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ESTIMATION OF LONG-RUN PROBABILITY OF ZERO OFFSPRING USING BRANCHING PROCESSES IN VARYING ENVIRONMENT

BY

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DECLARATION

Candidate's Declaration

I, Emmanuel Kojo Aidoo, hereby declare that this dissertation is the result of original research work, wholly conducted by myself under the supervision of Dr. Felix O. Mettle and Dr. Isaac Baidoo.

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Supervisors' Certification

We hereby certify that this research work was prepared from the candidate's own effort and supervised in accordance with the University of Ghana's guidelines on supervision of theses.

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DEDICATION

This work is dedicated to my supervisors and all the lecturers in the Department of Statistics and Actuarial Science, University of Ghana.

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ABSTRACT

There are many challenges associated with both young and ageing population. If a country experiences a younger population, there's a tendency for high unemployment rates and social vices. On the other hand, an ageing population typically results in a low labour force and high dependency ratios. Countries that tend to solve the problem of a young population initiate policies to control birth rates. However, these policies gradually lead to an ageing population before being revised, due to high costs associated with regular monitoring of population dynamics. Therefore, there is a need to develop a less costly method to monitor population dynamics and estimate the expected time to revise population policies. This study employed a more general theorem and a corollary based on ideas of probability generating functions in a branching process to come out with a method to solve the problem. The method was applied to both hypothetical and empirical data in the branching processes. The empirical data were obtained from Demographic and Health Surveys (DHS) for seven selected countries. The results from the study revealed that under certain closeness conditions, both constant and random environments yield similar results. Hence, using the method under the constant environment, which is easier, is a step in the right direction; otherwise, the proposed method for the random environment should be used. Burkina Faso recorded the youngest population, while Philippines recorded the least country with younger population. Results from the spectral analysis estimated that population policy for the selected countries should be revised between 34 to 40 years. The study recommended that the proposed model should be used to monitor population dynamics regularly. Also, population policies should be guided by appropriate time frame depending on the country's demographic characteristics.

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Chapter 1

INTRODUCTION

1.1 Background of the Study

Population dynamics (young or ageing) have been a significant source of worry to nations all over the world. This is because an ageing population has consequences on labour force participation, savings, and investments which eventually slows down economic growth (Grzenda, 2019; Reynaud & Miccoli, 2019; Maity & Sinha, 2020). According to Vincent (2010), the expected growth of the older adult population in the United States over the next 50 years will have an unprecedented impact on the health care system. There will be a high demand for health care workers to take care of aged people, with an associated decrease in the US labour force. On the other hand, the consequences of a relatively young populated country include unemployment, a tremendous amount of pressure on the environment and the resources of the country, as well as a high incidence of social vices such as armed robbery, prostitution and conflicts (Coale, 2017; Michoń, 2019). Most developing countries such as Ghana have a young population and, as a result, experience a high level of unemployment rate (Mutunga, Zulu, & De Souza, 2012).

To solve the problems associated with population dynamics (young and ageing), many countries have resorted to several solutions ranging from advocacy for the use of contraceptive methods to increasing immigration of young people (Mason et al., 2014; Mettle, Quaye, Appiah, & Aidoo, 2019). For example, Italy, Ireland, United States, New Zealand, Canada and Germany, which exhibit ageing populations, have implemented incentive packages to families with children by giving child support to parents (Mordechay, Gándara, & Orfield, 2019; Mettle et al., 2019). According to Seale, Awosika, and Lim (2021), the black race is enjoying these privileges and reproducing more offspring. Therefore, it is projected that by the year 2045, United States citizens will be made up of more black people and foreigners than indigenous whites. Other countries like Romania have imposed more taxes on families with no children (Bradshaw & Finch, 2002). The United States of America, upon realising the rate at which their population was ageing, introduced the US lotteries (King, Lulle, Sampaio, & Vullnetari, 2017). This policy allowed foreigners to come into the country to develop since their labour force was reducing.

