UNIVERSITY OF GHANA

MODELLING ASSET RETURNS IN A PORTFOLIO USING ORNSTEIN-UHLENBECK STOCHASTIC PROCESS

BY

EMMANUEL KWADZO ATSU-LAWLUVI

(10109037)

THIS THESIS IS SUBMITTED TO THE UNIVERSITY OF GHANA, LEGON, IN PARTIAL FULFILMENT OF THE REQUIREMENT FOR THE AWARD OF MPHIL IN ACTUARIAL SCIENCE DEGREE

JULY 2019

DECLARATION

This is certify that with the exception of references to other people's work which have been duly acknowledged, this thesis is entirely my research work produced under supervision and that neither part nor whole of it has been presented in this University or elsewhere for an award of a degree.

EMMANUEL KWADZO ATSU-LAWLUVI Date

SUPERVISORS:

Dr. E. N. N. NORTEY Date

(Principal Supervisor)

Dr. RICHARD MINKAH Date

(Co-supervisor)

ABSTRACT

The study explores the modelling of asset returns in portfolio as a stochastic process which exhibits mean reversion towards the long-term stationary mean. This can be thought of as if an asset return is connected to its long-run mean with a spring which pulls the asset return towards the long-run mean. The study investigates the stochastic nature of two assets returns by use of autoregressive and Ornstein-Uhlenbeck (OU) processes to illustrate the features of the assets weekly return series in Ghana from January 2011 to December 2017. The assets in the portfolio are assumed to include two major classes of investment: three-month Treasury bill and equity. The study elucidated from using the OU process to examine the mean reversion speed that the accumulated interest rate of equity approaches its long-run mean more quickly than the Treasury bill. The bivariate model revealed that the interest rate of each asset depends moderately on its interest rate of the past week and on a small portion of the interest rate of the other asset. For an investor to achieve an optimum portfolio of these assets, the investor should consider to invest in about 60% of equity and 40% of Treasury bill. One key recommendation is that the Ghana Stock Exchange should encourage the investing public to invest in stocks listed on the exchange because the rates of return are more stable.

DEDICATION

To the memory of my late dad, Victor.

ACKNOWLEDGEMENT

I would like to express my sincere gratitude to my supervisors, Dr. E. N. N. NORTEY and Dr. Richard MINKAH for their guidance, useful discussions and constructive criticisms as well as allowing me to explore an additional research objective. I am very thankful to the lecturers of the Department of Statistics & Actuarial Science, especially Dr. Gabriel KALLAH-DAGADU for his constant encouragement and inspiration.

Thanks also go to my family and all my friends for their support and prayers. Special thanks to Edong-ta Eloge Fortune AMAKBRE for his precious help. I am very grateful.

TABLE OF CONTENTS

Declaration ii

Abstract iii

Dedication iv

Acknowledgements v

Table of Contents vi

List of Figures ix

List of Tables x

List of Abbreviations xi

CHAPTER ONE: INTRODUCTION 1

1.1. Background of Study 2

1.2. Problem Statement 4

1.3. Research Objectives 6

1.4. Significance of the Study 6

1.5. Outline of Research Methodology 7

1.6. Data Source and Scope 8

1.7. Organisation of the Study 8

CHAPTER TWO: LITERATURE REVIEW 9

2.0. Introduction 9

2.1. Stochastic Investment Modelling 9

2.2. Asset Returns 16

2.2.1. Simple Return of Asset 16

2.2.2. Log Return of Asset 17

2.3. Asset Return Distribution 18

2.4. Some Issues about Statistical Estimation of Asset Returns 18

2.4.1. Stationarity 18

2.4.2. Covariances and Correlations 19

2.4.3. Autocorrelation 20

2.5. Portfolio 20

2.5.1. Portfolio Return 21
2.5.2. Portfolio Risk 22
2.6. Simple Autoregressive Models 22
2.6.1. AR(1) Model 23
2.7. Stochastic Processes 25
2.7.1. Brownian Motion 26
2.8. Use of the Ornstein Uhlenbeck Process in Asset Modelling 28
2.9. Empirical Evidence 29
2.10. Conclusion 34
CHAPTER THREE: METHODOLOGY 35
3.0. Introduction 35
3.1. Investment Models 35
3.2. Univariate Model 36
3.2.1. AR(1) Process 36
3.2.2. Ornstein-Uhlenbeck Process 37
3.2.3. Equivalent AR(1) and OU Process 39
3.2.4. Portfolio Return under Univariate Model 40
3.3. Multivariate Model 41
3.3.1. Multivariate AR(1) Process 42
3.3.2. Multivariate Ornstein-Uhlenbeck Process 43
3.3.3. Equivalent VAR(1) and Multivariate OU Process 45
3.3.4. Portfolio Return under Bivariate Model 46
3.4. Global Model 51
3.5. Data Description 52
3.6. Data Collection 54
3.7. Estimation Method 54
CHAPTER FOUR: EMPIRICAL RESULTS AND ANALYSIS 56
4.0. Introduction 56
4.1. Displaying Observed Assets Returns Data 56

4.2. Fitting the Models 59
4.2.1. Model Identification 59
4.2.2. Parameter Estimation 60
4.3. Equivalent AR(1) and OU Processes 65
4.4: Optimum Portfolio 71
4.5. The Ornstein-Uhlenbeck Position Process 72
4.6. Conclusion of Empirical Results 73
CHAPTER FIVE: CONCLUSION AND RECOMMENDATION 76
5.0. Introduction 76
5.1. Conclusion 76
5.2. Recommendation 79
5.3. Future Research Studies 79
REFERENCES 81
APPENDIX 84

LIST OF FIGURES

Figure 1: Time series plots of the weekly returns of the two assets 57

Figure 2: Scatterplot of the weekly returns of the two assets 58

Figure 3: Plots of ACF and PACF for the weekly asset returns 60

Figure 4: Plots of ACF and PACF for weekly return rates of portfolios 66

LIST OF TABLES

Table 4.1: Summary statistics for the two asset returns..58

Table 4.2: Empirical correlation matrix for the returns of the two assets.....................58

Table 4.3: AR(1) model output for Treasury Bill 61

Table 4.4: AR(1) model output for Equity...61

Table 4.5: Variance of univariate and bivariate models for the two assets 64

Table 4.6: Univariate OU velocity process: α and σ for the two assets 68

Table 4.7: Mean reversion for the assets in portfolio 68

Table 4.8: OU velocity process under the global model..70

Table 4.9: Mean reversion for portfolios 71

Table 4.10: Portfolio risk for different weights 72

LIST OF ABBREVIATIONS

ABM	Arithmetic Brownian motion
ACF	Autocorrelation function
AR	Autoregressive model
AR(1)	Autoregressive model of order one
ARCH	Autoregressive conditional heteroskedastic
GBM	Geometric Brownian motion
GSE	Ghana Stock Exchange
GSE-CI	Ghana Stock Exchange Composite Index
IID	Independently and identically distributed
ILN	Independent lognormal
ML	Maximum likelihood
PACF	Partial autocorrelation function
RSLN	Regime-switching lognormal process
SDE	Stochastic differential equation
TB	Treasury bill
OU	Ornstein-Uhlenbeck process
VAR	Vector (multivariate) autoregressive
VAR(1)	Multivariate autoregressive model of order one

CHAPTER ONE

INTRODUCTION

This study investigates the rates of return of assets in a portfolio as a stochastic process that depicts mean reversion towards a stationary point. Basically, estimation of assets returns in a multivariate framework is considered in this study. First, the Ornstein-Uhlenbeck stochastic process is used in this study to capture the mean reversion of the accumulated rates of returns for the two different assets included in the portfolio. Second, these assets returns are estimated by a bivariate autoregressive process of order one. From the estimated results in this bivariate discrete time framework, an optimum portfolio of assets is created.

Financial asset (simply referred to as asset or security) represents claims against the income or real assets of individuals and institutions issuing those claims at a future date. The collection of different assets held by an investor is called a portfolio. There are high chances that the assets held in a portfolio tend to correlate. To describe the co-movements of the assets in a portfolio, a stochastic investment model can be studied. A stochastic investment model attempts to describe how returns on assets behave over time. It is, thus, probable to utilise a stochastic investment model to estimate how investing in different assets could impact returns in portfolio over time. The background of study, problem statement, research objectives, significance of the study, outline of research methodology, data source and scope, and organisation of the study constitute the sections of this introductory chapter.

1.1 Background of Study

One of the most important and lasting question of financial time series modelling is whether asset prices are predictable. Perhaps, this is because the prices of assets in financial markets exhibit significant random fluctuations due to unexpected events. The fluctuations in prices of assets are usually as a result of unexpected set of new information, and as this new information is available to investors of securities, they instantly use the information and that lead to decisions as to buy or sell an asset. Future prices or returns on assets are uncertain or stochastic, but many investors still examine historical data of the returns on assets to see possible trends and patterns on which they make their investment decisions.

There is a considerable attention in the literature that covers the forecasting of the rates of returns on assets. Surprising from these loads of publications, there is no consensus as to the sources of forecasting the returns on assets, but there is the growing belief that forecasting is an important characteristic of many assets. Investors are therefore interested in estimating the impact of modelling returns on assets in a portfolio. In particular, insurers, pension funds, and mutual funds give greater consideration to implementing investment strategies that will maximise the returns on the funds invested among the different asset classes.

The price of a particular asset at any moment in time follows a stochastic process that depends on the state of the financial market (or world) it is traded. In finance and insurance, the focus is on the asset return process, so there is a need to investigate the return as long as the market exists. The effort of investment research focuses around pre-determination of return and measurement of risk. The investment risks mainly depend on macroeconomic factors. In considering these investment risks, it is essential for investors to formulate a stochastic model that reasonably and accurately describes the random fluctuations in the asset prices. This has resulted in the estimations and applications of stochastic investment models to help explain the movement in financial asset prices.

Modelling returns on assets plays significant roles in actuarial science. In actuarial analysis, a reliable stochastic investment model provides an actuary a clear picture and forecast for asset returns in both short and long investment horizons. Such forecasts are useful for pricing life insurance and pension products. Further, an accurate stochastic investment model for predicting asset return outcomes helps to estimate the future expenses associated with management of insurance and pension products. In addition, for actuarial analysis, a stochastic investment model can be useful in constructing an optimal portfolio and estimate the minimum capital requirement for selling insurance and pension policies.

Stochastic models have been used to model rates of return on assets or portfolio over time for many years. The deterministic model for forecasting rates of return on assets is simple to use, but stochastic models are now largely used to fit the rates of return on assets, because of the fact that the noise terms are considered. The groundwork for modelling rates of return of securities is by Markowitz (1952); and following the footsteps of the stochastic investment model by Wilkie (1986), many stochastic investment models have been developed. The Ornstein-Uhlenbeck process as a class of stochastic process is widely used in finance and insurance to model returns of assets. Vasicek (1977) applied the Ornstein-Uhlenbeck process to model the instantaneous interest rate over times. The Ornstein-Uhlenbeck process has other important applications such as forecasting currency exchange rates and commodity prices.

One common feature of these stochastic models, including the Ornstein-Uhlenbeck process, is that most of these models considered are univariate models (one dimensional); these studies concentrates on the interest rates of a single asset. However, the problem is that for a portfolio, the use of the univariate models means that each asset is modelled separately under the assumption that the assets returns are uncorrelated. This assumption is unreasonable for most portfolios as the assets may be correlated.

For a portfolio, each asset return has its own process, but importantly the returns of these assets may be correlated. In that case, to describe and forecast the return of the portfolio, a univariate model for each asset is inapt because the correlation between the assets cannot be captured. The overall return on a portfolio is greatly influenced by correlation between the assets. So, ignoring correlation in the asset return movement presents an error in the stochastic model that can lead to imprecise predicting. In view of this, a multivariate stochastic process is considered. The multivariate stochastic process provides a natural way to incorporate possible correlations of assets. It is also needed for optimal portfolio selection and risk management at the portfolio level.

1.2 Problem Statement

To the extent that the investors or portfolio managers are engaged in commitments whose returns lie in the future, the problem of constructing a comprehensive stochastic model that reasonably explains the movement of asset returns cannot be overemphasised. Many decades ago, the asset portfolios of insurance companies, pension funds, and financial institutions were mainly fixed interest assets (loans) and money market securities. The asset portfolios of these institutions have changed significantly as these institutions have also largely invested in equities than previously. These institutions hold portfolio made of several assets and some of these assets tend to be correlated. Therefore, it is inappropriate to use a stochastic investment model without considering the correlation between the assets. Hence, a multivariate stochastic model to explain and forecast the return rates on assets over time is needed.

Insurance buyers frequently ask what rate of return they earn on the money they save with a life insurer or a pension fund or a financial institution. The purchaser of pension fund or life insurance may choose from a vast array of companies and within each company the buyer is faced with a variety of plans which carry different premiums. Insurers determine these premiums on insurance policies that reflect expected investment returns; that is, insurers reflect expected investment income in premiums quoted to policyholders. The insurer collects a premium when the insurance contract is written and it is invested almost immediately. The horizon of these investments may be many years ahead. It is therefore desirable that an insurer obtain a reliable stochastic investment model that depicts the trends and patterns in which asset returns have behaved over time. It is also desirable for insurers or a financial institution to construct a stochastic investment model that is satisfactorily presentation of historical data and can produce stimulated returns close to the actual returns.

In the light of the above considerations for pension funds and insurance companies, it is useful that the rates of asset returns are correctly predicted. The simple proposition is that premiums collected by these institutions are invested to earn investment incomes which are then used to meet future contingencies. Pension funds and insurance companies that fail to maximise the returns on invested premiums may in the long run end up be insolvent. It is clear that one significant concern in actuarial analysis or asset portfolio management is the need for a stochastic investment model can satisfactorily describe and explain the long-term features of asset returns.

1.3 Research Objectives

The general objective of the study is to investigate the rates of return of two assets in a portfolio as a stochastic process that depicts mean reversion towards a stationary point. In essence, the specific objectives are threefold and are stated as follows:

- To examine the mean reversion for each asset return in the portfolio.
- To use a stochastic model to predict the assets returns in the portfolio.
- To identify an optimum portfolio of these assets.

1.4 Significance of the Study

Modelling rate of returns of assets has been studied for many years. Except a couple, these studies consider modelling of interest rates of one asset, and as such, a univariate stochastic process is probably adequate to explain the empirical behaviour of the rates of return. In a portfolio of assets, each asset return has its own process, but importantly the returns of these assets maybe correlated. To describe and forecast the overall portfolio return, it is essential that a multivariate stochastic process is used. But, using a multivariate stochastic process to explain the empirical behaviour of asset returns is arduous. Yet, the study attempts to use a multivariate stochastic process to construct a stochastic investment model (for the return rates of two major asset classes traded in

Ghana) for use by an investor. Essentially, the findings of the study may probably guide the investment community on how to achieve an optimal portfolio of assets.

A reliable and accurate stochastic investment model forms an important element in actuarial analysis that allows for possible simulations of asset returns in the future. It is necessary that stochastic models that can reasonably explain the movements in an asset returns are formulated for an actuarial use in an economy. Next to estimating the overall return on a portfolio, a stochastic model is necessary for actuarial uses that include projecting pension or long-term insurance payments, calculating required reserves to meeting future contingencies, and pricing insurance products. Finally, the desire is to formulate an optimal portfolio for investors. The construction of optimum portfolio is significant for the reason that an asset allocation strategy adopted by an investor has effect on the overall portfolio return. For example, pension funds pay crucial attention to the process of selecting an optimal investment strategy that benefits them and more specifically how they should split their funds into the different asset classes so that the risk in pension payments is minimised.

1.5 Outline of Research Methodology

A univariate AR(1) process estimated from the observed assets returns series is converted to its Ornstein-Uhlenbeck (OU) process under the principle of covariance. The OU process is then used to examine the mean reversion of each asset. Further, the estimated results given by the bivariate AR(1) process are used to identify an optimum portfolio. The parameters of the models are to be estimated by the method of maximum likelihood.

1.6 Data Source and Scope

The three-month Treasury bill and equity are selected to correspond to major asset classes based on default risk as risk-free asset and risky asset respectively. The interest rate on the three-month Treasury bill will be used to calculate the rate of return on the risk-free asset, and the data is collected from the research section of Delta Capital Ghana. The Ghana Stock Exchange Composite Index (GSE-CI) for the last trading day in a week is used to calculate the rates of return on equity. GSE-CI data is collated from the online indices of the Ghana Stock Exchange. The sample data run from 2011 to 2017, collected at weekly intervals.

1.7 Organisation of the Study

The rest of the study has the following chapters. Chapter 2 reviews the literature. Some known facts on stochastic processes for modelling asset returns as well as portfolio return and risk are reviewed in this chapter. Further, the basic properties of the autoregressive process of order one and some empirical evidences on the topic are provided. Methodology follows in Chapter 3, where more details of the investment models are discussed. It also describes the data. Chapter 4 covers the empirical results and analysis. Chapter 5 gives the conclusions and recommendations.

CHAPTER TWO

LITERATURE REVIEW

2.0 Introduction

The chapter focuses on a review of the literature on models for estimating asset returns. First, a review of the theoretical literature is carried on some classes of stochastic investment modelling, asset returns and some issues involved in statistical estimation of asset returns, portfolio, basic properties of the autoregressive process of order one, Brownian motion, and the use of the Ornstein-Uhlenbeck process in modelling asset returns. Then, the chapter concludes on a review of some empirical studies. It is important to mention that there are several studies on the use of stochastic model for the purpose of estimating asset returns. Most of these studies are, however, under the broad umbrella of univariate stochastic investment models.

2.1 Stochastic Investment Modelling

The time series analysis of a variable is to look for trends and patterns in the observations of the variable and derive 'rules' from them, and/or utilise all information extracted in the variable for a more precise prediction. The basic idea of a stochastic time series model is, thus, it takes useful historical data and combines it with the present to predict the future. Invariably, a statistical analysis of empirical assets returns gives valuable insights into the features of past experience that a stochastic model must capture.

Stochastic investment modelling involves extrapolating assets prices under a large number of equally likely randomly generated asset paths. A stochastic investment model attempts to describe how returns on assets behave over time. It is, thus, probable to utilise a stochastic model to estimate how investing in different assets could impact overall portfolio return over time. The pioneering paper in the area of modelling rates of return on an asset is by Markowitz (1952). Following this, Boyle (1976) presented a simple generalisation of the traditional compound interest and actuarial models to accommodate a stochastic return rate on assets. The investment model assumes that the return in a particular year is not related to the return in another year and that the probability distribution is unchanged over time. For most assets, this assumption is unrealistic. Though, equities may be quite unrelated from year to year.

The pioneering paper in the area of modelling rates of return is by Markowitz (1952), but the benchmark to stochastic model for modelling investment returns data is the Wilkie investment model. Wilkie (1986) built the first comprehensive stochastic investment model. The model covered annual series of inflation, share dividends, share dividend yields and long-term interest rates (the yield on consols) for the United Kingdom from 1919 to 1982. Wilkie used a first-order autoregressive stochastic process to stimulate and forecast inflation. The Wilkie's model portrays a cascade structure for the investment series. Inflation is assumed to influence the other investment series. Using the cascade modelling method, Wilkie interrelate inflation with the other series to develop a comprehensive model for use by insurance companies (life or general), pension funds, or any other financial institution.

The Wilkie investment model recorded some critics. Therefore, Wilkie (1995) updated and extended the Wilkie investment model to a more comprehensive model to include short-term interest rates, property rentals and yields, yields on index-linked stock, and wages (earnings) index; used monthly data instead of annual data; considered other countries in addition to the United Kingdom model; introduced a model for predicting currency exchange rates; extended some aspects of the model to several other countries; and used more complex time-series modelling techniques, largely the cointegrated and autoregressive conditional heteroskedastic (ARCH) models. Inflation series was then modelled as an AR-ARCH model.

