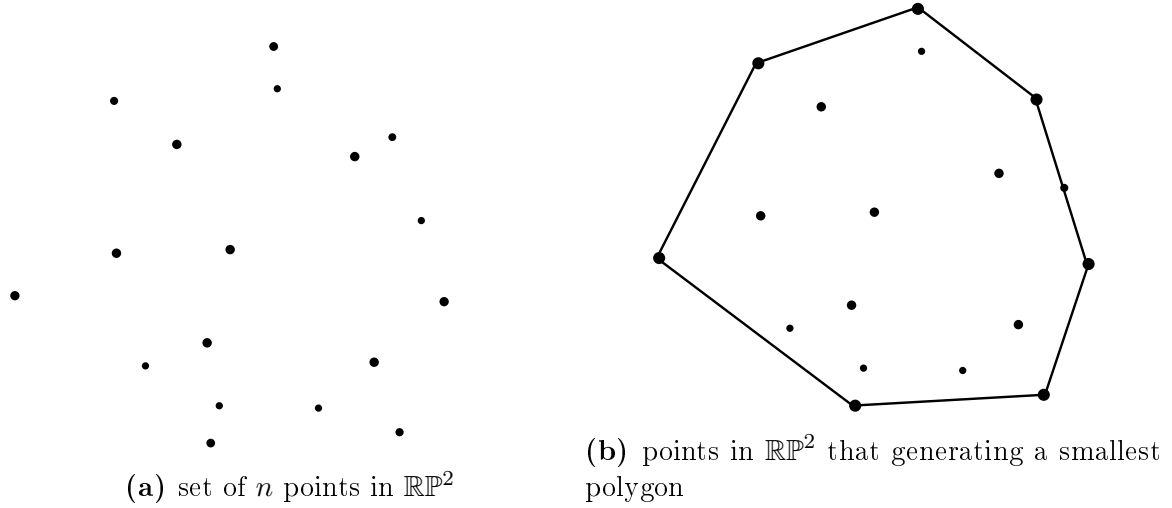


Figure A.1: Convex hull

is called **Convex Hull**.

Definition A.1.6. (Newton Polygon) Newton polygon of an algebraic curve

$$\sum a_{ij} z^i w^j = 0,$$

is the convex hull of

$$\{(i, j) \in \mathbb{Z}^2 | a_{ij} \neq 0\}.$$

A.2 Integrable System

An integrable system is a system with the property of concrete model used to solve differential equations in a closed form or expressions in terms of quadratures. In differential systems, we have Frobenius integrability and in classical theory of Hamiltonian dynamical system and also the notion of Liouville integrability. Solutions to equations are expressed using integrals.

Example

The Lax equation is an example of an integrable system. Let $I = I(t)$ and $L = L(t)$ be two time dependent matrices. **Lax pair** is a pair of time dependent matrices that satisfies a corresponding differential equation

$$\frac{dL}{dt} = [I, L]$$

called the **Lax equation** where $[I, L] = IL - LI$ is a commutator.

A.2.1 Zero locus of set of Polynomial

Suppose X is a field. Let $B = X[y_1, y_2, \dots, y_n]$ be the polynomial ring in n variables where $n \geq 1 \in \mathbb{N}$ such that $A \in B$ is a set. We define the zero locus of A as the set

$$V(A) = \{a \in X^n : g(A) = 0, \forall g \in A\}$$

Definition A.2.2. (Spectral Curve) Given a classical integrable system, the **spectral curve** is defined as the locus of zeros of the characteristic polynomial of the Lax matrix of the integrable system.

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