The study of population dynamics (young or ageing) now forms an integral part of every country and provides information on the rate of population growth to match the economy's space. This is because rapid population growth reduces per capita income, lowers the standard of living, plunges the economy into mass unemployment and underemployment, brings environmental damage, and puts a burden on existing social infrastructure (Sharifi & Hosseingholizadeh, 2019). Indeed, population dynamics is of paramount importance to society. When the population is increasing rapidly, humanity is faced with innumerable problems. Examples are the shortages of essential services like water, electricity, transport and communications, public health, and education. In addition, issues of migration and urbanisation are associated with a growing population which further leads to the law-and-order problem (Bhagat, 2018). Population dynamics models are valuable tools for understanding, explaining, and predicting the dynamics and persistence of biological populations. These models can also help to predict how populations respond to environmental changes such as global climate change. Branching processes provide the right avenue in analysing population dynamics (Palau & Pardo, 2018). Bansaye, Millan, and Smadi

(2013) define a branching process as a mathematical representation of the development of a population whose members reproduce and die, according to the law of chance. The members may be of different types, depending on their age, position, energy, or other factors, but they must be independent of each other. A branching process is also defined as a Markov process that models a population in which each individual in generation n produces some random number of individuals in generation $n + 1$, and individuals reproduce independently of each other (Le Page, Peigné, & Pham, 2018). Galton and Watson proposed the first model of branching processes to investigate the extinction of family names in the nineteenth century (González & Puerto, 2010; Andrade-Ines, Beaugé, Michtchenko, & Robutel, 2016).

Branching processes constitute a fundamental part of the theory of stochastic processes. The view of branching processes deals with the issue of exponential growth or decay of random sequences. Its central concept consists of a population made up of particles or individuals which independently produce descendants (Andrade-Ines et al., 2016). In the Galton-Watson branching processes, time is measured discretely in generations, and organisms are only one type. Each organism lives in one generation; during this period, it produces offspring contributing to the next generation (Grosjean & Huillet, 2018). Another assumption is that organisms reproduce independently according to the same probabilistic law (i.e. distribution of offspring in each generation is constant). Interaction between organisms is forbidden, hence independent. The Galton-Watson, processes experienced many weaknesses based on its assumption since the distribution of offspring may differ from generation to generation. Therefore, many extensions had been proposed, such as the branching process in varying environments (Zhang, 2017). A branching process in a varying environment is a modification of the constant environment branching process in which the reproduction laws (i.e. offspring distribution) vary from generation to generation (González, Kersting, Minuesa, & Puerto, 2019). Under the varying environment, we have an oscillating environment, random environment and varying environment that approaches a limit. An oscillating environment is a varying environment branching process where the offspring distribution varies from generation to generation but oscillates or swings back and forth in a regular rhythm. The random environment is when the offspring distribution varies from generation to generation according to some random mechanism (Kersting & Vatutin, 2017). A varying environment that approaches a limit is a situation where the offspring distribution varies from generation in a way that the probabilities of giving birth to offspring are monotone increasing or decreasing until it reaches a limit.

The probability of ultimate extinction has a link to ageing population. This is because probability of ultimate extinction is the probability that in future generations, there will be no offspring. According to Merchant et al. (2017) aged people have low reproduction abilities; therefore, an ageing population may not multiply for population expansion. In the same vein, the possibility of death rate is likely to be high since the probability of the aged dying is high compared to new birth (Merchant et al., 2017). Considering these characteristics of the aged population, the long-run probability of giving birth to zero offspring becomes higher. Hence if people are not giving birth, then in future, we aspect the population to be ageing. Therefore, most countries solve ageing problems by encouraging people to give more birth (Ge, Yang, & Zhang, 2018). By monitoring or examining the country's population dynamics, one can suggest a time frame to reverse population policy if it is being implemented. This study uses the generational probabilities of zero offspring as a proxy to monitor population dynamics (young or ageing) as in the case of Mettle et al. (2019) and Jagers (1995) where wavelength was estimated to suggect the time frame where population policy should be revised.