Stochastic time series modelling has since then received considerable attention. Many of these stochastic investment models (including the Wilkie investment models) are developed using the Box and Jenkins (1976) linear modelling techniques. In addition to the linear stochastic time series models, other non-linear approaches for building stochastic investment models have attracted appreciation. For example, Whitten and Thomas (1999) reviewed the Wilkie (1995) stochastic investment model and prior work in order to refine the model. They considered non-linear time series model to explain the investment series, where ARCH models and threshold model were used. They suggested a threshold autoregressive (TAR) process as a useful stochastic asset model to the Wilkie (1995) stochastic asset model.

Others have used different classes of continuous time processes in modelling the rates of return of securities. Parker (1995) used a linear second order stochastic differential equation to explain the interest rates on securities. Parker computed the first two moment functions of the interest rates and the interest rates accumulation function as these are needed for studying actuarial functions with random interest rates. The class of a shot noise processes with Poissonian times and Brownian magnitudes for modelling the rate of returns of securities was used by Chobanov (1999).

Hardy (2001) revealed that the traditional stochastic approaches for predicting the returns on long-term securities are based on the general idea that these returns have a geometric Brownian motion. Thus, an investment return during any discrete time scale is a log normal distribution and the returns in each different interval are unrelated. It is straightforward and tractable to use the independent lognormal (ILN) model to model the rates of return on securities. The ILN model presents a sensible explanation to the random behaviour of the returns when used in predicting returns on short-term securities, but it is less appealing for predicting returns on long-term securities. Empirical studies on rates of return on securities indicate that the ILN model, in particular, fails to include extremely movements of assets prices and random noise. To capture the random noise, the regime-switching lognormal process (RSLN) is used, where it assumed that the noise takes one of some discrete values, switching between these values at random.

The RSLN technique is simple like ILN model but it has added feature of precisely incorporates the extremely observed behaviour. The RSLN allows the price process of the security to randomly switch between (say) n states, where each state is characterised by different model parameters. The process that describes the state of the price process at a period is said to follow a Markov process; thus, the probability of changing state strictly depends on the current state, not on the history of the price process. The main reason for the use of the RSLN model is that the financial markets are likely to change from time to time. The financial market, for example, may switch between a stable low-risk state and a more unstable high-risk state. In particular, unstable high-risk state can result because of short-term economic or political uncertainties.

Hardy (2001) was first to use the Markovian regime-switching lognormal models to model returns on equities. The Standard and Poor (S&P 500) index and the Toronto Stock Exchange (TSE 300) index from 1956 to 1999, taken at monthly intervals, were utilised to fit the RSLN model. The fitted RSLN model was compared to other models, such as the generalised autoregressive and traditional lognormal models. The RSLN model was a much appealing fit to the S&P and TSE data compared to other models. However, she pointed out that estimated stock returns from the Wilkie stochastic investment model are much like the results from the lognormal model using the same data set.

The class of RSLN models is simple and parsimonious. The class of RSLN models provides a better fit to monthly return series of major stock markets. But, empirical studies have revealed significant stylised facts of asset return data, such as trends, asymmetries, seasonalities, jumps, non-linearity, and many others (example, see Cont, 2001) that it is not possible to have one particular class of stochastic investment models that can express and describe all stylised facts detected in the return series. Wong and Chan (2005) used the class of finite mixture Gaussian time series models to fit returns on long-term investments as an alternative to using the RSLN process to model returns on long-term investments. The mixture time series models are more apt in modelling tails and higher-order moments of asset return distribution. Wong and Chan (op cit) utilised the S&P 500 and TSE 300 indices (same dataset used by Hardy (2001)) to fit the returns series and compared the estimates to the RSLN and ILN models. It was revealed that the class of the RSLN models provides an overall good fit to the empirical returns data for the S&P 500 and TSE 300 indices.

Enlarging the set of stochastic models to fit returns on long-term investments, Lau and Siu (2008) used the class of the Bayesian infinite time series mixture models to fit returns on long-term investments. The Bayesian mixture models as a candidate for modelling returns on long-term investments present a flexible method to depict and explain important empirical features of the returns such as skewness, kurtosis of returns' distribution, and the conditional heteroskedasticity. It takes into account full information involved in the AR and ARCH processes with different orders by using Bayesian averaging or mixing. The model naturally incorporates a possible risk or uncertainty associated with it, provides a better way to partition and detects outliers of the investment returns. Lau and Siu (op cit) adopted a Bayesian sampling method based on a weighted Chinese restaurant process for clustering the asset returns to estimate the Bayesian mixture models. The TSE 300 index (the dataset as adopted in Hardy (2001) and Wong and Chan (2005)) was used to fit the models and the simulated results compared to the observed features of the TSE 300 data. The simulated results exhibited that the fitted BMAR and BMAR-ARCH models with Bayesian averaging can better capture some observed features such as the negatively skewed and heavy-tailed behaviours of the logarithmic returns and the probability of crash, of the TSE 300 data.

These studies are to broaden the set of stochastic time series models for predicting investment returns. One common feature about most of these studies is that the investment returns are modelled as a Gaussian process; as a result, the discounting function or the accumulation function has a lognormal distribution. Finally, these classes of stochastic models for modelling returns talked about are univariate models.

Nonetheless, the Ornstein-Uhlenbeck (OU) process is a popular stochastic model in finance and insurance used to model return rates on assets. Modelling the rates of returns of assets with OU process has been studied for many years (see Vasicek, 1977; Panjer & Bellhouse, 1980, 1981; Beekman & Fuelling, 1990, 1991; & Parker, 1993, 1994a, 1994b). The rates of return of a portfolio can be analysed and modelled by a separate univariate OU process for each asset included in the portfolio (that is, a univariate model) or a single multivariate OU process (that is, a multivariate model). The rationale of modelling with a multivariate process is to incorporate the correlations between the several assets returns in an entire portfolio, which is absent in the univariate modelling. Multivariate models are, thus, needed to study co-movements and spill over effects between several assets. The use of the class of multivariate OU model has been well studied in Wan (2010) and Qian (2010).

Formulating stochastic investment models for assets returns plays significant roles in actuarial science. Stochastic investment model is useful in determining premium for any kind of policy contract written by pension funds and life insurance companies. The pension fund or the life insurance company can have a practical estimate about mortality and expenses (probably using inflation to estimate expenses), but then adopt a reasonable investment plan. On the basis of simulated return paths of the portfolio held, the actuary can reasonably estimate the level of premium which will be sufficient to meet the sum assured. This means that an empirical premium frequency distribution can be derived.

A reliable and accurate stochastic investment model is useful in the valuation of insurance companies. Fundamentally, a stochastic model is needed to investigate the solvency of a life insurance company having portfolio of assets and of liabilities. Indeed, one key task of actuary is ensuring that an insurer will not run out of assets or become ruined by not having sufficient assets to satisfy a statutory minimum valuation basis.

Similarly, stochastic investment model is an important tool in pension fund for investigating the fund's solvency, or the adequacy of any chosen contribution rate or the effect of investment strategies adopted. In simple terms, a stochastic model is necessary tool for the estimation of contingency reserves of life insurance companies and pension funds.

2.2 Asset Returns

For lucid explanations, let us fix some notations for this study. Expressing return on an asset as a set of random variable over time gives a time series $\{X_t\}$. In this study, the price of an asset at time t is denoted as S_t and the return period as dt, which can range from probably a few seconds to a year or over. The price S_t of a particular asset in any time moment $t \geq 0$ is a random variable that depends on the state of financial market and/or on the macroeconomic environment. Security prices are the fundamental market observables but the object of most investors and empirical studies is the so-called asset returns. This is attributable to the stationarity assumption of the asset return process. The return on asset can be calculated from many different formulas, but here the focus is on log-returns in modelling the assets returns.

2.2.1 Simple Return of Asset

The simple net return of an asset for one-period dt is computed as:

$$X_t = \frac{S_t - S_{t-1}}{S_{t-1}} \tag{2.1}$$

where S_t is specifically the end-of-period asset price (or market index) at time t. This (above) equation means that the gross asset return at time t can be defined as $1+X_t$. If an asset is held for n periods dt, then the n-period simple return can be computed as:

$$X_n(t) = \frac{S_t - S_{t-n}}{S_{t-n}} \tag{2.2}$$

Equivalently, Equation (2.2) can be stated as:

$$S_t = S_{t-n}\left(1+X_n(t)\right) \tag{2.3}$$

2.2.2 Log-Return of Asset

The log-return Y_t of an asset is given as:

$$Y_t = \log\left(\frac{S_t}{S_{t-1}}\right) = \log(S_t) - \log(S_{t-1}) \tag{2.4}$$

The log-return is simple to calculate. But, the important issue in considering logarithmic investment return series is the ability of the model to include extreme movements of the rates of return on an asset. The log-return (Y_t) is closely related to the simple return (X_t) by

$$Y_t = \log\left(\frac{S_t}{S_{t-1}}\right) = \log\left(\frac{(S_t - S_{t-1}) + S_{t-1}}{S_{t-1}}\right)$$

$$Y_t = \log(1+X_t)$$

$$Y_t = X_t - \frac{1}{2}X_t^{\,2} + \frac{1}{3}X_t^{\,3} - \frac{1}{4}X_t^{\,4} + \cdots \tag{2.5}$$

2.3 Asset Return Distribution

There is no agreement as to a particular fashion that asset prices should exhibit. Nevertheless, the random variations of most assets prices share certain common nontrivial statistical properties, called stylised facts (Cont, 2001). Three of these statistical properties are:

- **Gaussian distribution:** the distribution of return varies at different time ranges. But, by increasing the time range for which the assets return is calculated, the distribution appears more like a Gaussian distribution.
- **Slow decay of autocorrelation in absolute returns:** the autocorrelation function of absolute asset returns decay at a slow pace over time; that is, long-range dependence.
- **Gain/loss asymmetry:** except for exchange rates, a large downward movement in stock prices and indices can be observed but not equally large upward movement.

2.4 Some Issues about Statistical Estimation of Asset Returns

Some issues are essential when interpreting statistical estimation of asset returns. Some of these issues relevant to this study are summarised as follows.

2.4.1 Stationarity

Observed returns on an asset do not essentially reflect future performance of that asset. But to estimate the moments of asset returns, the asset returns should be stable over time. In simple terms, to analyse asset returns, stationarity is required. An asset return series X_t is has a strict stationarity if the joint distribution of ($X_{t_1}, X_{t_2}, \ldots, X_{t_k}$) is equal under different time scales, where k is any positive integer. This condition is difficult to prove empirically (Tsay, 2005). So, a weaker form of stationarity is used. An asset return series X_t has a weak stationarity if its mean and its covariance do not depend on time. Specifically, X_t is weakly stationary if $E(X_t) = \mu$ for all t and $Cov(X_t, X_{t-\ell}) = \gamma_\ell$, where ℓ is an arbitrary integer. Asset returns are generally assumed to be weakly stationary. This assumption enables us to draw inferences about future outcomes, such as prediction (Tsay, 2005).

2.4.2 Covariances and Correlations

The covariance matrix is used for analysing the dependence of asset returns in a portfolio. For two assets returns, the covariance matrix $\boldsymbol{C}_{ij}$ is:

$$\boldsymbol{C}_{ij} = \mathrm{cov}\left(X_i(t,T), X_j(t,T)\right) \tag{2.6}$$

In general, the covariance between asset return X_i and asset return X_j is given as:

$$\sigma_{ij} = E\left\{\left[X_i - E(X_i)\right]\left[X_j - E(X_j)\right]\right\} \tag{2.7}$$

The covariance of these two assets returns may also be computed as the multiplication of the two assets' volatilities and the correlation coefficient (ρ_{ij}) between them:

$$\sigma_{ij} = \sigma_i \sigma_j \rho_{ij} \tag{2.8}$$

It is clear from Equation (2.8) that the covariance σ_{ij} can vary, either because the correlation coefficient between the assets changes or because an asset volatility (σ_i) changes. The correlation coefficient is a measure of the strength of linear dependence between these two assets returns. It can be shown that $\rho_{ij} \in [-1,1]$ and $\rho_{ij} = \rho_{ji}$. The correlation coefficient has these special values:

$$\rho_{ij} = \begin{cases} \pm 1 \text{ perfectly correlated} \\ \;\;0 \text{ perfectly uncorrelated} \end{cases}$$

If ρ_{ij} is close to +1 or -1, then the assets returns are more closely related. The two assets returns are unrelated if $\rho_{ij} = 0$, so if it is close to 0, it indicates a weaker correlation between the two assets returns.

2.4.3 Autocorrelation

Autocorrelation is a measure of the linear dependence between X_t and its past values $X_{t-\ell}$. Thus, the correlation between an asset return series X_t and its lagged return series $X_{t-\ell}$ over successive time scales is described by autocorrelation. In simple terms, the correlation coefficient between X_t and $X_{t-\ell}$ is referred to as the lag-ℓ *autocorrelation* of X_t. If the autocorrelation of the asset returns is zero, then the return series can be modelled as a random walk, that is, the asset returns are said to be independent random variables.

2.5 Portfolio

Portfolio is a collection of different assets held by an investor. In selecting these assets to form the portfolio, the key assumption is that an investor is a risk averse; given an option to select between two assets with same rates of return, the investor will select the asset with the least volatility. However, some investors are not risk averse. To construct a portfolio, investors must evaluate the risk involved in buying each asset under consideration. Decades past, there was no explicit expression to measure risk. So, in the 1950s, a basic portfolio model was built by Markowitz (1952), where he provided the expressions for expected return rate and expected risk measure for a portfolio of financial assets based on some assumptions. He actually showed that the variance of the rate of return is an important measure of portfolio risk. This conventional risk measure formula shows the need to diversify assets to minimise a portfolio total risk and also shows how investors are to diversify assets effectively.

The Markowitz portfolio model was built on some assumptions concerning how investors behave. Four such assumptions crucial to this study are:

- Investors consider the different investment options as each option having a probability distribution of expected returns over some investment period.
- Investors determine a portfolio risk on the basis of the dispersion of expected returns.
- Investors' asset selection decision is based solely on expected return and risk.
- For a certain level of risk, investors prefer the assets with higher returns to lower returns. Likewise, for a certain expected return, investors prefer asset with less risk to more risk.

These assumptions implicitly implies that the combination of asset that gives the highest expected return for a given amount of risk is known as the investor's optimal portfolio.

2.5.1 Portfolio Return

For simplicity of exposition, X_i is denoted as the return on asset i and its expected value is represented as μ_i. The percent invested in asset i is ω_i. The overall portfolio return (X) is:

$$X = \sum \omega_i X_i \tag{2.9}$$

Each asset return (X_i) and thus the overall portfolio return (X) are treated as random variables. That is, investors are assumed to really behave like they have probability values of these variables. The ω_i's are arbitrary are set by an investor; ω_i is a percent-weight of each asset included in the portfolio, so $\sum \omega_i = 1$; and further assumes $\omega_i \geq 0$ for all i. The overall portfolio return (X) is then a weighted sum of random variables. The expected return on the portfolio, therefore, is:

$$E[X] = \sum \omega_i E[X_i] = \sum \omega_i \mu_i \tag{2.10}$$

2.5.2 Portfolio Risk

The basic formula for measuring portfolio risk that investors consider when combining assets is provided by Markowitz (1952). For *n* assets in a portfolio, the variance is:

$$\sigma^2(X) = \sum_{i=1}^{n} \omega_i^2 \sigma_i^2 + 2\sum_{i=1}^{n}\sum_{j>1}^{n} \omega_i \omega_j \sigma_{ij} \tag{2.11}$$

Taking square root of the expression in Equation (2.11) gives the standard deviation of the portfolio, which is the portfolio risk. For two assets in a portfolio, the variance is:

$$\sigma^2(X) = \omega_1^2 \sigma_1^2 + \omega_2^2 \sigma_2^2 + 2\omega_1 \omega_2 \sigma_{12} \tag{2.12}$$

From Equation (2.12), it can be deduced that if the assets included in the portfolio have equal variance, then the portfolio has a minimum variance (risk) when the percent invested in each asset is equal. This proposition is proved in the appendix. Hence, the formula to determine the percent weight of one asset (say the first asset) to achieve a minimum portfolio variance for a portfolio of two assets can be given as:

$$\omega_1 = \frac{\sigma_2^2 - 2\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2} \tag{2.13}$$

2.6 Simple Autoregressive Models

One important approach for statistical modelling of modelling asset return series is the autoregressive processes. An autoregressive (AR) model is a stochastic process that is a linear combination of its past values plus a noise term. Suppose a simple model given as:

$$X_t = \phi_0 + \phi_1 X_{t-1} + a_t \qquad (2.14)$$

where the white noise (a_t) has mean 0 and variance σ_a^2. The form of Equation (2.14) is the same as a complete simple linear regression equation; X_t is the dependent variable and X_{t-1} is the explanatory variable. Equation (2.14) is called an autoregressive model of order one or simply an AR(1) model. Condition on the past asset return X_{t-1}, this gives:

$$E\left(X_t \mid X_{t-1}\right) = \phi_0 + \phi_1 X_{t-1} \qquad Var\left(X_t \mid X_{t-1}\right) = Var\left(a_t\right) = \sigma_a^2$$

The AR(1) model can be generalised to an AR(*p*) model as:

$$X_t = \phi_0 + \sum_{i=1}^{p} \phi_i X_{t-i} + a_t = \phi_0 + \phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots + \phi_p X_{t-p} + a_t \qquad (2.15)$$

Equation (2.15) means that the p values X_{t-i} $(i \geq 1)$ simultaneously give the conditional expected value of X_t given the previous data. The AR(1) model is effectively used in this thesis, so its basic properties are explored.

2.6.1 AR(1) Model

Suppose an asset return series X_t has a weak stationarity, then at each time t, $E\left(X_t\right) = \mu$, $Var\left(X_t\right) = \gamma_0$, and $Cov\left(X_t, X_{t-j}\right) = \gamma_j$, where μ and γ_0 are constant and γ_j is a function of j. The mean, variance, and autocorrelations of the asset return can be obtained. Taking the expectation of Equation (2.15) and because $E\left(a_t\right) = 0$, the following result is obtained:

$$E\left(X_t\right) = \phi_0 + \phi_1 E\left(X_{t-1}\right)$$

Under the stationarity condition, $E(X_t) = E(X_{t-1}) = \mu$ and hence

$$E(X_t) = \mu = \frac{\phi_0}{1-\phi_1}. \qquad (2.16)$$

Equation (2.16) has two implications for X_t: $E(X_t)$ exists if $\phi_1 \neq 1$ and $E(X_t) = 0$ if and only if $\phi_0 = 0$. If the AR(1) process is stationary, we have $\phi_0 = (1-\phi_1)\mu$, which means Equation (2.14) can be restated as:

$$X_t - \mu = \phi_1(X_{t-1} - \mu) + a_t \qquad (2.17)$$

By repeated substitutions, Equation (2.17) implies that

$$X_t - \mu = a_t + \phi_1 a_{t-1} + \phi_1^2 a_{t-2} + \cdots = \sum_{i=0}^{\infty} \phi_1^i a_{t-i} \qquad (2.18)$$

This relation implies that $X_t - \mu$ is a linear function of a_{t-i} for $i \geq 0$. Using this property and the independence of a_t, $E[(X_t - \mu)a_{t+1}] = 0$. Thus, from the stationarity assumption, $Cov(X_{t-1}, a_t) = E[(X_{t-1} - \mu)a_t] = 0$. Taking the square, and then the expectation of Equation (2.17), this gives

$$Var(X_t) = \phi_1^2 Var(X_{t-1}) + \sigma_a^2,$$

where σ_a^2 is the variance of a_t. The stationarity assumption means $Var(X_t) = Var(X_{t-1})$, so

$$Var(X_t) = \frac{\sigma_a^2}{1-\phi_1^2}, \; \phi_1^2 < 1$$

From this prior expression and the assumed weak stationarity for the AR(1) model means $-1 < \phi < 1$. Thus, the required condition for AR(1) model to be weakly stationary is $|\phi_1| < 1$. If the AR(1) model in Equation (2.14) is a weak stationary model, then the autocorrelation function of X_t satisfies

$$\rho_\ell = \phi_1 \rho_{\ell-1}, \ \ell \geq 0 \tag{2.19}$$

Given that $\rho_0 = 1$, Equation (2.19) gives $\rho_\ell = \phi_1^\ell$. This indicates that the autocorrelation of a weakly stationary AR(1) process decays exponentially with rate ϕ_1 and its starting value is $\rho_0 = 1$. For a positive ϕ_1, the ACF plot of an AR(1) process exhibits a nice exponential decay, but for a negative ϕ_1, the ACF plot exhibits two alternating exponential decays with rate ϕ_1^2.