1.2 Problem Statement

There are many challenges associated with both young and ageing populations. Countries that tend to solve the problem of a young population initiate policies that will control birth rates. However, these policies gradually lead to an ageing population before being revised, due to the high costs associated with regular monitoring of population dynamics (Zeng & Hesketh, 2016). As a result, some countries have experienced a cyclical age structure in a population where at a certain time the population becomes younger and then ages at another stage (Coale, 2017). For instance, in 1980, China experienced an ageing population of 0.5% and recently in 2020 had 17.4% of its population ageing. According to Peng (2011), China's ageing population is a significant threat to its future as it is projected to increase to 28% in 2040 if the policy is not revised. Therefore, every country needs to monitor the dynamics of its population periodically. In developing countries, population dynamics are monitored from population censuses. However, a population census is typically not conducted regularly due to its associated cost Mettle et al. (2019), suggested that given country parity data, one can monitor the country's population dynamics (young or ageing) regularly using branching processes in a constant environment. However, the application was done in a constant environment.

The branching process in a constant environment assumes that the distribution of offspring does not change from generation to generation. However, individuals in populations are subject to their surrounding environment, which varies. The survival of plants and animals solely depends on environmental conditions such as food and water availability and temperature. Under favourable environmental conditions, the number of offspring increases. However, under unfavourable conditions, the number of offspring declines and it is expected that the distribution of offspring changes from time to time (Li et al., 2021) . Therefore, it is prudent to model branching processes under a varying environment. This study, as a result, develops a model that will monitor the dynamics of a population using branching processes in a varying environment and estimate the expected time to revise population policies.

1.3 Objectives of the Study

The study's main objective is to develop a method to monitor population dynamics (young or ageing) using a branching process in varying environments. The specific objectives are

1. To determine the long-run probability of zero offspring for population subject to
   a. Oscillating environment, when the offspring distribution oscillates or swings back and forth in a regular rhythm.
   b. Varying environment where the offspring distribution goes to a limit.
2. To determine the condition under which estimations of the long-run probability of zero offspring in a random environment are similar to constant environment.
3. To analyse the probability of zero offspring in multi-type branching processes with respect to gender.
4. To determine the expected time to revise population dynamics policies based on parity data from selected countries using spectral analysis.

1.4 Research Question

The thesis is guided by the following research questions

1. What is the long-run probability of zero offspring for population subject to

   a. An oscillating environment?

   b. A varying environment where the offspring distribution goes to a limit?

2. What is the condition under which estimations of the long-run probability of zero offspring in a random environment are similar to a constant environment?

3. What is the probability of zero offspring in multi-type branching processes concerning gender?

4. What is the expected time to revise population dynamics policies based on parity data from selected countries?

1.5 Relevance of the Study

This study is very significant, as it will inform stakeholders on the right decisions to make in managing population dynamics (young or ageing) regarding knowing the appropriate offspring distribution to balance the population dynamics. Population dynamics is one of the fundamental areas of ecology, forming the basis for studying more complex communities and many applied questions. Understanding population dynamics is the key to understanding the relative importance of competition for resources and predation in structuring ecological communities, which is a central question in ecology. This study will help biologists and demographers with the appropriate planning. In the case of demographers, the findings from this research will inform proper implementation of good socio-economic policies for a nation. This is because the young population needs socio-economic oriented policies in line with job creation and education, among many others. On the other hand, the ageing population will require good socio-economic policies such as homes for the aged and social welfare development centres for the aged.

Achieving the first objectives of this study implies that in any mobile population (be it human or not), if one can control the fertility rates or offspring distributions, then the implication is that one can also control the long-run population dynamics given that the survival rates do not change significantly. This is because the fertility rates of human populations, unlike survival rates, can be controlled to some extent. Therefore, if the government targets a certain long-run probability of giving birth, it can work towards a certain offspring distribution to achieve such a goal.

The second objective provides the method to use parity data of women in their terminal age to obtain the probability of ultimate extinction, which serves as proxy for analysing population dynamics. This is very relevant because it provides a less costly and more convenient way of monitoring the dynamics of the population (young or ageing) since parity data can be easily obtained. The results of the second objective explain why it is important to consider the random environment in estimating the probability of ultimate extinction instead of the constant environment that is often applied. In addition, it also provides the conditions under which estimation of the long-run probability of zero offspring is as that of the constant environment. This will informed the researcher of the appropriate condition under which to apply branching processes in a constant environment (which is easier) rather than a varying environment.

The results of the third objective give information on how a country will monitor its aged population by gender. These will help a country know which gender is ageing to formulate a gender-specific policy for the aged population. This information is very relevant because, even though there are common needs for both aged males and females, there are some gender-specific needs. In most cases, especially in Africa, men are the breadwinners and have a higher income than females (Akanle & Nwaobiala, 2020). Therefore, most aged women face more economic problems and are more likely to be exposed to inadequate housing (physical determinant), societal violence (social determinant) and not eating nutritious foods (behavioural determinant) than the aged males (Beard et al., 2016).

Developing a methodology to estimate the expected time to population policy revision is very important. As stated earlier, most population policies are not guided by a time frame. Hence countries implement their population policies without any time-bound but later revised them when they sense a negative effect, as in the case of China. Therefore, these results will contribute immensely to the knowledge of population and demography studies.

1.6 Scope of the Study

The study looked at the probability of ultimate extinction of hypothetical populations and parity data from seven selected countries, subject to the same offspring distribution all the time or offspring distributions that vary from time to time.

1.7 Sources of Data

The data used in the study are parity data from women in the terminal reproductive age in seven countries, namely Ghana, Burkina Faso, Bangladesh, Philippines, Colombia, Indonesian and Mali from the Demographic and Health Surveys (DHS). The surveys are done with five years intervals. We also used hypothetical population on offspring distributions.

Nine hypothetical populations were generated from the hypothetical data, with the probability mass function of offspring being a Poisson distribution. The study looked at the various situations of offspring distribution, such as when the offspring distribution remains constant for each generation; when the offspring distribution oscillates; when the offspring distribution varies but approaches a limit; and when the offspring distribution differs from generation to generation by a random fashion.

1.8 Limitation of the Study

The study used offspring distribution to analyse the dynamics of a given population subject to a normal survival rate. However, there may be issues like natural disasters such as floods and earthquakes that can lead to the death of many people. Also, there are artificial disasters such as fire outbreaks, road accidents and pandemics. These disasters can easily decrease the population size beyond expectation. Thus, the limitation of the study is premised on the assumption of a normal survival rate with no possibility of disasters that could suddenly wipe out the population. In addition, the estimation of the generational probability of offspring is under close population where there is no immigration and emigration.

1.9 Brief Methodology

This study used the method of branching processes in constant and random environments and spectral analysis to achieve the objectives. The study developed a methodology for the analysis of branching processes in a random environment. Details of the methodology used and analysis are provided in Chapter four.

1.10 Organization of the Study

The thesis consists of nine chapters, and each of the chapters is introduced as follows; Chapter one introduces the entire thesis. It consists of background, problem statement, research questions, objectives, and significance of the study. The chapter gives the limitation of the thesis, source of data and a brief methodology. Chapter two gives a general overview of the literature reviewed. It talks about the various concepts of the branching processes. The chapter contains information about the history of branching processes and the various application of the branching processes. Chapter two also reviewed works on population dynamics. The importance of the literature review is to indicate the existence of current research that have attempted to apply branching processes so that after reviewing the literature, the author identifies the loopholes in the current body of knowledge. This gives a clear justification of the thesis, which seeks to fill in the gap. Chapter three provides the mathematics descriptions of the branching process. Chapter four discusses the research methods, including the source of data collected and the various methods used to achieve the objectives. Chapter five entail analyses based on the study's specific objectives and their related research questions. Chapter six constitutes a summary of the key findings, conclusions and ends with the study's recommendations.