2.7 Stochastic Processes

The modelling of random asset returns is based on stochastic processes. Stochastic process is a set of observed random variables over time; so X_t is a random variable (asset return) that models the value of the stochastic process at time t. In statistical modelling, a sample of data from the process being modelled is collected. If X_t models asset return at time t, and the asset returns for last 12 trading weeks are observed, these data can be used to describe the process and analyse the nature of its past behaviour. The data may also be used to estimate the parameters of a class of stochastic process models. The estimated model can then be used to predict the future behaviour of the asset return.

It is the dependence between the random variables in the set that allows to make predictions by extrapolating past patterns into the future. For instance, a stochastic model might say that the current value of X_t depends on the values of the two past trading weeks X_{t-1} and X_{t-2}. If this dependence does not change with t, then the model could be use to predict future values of X_t.

2.7.1 Brownian Motion

Before describing the Brownian motion as a stochastic process, a brief background to it is considered. Brownian motion gets its name from the British botanist Robert Brown who in the nineteenth century (1827) observed that tiny pollen grains move in small but incessant and random fashion. He posed a mathematical problem of describing the observed movement, but did not solve it. It was in the twentieth century (the 1920s) that Wiener (1923) gave the mathematical basis for Brownian motion as a class of stochastic process. For this reason, Brownian motion is mostly known as a Wiener process.

The Wiener process is a simple stochastic process from which an understanding of a system that fluctuates is obtained. Brownian motion is a simple class of continuous stochastic process often utilised to model random behaviour evolving over time. Example of such random behaviour is the variations in the price of an asset. Mathematically, Brownian motion or Wiener process, commonly denoted $W(t)$, is defined as a stochastic process, for $t \geq 0$, if it satisfies the following properties:

- $W(t)$ is continuous function of t with $W(0)=0$.
- $W(t+m)-W(m) \sim N(0,t)$ for $0 \leq m < t$.
- $W(t+m)-W(m)$ is independent of any details of the process for periods before m.

Brownian motion can be geometric or arithmetic. This is illustrated as follows. Different assets have different amounts of fluctuation. But, assume that an asset (in particular, Treasury bills) has no volatility, this result in accumulated value such as

$S(t) = S_0 e^{rt}$, where r is a (constant) continuously compounded interest rate. Differentiating gives $\frac{dS}{dt} = rS$. This results in the ordinary differential equation for the asset price process as:

$$dS_t = rS_t dt \tag{2.20}$$

One other main feature of the differentiation implies $\frac{dS}{S} = r \cdot dt$, thus the relative change in the asset price can be stated as $\frac{\Delta S}{S} = r\Delta t$. It can be assumed that this relative change has both a deterministic part and an additional stochastic part:

$$\frac{\Delta S}{S} = \alpha \Delta t + \sigma \Delta W$$

Rearranging and writing in terms of differentials, the asset satisfies the linear SDE:

$$dS_t = \alpha S_t dt + \sigma S_t dW_t \tag{2.21}$$

where α is a constant return rate of the asset price (the asset drift), σ is a constant asset volatility, W_t is a Wiener process (Brownian motion) and S_0 is the initial value (assumed to be $S_0 > 0$). Equation (2.21) is said to be a geometric Brownian motion (GBM).

On the other hand, the arithmetic Brownian motion (with drift) of an asset return X_t can be given as the solution of

$$dX_t = \alpha dt + \sigma dW_t \tag{2.22}$$

with initial value $X_0 = x_0$. This arithmetic Brownian motion (ABM) can be obtained by integrating the SDE:

$$X_t = x_0 + \alpha t + \sigma W_t \tag{2.23}$$

It is possible to use the ABM to obtain the GBM; defining $S_t = e^{X_t}$ and using Itô's lemma:

$$dS_t = \left(\alpha + \frac{\sigma^2}{2}\right) S_t dt + \sigma S_t dW_t \qquad (2.24)$$

Without loss of generality and using $\mu = \alpha + \frac{\sigma^2}{2}$, the GBM is described as:

$$dS_t = \mu S_t dt + \sigma S_t dW_t \qquad (2.25)$$

with initial value S_0. The solution can then be determined by using the solution of the ABM

$$S_t = S_0 \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right\} \qquad (2.23)$$

Qualitative behaviour of this solution is as follows:

- If $\mu > \frac{\sigma^2}{2}$ then $S_t \to \infty$ when $t \to \infty$, a.s.
- If $\mu < \frac{\sigma^2}{2}$ then $S_t \to 0$ when $t \to \infty$, a.s.
- If $\mu = \frac{\sigma^2}{2}$ then S_t fluctuates between arbitrary large and small values as $t \to \infty$, a.s.

2.8 Use of the Ornstein Uhlenbeck Process in Asset Modelling

The class of stochastic process used to describe the characteristic of a process to revert to the mean is the Ornstein-Uhlenbeck (OU) process. This process is a model describing the velocity process of a Brownian motion. Uhlenbeck and Ornstein (1930) proposed the Ornstein-Uhlenbeck process, instead of the Brownian motion, in a physical modelling situation, where a mean-reverting tendency is captured in order to explain the situation modelled. In contrast to the Brownian motion where the drift term is assumed constant, the drift term of the OU process is determined by the current value of the process: the drift is positive if the current value of the process is less than the long-run mean and it is negative if the current value of the process is greater than the long-run mean. In general, the advantage of the OU process is that its sample functions have a tendency to revert to the initial position, a situation characterised by many interest rate situations. The OU process as a class of continuous stochastic process is parsimonious but sufficient to include a wide range of processes.

2.9 Empirical Evidence

This section presents evidence from modelling and forecasting rate of returns on assets using the OU stochastic process. The process is best identified with the Vasicek (1977) interest rate model. In fact, the OU process is widely utilised for modelling assets returns, but has other useful applications such as modelling commodity prices and currency exchange rates. For example, Chaiyapo and Phewchean (2017) used the OU process to price Thai commodity.

Panjer and Bellhouse (1980, 1981) studied stochastic modelling of the rates of return. The autoregressive, OU and white noise processes were the univariate models used in these papers to model the rates of return in continuous and discrete times. These stochastic models for the rates of return were used in computing net single premiums for both life annuity and whole life insurance policies. In the first paper, the results were derived under stationarity process, and in the second paper, the results were derived under a conditional autoregressive model for the rates of return.

Beekman and Fuelling (1990, 1991) introduced a model for use when the rates of return and future lifetimes are stochastic, for some annuities. In these papers, the OU process was utilised to model the accumulation function of interest rate, where they derived the mean and variance of life annuity functions. The motivation in these papers was deriving the moments to help estimate contingency reserves for a likely unfavourable interest and mortality occurrence for collections of life annuity policies.

Parker (1993, 1994), also, provided methods to determine the first three moments of the discount value of accumulated values for a portfolio of insurance or annuity contracts by assuming interest rates follow white noise or OU processes. In 1995, Parker then used a second order stochastic differential equation to modelled interest rates and provided the first three moments of certain annuities.

These univariate models are limited in that they failed to capture the correlation between individual assets in a portfolio, so a multivariate model is also studied. Wan (2010) modelled interest rates of three assets by both univariate and multivariate OU processes. He fitted both univariate and multivariate AR(1) models to the daily observed interest rates of 10-year long-term bond (a low-risk asset), three-month Treasury bill (a moderate-risk asset), and S&P 500 Index (a high-risk asset) for 35 years of the US market spanning from early 1974 to June 2009. These models were converted to corresponding OU processes to model accumulated interest rates. The results from the univariate model showed that the interest rates for the bond and the Treasury bill were highly dependent on the previous day's interest rates. But, the equity's rate of return for one day was weakly related to its rate of return for the previous day. Between the bond and the Treasury bill, the bond experienced lesser risk. However, the risk of the equity was much high these assets.

In the multivariate model, interest rates for both bond and Treasury bill have effect on the return rates on equity. On the other side, the rate of return of equity did not affect the returns on bond and Treasury bill. For long-term bond, the current interest rate depends highly on its past day's interest rate and on a marginal portion of interest rate of the Treasury bill for the previous day. For equity, its return rate for the previous day has approximately no effect on the bond. The current interest rate of Treasury bill depends much on its interest rate for the past day and on a minimal percent of the bond's previous day interest rate, with close to zero impact from the equity. The current rate of return for equity is explained by the previous day interest rates of both bond and Treasury bill; but more dependent on bond than Treasury bill. Compared to the other two assets, the current interest rate of equity is marginally explained by its interest rate for the previous day. Wan (2010) mentioned that it is very difficult to use the multivariate OU process to model interest rates, and also emphasised that there is much that needs to be studied about the use of the multivariate OU process in modelling interest rate rates. From the univariate OU process, the result revealed the accumulated return rate on equity reverts faster to the long-run mean than the bond and the Treasury bill.

Wan (2010) simulated 5000 sets of independent realisations. The result revealed that the three assets in shared one characteristic; the multivariate model takes extended time to revert to the long-run mean. In the short run, each model takes distinct time to revert to the long-run mean. For instance, the rate of return on equity takes almost 50 years to revert to the long-run mean, but not to a day to revert to the long-run mean in the univariate model. The interest rates were actually simulated to price annuities under several asset allocation strategies. For the asset allocation strategy to be adopted to realise the lowest price of annuity for a man aged 65, the models showed varied outcomes.

Qian (2010) also used three different methods under AR(1) and OU processes to fit assets returns in a portfolio. In the first method, each asset return is modelled by a univariate process, called the univariate model. The global model, the second method, computes the overall portfolio return with a pre-assigned percent on each asset, in which a single univariate process is used to estimate the combined return. In the third method, a multivariate model is used to model the assets returns. The daily return rates on ten-year Constant Maturity Treasury bills (as long-term bond), three-month Treasury bills (as short-term asset), and S&P 500 Index (as equity) in the US spanning from January 1974 to December 2007 were used. The interest rates of equity were highly dispersed and weakly correlated. The author decided to summarise the daily interest rate of each asset into an annual rate in order to reduce the noise. These observed annual rates were utilised to fit the AR(1) processes.

In the univariate model, the result showed that the current annual interest rates of the bond and Treasury bill depend greatly on the their interest rates of the past year. For equity, the current interest rate has a weak dependence on its interest rate for the past year. The three AR(1) processes were all stationary, and thus, were converted to their corresponding OU processes. From the OU processes, the result indicated that the conditional expected values of bond, Treasury bill and equity will respectively takes about 42, 19 and 1 year to revert 95% closer to their long-run equilibrium points.

To clearly examine the relationships among the assets in the multivariate context, three separate bivariate AR(1) processes were estimated taking two assets at a time. The three-dimensional AR(1) process was also estimated. In the bivariate AR(1) model for bond and Treasury bill, the results showed that the bond's interest rate depends on its last year interest rate (56.5%) and on last year interest rate (35%) of Treasury bill (35%). But, the Treasury bill's interest rate is much correlated with its past year interest rate (88%) and weakly correlated with last year interest rate of the bond.

For the other two bivariate AR(1) processes involving equity, the results showed that the interest rate on equity is highly dependent on the past year interest rates of bond or Treasury bill but not on the last year interest rate of the equity itself. Conversely, the last year interest rate of equity has negligible effect on the rates of returns on these instruments.

The three-dimensional AR(1) process exactly describes the same relationships as the bivariate AR(1) process involving the bond and Treasury bill. However, it revealed that the interest rate of the equity has mixed relationship forms using the past interest rates of bond and Treasury bill. It was thus complicated to directly explain how the rate of return of equity is related to the last year interest rates of the other instruments in the three-dimensional AR(1) process.

In the global model studied, the assets were combined for three portfolio cases: each portfolio altered by the percent weight of the assets. The results from these three AR(1) processes showed that the return and the risk of the portfolio increase if the portfolio has more percent invested in equity. Further, as more equity is included in the portfolio, the current rate of return has a weaker correlation with the rate of return for the past year.

The results of the three investment models were compared in the study and it was concluded that to for accurate estimations of the return rates, the multivariate model should be used. Qian (2010) also illustrated how the estimated results can be used to find the prices of annuities and optimise asset investment strategies. The stochastic models gave varied conclusions for these two applications.

2.10 Conclusion

From the review of literature, it can be concluded that different univariate stochastic processes have been utilised to capture the most distinct features of the return process of an asset. In the multivariate context, modelling asset returns is arduous task. Besides, capturing the individual dynamics of the assets, the multivariate model needs to reproduce the correlations between the assets involved in the portfolio. For instance, knowing the correlation structure is critical to optimum portfolio construction or asset allocation strategy. The next chapter elaborates on the statistical techniques and properties of the AR(1) and OU stochastic processes.

CHAPTER THREE

METHODOLOGY

3.0 Introduction

The approach of the autoregressive and the Ornstein-Uhlenbeck processes is well followed in this chapter. The formulas and methods for the AR(1) and OU processes described here closely follow Wan (2010) and Qian (2010). The OU process is an AR(1) process studied in continuous time. In modelling rates of return, either AR(1) or OU process can be used. However, the AR(1) process is first used for analysing the series because, in practice, series are collected at a certain discrete time interval. Then based on need, the AR(1) process can be converted to the OU process to study the return rates in different time intervals. Further, the chapter explains how to convert the AR(1) processes to OU processes under the covariance equivalence principle, which states that the first two moments of each process must be equal at all time. The data for the study is also expounded.

3.1 Investment Models

Three different stochastic models for fitting the assets return in a portfolio are studied. First, a univariate model which explains the each asset return in the portfolio by a univariate stochastic process is discussed. Second, instead of fitting a separate univariate process to each asset in the portfolio, the return rates of the assets in the portfolio are combined based on the percents invested in each asset. This is called a global model. Third, a multivariate model that describes a multivariate process to the assets returns in the portfolio is analysed. The accumulation functions of the rate of returns are also derived because it is used to determine the future value or present value of a portfolio and also crucial for pricing insurance products. In the discussion, transaction costs involved in investing are ignored.

3.2 Univariate Model

Many studies have focused on the univariate AR(1) and OU processes for modelling rates of returns on assets, prices of commodities (such as oil prices), and currency exchange rates. This section gives the parametric relations between these stochastic processes.

3.2.1 AR (1) Process

From Equation (2.17), the univariate AR(1) process for the asset returns is stated as

$$X_t - \mu = \phi\left(X_{t-1} - \mu\right) + a_t, \quad t = 1, 2, 3, \ldots \tag{3.1}$$

where μ is the mean of the asset returns, ϕ is the correlation coefficient between the current asset return and previous asset return, and a_t is a random error with a key assumption that the a_t's are independent and follow identical normal distribution. If $\phi = 1$, the asset return process is a random walk, and if $\phi = 0$, it is a white noise. If $0 < \phi < 1$, the asset return process satisfies mean-reverting and stationarity condition, and an OU process can be estimated.

Assuming that the process is mean-reverting and stationary and given initial value as X_0, some relations for the AR(1) process can be obtained. For $t = 1, 2, 3, \ldots$,

$$X_t - \mu = \phi^t\left(X_0 - \mu\right) + \sum_{j=0}^{t-1} \phi^j a_{t-j} \tag{3.2}$$

Therefore,

$$E\left(X_t \mid X_0\right) = \mu + \phi^t\left(X_0 - \mu\right) \tag{3.3}$$

$$Var\left(X_t \mid X_0\right) = \frac{1-\phi^{2t}}{1-\phi^2}\sigma_a^2 \tag{3.4}$$

$$Cov\left(X_s, X_t \mid X_0\right) = \phi^{|t-s|}\left(\frac{1-\phi^{2\min(t,s)}}{1-\phi^2}\right)\sigma_a^2 \tag{3.5}$$

The assumption that the AR(1) process is stationary means that its first two moments will always exist as $t \to \infty$:

$$\lim_{t\to\infty} E\left(X_t \mid X_0\right) = \mu \tag{3.6}$$

$$\lim_{t\to\infty} Var\left(X_t \mid X_0\right) = \frac{\sigma_a^2}{1-\phi^2} \tag{3.7}$$

3.2.2 Ornstein-Uhlenbeck Process

It is generally assumed that asset return tends to revert to its long-run stationary point, usually its historical average value. If the tendency of reversion towards a stationary point is needed, the OU process can be used. That is, the OU process is a stochastic process that tends to measure the speed of mean reversion, an important feature of an asset return. Its velocity and position processes are studied below.

OU Velocity Process

The OU velocity process is sometimes called the Vasicek model. The stochastic process X_t is a univariate Gaussian OU process if it satisfies the SDE:

$$dX_t = -\alpha\left(X_t - \mu\right)dt + \sigma dW_t, \quad t \ge 0 \tag{3.8}$$

where α, μ, and σ are all strictly non-negative constants and W_t is a standard Weiner process. In Equation (3.8), μ is the asymptotic mean return rate of the asset (the long run mean, the asset return rate tends to revert to). The parameter α is the speed of reversion (growth or decay rate), and the larger the absolute value, the faster the reversion is. If more time elapses, the process should be closer to its stationary point. The parameter σ measures the instantaneous volatility of the asset return rate. The larger this value, the higher the asset return rate's volatility. The solution of this SDE is:

$$X_t - \mu = e^{-\alpha t}\left(X_0 - \mu\right) + \sigma \int_0^t e^{-\alpha(t-s)} dW_s, \quad t \geq 0 \qquad (3.9)$$

Given the asset return rate at time 0 (X_0), the following are obtained:

$$E\left(X_t \mid X_0\right) = E\left[\mu + e^{-\alpha t}\left(X_0 - \mu\right) + \sigma \int_0^t e^{-\alpha(t-s)} dW_s\right] = \mu + e^{-\alpha t}\left(X_0 - \mu\right) \qquad (3.10)$$

$$Var\left(X_t \mid X_0\right) = E\left[\left(\sigma \int_0^t e^{-\alpha(t-s)} dW_s\right)^2\right] = \frac{\sigma^2}{2\alpha}\left(1 - e^{-2\alpha t}\right) \qquad (3.11)$$

$$Cov\left(X_s, X_t \mid X_0\right) = e^{-\alpha(t+s)}\left(\frac{e^{2\alpha \min(t,s)} - 1}{2\alpha}\right)\sigma^2 \qquad (3.12)$$

$$\lim_{t \to \infty} E\left(X_t \mid X_0\right) = \mu \qquad (3.13)$$

$$\lim_{t \to \infty} Var\left(X_t \mid X_0\right) = \frac{\sigma^2}{2\alpha} \qquad (3.14)$$

OU Position Process

The OU process gives the formula that is use to determine the accumulated rate of return on an asset. The OU position process, Y_t, is derived as an integration of the process X_t:

$$Y_t = Y_0 + \int_0^t X_s ds \qquad (3.15)$$

In the financial environment, Y_0 is usually 0 and given that X_0 is a known value, Y_t is a Gaussian process as well. The first two moments of Y_t are given as:

$$E\left(Y_t \mid X_0\right)=E\left[\int_0^t X_s ds\right]=\int_0^t E\left[X_s ds\right]=\frac{1-e^{-\alpha t}}{\alpha}\left(X_0-\mu\right)+\mu t \tag{3.16}$$

$$Var\left(Y_t \mid X_0\right)=\frac{\sigma^2}{\alpha^2}+\frac{\sigma^2}{2\alpha^3}\left(-3+4e^{-\alpha t}-e^{-2\alpha t}\right) \tag{3.17}$$

$$Cov\left(Y_s,Y_t \mid X_0\right)=\frac{\sigma^2}{\alpha^2}\min\left(s,t\right)+\frac{\sigma^2}{2\alpha^3}\left[-2+2e^{-\alpha t}+2e^{-\alpha s}-e^{-\alpha|t-s|}-e^{-\alpha(t+s)}\right] \tag{3.18}$$

3.2.3 Equivalent AR(1) and OU Process

The parametric equations between the AR(1) process and the OU process can be found by matching the covariance for each process at all time. Suppose the process is observed at a time interval of Δ starting from 0, then the covariance between the asset return at time t and k past periods ($t\Delta-k\Delta$) is:

$$Cov\left(X_{t\Delta},X_{t\Delta-k\Delta} \mid X_0\right)=\frac{\sigma^2}{2\alpha}e^{-\alpha k\Delta}\left(1-e^{-2\alpha\left(t\Delta-k\Delta\right)}\right) \tag{3.19}$$

Now, using Equation (3.5), the covariance in the form of Equation (3.19) for a conditional AR(1) process is:

$$Cov\left(X_t,X_{t-k} \mid X_0\right)=\frac{\sigma_a^2}{1-\phi^2}\phi^k\left(1-\phi^{-2\left(t-k\right)}\right) \tag{3.20}$$

From Equations (3.19) and (3.20), these results are derived:

$$\phi=e^{-\alpha\Delta} \quad \text{or} \quad \alpha=\frac{-\ln\phi}{\Delta} \tag{3.21}$$

and

$$\frac{\sigma_a^2}{1-\phi^2}=\frac{\sigma^2}{2\alpha}. \tag{3.22}$$

It is required that σ, σ_a, ϕ, and α in Equation (3.22) have equal time scale. Equally, the relation between ϕ and α can be determined by matching the first moment of Equation (3.3) to that of Equation (3.10). From these equations, $\phi=e^{-\alpha\Delta}$. It is clear from their parametric relations that to satisfy Equation (3.21), $\phi > 0$ is required. This study is limited to stationary processes, so it is also required that $\phi < 1$ and $\alpha > 0$ to satisfy Equation (3.22).