Chapter 2

LITERATURE REVIEW

2.1 Introduction

This section provides the relevant literature on population dynamics and the branching process. It comprises of both theoretical and empirical studies on branching processes.

2.2 Population Dynamics

The section of ecology that handles one or more species, focusing on the time and space disparities of their population size and density, is known as Population dynamics (Chaloupka & Musick, 2017). Population dynamics, according to Juni (2015) can be viewed as describing the variation in the number, surge or drop of the population of a particular organism quantitatively. This includes analyzing data that is effective for evaluating and estimating future trends using the required statistical procedure (Vyas, Golub, Sussillo, & Shenoy, 2020). Population dynamics is vital for biological regulation, one as a concept and the other as quantifying it (Charlebois & Balázsi, 2019; Lee-Six et al., 2018; Vyas et al., 2020). Population dynamics can be defined as changes in the size and age structure of a population. It can be inferred that population dynamics are the changes in size, the age structure of a population. In the subsequent sections, we will discuss ageing population and young population and the consequences that come with them.

2.2.1 Ageing Population and its Effects

Population ageing is characterized by the increment in the population's median age, attributed to plunging fertility rate and high life expectancy (Juni, 2015; Bloom, Canning, & Fink, 2010). Many developed countries such as Japan, France, Germany, and The UK are experiencing an ageing population due to a risen life expectancy and low birth rates (Zubiashvili & Zubiashvili, 2021; Yenilmez, 2015). At present, the aged population is at its peak, a feat which is the first in history. The United nation anticipates that the frequency at which the 21st-century population ages is bound to surpass the preceding century (Juni, 2015). The number of people aged 60 years and over has tripled since 1950 and reached 600 million in 2000 and surpassed 700 million in 2006. United Nations report projected that the ageing population will reach 2.1 billion by 2050 (Sidorenko, 2007).

2.2.2 Consequences of Ageing Population

Ageing population has many consequences on the socio-economic and health of the people. According to Pettinger (2013), Jowit (2013) and Organization (2013) the negative impact of ageing population includes but is not limited to higher savings for a pension that may reduce capital formation, shortage of workers, chronic and degenerative disease, higher taxes, increased government spending on health care and pensions. Detailed discussions of the effects of ageing are explained below.

Effects of Ageing on Population Growth and Expansions

According to the Malthusian theory of population growth, population growth will increase at an exponential rate. However, one of the core determinants of the population growth of a country is fertility. Countries with high ageing populations have a high likelihood of experiencing low birth rates due to fertility issues. According to Merchant et al. (2017), the ageing population problem in Singapore is driven by a fall in the fertility rate. This can be attributed to the fact that childbirth at an older age, especially women, are rare. Invariably, a country's population with a decline in birth have the probability of shrinking in the long-run. For example, the total fertility rate in Singapore in 1960 stood around six children per woman. Extreme intervention from the government in the form of a campaign was executed to reduce the total fertility rate. The campaign has proven to be successful as Singapore recorded a 0% population growth rate (Merchant et al., 2017). It is evident that aged people have low reproduction abilities; therefore, an ageing population may not multiply for population expansion. In the same vein, the possibility of death rate is likely to be high since the probability of the aged dying is high compared to new birth. All these characteristics of the aged population will, in the long-run, affect population growth and expansion.

Effects of Ageing on Economic Growth

Economies are affected as the population ages; such an effect can be seen in slow GDP growth. The active-age class pay more to support the aged. Likewise, the government allocates budgets extensively for the older population. An ageing population weakens the labour force, potentially leading to stalling if firms are unable to invest. Many empirical studies have shown that in the United States and Europe, GDP growth has gradually fallen due to the ageing population. As the population ages, the relative number of workers reduces whilst that of the high-consuming elderly rises. This is evident in countries such as Japan, Sweden, and the United States (Béjot & Yaffe, 2019).

A lesser percentage of children may partially offset the majority of the expenditure of the elderly, this notwithstanding, if fertility rates begin to rise, there may be an increment in the proportion of both children and aged in the population, consequentially leading to pressure on the working class. A society with an ageing population inadvertently puts pressure on the budget since there is an imbalance between the number of workers and the number of consumers, the latter being greater. This can be assessed by quantifying the ratio of support to the total number of the working class.

The government also provides for the aged in innumerable ways; this ranges from pension to in-kind public transfers, which are paid by taxing the adult population and long-term health care. Younger family members also offset the expenditure of the aged by providing net support. An example is East Asia, where the elderly receive more support from their families than they can give in return, this is the opposite in the other places within Asia, such as Japan and Korea, Europe and the Americas where you find the older generation averagely giving more support to their younger family members compared to what they receive. Generally, the majority of the elderly pay for their expenditures themselves whilst the working-class pay for the rest in exorbitant taxes (Acemoglu & Restrepo, 2017).

Public sector transfers for pensions, health care, and long-term care becomes a problem because although the elderly fund the above from taxes levied, a large chunk is from public budgets. Experts have projected that these programs and interventions are not viable unless changes are made to the fundraising process. This can be effective if the benefits are reduced, or taxes are raised, or both are implemented.

In terms of productivity payoff, if there is a slow expansion of the labour force and constant saving rate, then the machines, infrastructure, and production that comprises the country's per-capita income would soar. This implies that productivity will increase, automatically increasing wages whilst decreasing interest rates. According to Bloom, Canning, Kotschy, Prettner, and Schünemann (2019), there was a decline in productivity during 2015–2018 as compared with 1975–2015; this indicates a sizeable increase in the worker-capital ratio. According to Lee and Mason (2011), the elderly have more assets than the younger generation and are net savers in the United States and several countries. Prolonged longevity coupled with a low rate of fertility boost savings, fortify private savings. This is, however, not the case because there is no increment in the capital per worker in some situations. Governments that are forced to borrow more funds to offset the expenditure of the ageing population would eventually lose out on private capital investment funds. Also, the worker-capital ratio rises and interest rates decrease automatically, and the working-class may opt to pay less, having a negative impact on the increment of the capital. Finally, those who tend to invest will do so for a higher return on investment in the foreign capital markets. This is typical in developing and emerging economies because these economies have a younger population; therefore, the return on investment may be higher. It is worthy to note that, in such a case, although returns on foreign investment may be on the rise, it does not benefit the domestic worker. An ageing population can result in a long-term productivity slowdown. Béjot and Yaffe (2019) explained that as interest rates plunge, if Firms think factors such as declining population, labour force and lower productivity will stall expenditure growth, then Firms will consider the option of substantially cutting down on investment into the domestic economy. Lower productivity refers to a part of the economy growth inexplicable by capital and labour increases reflecting underlying factors such as technology. If firms decide to be pessimistic and central banks drive interest rates below zero, the economy could stall with high unemployment for many years (Béjot & Yaffe, 2019). Some economists attribute Japan's stalled economic growth in recent times and Europe's inability to restore its economy following the global financial crisis to these factors (Baldwin & Teulings, 2014).

According to Lee and Mason (2011), as the population ages, it slows down GDP and national income growth, but measuring its effect on individuals on the scale of per capita income and expenditure may be an entirely different issue. A greying population implies that the people being old-age dependent cannot rely on their assets or labour to support themselves, meaning more old-age dependency. Although the population may not support itself, it may increase worker per capita, increasing productivity and minimum wage, especially in cases where the debt owed by the government does not push out capital investments. There are no simple answers to whether or not an ageing population is good for the economy. The ageing pace of the population and the effectiveness of adjusting policies to suit the current realities of demographic change will inform the gravity of the problem.