3.2.4 Portfolio Return under Univariate Model

The overall portfolio return in time t is sum of each asset return where each asset's percent-weight forming the portfolio is considered. For only two assets in a portfolio, we have two OU processes as:

$$X_{i,t} - \mu_i = e^{-\alpha_i t}\left(X_{i,0} - \mu_i\right) + \sigma_i \int_0^t e^{-\alpha_i (t-s)} ds \qquad (3.23)$$

where $X_{i,t}$ is the instantaneous return rate of asset i $(i = 1, 2)$ at time t; $X_{i,0}$ is the instantaneous return rate of asset i at initial time 0; α_i and σ_i are the parameters of the univariate OU process to explain the behaviour of the return rate of asset i. The overall portfolio return can then be calculated as a combination of the rates of return of the two assets taking into consideration their corresponding percentage weights in the portfolio as:

$$X_{P,t} = \sum_{i=1}^{2} \omega_i \cdot X_{i,t} = \omega_1 \cdot X_{1,t} + \omega_2 \cdot X_{2,t} \qquad (3.24)$$

where $X_{P,t}$ is the accumulated rate of return of the portfolio, ω_i is the percent-weight of each asset in the portfolio, and $\sum_{i=1}^{2} \omega_i = 1$. Further, $\forall t$, this gives:

$$E\left(X_{P,t}\right) = \omega_1 \cdot \left[e^{-\alpha_1 t}\left(X_{1,0} - \mu_1\right) + \mu_1\right] + \omega_2 \cdot \left[e^{-\alpha_2 t}\left(X_{2,0} - \mu_2\right) + \mu_2\right] \qquad (3.25)$$

$$\sigma^2\left(X_{P,t}\right) = \omega_1^2 \cdot Var\left(X_{1,t}\right) + \omega_2^2 \cdot Var\left(X_{2,t}\right) \qquad (3.26)$$

$$Cov\left(X_{P,t}, X_{P,s}\right) = \omega_1^2 \cdot Cov\left(X_{1,t}, X_{1,s}\right) + \omega_2^2 \cdot Cov\left(X_{2,t}, X_{2,s}\right) \qquad (3.27)$$

The use of a separate univariate model to fit each asset return in the portfolio is based on the idea that there is no correlation between these two assets. Thus, in Equation (3.25), $Cov\left(X_{1,t}, X_{2,t}\right)=0$. From Equation (3.15), we know that the accumulation function of X_t is defined as $Y_t = Y_0 + \int_0^t X_s ds$. Based on the univariate model, the following formulas for the accumulation functions of the return on the portfolio ($Y_{P,t}$) are obtained:

$$Y_{P,t} = \omega_1 \cdot Y_{1,t} + \omega_2 \cdot Y_{2,t} \tag{3.28}$$

$$E\left(Y_{P,t}\right) = \omega_1 \cdot E\left(Y_{1,t}\right) + \omega_2 \cdot E\left(Y_{2,t}\right) \tag{3.29}$$

$$Var\left(Y_{P,t}\right) = \omega_1^2 \cdot Var\left(Y_{1,t}\right) + \omega_2^2 \cdot Var\left(Y_{2,t}\right) \tag{3.30}$$

$$Cov\left(Y_{P,t}, Y_{P,s}\right) = \omega_1^2 \cdot Cov\left(Y_{1,t}, Y_{1,s}\right) + \omega_2^2 \cdot Cov\left(Y_{2,t}, Y_{2,s}\right) \tag{3.31}$$

where $Y_{i,t} = Y_{i,0} + \int_0^t X_{i,s} ds$ for $i = 1, 2$.

The univariate stochastic model is sufficient to use to fit the rates of returns on assets in a portfolio if the assets are highly uncorrelated. In modern financial market, there are high chances that the assets in a portfolio are correlated. It is, thus, inappropriate to use a univariate model if the assets are really correlated.

3.3 Multivariate Model

The stochastic processes for modelling different assets returns included in a portfolio by the multivariate AR(1) and OU models are considered in this section. The rationale of introducing a multivariate model is to incorporate the correlations between asset returns, which is absent in the univariate model. The parametric relations are also studied.

3.3.1 Multivariate AR (1) Process

The multivariate AR(1) process, denoted as VAR(1), can be given as:

$$\begin{bmatrix} X_{1,t}-\mu_1 \\ X_{2,t}-\mu_2 \\ \vdots \\ X_{n,t}-\mu_n \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{bmatrix} \begin{bmatrix} X_{1,t-1}-\mu_1 \\ X_{2,t-1}-\mu_2 \\ \vdots \\ X_{n,t-1}-\mu_n \end{bmatrix} + \begin{bmatrix} a_{1,t} \\ a_{2,t} \\ \vdots \\ a_{n,t} \end{bmatrix} \tag{3.32}$$

In vector form, the VAR(1) process is restated as:

$$\underline{\boldsymbol{X}}_t - \underline{\boldsymbol{\mu}} = \underline{\boldsymbol{\Phi}}\left(\underline{\boldsymbol{X}}_{t-1} - \underline{\boldsymbol{\mu}}\right) + \underline{\boldsymbol{a}}_t, \quad t = 1, 2, \ldots \tag{3.33}$$

In Equation (3.33), $\underline{\boldsymbol{\Phi}}$ is a square matrix of numerical values that describes the dependences between $\underline{\boldsymbol{X}}_{k,t}$ and $\underline{\boldsymbol{X}}_{s,t-1}$ $(t \geq 1)$. Vector $\underline{\boldsymbol{a}}_t$ has a multivariate Gaussian distribution with mean $\underline{\boldsymbol{\mu}} = 0$ and covariance matrix $\underline{\boldsymbol{\Sigma}}_a$. For an initial value given as vector $\underline{\boldsymbol{X}}_0$. The VAR(1) process can be stated as:

$$\underline{\boldsymbol{X}}_t - \underline{\boldsymbol{\mu}} = \underline{\boldsymbol{\Phi}}^t\left(\underline{\boldsymbol{X}}_0 - \underline{\boldsymbol{\mu}}\right) + \sum_{j=0}^{t-1} \underline{\boldsymbol{\Phi}}^j \underline{\boldsymbol{a}}_{t-j}, \quad t \geq 1 \tag{3.34}$$

From Equation (3.34), the conditional mean and variance of $\underline{\boldsymbol{X}}_t$ are as follows:

$$E\left(\underline{\boldsymbol{X}}_t \mid \underline{\boldsymbol{X}}_0\right) = \underline{\boldsymbol{\mu}} + \underline{\boldsymbol{\Phi}}^t\left(\underline{\boldsymbol{X}}_0 - \underline{\boldsymbol{\mu}}\right) \tag{3.35}$$

$$Var\left(\underline{\boldsymbol{X}}_t \mid \underline{\boldsymbol{X}}_0\right) = \sum_{i=0}^{t-1} \underline{\boldsymbol{\Phi}}^i \underline{\boldsymbol{\Sigma}}_a \left(\underline{\boldsymbol{\Phi}}^i\right)^T \tag{3.36}$$

The conditional covariance of Equation (3.34) is derived as:

$$\begin{aligned} Cov\left(\underline{\boldsymbol{X}}_t, \underline{\boldsymbol{X}}_{t-k} \mid \underline{\boldsymbol{X}}_0\right) &= Cov\left(\underline{\boldsymbol{\Phi}}^t\left(\underline{\boldsymbol{X}}_0 - \underline{\boldsymbol{\mu}}\right) + \sum_{j=0}^{t-1} \underline{\boldsymbol{\Phi}}^j \underline{\boldsymbol{a}}_{j-i},\ \underline{\boldsymbol{\Phi}}^{t-k}\left(\underline{\boldsymbol{X}}_0 - \underline{\boldsymbol{\mu}}\right) + \sum_{i=0}^{t-k-1} \underline{\boldsymbol{\Phi}}^i \underline{\boldsymbol{a}}_{t-k-i} \,\middle|\, \underline{\boldsymbol{X}}_0\right) \\ &= Cov\left(\sum_{j=0}^{t-1} \underline{\boldsymbol{\Phi}}^j \underline{\boldsymbol{a}}_{t-j},\ \sum_{i=0}^{t-k-1} \underline{\boldsymbol{\Phi}}^i \underline{\boldsymbol{a}}_{t-k-i}\right) \\ &= \sum_{j=0}^{t-1} \sum_{i=0}^{t-k-1} Cov\left(\underline{\boldsymbol{\Phi}}^j \underline{\boldsymbol{a}}_{t-j},\ \underline{\boldsymbol{\Phi}}^i \underline{\boldsymbol{a}}_{t-k-i}\right) \end{aligned}$$

$$= \sum_{j=0}^{t-1} \sum_{i=0}^{t-k-1} E\left(\left(\underline{\mathbf{\Phi}}^{j} \underline{\boldsymbol{a}}_{t-j} \right) \left(\underline{\mathbf{\Phi}}^{i} \underline{\boldsymbol{a}}_{t-k-i} \right)^{T} \right)$$

$$Cov\left(\underline{\boldsymbol{X}}_{t}, \underline{\boldsymbol{X}}_{t-k} \mid \underline{\boldsymbol{X}}_{0} \right) = \sum_{i=0}^{t-k-1} \underline{\mathbf{\Phi}}^{k+i} \underline{\mathbf{\Sigma}}_{a} \left(\underline{\mathbf{\Phi}}^{i} \right)^{T} \qquad (3.37)$$

In Equation (3.37), $\left(\underline{\mathbf{\Phi}}^{i} \right)^{T}$ is defined as the transpose matrix of $\underline{\mathbf{\Phi}}^{i}$.

3.3.2 Multivariate Ornstein-Uhlenbeck Process

The multivariate OU process for modelling rates of return is useful for describing the multivariate dynamics of the assets involved in the portfolio.

Multivariate OU Velocity Process

The multivariate OU process is stated in the form of SDE as:

$$d\underline{\boldsymbol{X}}_{t} = \underline{\mathbf{A}}\left(\underline{\boldsymbol{X}}_{t} - \underline{\boldsymbol{\mu}} \right) dt + \underline{\boldsymbol{\sigma}} d\underline{\boldsymbol{W}}_{t}, \quad t \geq 0 \qquad (3.38)$$

In this expression, $\underline{\mathbf{A}}$ is a fully generic square matrix that describing the continuous change of every asset return depending on its current level and the levels of other assets returns; $\underline{\boldsymbol{\mu}}$ is a vector of the expected values; $\underline{\boldsymbol{\sigma}}$ is a diffusion matrix indicating dispersion of the process); and $\underline{\boldsymbol{W}}_{t}$ is a vector of independent Gaussian Brownian motions. Equation (3.38) can be restated in matrix representation as:

$$d\begin{bmatrix} X_{1,t} \\ X_{2,t} \\ \vdots \\ X_{n,t} \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{bmatrix} \begin{bmatrix} X_{1,t-1} - \mu_{1} \\ X_{2,t-1} - \mu_{2} \\ \vdots \\ X_{n,t-1} - \mu_{n} \end{bmatrix} dt + \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_{nn} \end{bmatrix} d\begin{bmatrix} W_{1,t} \\ W_{2,t} \\ \vdots \\ W_{n,t} \end{bmatrix} \qquad (3.39)$$

The solution of the SDE in Equation (3.38) is

$$\underline{\boldsymbol{X}}_{t} - \underline{\boldsymbol{\mu}} = e^{\underline{\boldsymbol{A}}t} \left(\underline{\boldsymbol{X}}_{0} - \underline{\boldsymbol{\mu}} \right) + \int_{0}^{t} e^{\underline{\boldsymbol{A}}(t-s)} \underline{\boldsymbol{\sigma}} d\underline{\boldsymbol{W}}_{s}, \quad t \geq 0 \qquad (3.40)$$

In Equation (3.40), $\underline{\mathbf{A}}$ is the time (speed) that the process times to revert to its long run mean from a given start value. Here, the speed of the mean reversion is a pooled effect of all assets returns involved in the portfolio; and $\underline{\boldsymbol{\sigma}}$ measures the instantaneous volatility from all assets returns.

For an initial value of $\underline{\boldsymbol{X}}_0$, the following results are obtained:

$$E\left(\underline{\boldsymbol{X}}_t \mid \underline{\boldsymbol{X}}_0\right)=\underline{\boldsymbol{\mu}}+e^{\underline{\boldsymbol{A}}t}\left(\underline{\boldsymbol{X}}_0-\underline{\boldsymbol{\mu}}\right) \tag{3.41}$$

$$Var\left(\underline{\boldsymbol{X}}_t \mid \underline{\boldsymbol{X}}_0\right)=e^{\underline{\boldsymbol{A}}t}\left[\int_0^t\left(e^{\underline{\boldsymbol{A}}u}\right)^{-1}\underline{\boldsymbol{\Sigma}}_{OU}\left(\left(e^{\underline{\boldsymbol{A}}u}\right)^{-1}\right)^T du\right]\left(e^{\underline{\boldsymbol{A}}t}\right)^T \tag{3.42}$$

$$Cov\left(\underline{\boldsymbol{X}}_s,\underline{\boldsymbol{X}}_t \mid \underline{\boldsymbol{X}}_0\right)=e^{\underline{\boldsymbol{A}}s}\left[\int_0^{\min(s,t)}\left(e^{\underline{\boldsymbol{A}}u}\right)^{-1}\underline{\boldsymbol{\Sigma}}_{OU}\left(\left(e^{\underline{\boldsymbol{A}}u}\right)^{-1}\right)^T du\right]\left(e^{\underline{\boldsymbol{A}}t}\right)^T \tag{3.43}$$

where $\underline{\boldsymbol{\Sigma}}_{OU}=\underline{\boldsymbol{\sigma}}\cdot\underline{\boldsymbol{\sigma}}^T$

Multivariate OU Position Process

The integration of $\underline{\boldsymbol{X}}_t$ gives the multivariate OU position process $\underline{\boldsymbol{Y}}_t$:

$$\underline{\boldsymbol{Y}}_t=\underline{\boldsymbol{Y}}_0+\int_0^t\underline{\boldsymbol{X}}_s ds \tag{3.44}$$

Equally, considering the SDE

$$d\begin{bmatrix}\underline{\boldsymbol{X}}_t\\ \underline{\boldsymbol{Y}}_t\end{bmatrix}=\underline{\boldsymbol{B}}\cdot\begin{bmatrix}\underline{\boldsymbol{X}}_t-\underline{\boldsymbol{\mu}}\\ \underline{\boldsymbol{Y}}_t-\underline{\boldsymbol{\mu}}t\end{bmatrix}dt+\underline{\boldsymbol{\sigma}}_Y d\begin{bmatrix}\underline{\boldsymbol{W}}t\\ \underline{\boldsymbol{W}}t\end{bmatrix} \tag{3.45}$$

where $\underline{\boldsymbol{B}}=\begin{bmatrix}\underline{\boldsymbol{A}} & \underline{\mathbf{0}}\\ \underline{\boldsymbol{E}} & \underline{\mathbf{0}}\end{bmatrix}$, $\underline{\boldsymbol{E}}$ is identity matrix of size n, and $\underline{\boldsymbol{\sigma}}_Y=\begin{bmatrix}\underline{\boldsymbol{\sigma}} & \underline{\mathbf{0}}\\ \underline{\mathbf{0}} & \underline{\mathbf{0}}\end{bmatrix}$.

The solution of Equation (3.45) is:

$$\begin{bmatrix}\underline{\boldsymbol{X}}_t-\underline{\boldsymbol{\mu}}\\ \underline{\boldsymbol{Y}}_t-\underline{\boldsymbol{\mu}}t\end{bmatrix}=e^{\underline{\boldsymbol{B}}t}\begin{bmatrix}\underline{\boldsymbol{X}}_0-\underline{\boldsymbol{\mu}}\\ \underline{\boldsymbol{Y}}_0\end{bmatrix}+\int_0^t e^{\underline{\boldsymbol{B}}(t-s)}\underline{\boldsymbol{\sigma}}_Y d\underline{\boldsymbol{W}}_s \tag{3.46}$$

3.3.3 Equivalent VAR(1) and Multivariate OU Process

It is difficult to derive the parametric equations in the multivariate process. To obtain these relations, however, the covariance equivalence that exists between the matrices $\underline{\mathbf{A}}, \underline{\sigma}$ and $\underline{\mathbf{\Phi}}, \underline{\Sigma}_a$ must be found. Supposing an VAR (1), the matrices $\underline{\mathbf{A}}$ and $\underline{\sigma}$ must be found to estimate the multidimensional OU process.

Matrix $\underline{\mathbf{A}}$

From the covariance principle, the first moment of Equation (3.41) should be equal to that of Equation (3.35). $\forall t$, this means

$$E\left(\underline{\boldsymbol{X}}_t \mid \underline{\boldsymbol{X}}_0\right) = \underline{\boldsymbol{\mu}} + e^{\underline{\boldsymbol{A}}t}\left(\underline{\boldsymbol{X}}_0 - \underline{\boldsymbol{\mu}}\right) = \underline{\boldsymbol{\mu}} + \underline{\mathbf{\Phi}}^t\left(\underline{\boldsymbol{X}}_0 - \underline{\boldsymbol{\mu}}\right),$$

which implies that

$$e^{\underline{\boldsymbol{A}}} = \underline{\mathbf{\Phi}} \tag{3.47}$$

Estimating $e^{\underline{\boldsymbol{A}}}$ is sufficient for the velocity process, but $\underline{\mathbf{A}}$ is needed if the multivariate OU position process is of interest. The eigenvalues and eigenvectors of matrix $\underline{\mathbf{A}}$ are largely used to obtain $e^{\underline{\boldsymbol{A}}t}$. If the eigenvalues of $\underline{\mathbf{A}}$ are $\mu_1, \mu_2, \dots, \mu_n$ with corresponding eigenvectors $\left([v_{11}, v_{21}, \dots, 1]^T, [v_{12}, v_{22}, \dots, 1]^T, \dots, [v_{1n}, v_{2n}, \dots, 1]^T\right)$, then

$$E\left(\underline{\boldsymbol{X}}_t \mid \underline{\boldsymbol{X}}_0\right) = e^{\underline{\boldsymbol{A}}t}\underline{\boldsymbol{X}}_0 = \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} c_1 e^{u_1 t} \\ c_2 e^{u_2 t} \\ \vdots \\ c_n e^{u_n t} \end{bmatrix} \tag{3.48}$$

where $c_1, c_2, \dots, c_n$ are constants. From Equation (3.48) with $t = 0$, this gives

$$E\left(\underline{\boldsymbol{X}}_t \mid \underline{\boldsymbol{X}}_0\right) = \underline{\boldsymbol{X}}_0 = \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \tag{3.49}$$

The simultaneous linear equation in Equation (3.49) is solved to get $c_1, c_2, \ldots, c_n$. Equation (3.49) gives $\underline{\boldsymbol{c}} = \underline{\boldsymbol{V}}^{-1}\underline{\boldsymbol{X}}_0$. Putting $\underline{\boldsymbol{c}}$ into Equation (3.48) and matching the coefficients of $\underline{\boldsymbol{X}}_0$ on each row on both sides of the matrices gives the expression $e^{\underline{\boldsymbol{A}}t}$.