Effects of an Ageing Population on Disease Patterns and Prevalence

Factors affecting individual health can be broadly classified into demographic, socio-economic, lifestyle behaviour, and social factors (Tran et al., 2018). The factors, as mentioned earlier, can be grouped into internal and external factors. Internal factors include personality and the mindset one has about physical deterioration, while external factors deal with economic level change, regional environmental change and retirement, leading to a decline in social status. Many researchers focused on individuals' socio-economic levels, classifying social class according to occupation, educational level, and income. These studies had confirmed that there were differences in mortality and morbidity across social categories, shedding light on the importance of socio-economic factors the working status might also affect health (Alferink et al., 2019). Compared to regular workers, involuntary non-regular workers had a low job and life satisfaction, which could have an adverse impact on the overall quality of life, including health (Chang, Skirbekk, Tyrovolas, Kassebaum, & Dieleman, 2019). Work hours also influenced workers' health status. Long work hours had detrimental effects on workers' health. Non-regular workers had longer work hours than regular workers. Because of this, they could not seek timely medical treatment, which in turn leads to job stress, ultimately having adverse effects on their health (Lonnie et al., 2018).

Population ageing was a key factor contributing to the elevation of medical expenditure because per capita medical expense in the elderly was higher than in other age groups. South Korea was experiencing unprecedentedly rapid population ageing, which has elevated medical expenditure (Béjot & Yaffe, 2019). However, relevant previous studies had reported that non-demographic causes, as opposed to the ageing population, account for a greater proportion of the increase in medical expenditure (Béjot & Yaffe, 2019). The proportion of demographic caused in the increase of medical expenditure in Korea had been reported to be 10% in an OECD study (11), 13.8% (12), and 28.7% (13). Despite the fact that population ageing was a potential bidirectional factor that increases and suppresses medical expenditure, few studies had used longitudinal data to confirm that the importance of medical cost changes over time.

Challenges in Social Insurance and Pension System

The number of countries affected by ageing in the population is increasing, raising concerns over the future sustainability and extension of social security coverage. More often than not, a large portion of the population is excluded from receiving benefits (Béjot & Yaffe, 2019). Governments are introducing policy changes because it has come to the fore that there is heavy reliance on social security contributions, which puts a burden on the working class. This effect may have triggered a poor response from the labour supply. On the tangent of benefits, many instances show a large part of the working class facing the possibility of a long lifetime of working than anticipated or having low pay-outs for pension. Granted the proportion of ageing, irrespective of the fact that many countries have initiated reforms, it has been established that the GDP percentage will shoot up over the next fifty years, thereby placing a significant burden on national budgets.

2.2.3 Younger Population Distribution and its Effects

Young population is defined as those people who are within the range of 15 years to 24 years. However, another school of thought refers to the young population as the population with the majority of its age structure younger than 30 years (Madsen, Daumerie, & Hardee, 2010).

The percentage of the younger population in Africa is almost 60% making Africa the world youngest continent, and this can be attributed to high fertility and decline in child mortality from various social and health interventions. Besides, it is predicted that Africa young population growth is expected to grow by 42% by 2030 (Taylor, 2016).

However, regarding the young population, Ghana reflects underlying high annual growth of 2.5%, with the age structure constituting about 57% of the entire population (Ghana Demographic profile, 2020). A greater percentage of developing countries populations are young or within these age structures. However, this structure of a population poses some consequences in the long-run. The following section discussed the implications of the young population.

2.2.4 Consequences of Young Population Distribution

The existence of a large young population is not in itself automatically a negative element for any country. A young population can add to vigour and productivity (Bhandari & Mishra, 2018). In the same vein, if the society lacks the social amenities and infrastructure to integrate, employ and care for a growing population, the potential demographic benefits of the youthful population instead become a severe drain on the resources of the state and form a dangerous element of the society (Fuller, Carleton, Fadool, Spady, & Travis, 2004).