To obtain $\underline{\mathbf{A}}$, the eigenvalues and eigenvectors of matrix $\underline{\mathbf{\Phi}}$ must also be calculated. Suppose that matrix $\underline{\mathbf{\Phi}}$ has n eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. If $e^{\underline{\boldsymbol{A}}} = \underline{\mathbf{\Phi}}$, then the eigenvalues of $\underline{\mathbf{A}}$ are $\mu_i = \ln(\lambda_i)$ $(i = 1, 2, \ldots, n)$ with its eigenvectors as same as those of $\underline{\mathbf{A}}$; this gives the simultaneous linear equation:

$$\mu \underline{\boldsymbol{v}} = \underline{\boldsymbol{A}}\underline{\boldsymbol{v}} \tag{3.50}$$

where μ is one eigenvalue of $\underline{\mathbf{A}}$ with $\underline{\boldsymbol{v}}$ as its eigenvector. $\underline{\mathbf{A}}$ is expressed as:

$$\underline{\boldsymbol{A}} = \underline{\boldsymbol{V}}\,\underline{\mathbf{\Lambda}}\,\underline{\boldsymbol{V}}^{-1} \tag{3.51}$$

In Equation (3.51), $\underline{\mathbf{\Lambda}}$ is defined as a diagonal matrix formed by eigenvalues of $\underline{\mathbf{A}}$ and each column in the $n \times n$ matrix $\underline{\boldsymbol{V}}$ is the resultant eigenvector. If all the eigenvalues of $\underline{\mathbf{\Phi}}$ lie between 0 and 1, then the multivariate OU process is stationary.

Matrix $\underline{\boldsymbol{\sigma}}$

To get matrix $\underline{\boldsymbol{\sigma}}$, first solve for $\underline{\mathbf{\Sigma}}_{OU} = \underline{\boldsymbol{\sigma}} \cdot \underline{\boldsymbol{\sigma}}^T$ where $\underline{\boldsymbol{\sigma}}$ can be defined as a lower triangular matrix in order to make it easily solved using the idea of Cholesky decomposition.

3.3.4 Portfolio Return under Bivariate Model

In this study, one main object is to use a bivariate AR(1) process to fit the two assets returns involved in the portfolio. Based on need, the estimated bivariate AR(1) process may be converted to its bivariate OU process. In the bivariate OU process, the mean and the variance of $\underline{\boldsymbol{X}}_t$ are obtained as follows.

First, a bivariate AR(1) process is fitted on the historical assets returns in the portfolio to explain the instantaneous interest rate $\underline{\boldsymbol{X}}_t$ as:

$$\begin{bmatrix} X_{1,t} \\ X_{2,t} \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} X_{1,t-1} \\ X_{2,t-1} \end{bmatrix} + \begin{bmatrix} a_{1,t} \\ a_{2,t} \end{bmatrix}, \quad t \geq 1 \tag{3.52}$$

where $\underline{\boldsymbol{a}}_t$ has a bivariate Gaussian distribution with $\underline{\boldsymbol{\mu}} = 0$ and a 2×2 covariance matrix $\underline{\boldsymbol{\Sigma}}_a$. Suppose initial value of $\underline{\boldsymbol{X}}_0$, then

$$\begin{bmatrix} X_{1,t} \\ X_{2,t} \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}^t \cdot \begin{bmatrix} X_{1,0} \\ X_{2,0} \end{bmatrix} + \sum_{j=0}^{t-1} \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}^j \cdot \begin{bmatrix} a_{1,t-j} \\ a_{2,t-j} \end{bmatrix} \tag{3.53}$$

Second, the bivariate AR(1) process is used to compute the parameter $\underline{\mathbf{A}}$ of the equivalent bivariate OU process. Explicit expression for $e^{\underline{A}t}$ is also obtained. The eigenvalues (λ_1, λ_2) and the corresponding eigenvectors ($[v_{11},1]^T, [v_{12},1]^T$) of matrix $\hat{\underline{\mathbf{\Phi}}}$ is solved numerically. From Section (3.3.3) and Equation (3.47), the eigenvalues of $\underline{\mathbf{A}}$ are $\mu_1 = \ln(\lambda_1), \mu_2 = \ln(\lambda_2)$. The eigenvectors of matrices $\underline{\mathbf{A}}$ and $\underline{\mathbf{\Phi}}$ are the same. So, matrix $\underline{\mathbf{A}}$ is solved using Equation (3.51), that is:

$$\underline{\boldsymbol{A}} = \underline{\boldsymbol{V}}\,\underline{\boldsymbol{\Lambda}}\,\underline{\boldsymbol{V}}^{-1} = \begin{bmatrix} v_{11} & v_{12} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \ln(\lambda_1) & 0 \\ 0 & \ln(\lambda_2) \end{bmatrix} \begin{bmatrix} v_{11} & v_{12} \\ 1 & 1 \end{bmatrix}^{-1} \tag{3.54}$$

If matrix $\underline{\mathbf{A}}$ is obtained, its eigenvalues and eigenvectors are used to derive the explicit expression for $e^{\underline{A}t}$. This is sketch as follows. Using Equation (3.48):

$$E(\underline{\boldsymbol{X}}_t \mid \underline{\boldsymbol{X}}_0) = e^{\underline{A}t}\,\underline{\boldsymbol{X}}_0 = \begin{bmatrix} v_{11} & v_{12} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 e^{u_1 t} \\ c_2 e^{u_2 t} \end{bmatrix} = \begin{bmatrix} v_{11} & v_{12} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \lambda_1^t \\ c_2 \lambda_2^t \end{bmatrix} \tag{3.55}$$

where c_1, c_2 are constants. For $t=0$, Equation (3.53) becomes

$$\underline{\boldsymbol{X}}_0 = \begin{bmatrix} X_{1,0} \\ X_{2,0} \end{bmatrix} = \begin{bmatrix} v_{11} & v_{12} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \tag{3.56}$$

From Equation (3.56), $\underline{\boldsymbol{c}} = \underline{\boldsymbol{V}}^{-1} \underline{\boldsymbol{X}}_0$. So, c_1, c_2 can be expressed in term of $\underline{\boldsymbol{X}}_0$ and $\underline{\boldsymbol{V}}$ as:

$$c_1 = \frac{-X_{1,0} + v_{12} X_{2,0}}{\beta} \tag{3.57}$$

$$c_2 = \frac{X_{1,0} - v_{11} X_{2,0}}{\beta} \tag{3.58}$$

where $\beta = v_{12} - v_{11}$; β is the determinant of the eigenvector matrix $\underline{\boldsymbol{V}}$. Now, if putting c_1, c_2 into Equation (3.55), the formula for $e^{\underline{\boldsymbol{A}}t}$ can be obtained as:

$$e^{\underline{\boldsymbol{A}}t} = \frac{1}{\beta} \begin{bmatrix} v_{12}\lambda_2^t - v_{11}\lambda_1^t & v_{11}v_{12}\left(\lambda_1^t - \lambda_2^t\right) \\ \lambda_2^t - \lambda_1^t & v_{12}\lambda_1^t - v_{11}\lambda_2^t \end{bmatrix} \tag{3.59}$$

Engaging in symbolic calculation and using Equation (3.43), $OU_{cov} = Cov\left(X_s, X_t\right)$ can be computed. Letting $\underline{\boldsymbol{\Sigma}}'_{OU} = \begin{pmatrix} \Sigma_{OU_{11}} \\ \Sigma_{OU_{12}} \\ \Sigma_{OU_{21}} \\ \Sigma_{OU_{22}} \end{pmatrix}$ and simplifying, the elements of OU_{cov} are:

$$OU_{cov}[1,1] = \frac{\underline{\boldsymbol{\tau}} \cdot \underline{\boldsymbol{\pi}}_{11} \cdot \underline{\boldsymbol{\Sigma}}'_{OU}}{2\log(\lambda_1)\log(\lambda_2)\left(\log(\lambda_1)+\log(\lambda_2)\right)(v_1 - v_2)^2}$$

$$OU_{cov}[1,2] = \frac{\underline{\boldsymbol{\tau}} \cdot \underline{\boldsymbol{\pi}}_{12} \cdot \underline{\boldsymbol{\Sigma}}'_{OU}}{2\log(\lambda_1)\log(\lambda_2)\left(\log(\lambda_1)+\log(\lambda_2)\right)(v_1 - v_2)^2}$$

$$OU_{cov}[2,1] = \frac{\underline{\boldsymbol{\tau}} \cdot \underline{\boldsymbol{\pi}}_{21} \cdot \underline{\boldsymbol{\Sigma}}'_{OU}}{2\log(\lambda_1)\log(\lambda_2)\left(\log(\lambda_1)+\log(\lambda_2)\right)(v_1 - v_2)^2}$$

$$OU_{cov}[2,2] = \frac{\underline{\boldsymbol{\tau}} \cdot \underline{\boldsymbol{\pi}}_{22} \cdot \underline{\boldsymbol{\Sigma}}'_{OU}}{2\log(\lambda_1)\log(\lambda_2)\left(\log(\lambda_1)+\log(\lambda_2)\right)(v_1 - v_2)^2}$$

where

$$\underline{\tau}^T = \begin{pmatrix} \lambda_1^s \lambda_2^t \log(\lambda_1)\log(\lambda_2) \\ \lambda_1^t \lambda_2^s \log(\lambda_1)\log(\lambda_2) \\ \lambda_1^{t-s} \log(\lambda_1)\log(\lambda_2) \\ \lambda_2^{t-s} \log(\lambda_1)\log(\lambda_2) \\ \lambda_1^{t+s} \log(\lambda_1)\log(\lambda_2) \\ \lambda_2^{t+s} \log(\lambda_1)\log(\lambda_2) \\ \lambda_1^{t-s} (\log(\lambda_2))^2 \\ \lambda_2^{t-s} (\log(\lambda_1))^2 \\ \lambda_1^{t+s} (\log(\lambda_2))^2 \\ \lambda_2^{t+s} (\log(\lambda_1))^2 \end{pmatrix}$$

$$\underline{\pi}_{11} = \begin{pmatrix} -2v_1v_2 & 2v_1^2v_2 & 2v_1v_2^2 & -2v_1^2v_2^2 \\ -2v_1v_2 & 2v_1v_2^2 & 2v_1^2v_2 & -2v_1^2v_2^2 \\ v_1(2v_2 - v_1) & v_1v_2(v_1 - 2v_2) & -v_1^2v_2 & v_1^2v_2^2 \\ v_2(2v_1 - v_2) & v_1v_2(v_2 - 2v_1) & -v_1v_2^2 & v_1^2v_2^2 \\ v_1^2 & -v_1^2v_2 & -v_1^2v_2 & v_1^2v_2^2 \\ v_2^2 & -v_1v_2^2 & -v_1v_2^2 & v_1^2v_2^2 \\ -v_1^2 & v_1^2v_2 & v_1^2v_2 & -v_1^2v_2^2 \\ -v_2^2 & v_1v_2^2 & v_1v_2^2 & -v_1^2v_2^2 \\ v_1^2 & -v_1^2v_2 & -v_1^2v_2 & v_1^2v_2^2 \\ v_2^2 & -v_1v_2^2 & -v_1v_2^2 & v_1^2v_2^2 \end{pmatrix}$$

$$\underline{\pi}_{12} = \begin{pmatrix} -2v_1 & 2v_1^2 & 2v_1v_2 & -2v_1^2v_2 \\ -2v_2 & 2v_2^2 & 2v_1v_2 & -2v_1v_2^2 \\ (2v_2 - v_1) & -v_2(2v_2 - v_1) & -v_1v_2 & v_1v_2^2 \\ (2v_1 - v_2) & -v_1(2v_1 - v_2) & -v_1v_2 & v_1^2v_2 \\ v_1 & -v_1v_2 & -v_1v_2 & v_1v_2^2 \\ v_2 & -v_1v_2 & -v_1v_2 & v_1^2v_2 \\ -v_1 & v_1v_2 & v_1v_2 & -v_1v_2^2 \\ -v_2 & v_1v_2 & v_1v_2 & -v_1^2v_2 \\ v_1 & -v_1v_2 & -v_1v_2 & v_1v_2^2 \\ v_2 & -v_1v_2 & -v_1v_2 & v_1^2v_2 \end{pmatrix}$$

$$\underline{\boldsymbol{\pi}}_{21} = \begin{pmatrix} -2v_2 & 2v_1v_2 & 2v_2^2 & -2v_1v_2^2 \\ -2v_1 & 2v_1v_2 & 2v_1^2 & -2v_1^2v_2 \\ v_1 & -v_1v_2 & v_1\left(v_2-2v_1\right) & v_1v_2\left(2v_1-v_2\right) \\ v_2 & -v_1v_2 & v_2\left(v_1-2v_2\right) & v_1v_2\left(2v_2-v_1\right) \\ v_1 & -v_1v_2 & -v_1v_2 & v_1v_2^2 \\ v_2 & -v_1v_2 & -v_1v_2 & v_1^2v_2 \\ -v_1 & v_1v_2 & v_1v_2 & -v_1v_2^2 \\ -v_2 & v_1v_2 & v_1v_2 & -v_1^2v_2 \\ v_1 & -v_1v_2 & -v_1v_2 & v_1v_2^2 \\ v_2 & -v_1v_2 & -v_1v_2 & v_1^2v_2 \end{pmatrix}$$

$$\underline{\boldsymbol{\pi}}_{22} = \begin{pmatrix} -2 & 2v_1 & 2v_2 & -2v_1v_2 \\ -2 & 2v_2 & 2v_1 & -2v_1v_2 \\ 1 & -v_2 & \left(v_2-2v_1\right) & -v_2\left(v_2-2v_1\right) \\ 1 & -v_1 & \left(v_1-2v_2\right) & -v_1\left(v_1-2v_2\right) \\ 1 & -v_2 & -v_2 & v_2^2 \\ 1 & -v_1 & -v_1 & v_1^2 \\ -1 & v_2 & v_2 & -v_2^2 \\ -1 & v_1 & v_1 & -v_1^2 \\ 1 & -v_2 & -v_2 & v_2^2 \\ 1 & -v_1 & -v_1 & v_1^2 \end{pmatrix}$$

These expressions for the solution are lengthy and difficult. So, $\underline{\boldsymbol{\Sigma}}_{OU}$ (given as $\underline{\boldsymbol{\Sigma}}_{OU} = \underline{\boldsymbol{\sigma}} \cdot \underline{\boldsymbol{\sigma}}^T$) is solved numerically. Letting $s = t = 1$, this gives the $Var\left(\underline{\boldsymbol{X}}_1\right)$ of the OU process, which is then equated to the covariance matrix of the bivariate AR(1) process element by element. This can result in a simultaneous linear equation involving four equations to determine the four elements in $\underline{\boldsymbol{\Sigma}}_{OU}$. If there is unique solution to this system of linear equations, $\underline{\boldsymbol{\Sigma}}_{OU}$ can be calculated and $\underline{\boldsymbol{\sigma}}$ of OU process can be determined by a Cholesky decomposition from $\underline{\boldsymbol{\Sigma}}_{OU}$.

Once the estimated parameters of the model and the values for matrices $\underline{\boldsymbol{A}}$ and $\underline{\boldsymbol{\sigma}}$ are obtained, the overall portfolio return at general time t can be estimated:

$$X_{P,t} = \sum_{i=1}^{2} \omega_i \cdot X_{i,t} = \omega_1 \cdot X_{1,t} + \omega_2 \cdot X_{2,t} \tag{3.60}$$

where $X_{i,t}$ signifies the ith row of $\underline{\boldsymbol{X}}_t$, given by the bivariate OU process. The first two moments of the accumulated rate of return for the overall portfolio at time t are derived as:

$$E\left(X_{P,t}\right) = \omega_1 \cdot E\left(X_{1,t}\right) + \omega_2 \cdot E\left(X_{2,t}\right) \tag{3.61}$$

$$Var\left(X_{P,t}\right) = \omega_1^2 \left[Var\left(\underline{\boldsymbol{X}}_t\right)\right]_{11} + \omega_2^2 \left[Var\left(\underline{\boldsymbol{X}}_t\right)\right]_{22} + 2\omega_1\omega_2 \left[Var\left(\underline{\boldsymbol{X}}_t\right)\right]_{12} \tag{3.62}$$

$$Cov\left(X_{Pt}, X_{Ps}\right) = \omega_1^2 Cov\left(\underline{\boldsymbol{X}}_t, \underline{\boldsymbol{X}}_s\right)_{11} + \omega_2^2 Cov\left(\underline{\boldsymbol{X}}_t, \underline{\boldsymbol{X}}_s\right)_{22} + 2\omega_1\omega_2 Cov\left(\underline{\boldsymbol{X}}_t, \underline{\boldsymbol{X}}_s\right)_{12} \tag{3.63}$$

where $E\left(X_{i,t} \mid X_{i,0}\right)$ signifies the ith row of $[E\left(\underline{\boldsymbol{X}}_t \mid \underline{\boldsymbol{X}}_0\right)]$, obtained from Equation (3.41). $\left[Var\left(\underline{\boldsymbol{X}}_t \mid \underline{\boldsymbol{X}}_0\right)\right]_{ij}$ is the element in row i and column j of $Var\left(\underline{\boldsymbol{X}}_t \mid \underline{\boldsymbol{X}}_0\right)$, and $Var\left(\underline{\boldsymbol{X}}_t \mid \underline{\boldsymbol{X}}_0\right) = \sum_{i=0}^{t-1} \underline{\boldsymbol{\Phi}}^i \underline{\boldsymbol{\Sigma}}_a \left(\underline{\boldsymbol{\Phi}}^i\right)^T$ in Equation (3.36). $\left[Cov\left(\underline{\boldsymbol{X}}_t, \underline{\boldsymbol{X}}_s \mid \underline{\boldsymbol{X}}_0\right)\right]_{ij}$ is the element in row i and column j of the 2×2 matrix $Cov\left(\underline{\boldsymbol{X}}_t, \underline{\boldsymbol{X}}_s \mid \underline{\boldsymbol{X}}_0\right)$, and $Cov\left(\underline{\boldsymbol{X}}_t, \underline{\boldsymbol{X}}_s \mid \underline{\boldsymbol{X}}_0\right)$ is solved from the AR(1) process of Equation (3.36) or OU process of Equation (3.43).

The conditional values of the mean and variance of the accumulated return rate of an overall portfolio are conveniently calculated using the equations in the AR(1) process. But, in practice, one may want to use the OU position process for these values. This is due to the assumption that the rate of return is accumulated in a continuous way.

3.4 Global Model

For this method, the two assets in the portfolio are combined as one asset and the combined rate is fitted by a univariate OU process. The combined rates at each moment are considered as the historical return rates used to fit a univariate AR(1) process from which a univariate OU process can be estimated. Then, one can estimate α_P and σ_P of the OU process as:

$$X_{P,t} - \mu_P = e^{-\alpha_P t}\left(X_{P,0} - \mu\right) + \sigma_P \int_0^t e^{-\alpha_P (t-s)} dW_s \tag{3.64}$$

It can be seen that $X_{P,t}$ is univariate OU process, so the expressions for the mean and variance of $X_{P,t}$ and $Y_{P,t}$ are like the univariate model.

3.5. Data Description

The variables selected in this study correspond to major asset classes as risky asset and risk-free asset. These two assets will form the universe of the securities held in the portfolio. The two assets are Treasury bill and equity. Treasury bill represents the risk-free asset and equity represents the risky asset. Treasury bills (TBs) are short-term debt instruments issued by a government. TBs pay a predetermined sum at maturity (called par or face value) and have no periodic interest payments. TBs effectively pay interest by originally selling at a discount; at a price less than the predetermined sum to be paid at maturity. The par value minus the discount price is the gain or income. TB is the safest money market instrument because the probability of default is zero. Government is largely capable of paying off its debt obligations, because it can raise taxes or even print more money in order to pay off its debts. By law, TBs must have an original maturity of one year or less; hence they are typically issued in maturities of 91 days (13 weeks), 182 days (26 weeks), and 364 days (52 weeks). TBs of the government of Ghana are issued in 91- and 182-day maturities.

Equity (stock) is an investment instrument that represents a fractional ownership interest in a corporation. It gives the holder the right for a claim on a portion of the corporation's earnings and assets. There are two basic types of stock: common stock and preferred stock. Shares of stocks are bought and sold on stock markets (called stock exchange, a regulated marketplace for the trading of stocks). This buying and selling activity gives rise to stock prices. Stocks are only traded on days called trading days, which are weekdays. Thus, stocks are not traded on weekends or holidays, and so, there can be a varied number of trading days per year. The price of a stock can change as many times each trading day. Nonetheless, the price is typically quoted in time series of stock prices as the price it has at the close of each trading day.

The performance of a group of stocks can be measured by a stock market index (I_t); the index calculates the collective movements of a group of stocks. In essence, a stock index provides an overview of the performance of a group of stocks and also guides investors in decision making. Market capitalization (cap) of a company, stock price times the number of its outstanding shares, affects the stock index. Companies with high market capitalization add much to the stock index than those with low market capitalization. To avoid a stock with a very high market capitalization accounting for large movements of the index, an equally weighted index is calculated to provide a better measure for the performance of a group of stocks. The GSE-CI is a stock index which measures the performance of all listed stocks on the exchange. It is a capitalization-weighted index; thus, the contribution of each stock to the index is not equal but depends on the market capitalization of each listed company. The calculation of the GSE-CI takes into consideration all listed stocks (ordinary stocks) on the exchange except stocks listed on other stock markets. Each stock index has a base, which is set arbitrarily as the starting value. All future index values are compared against the base index to determine the overall performance of the stock market. The GSE was incorporated in July 1989 with trading commencing in 1990 with index value of 10,000 points. The GSE-CI was rebased on December 31, 2010 with index value of 1,000 points.

3.6 Data Collection

The two secondary time series data for the study run from January 2011 to December 2017; collected at a weekly frequency, giving a sample size of 364 observations. The rationale for the 2011 start date is that the Ghana Stock Exchange Composite Index was rebased at the end of December 2010. The data set for the three-month Treasury bill (risk-free asset) is collected from the research section of Delta Capital Ghana. The interest rate on the three-month Treasury bill is used to calculate the rate of return on risk-free asset. The rate of return on equity (the risky asset) is calculated using the GSE-CI. Closing index prices are used which do include dividends. So, if the GSE-CI closed at I_t on week t and at I_{t-1} on previous week, then the rate of return for equity on week t is calculated as $\log(I_t)-\log(I_{t-1})$. The GSE-CI data set is collected from the online dataset of the Ghana Stock Exchange.

3.7 Estimation Method

If an asset price is observed at high frequencies, large data set can be collected to estimate the parameters of the price process. High-frequency data can cause significant bias in parameter estimation because such data is greatly affected by the microstructure disturbance in the market. For this reason, the maximum likelihood (ML) method which is robust to the market disturbances is used in this study to estimate the models' parameters.

CHAPTER FOUR

EMPIRICAL RESULTS AND ANALYSIS

4.0 Introduction

The chapter presents the empirical analysis carried out on the real data gathered for the purposes of the study. Using historical data, the autoregressive models of order one (in forms of univariate, multivariate, global) for the weekly return rates of the two assets in the portfolio are estimated in discrete time. If these AR(1) models are mean-reverting, they can be converted to OU processes based on need. The general concepts of the stochastic processes considered in Chapters 3 and the discussions are utilised in this chapter.

4.1 Displaying Observed Assets Returns Data

When taking observations on a process, one obtains a sample of each of the random variables in the set making up the stochastic process. In displaying the assets returns data, this is normally to convey information about three features: to give an idea of the means of the assets returns in the data set; to give an idea of the variances of the assets returns in the data set (the volatility of the rates of return); and to give an idea of the covariances between the assets returns in the data set (the relationships between the assets returns). This can simply be done by plotting the assets returns (X_t) against time (t).

Figure 1 shows the observed weekly rates of returns of the two assets that are considered. The data consist of 364 weekly observations between January 2011 and December 2017. First, looking at the graphs in the time-series plot, it seems that each asset interest rate series sometimes rises and sometimes falls but that each asset return rate series seems to revert to some mean level. Each graph in Figure 1 suggests a stationary process with a constant mean and variance. Second, it is unclear if these rates of return of the assets are correlated or not. This accounts for the key object of studying the bivariate model to fit the assets returns as an improvement over using the univariate model to fit the assets returns.

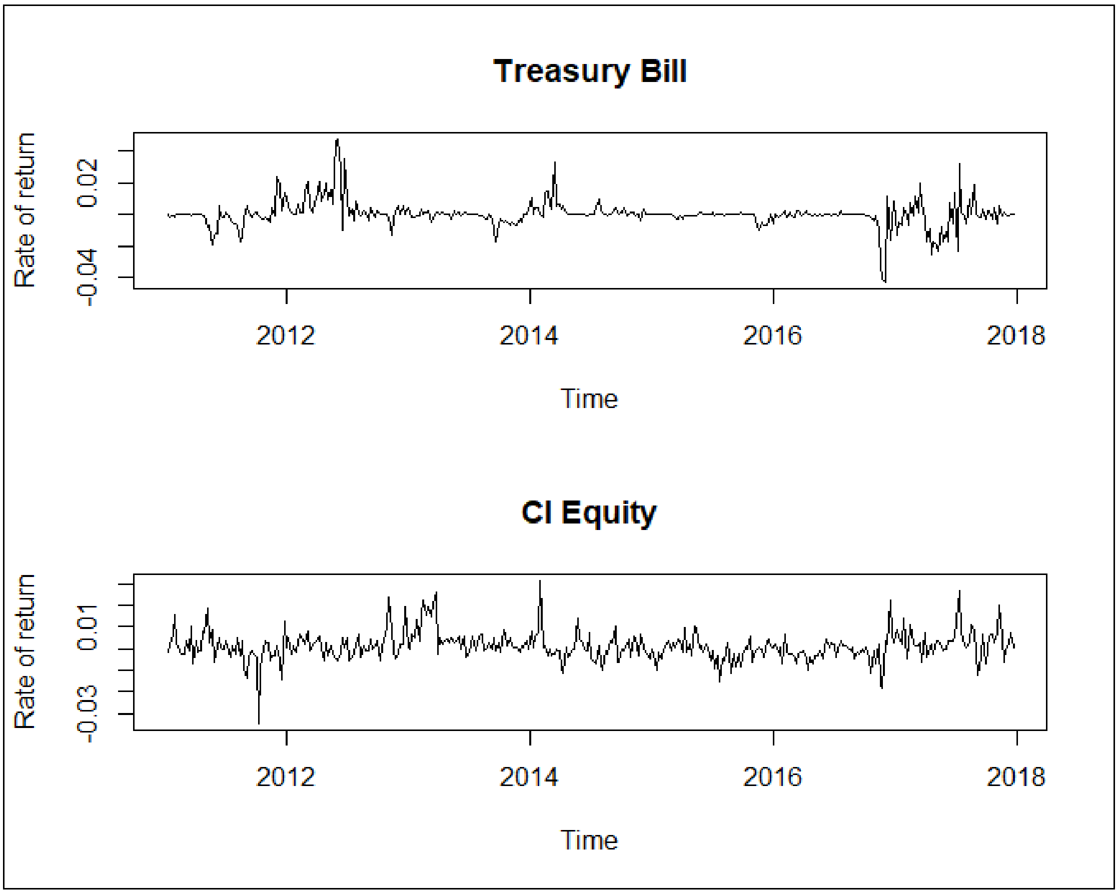

Figure 1: Time series plots of the weekly returns of the two assets

It is also difficult to see from the graphs in Figure 1 which asset return has the highest volatility for the period. However, Table 4.1 gives summary statistics of the assets returns where it can be seen that the weekly interest rate of the Treasury bill has the highest volatility when compared to that of the equity. The results obtained from Table 4.1 depicts that Treasury bill has average weekly return rate of 0.010123%, but its weekly return rate ranges between –4.279472% and 4.7312089% from January 2011 to December 2017. Similarly, the equity has average weekly return rate of 0.113069%, but the weekly return rate ranges between –3.465042% and 3.1513412% from January 2011 to December 2017.

Table 4.1: Summary statistics for the two asset returns

Asset	Mean	Minimum	Maximum	Std Dev.	Skewness
TB	0.0001012	-0.042795	0.0473121	0.00858065	0.57189194
Equity	0.0011307	-0.034650	0.0315134	0.00674804	0.54032680

Because dependence plays a significant role in using a multivariate stochastic model to estimate, the correlation between the two different assets is shown in Table 4.2. Figure 2 shows the scatterplot of the weekly return rates of the two assets in the portfolio.

Table 4.2: Empirical correlation matrix for the returns of the two assets

Asset	Treasury Bill	Equity
Treasury Bill	1	0.023919969
Equity	0.023919969	1

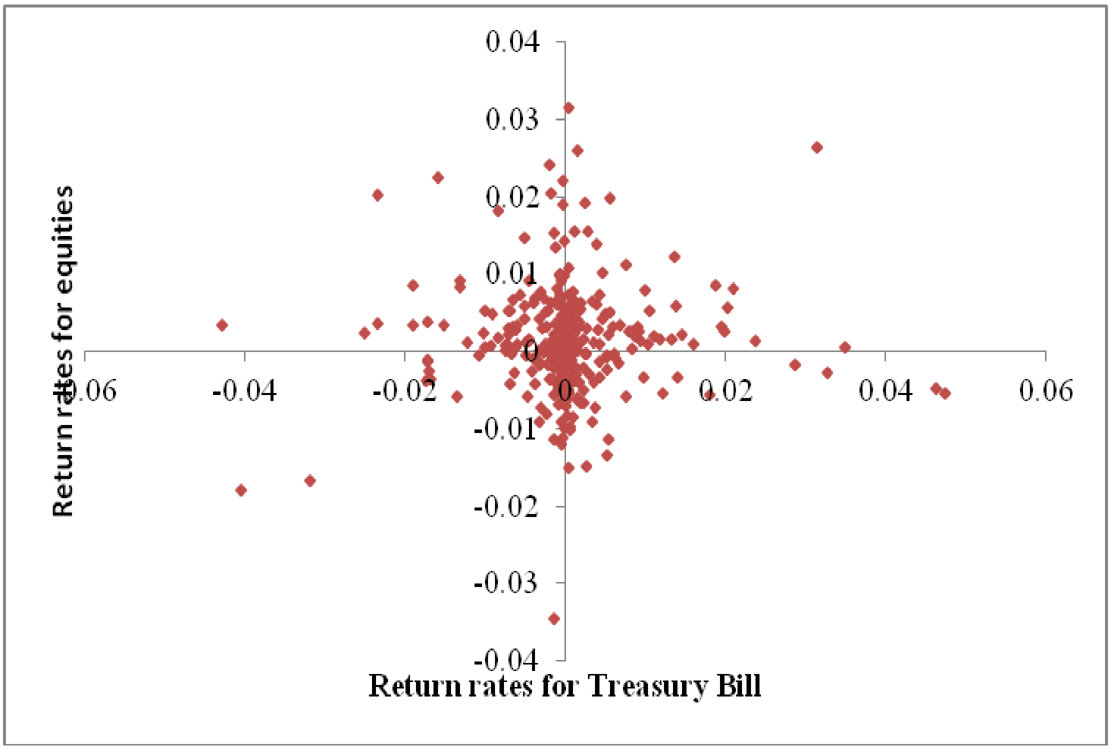

Figure 2: Scatterplots of the weekly returns of the two assets

From Figure 2, the weekly rates of returns are not strongly correlated. This is confirmed by the estimated result in Table 4.2 that shows that the returns on the two assets are correlated, even though the correlation coefficient ($\rho_{SE} = 0.0239$) appears statistically to be marginal.

4.2 Fitting the Models

This section is divided into two major parts: checking for the proposed AR(1) models and analysing the estimated models. The estimated models are often displayed in equations. From the results of the estimated models under the OU processes, the speed of the mean reversion for the two assets is discussed.

4.2.1 Model Identification

To achieve the objectives, the first step is to choose an AR(1) process to fit the observed returns in discrete time. To see if an AR(1) model is appropriate, the autocorrelation function (ACF) plot and the partial autocorrelation function (PACF) plot for the weekly assets returns rates depicted in Figure 3 is examined. In Figure 3, it can be seen that the ACF for the Treasury bill decays exponentially; but the autocorrelation still exist even after a 12-week lag. Thus, the autocorrelation for the Treasury bill is significant after a three-month lag. In the PACF plot for the Treasury bill, the first partial autocorrelation coefficient is approximately 0.5 and the others are relatively small; it can be said that the PACF of the Treasury bill has a significant spike at lag 1. So, the ACF and PACF plots suggest that an autoregressive process of order one seems a reasonable model to use to the interest rates for the Treasury bill.

The ACF of equity's returns also tails off exponentially. In the PACF plot for the equity, it can also be seen there is a significant spike at lag 1. This suggests that the process to fit the rates of return of equity is likely to be generated by an AR(1) model as well. Goodness-of-fit test is not conducted, but it is assumed that univariate and bivariate AR(1) processes are appropriate.

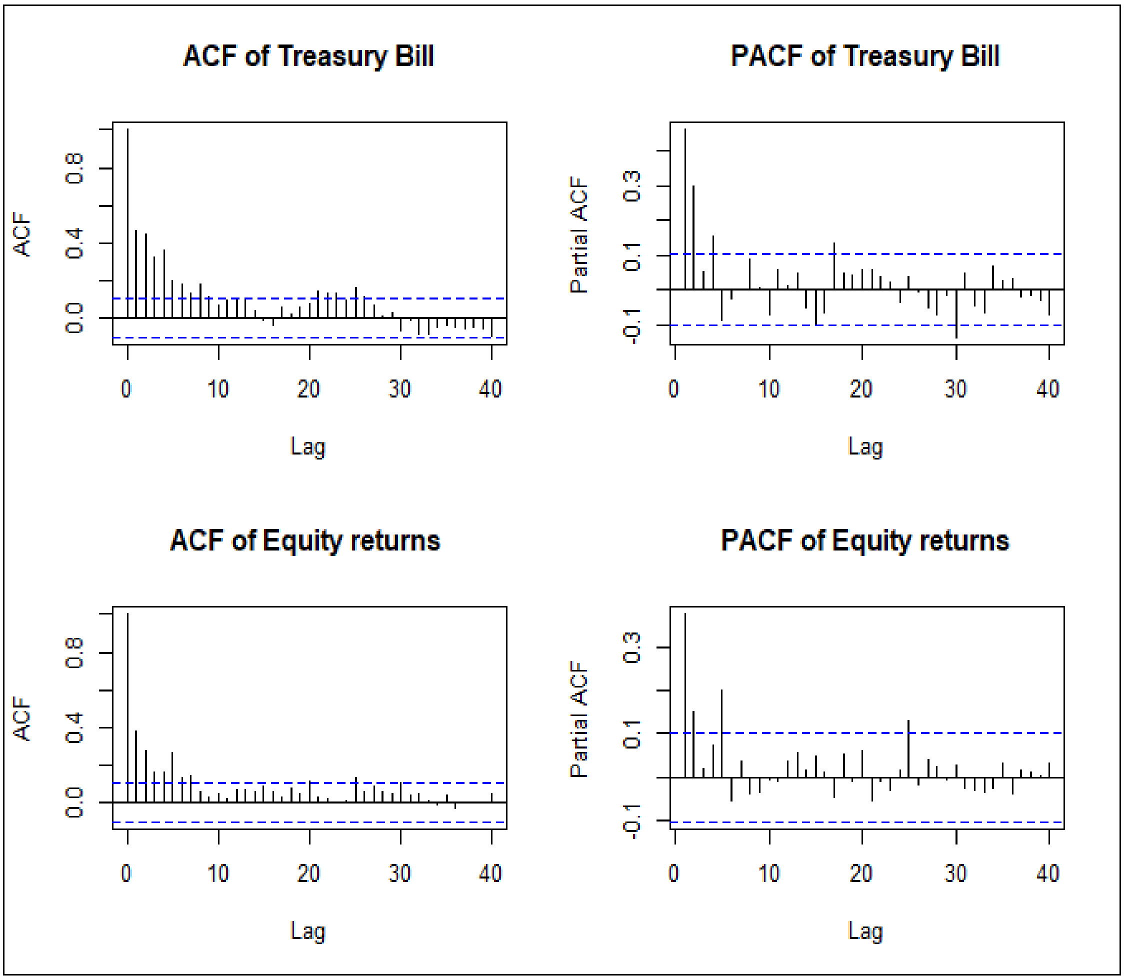

Figure 3: Plots of ACF and PACF for the weekly assets returns

4.2.2 Parameter Estimation

Using the maximum likelihood (ML) estimation procedure in *R* package, the parameters of two univariate AR(1) models, bivariate AR(1) model, and three global AR (1) models are first estimated in that sequence at weekly interval as that is the interval of the data set. Second, these discrete time AR(1) processes are then converted to their continuous time analogues, the OU processes. The mean reversion of the asset returns from the univariate OU processes is then studied. The estimates for the two assets using the three models are presented.

Estimated Parameters of Univariate Model

Table 4.3 and Table 4.4 give the model outputs for the two separate AR(1) models.

Table 4.3: AR(1) model output for Treasury Bill

Treasury Bill	Coefficient	Standard Error
Intercept	0.00010124	0.00073738
AR(1)	0.45921717	0.04640834

Table 4.4: AR(1) model output for Equity

Equity	Coefficient	Standard Error
Intercept	0.00112807	0.00052489
AR(1)	0.37405980	0.04849968

Each table presents the coefficients and standard errors of the AR model for each asset return. From these results and expressing the AR(1) in the form as stated in Equation (3.1), the estimated AR(1) stochastic processes for the risk-free asset (Treasury bill) and the risky asset (Equity) are presented in Equations (4.1) and (4.2) as follows:

- **Treasury Bill**

$$X_{S(t)} - 0.00018721008 = 0.45921717\left(X_{S(t-1)} - 0.00018721008\right) + a_{S(t)} \qquad (4.1)$$

where 0.00018721008 is the Treasury bill's long-run mean weekly return rate and the random error term $a_{S(t)} \sim N\left(0, 5.7856572\text{e-}05\right)$.

- **Equity**

$$X_{E(t)} - 0.00180220091 = 0.37405980\left(X_{E(t-1)} - 0.00180220091\right) + a_{E(t)} \qquad (4.2)$$

where 0.00180220091 is the equity's long-run mean weekly return rate and the random error term $a_{E(t)} \sim N\left(0, 3.9027209\text{e-}05\right)$.

Between these two assets, the Treasury bill (risk-free asset) has the least mean weekly rate of return (0.018721%) in the long run and equity (risky asset) has the highest mean weekly rate of return (0.18022%) in long run. The risks of weekly rates of returns for these assets are measured by the volatility. Equity that has the least risk (3.9027209×10^{-5}) compared to the risk (5.7856572×10^{-5}) of Treasury bill. The implication of this result is that the return rates for equity are more stable compared to the return rates of the Treasury bill. However, this result refutes the hypothesis in the literature that the returns on assets tend to be positively correlated with the risks involved.

From the estimated AR(1) model shown in Equation (4.1), the centred interest rate of Treasury bill at week t is a sum of 0.45921717 times of previous week's Treasury bill interest rate and a random term. This means that the three-month TB interest rate depends on 46% of its interest rate of the past week. In the estimated AR(1) model in Equation (4.2), the centred return rate of equity at week t is a sum of 0.37405980 times of past week's equity return and a random term. Thus, the rate of return on equity for the current week depends on 37% of its interest rate of the past week. It can be concluded that there is dependence between the return rate for the current week and the past week for each asset in the portfolio. But, when compared, the current weekly rate of return for Treasury bill is more depended on its rate of return of the past week than that of equity.

Estimated Parameters of Bivariate Model

The estimated bivariate model for the two assets is represented in Equation (4.3) as follows:

$$\begin{bmatrix} X_{St} - 0.0001872 \\ X_{Et} - 0.0018022 \end{bmatrix} = \begin{bmatrix} 0.460064347 & 0.02174525 \\ -0.029140549 & 0.37571820 \end{bmatrix} \begin{bmatrix} X_{S(t-1)} - 0.0001872 \\ X_{E(t-1)} - 0.0018022 \end{bmatrix} + \begin{bmatrix} \hat{a}_{St} \\ \hat{a}_{Et} \end{bmatrix} \quad (4.3)$$

The covariance matrix for $\begin{bmatrix} \hat{a}_{St} \\ \hat{a}_{Et} \end{bmatrix}$ is $\underline{\mathbf{\Sigma}}_a = \begin{bmatrix} 5.8477148\text{e-}05 & 1.7721875\text{e-}06 \\ 1.7721875\text{e-}06 & 3.9374954\text{e-}05 \end{bmatrix}$.

From the estimated bivariate AR(1) process, the centred return rate of Treasury bill at week t is 0.460064347 times the past week's Treasury bill return rate and 0.02174525 times the past week's equity return rate plus a random term. The Treasury bill depends on 46% of its interest rate for the past week and on 2% of the return rate of equity for the past week. Therefore, it can be stated that the interest of the three-month TB is dependent on its previous week's interest rate and on a small percent from the previous week's equity's return. The centred return rate of equity at week t is 0.37571820 times the past week's equity return and –0.029140549 times the past week's Treasury bill return plus a random term. The return rate for equity depends on 38% of the equity's return rate in last week and almost –3% of the interest rate of the Treasury bill last week. Equity is thus affected by the previous week's return of itself and inversely affected by the Treasury bill. This seems plausible resulting from the fact that the three-month Treasury bill interest rate is a crucial factor considered in setting the interest rates of most instruments especially money market instruments. Interest rates on Treasury bills tend to be the anchor for all other money market interest rates. Following an increase in interest rate for especially three-month Treasury bill, the interest rates on most other instruments also rise. In particular, risk-averse investors are likely to discount some holdings in equities (risky asset) to buy a safe instrument in form of Treasury bill or any other money market instrument following an interest rate rise in the three-month Treasury bill.

The univariate model fits the assets returns independently, but the bivariate model considers the covariance between the assets returns in the portfolio. This makes the variance of a bivariate model more than the variance of a univariate model. If an asset return is highly uncorrelated with the other assets returns, then it is sufficient to use a univariate model to fit the assets returns. But if the assets returns are truly correlated, the existence of these correlations cannot be captured by a univariate model. Table 4.5 shows the variance for the two assets with different models. From this table, it can be seen that the variance in the univariate model is lower compared to the bivariate model, because the bivariate model is incorporating some covariances with other assets.

Table 4.5: Variance of univariate and bivariate models for the two assets

Asset	Univariate Model	Bivariate Model
Treasury Bill	5.7856572e-05	5.8477148e-05
Equity	3.9027209e-05	3.9374954e-05

Estimated Parameters of Global Model

In the proposed global model for fitting the return on the portfolio, the rates of return of the two assets are split into three classes of asset portfolios where the rate of return on each portfolio is estimated by a univariate AR(1) model. The components of these asset portfolios under the global model are constructed as follows: Portfolio 1 is assumed to be 70% invested in Treasury bill and 30% invested in equity, Portfolio 2 is 60% invested in Treasury bill and 40% invested in equity, and Portfolio 3 is 30% invested in Treasury bill and 70% invested in equity. The ACF and PACF of each asset portfolio is plotted in Figure 4. In Figure 4, the ACF of each asset portfolio tails off exponentially. But in the PACF plot of each asset portfolio, it can be seen there is a significant spike at lag 1. It is, therefore, reasonable to model each asset portfolio using an AR(1) model. The univariate AR(1) models for the three portfolios are:

- **Portfolio 1**

$$X_{P_1,t} - 0.00073636459 = 0.44414764\left(X_{P_1,t-1} - 0.00073636459\right) + a_{P_1,t} \qquad (4.4)$$

where 0.00073636459 is the long-run weekly mean return of Portfolio 1 and the random error term $a_{P_1,t} \sim N\left(0, 3.258497\text{e-}05\right)$.

- **Portfolio 2**

$$X_{P_2,t} - 0.00090183956 = 0.43246003\left(X_{P_2,t-1} - 0.00090183956\right) + a_{P_2,t} \qquad (4.5)$$

where 0.00090183956 is the long-run weekly mean return of Portfolio 2 and the random error term $a_{P_2,t} \sim N\left(0, 2.7903467\text{e-}05\right)$.

- **Portfolio 3**

$$X_{P_3,t} - 0.00133514407 = 0.38585654\left(X_{P_3,t-1} - 0.00133514407\right) + a_{P_3,t} \qquad (4.6)$$

where 0.00133514407 is the long-run weekly mean return of Portfolio 3 and the random error term $a_{P_3,t} \sim N\left(0, 2.5035617\text{e-}05\right)$.

The estimated results in Equations (4.4), (4.5), and (4.6) illustrate that as the percent of the portfolio invested in equity increases, the average return rate of the asset portfolio increases, while the volatility of the asset portfolio decreases. Also, the weekly dependence decreases when the percent-weight of equity included in the portfolio is increased. Further, as more percent of equity are included in the portfolio, the estimated global model behaves like the estimated univariate model for equity. Equally, if the percent-weight of the Treasury bill is increased, the estimated global method gets closer to the estimated univariate model for Treasury bill.

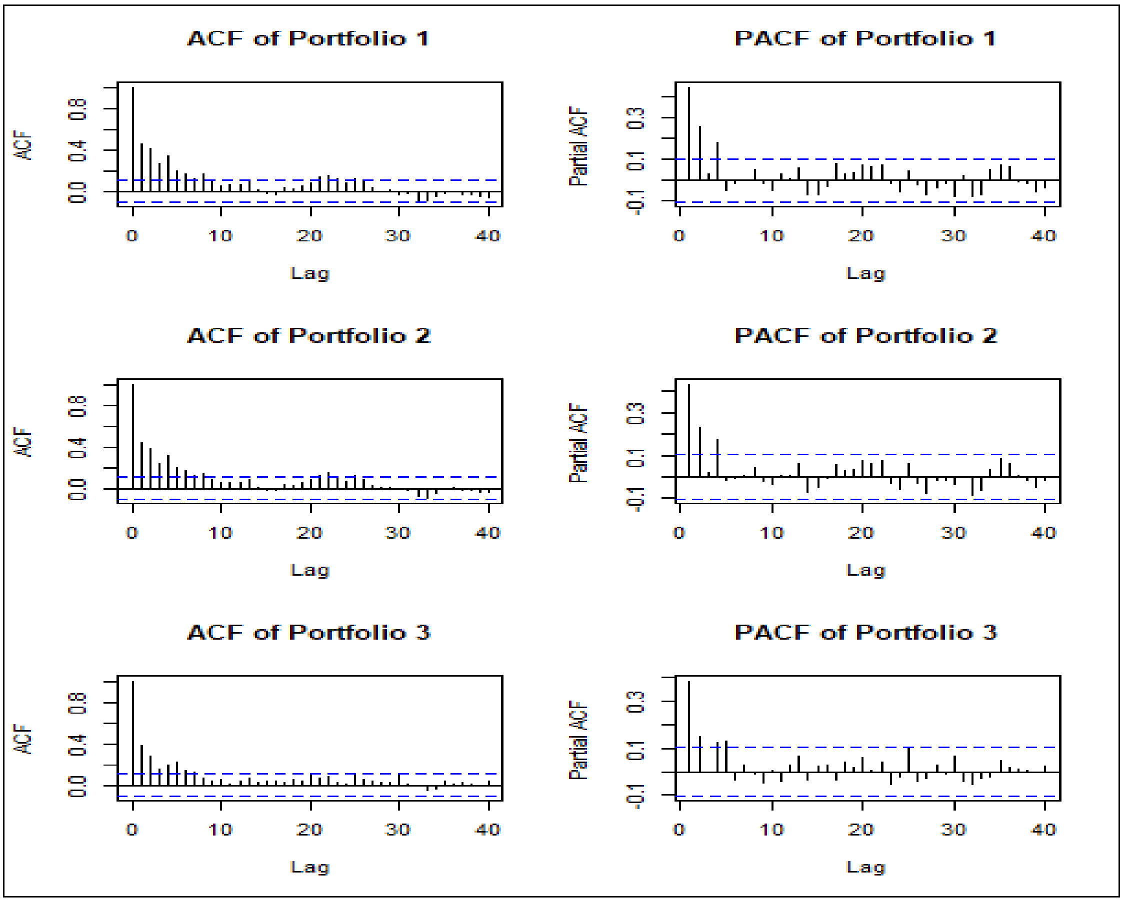

Figure 4: Plots of ACF and PACF for weekly return rates of portfolios

4.3. Equivalent AR(1) and OU Processes

The AR(1) models for the univariate, bivariate, and global processes are estimated and presented in the last section. The estimated results in Equations (4.1) and (4.2) indicate that all univariate AR(1) processes are stationary and mean-reverting; ϕ for each asset lies between 0 and 1. This condition is sufficient to use to convert the AR(1) processes to the corresponding OU processes. From the OU processes, the mean reversion is studied. To estimate the univariate OU process, α and σ must be computed. The general expression is:

$$X_t - \mu = e^{-\alpha t}\left(X_0 - \mu\right) + \sigma \int_0^t e^{-\alpha(t-s)} dW_s \qquad (4.7)$$

From Section 3.2, it is given that $\phi = e^{-\alpha}$, which means $\alpha = -\ln\phi$. The parameter σ is also much easier to determine; it is given as:

$$\sigma = \sqrt{\frac{2\alpha\sigma_a^2}{\left(1-\phi^2\right)}} \tag{4.8}$$

In Equation (4.8), σ_a and ϕ come from the AR(1) process. In the univariate models, is found that $0<\phi<1$ for all the assets, so each AR(1) process is converted to its corresponding OU process. The estimated univariate OU stochastic processes for the risk-free asset (TB) and the risky asset (Equity) are respectively presented in Equations (4.9) and (4.10).

- **Treasury Bill**

$$X_{St} - 0.00018721008 = e^{-\alpha_S t}\left(X_0 - 0.00018721008\right) + \sigma_S \int_0^t e^{-\alpha_S(t-s)} dW_s \tag{4.9}$$

where $\alpha_S = 0.7782320436$ and $\sigma_S = 0.01068253886$.

- **Equity**

$$X_{Et} - 0.00180220091 = e^{-\alpha_E t}\left(X_0 - 0.00180220091\right) + \sigma_E \int_0^t e^{-\alpha_E(t-s)} dW_s \tag{4.10}$$

where $\alpha_E = 0.9833396013$ and $\sigma_E = 0.00944671054$.

The results of the parameters for the OU processes in Equations (4.9) and (4.10) can be summarised; and this is shown in Table 4.6. In the OU process, the parameter α gives the how fast it takes the process to go back to the long-run mean (μ). Specifically, the larger the absolute value of α is, the faster the process reverts to the long-run mean. The summarised results in Table 4.6 show the rate of return for equity reverts to the long-run average very fast than the rate of return for Treasury bill. The current interest rate of equity will revert very fast to its long-run mean than the current interest rate of Treasury bill. The parameter σ describes the volatility of the return rate of the asset. The larger its value, the higher the volatility of the asset return rate. From the results in Table 4.6, the return rate of Treasury bill has higher volatility compared to that of the return rate of equity.

Table 4.6: Univariate OU velocity process: α and σ for two assets

Asset	α	σ
Treasury Bill	0.7782320436	0.01068253886
Equity	0.9833396013	0.00944671054

Generally, to express the length of time a process takes to revert to its long-run mean, $\frac{1}{\alpha}$ is used, which explains that after $\frac{1}{\alpha}$ units of time the process to be about 63% $(1-e^{-1})$ close to its long-run mean. It will, thus, take $\frac{3}{\alpha}$ (or $\frac{5}{\alpha}$) weeks to determine the distance that the asset return takes to revert to its long-term mean by 95% (or 99%) on average. From Table 4.7, the results indicate that it will take about 4 and 3 weeks for the conditional expected values of Treasury bill and equity to revert approximately 95% close to corresponding long-run mean; and also about 6 and 5 weeks for the conditional expected values of Treasury bill and equity to revert approximately 99% close to their long-run mean.

Table 4.7: Mean reversion for the assets in portfolio

Mean reversion	**63%**	**95%**	**99%**
Treasury Bill	1.2850	3.8549	6.4248
Equity	1.0169	3.0508	5.0847

It is needed to compute $e^{\underline{A}}$ and $\underline{\sigma}$ for the bivariate OU process. In Chapter 3, $e^{\underline{A}}$ is easily determined as $e^{\underline{A}} = \underline{\Phi}$. Hence,

$$e^{\underline{A}} = \begin{bmatrix} 0.460064347 & 0.02174525 \\ -0.029140549 & 0.37571820 \end{bmatrix} \tag{4.11}$$

Estimating $e^{\underline{A}}$ is sufficient for the bivariate OU velocity process, but $\underline{\mathbf{A}}$ is needed if the bivariate OU position process is of interest. Matrix $\underline{\mathbf{A}}$ is find by using the eigenvalues and eigenvectors of $\underline{\Phi}$. The eigenvalues and the corresponding eigenvectors of matrix $\underline{\mathbf{A}}$ are:

Eigenvalues:

$$\lambda_1 = 0.4517276395, \quad \lambda_2 = 0.3840549075 \tag{4.12}$$

Eigenvectors:

$$\begin{bmatrix} 0.9337313461 \\ -0.3579745428 \end{bmatrix}, \quad \begin{bmatrix} -0.2750516775 \\ 0.9614294434 \end{bmatrix} \tag{4.13}$$

In Equation (4.12), because $0 < \lambda_i < 1$, the bivariate OU process is stationary. Using Equation (3.13), this gives

$$\underline{\boldsymbol{A}} = \begin{bmatrix} -0.8316048988 & 0.03448181215 \\ 0.1342622265 & -0.9200406977 \end{bmatrix} \tag{4.14}$$

In Equation (4.14), $\underline{\mathbf{A}}$ relates to the time that it takes the process to revert to its long-run mean from a particular start value. The speed in the bivariate model is a pooled effect of the rates of returns of each asset involved in the system. Depending on the rate of return of the asset itself for the previous week, and starting from a given value, the rate of return for equity will revert to its long-term mean very fast than the rate of return for Treasury bill after an extended time in the multivariate process. The equity depending on the rate of return of Treasury bill last week reverts faster to its long-term mean than that of the Treasury bill as depending on the return rate of equity last week.

From the estimated results in Equations (4.4), (4.5) and (4.6), the AR(1) processes under the global modal are all converted to their corresponding OU processes because these processes are each stationary and mean-reverting ($0 < \phi < 1$).

- **Portfolio 1**

$$X_{P_1,t} - 0.000736365 = e^{-\alpha_{P_1} t}\left(X_{P_1,0} - 0.000736365\right) + \sigma_{P_1}\int_0^t e^{-\alpha_{P_1}(t-s)}dW_s \tag{4.15}$$

where $\alpha_{P_1} = 0.8115982493$ and $\sigma_{P_1} = 0.00811724561$.

- **Portfolio 2**

$$X_{P_2,t} - 0.00090184 = e^{-\alpha_{P_2} t}\left(X_{P_2,0} - 0.00090184\right) + \sigma_{P_2}\int_0^t e^{-\alpha_{P_2}(t-s)} dW_s \quad (4.16)$$

where $\alpha_{P_2} = 0.8382653731$ and $\sigma_{P_2} = 0.00758569474$.

- **Portfolio 3**

$$X_{P_3,t} - 0.00133514 = e^{-\alpha_{P_3} t}\left(X_{P_3,0} - 0.00133514\right) + \sigma_{P_3}\int_0^t e^{-\alpha_{P_3}(t-s)} dW_s \quad (4.17)$$

where $\alpha_{P_3} = 0.9522896366$ and $\sigma_{P_3} = 0.00748487622$.

The results of the parameters for these OU processes in Equations (4.15), (4.16) and (4.17) are summarised and presented in Table 4.8.

Table 4.8: OU velocity process under the global modal

Portfolio	α_P	σ_P
Portfolio 1	0.8115982493	0.00811724561
Portfolio 2	0.8382653731	0.00758569474
Portfolio 3	0.9522896366	0.00748487622

From Table 4.8, the rate of return of Portfolio 3 has a larger reverting speed coefficient ($\alpha = 0.95$) than that of the rates of return for Portfolio 2 ($\alpha = 0.83$) and Portfolio 1 ($\alpha = 0.81$). To be clear, the portfolio rate of return reverts to the long-run mean very fast when the percent of equity in the portfolio increases. This can be attributed to the larger reverting speed of the equity as depicted in Table 4.5. Even for Portfolio 1 with only 30% invested in equity, the reverting speed coefficient ($\alpha = 0.81$) is larger than the reverting speed coefficient ($\alpha = 0.78$) of investing solely in Treasury bills.

On the hand, as more percent of the portfolio are invested in equity, the volatility of the rate of return in the portfolio reduces. From Table 4.9, the expected value of Portfolio 3 takes about 3 weeks and 5 weeks to revert 95% and 99% closer to their corresponding long-run mean, while both the expected values of Portfolio 1 and

Portfolio 2 take 4 weeks and 6 weeks to revert 95% and 99% closer to the corresponding long-run mean.

Table 4.9: Mean reversion for asset portfolio

Mean reversion	63%	95%	99%
Portfolio 1	1.2321	3.6964	6.1607
Portfolio 2	1.1929	3.5788	5.9647
Portfolio 3	1.0501	3.1503	5.2505

4.4 Optimum Portfolio

The covariance matrix, $\underline{\mathbf{\Sigma}}_a = \begin{bmatrix} 5.8477148\text{e-}05 & 1.7721875\text{e-}06 \\ 1.7721875\text{e-}06 & 3.9374954\text{e-}05 \end{bmatrix}$, is used here to identify the optimum portfolio. In Chapter 2, it is stated that the combination of assets that has the less risk for a given expected return is the optimum portfolio. Table 4.10 shows the risks calculated for the case of changing the percent weights of the assets in the portfolio.

From Table 4.10, it is seen that increases in percent weights of equity, the risk of the portfolio decreases, but to where about 60% is invested in equity. With more than 60% invested in equity, the risk of the portfolio starts to increase. It can be said that for a given expected return of the portfolio, the minimum portfolio risk ($\sigma\left(X_{P,t}\right) = 0.004937811$) occurs when about 60% of the portfolio is made up of equity. Using the expression in Equation (2.13), the weight for Treasury bill is estimated to be 0.381717078938; that is, about 40% should be invested in Treasury bill.

Table 10: Portfolio risk for different weights

Cases	ω_{TB}	ω_E	$\sigma\left(X_{P,t}\right)$
1	0.00	1.00	0.006274947
2	0.05	0.95	0.005987524
3	0.10	0.90	0.005726908
4	0.15	0.85	0.005496913
5	0.20	0.80	0.005301524
6	0.25	0.75	0.005144687
7	0.30	0.70	0.005030009
8	0.35	0.65	0.004960415
9	0.40	0.60	0.004937811
10	0.45	0.55	0.004962840
11	0.50	0.50	0.005034791
12	0.55	0.45	0.005151699
13	0.60	0.40	0.005310595
14	0.65	0.35	0.005507846
15	0.70	0.30	0.005739501
16	0.75	0.25	0.006001575
17	0.80	0.20	0.006290268
18	0.85	0.15	0.006602089
19	0.90	0.10	0.006933919
20	0.95	0.05	0.007283023
21	1.00	0.00	0.007647035

4.5 The Ornstein-Uhlenbeck Position Process

The accumulation function of the instantaneous asset return rate (Y_t) in the univariate OU process is described in Chapter 3. Using Equations (3.16) and (3.17) and the results in Table 4.6, the conditional expected value and variance for the accumulated rate of return for each asset at time t are estimated as:

- **Treasury Bill**

$$E\left(Y_{St} \mid X_0\right) = \frac{1-e^{-0.77823204t}}{0.77823204}\left(X_0 - 0.0001872\right) + 0.0001872t \qquad (4.18)$$

$$Var\left(Y_{St} \mid X_0\right) = 1.88422\text{e-}04 + 1.210582\text{e-}04\left(-3 + 4e^{-0.778232t} - e^{-2(0.778232)t}\right) \qquad (4.19)$$

- **Equity**

$$E\left(Y_{Et} \mid X_0\right) = \frac{1-e^{-0.98333960t}}{0.98333960}\left(X_0 - 0.00180220\right) + 0.00180220t \qquad (4.20)$$

$$Var\left(Y_{Et} \mid X_0\right) = 9.22899\text{e-}05 + 4.69268\text{e-}05\left(-3 + 4e^{-0.9833396t} - e^{-2(0.9833396)t}\right) \qquad (4.21)$$

4.6 Conclusion of Empirical Results

The findings and empirical results realised from the sections of this chapter are concluded in this section. To recap, the key objects of this study are to investigate the mean reversion speed for each asset return in the portfolio, to use a multivariate model to predict the assets returns in the portfolio, and to identify an optimum portfolio of these assets. The analysis of the series employed in this study revealed that all assets returns are stationary and mean-reverting. The asset portfolio included two assets: three-month Treasury bill and equity.

Evidence from the descriptive analysis indicates that the return rates for the assets appear to be stationary. This is supported by both the univariate and bivariate autoregressive processes of order one that revealed the existence of stationarity and mean-reverting.

Second, for the two assets included in the portfolio, Treasury bill (risk-free asset) has the lowest average weekly rates of return and equity (risky asset) has the highest average weekly rates of return over the long-term. For risk as measured by volatility, Treasury bill has more volatility in the weekly rates of return than that of equity. This means that the return rates for equity are more stable compared to the return rates of the Treasury bill. This result refutes the hypothesis that the larger the risk, the higher the possible profit of the asset. In the literature, the returns on assets tend to be positively correlated with the risks (volatilities) involved.

Third, dependence plays an important role in a bivariate stochastic model, so the correlation between the two different assets is estimated. The empirical result shows that the weekly returns on the two assets are correlated, even though the correlation coefficient appears statistically to be marginal.

Fourth, the study revealed that both the rates of returns on Treasury bill and equity are (moderately) dependent on their rates of returns for the past week. Also, from the bivariate model, the result depicted that the interest rate of equity for the past week has a small positive effect on the current rate of return of Treasury bill. In contrast, the rate of return of Treasury bill in the past week has a small negative effect on the current rate of return of equity.

Fifth, a rise in interest rate of the three-month Treasury bill during the week has an inverse effect on the rate of return of equity for the following trading week. The three-month interest rate is a crucial factor considered in setting the interest rates of most instruments especially money market instruments. For most countries, the short-term interest rate (interest rate on three-month Treasury bill) tends to be an anchor for all other money market interest rates. Following a rise in the three-month Treasury bill interest rate, the interest rates on most other instruments also rise. Investors are, therefore, likely to trade in Treasury bills and/or other money market instruments; so there will be fall in demand for equities which can lead to a decrease in the rates of return for equities. Or, some investors are likely to discount holdings in equities to buy a safe instrument in form of Treasury bill and/or other money market instruments due to a rise in the interest rate of Treasury bill. Most investors are risk averse.

Further, the study elucidated from the three models (univariate, global, and bivariate) of using the OU process to examine the mean reversion speed that the accumulated interest rate of equity has a higher reverting coefficient than that of the Treasury bill. The interest rate of the equity approaches its long-term stationarity more quickly. It takes about few weeks for the conditional expected value of equity to revert closely to its long-run mean, but for Treasury bill, it will need about an additional week to also revert closely to its long-run mean.

One other key result revealed in this study is that to obtain an optimum portfolio of these two assets, an investor should consider investing in about 60% in equity. For a given level of an expected portfolio return, a portfolio made of 40% invested in the three-month Treasury bill and 60% in equity will give the minimum portfolio risk.

Finally, the study derived the mean and variance of the accumulated rate for investing in Treasury bills and equities in general time t, though expenses involved in trading were ignored. Thus, a mathematical framework for accumulation functions of rates of returns for the two assets in the portfolio has been proposed to allow for many investing applications.

CHAPTER FIVE

CONCLUSION AND RECOMMENDATION

5.0 Introduction

The concluding observations, recommendations and possible considerations for future studies are provided in this final chapter of the thesis.

5.1 Conclusion

The main goal of the study is fitting the rates of return of assets in portfolio as stochastic processes showing mean reversion toward their stationarity points. The study used two different assets: three-month Treasury bill and equities. The interest rates on the Treasury bill and the Ghana Stock Exchange Composite Index were respectively used to calculate the rates of return on Treasury bill (risk-free asset) and equities (risky asset). The sample period for the study runs from January 2011 to December 2017 at a weekly frequency. The autocorrelations of the assets returns were investigated. This was helpful in specifying stochastic investment models that explain the movements in the asset returns to include autocorrelation, by fitting an autoregressive process of order one. These discrete stochastic processes were then converted to equivalent continuous stochastic processes, called Ornstein-Uhlenbeck. The maximum likelihood was used for estimating the parameters of these models. The main conclusions are as follows:

The first objective of the study is considering the property of mean reversion of the two assets returns in the portfolio. The OU process is often used in finance to capture time series data that depict mean reversion; tend to move to their mean values over time. The empirical results of the OU processes show that mean reversion appears to be in financial data. From the OU processes, the interest rate of the equity has a larger reverting speed coefficient compared to that of the Treasury bill. The interest rate of the equity approaches its long-term stationarity more quickly. The empirical results of the univariate OU process indicate that it will take about 3 weeks for the conditional expected value of equity to revert to 95% closer to its long-run mean, but about 4 weeks for the conditional expected value of Treasury bill to revert to 95% closer to its long-run mean. It will take about 5 weeks for the conditional expected value of equity to revert to 99% closer to its long-run mean, but about 6 weeks for the conditional expected value of Treasury bill to revert to closely 99% to its long-run mean. In the global OU process, the overall portfolio return goes towards the long-term mean very fast when the percent of equity invested in the portfolio increases.

The second objective is estimating a bivariate model to explain the individual movements and the co-movements of the assets returns in the portfolio. The interest rate for the three-month Treasury bill depends moderately on its previous week's interest rate as well as on a small portion from the previous week's equity's return. The rate of return for Treasury bill depends on 46% of the Treasury bill's return rate of last week and on 2% of the return rate for equity last week. On the other side, the rate of return of equity for the current week is positively influenced by the rate of return of equity for the past week and inversely influenced by the Treasury bill. The rate of return for equity depends on 38% of the equity's return rate in last week and almost –3% of the return of the Treasury bill last week. The bivariate model has been able to capture the serial dependences between the assets involved in the portfolio, though the coefficients are very small.

The third object is to identify the best combination of the two assets (three-month Treasury bill and equity) that gives the lowest volatility; that is to find the optimum portfolio. Twenty-one (21) cases of changing the weights of the portfolio were considered. The findings revealed that as the equity's percent-weight increases, the risk of the portfolio reduces, but to the point where approximately 60% are invested in equity. After this, the portfolio starts to experience increase in volatility as more and more percent of equity is included. Hence, to achieve the optimum portfolio of these assets, an investor should consider investing 40% of his/her investment funds into the three-month Treasury bill and 60% into stocks. But, in considering forming the optimum portfolio, an investor might also wish to consider the kind(s) of stocks trading on the Ghana Stock Exchange to invest in.

Different stochastic approaches for investment modelling are used in modern-day financial analysis to explore the underlying dynamics of assets. Statistical modelling and inferences within this aspect is an important concern because pricing errors in assets could lead to serious economic losses for an investor. In this thesis, statistical estimation motivated by bivariate model is fitted, though the rates of return of assets in the portfolio are also modelled by univariate model and global model. The univariate stochastic process for modelling the asset returns do not take into account the correlation between the individual assets in the portfolio and this can result in imprecise estimate of the total returns on the asset portfolio. The global model for fitting the rates of return in the portfolio combined the two assets and described the asset return rates as a univariate OU process where it was also unable to capture the co-movement of the assets of the portfolio. The advantage of fitting the bivariate model for asset returns is that, it captured the correlation between the two assets in the portfolio. This means that the bivariate model presents a more precise estimate of the overall return of the portfolio. The empirical results in the bivariate model explained that for each individual asset in the portfolio, there is a correlation between the current interest rate and the previous interest rate. Most importantly, the model showed that the rate of return of each asset depends marginally on the rate of return of the other asset.

5.2 Recommendation

From the conclusions and to recommend for effective investing decisions, it is important to empirically investigate the predictive ability of the bivariate model. But, it must be noted that it is largely unsatisfactorily to clearly describe the random behaviour of interest rates of assets in a univariate stochastic model. The study showed that the interest rates on the assets are correlated, though it appears statistically to be marginal, can have serious consequence. Therefore, it is recommended that in practice, investors should use a multivariate stochastic investment model in order to obtain a more estimate of assets returns process because there are high chances in modern day that the returns on assets tend to be correlated. It is also recommended that the Ghana Stock Exchange should encourage the investing public to invest in stocks listed on the exchange since the returns are more stable.

5.3 Future Research Studies

There are many opportunities that the topic can be extended or enriched. In this study, the portfolio consisted of the three-month Treasury bill and the GSE-CI which respectively represented a risk-free asset and a risky asset. It will be useful that long-term bonds, Treasury bills, and equities are analysed if data can be obtained for these variables. Equally, it is also important for a study to use Ornstein-Uhlenbeck stochastic processes to model the rates of returns on other assets such as mutual funds or any investment instrument held by pension fund or insurance companies or any financial institution and then analyse the results. Such analysis plays significant roles in actuarial analysis; it can help in pricing life insurance and pension products, can be useful in constructing optimal portfolio and can help in estimating the minimum capital requirement for selling insurance and pension policies.

The sample period involved for the empirical investigation means weekly data set is obtained for large data points. Most insurance products are sold on annual basis; this makes it simpler to use annual intervals for simulation purposes. But, in particular, pension and life insurance products are for long-term; the premiums on these products are usually collected monthly or quarterly. Besides, portfolios performances are often reviewed quarterly, so it is necessary to explore quarterly data in modelling the rates of returns of the portfolios. One other possible avenue for enriching this study is to demonstrate how to use the estimated results of the stochastic models to price annuities sold by an insurer or pension funds.

REFERENCES

Baran, S., Pap, G., & van Zuijlen, M. C. A. (2003). Estimation of the mean of stationary and nonstationary Ornstein-Uhlenbeck processes and sheets. *Computers and Mathematics with Applications, 45,* 563-579.

Beekman, J. A., & Fuelling, C. P. (1990). Interest and mortality randomness in some annuities. *Insurance: Mathematics and Economics, 9,* 185-196.

Beekman, J. A., & Fuelling, C. P. (1991). Extra randomness in certain annuity models. *Insurance: Mathematics and Economics, 10,* 275-287.

Boudreault, M., & Panneton, C. (2009). Multivariate models of equity returns for investment guarantees valuation. *North American Actuarial Journal, 13,* (1), 36-53.

Box, G. E., Jenkins, G. M., & Reinsel, G. C. (2008). *Time series analysis* (4th ed.). New York: John Wiley & Sons.

Box, G. E. P., & Jenkins, G. M. (1976): *Time series analysis, forecasting and control*. San Francisco: Holdon Day.

Boyle, P. P. (1976). Rates of return as random variables. *The Journal of Risk Insurance*, *43*, (4), 693-713.

Chaiyapo, N., & Phewchean, N. (2017). An application of Ornstein-Uhlenbeck process to commodity pricing in Thailand. *Advances in Difference Equations, 179*

Chan, W. S. (2002). Stochastic investment modelling: a multiple time-series approach. *British Actuarial Journal, 8,* (3), 545-591.

Chiang R., Liu, P., & Okunev, J. (1995). Modelling mean reversion of asset prices towards their fundamental value. *Journal of Banking & Finance, 19,* 1327-1340.

Chobanov G. (1999). Modelling financial asset returns with shot noise processes. *Mathematical and Computer Modelling, 29,* 17-21.

Cont, R. (2001). Empirical properties of asset returns: stylized facts and statistical issues, *Quantitative Finance, 1,* 223-236.

Doob, J. L. (1942). The Brownian movement and stochastic equations. *Annals of Mathematics, Second Series, 43,* (2), 351-369.

Fasen, V. (2013). Statistical estimation of multivariate Ornstein-Uhlenbeck processes and applications to co-integration. *Journal of Econometrics, 172,* 325-337.

Hardy, M. R. (2001). A regime-switching model for long-term stock returns. *North American Actuarial Journal, 5,* (2), 41 - 53.

Lau, J. W., & Siu, T. K. (2008). Modelling long-term investment returns via Bayesian infinite mixture time series models. *Scandinavian Actuarial Journal, 4,* 243 - 282.

Markowitz, H. (1952). Portfolio selection. *The Journal of Finance, 7,* (1), 77-91.

Meucci, A. (2010). *Review of statistical arbitrage, cointegration, and multivariate Ornstein-Uhlenbeck.* Retrieved from http://ssrn.com/abstract=1404905

Önalan, Ö. (2009). *Financial modelling with Ornstein-Uhlenbeck processes driven by Lévy process.* Proceedings of the World Congress on Engineering II, July 1 - 3, 2009. London, United Kingdom.

Panjer, H. H., & Bellhouse, D. R. (1980). Stochastic modelling of interest rates with applications to life contingencies. *The Journal of Risk and Insurance, 47,* (1), 91-110.

Panjer H. H. & Bellhouse, D. R. (1981). Stochastic modelling of interest rates with applications to life contingencies. Part II. *The Journal of Risk and Insurance, 48,* (4), 628-637.

Parker, G. (1993). Distribution of the present value of future cash flows. *Proceedings of the 3rd AFIR International Colloquium, Rome,* 831-843

Parker, G. (1994a). Two stochastic approaches for discounting actuarial functions. *ASTIN Bulletin, 24,* (2), 167-181.

Parker, G. (1994b). Limiting distribution of the present value of a portfolio. *ASTIN Bulletin,*
94, (1), 47-60.

Parker, G. (1995). A second order stochastic differential equation for the force of interest. *Insurance: Mathematics and Economics, 16*, 211-224.

Pastor, L. (2000). Portfolio selection and asset pricing models. *The Journal of Finance, 55,* (1), 179-223.

Qian, X. (2010). *Analysing investment return of asset portfolio with multivariate Ornstein-Uhlenbeck processes.* (MSc Project). Simon Fraser University, Burnaby.

Schöbel, R., & Zhu, J., (1998). Stochastic volatility with an Ornstein-Uhlenbeck process: an extension. *Tübinger Diskussionsbeiträge, 139,* Eberhard Karls Universität Tübingen, Wirtschaftswissenschaftliche Fakultät, Tübingen

Thomson, R. J. (1996). Stochastic investment modelling: the case of South Africa. *British Actuarial Journal, 2,* (3), 765-801.

Tee, E., & Ofosu-hene, E. D. (2017). Stochastic asset models for actuarial use in Ghana. *International Journal of Statistics and Actuarial Science, 1,* (4), 101-111.

Tsay, R. S. (2005). *Analysis of financial time series* (2nd ed.). New York: John Wiley & Sons.

Uhlenbeck, G. E., & Ornstein, L. S. (1930). On the theory of the Brownian motion. *Physical Review, 36*, 823-841.

Vasicek, O. (1977). An equilibrium characterization of the term structure. *Journal of Financial Economics*, *5,* 177-188.

Vatiwutipong, P. & Phewchean, N. (2017). *Joint distribution of the multivariate Ornstein-Uhlenbeck process.* Proceedings of the 22nd Annual Meeting in Mathematics, Dept. of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai, Thailand.

Wan, Z. (2010). *Modelling investment returns with a multivariate Ornstein-Uhlenbeck process*. (MSc Project). Simon Fraser University, Burnaby.

Whitten, S. P. & Thomas, R. G. (1999). A non-linear stochastic asset model for actuarial use. *British Actuarial Journal, 5,* 919-953.

Wiener, N. (1923). Differential Space. *Journal of Mathematical Physics*, *2*, 131-174.

Wilkie, A. D. (1986). A stochastic investment model for actuarial use. *Transactions of the Faculty of Actuaries, 39*, 341-403.

Wilkie, A. D. (1986). Some applications of stochastic investment models. *Journal of the Actuaries Students' Society*, *29,* 25-51.

Wilkie, A. D. (1995). More on a stochastic asset model for actuarial use. *British Actuarial Journal*, *Vol. 1,* 777-964.

Wong, C. S., & Chan, W. S. (2005). Mixture Gaussian time series modelling of long-term market returns. *North American Actuarial Journal, 9,* (4), 83-94.

APPENDIX

Minimum Portfolio Variance

Preposition: Suppose the two assets in portfolio have equal variance, then the minimum portfolio risk occurs when the percent weight of each asset is equal.

If $\sigma_1^2 = \sigma_2^2 = \sigma^2$, then Equation (2.12) gives

$$\sigma^2(X) = \omega_1^2\sigma^2 + (1-\omega_1^2)\sigma^2 + 2\omega_1(1-\omega_1)\rho_{12}\sigma^2$$

$$\sigma^2(X) = \sigma^2\left[2\omega_1^2 + 1 - 2\omega_1 + 2\omega_1\rho_{12} - 2\omega_1^2\rho_{12}\right]$$

$\sigma^2(X)$ is a minimum if it satisfies the first order and second order conditions for minimisation.

For $\sigma^2(X)$ to be a minimum,

$$\frac{\partial\sigma^2(X)}{\partial\omega_1} = \sigma^2\left[4\omega_1 - 2 + 2\rho_{12} - 4\omega_1\rho_{12}\right] = 0$$

Assuming that $\sigma^2 > 0$,

$$4\omega_1 - 2 + 2\rho_{12} - 4\omega_1\rho_{12} = 0$$

$$4\omega_1(1-\rho_{12}) - 2(1-\rho_{12}) = 0$$

$$\omega_1 = \frac{2(1-\rho_{12})}{4(1-\rho_{12})} = \frac{1}{2}$$

The second order condition is satisfied because $\dfrac{\partial^2\sigma^2(X)}{\partial\omega_1^2} = 4\sigma^2(1-\rho_{12}) > 0$.

We have shown that if $\sigma_1^2 = \sigma_2^2$, the portfolio risk will be minimised by choosing $\omega_1 = \omega_2 = \dfrac{1}{2}$ regardless of the numerical value of ρ_{12}, except $\rho_{12} = +1$. Given $\rho_{12} = +1$, $\sigma^2(X) = \sigma_1^2 = \sigma_2^2$.