In developing countries especially, consequences of a country population being younger have a burden on the economic growth of the nation (Béjot & Yaffe, 2019). Béjot and Yaffe (2019), state that the young population reduces capital formation and decreases the worker per capital by creating other uses for an add-on to capital resources. This is because they pressurize the already limited natural resources.

Reduction of Private and Public Capital Formations

Capital accumulation is the dynamic that drives one to make profits. The concept deals with investing money or any other financial asset to increase the monetary value of the said assets, bringing in profit in various forms. According to Karl Marx's economic theory, capital accumulation is the operation whereby profits are reinvested into the economy, increasing capital. Another school of thought is of the view that capital formation acquiring and stocking valuable assets increase one's wealth or create wealth further. The central bank of Nigeria (2007) defines capital formation as the total change in the value of fixed assets in the economy in addition to fixed assets either for replacing or adding to stock (Ugochukwu & Chinyere, 2013). It refers to the increase in the fixed capital stock of the capital formed. However, to gather extra capital, the government of countries must create savings and investments with positive returns. This can emanate from the creation of government policies that suit the situation and encourage household savings.

Accumulation of household savings can consequently be used in the rapid production of capital goods and the government running a policy on the surplus budget for investment in the capital goods produced for profit. Young rapid population growth has a diverse effect on both private and public capital accumulation; there is nothing that increases the saving culture of the population, thereby allowing shallowing capital effect and rapid population growth to lower the ratio of capital to labour. Also, a spike in population growth brings about increasing demand in expenditures by the government in several non-profit aspects of the economy such as education infrastructure and health delivery infrastructure, security systems, among others. This consequently makes funds for productive and growth-oriented investments be diverted into the non-profitable ones as expenditures.

Rural Urban Migration Drifting and Brain Drain

The concept of satisfaction is probably universal. According to Bellenbaum (1995), notions of satisfaction have always been figured in conceptions of the good life. In other words, subjective enjoyment is seen as a fruit of moral virtue and hence as an aspect of quality-of-life. Besides, due to the quest for satisfaction, mostly in developing countries, the majority of the youth have drifted or migrated to the urban areas. According to Dingle (2014), migration is seasonal to and fro movement of populations between regions where conditions are favourable or unfavourable. Besides, Brain drain in contemporary literature reflects the emigration of the highly skilled labour force. Invariably, the drift and brain drain to the urban and developed areas is due to the search for comfortability and satisfaction in terms of jobs, education, security and other greener pastures. Drifting and migration to urban areas are currently common in developing countries, making their capital towns face challenges such as slums, the spread of sickness and diseases, pressure on social amenities, high cost of living, and armed robbery. A population with a dense youth age experiences migration rendering agricultural participation in the rural areas poor.

However, its demographic impact, large-scale, rural-urban migration affects both the patterns of urban growth at the destination and land cover and land use in the region of migrants' origin. In 2016, it was estimated that by 2030, the gross number of urban populations in developing countries would double as compared to 2015. Besides, rural-urban migration accounts for 20% of the gross internal flows in India (Hoffmann, Konerding, Nautiyal, & Buerkert, 2019). In the same vein, the young population structure constitutes the workforce, and the majority of the skilled labour in quest of greener pasture emigrate to different countries. These, in the long-run, affects the origin of the migrant negatively.

Labour Unemployment and Availability Conditions in Future Terms

One of the demographic variables that play an essential role in the supply of labour is the population's age structure; age structure is a product of the rate of growth of the people and its distribution. An imbalance between the supply of labour and the demand for it give rise to unemployment and underemployment. For countries with young populations, the supply of labour exceeds the demand for labour, leading to unemployment and cheap labour.

However, the unemployment situation among the young population has become a significant concern in recent times due to the issues that come with it. To explain the theory behind the consequences of unemployment, most researchers refer to Jahoda's (1982) theory of functional approach (Stiglbauer & Batinic, 2012). Jahoda (1982) theory of the functional method looks at what unemployment means with the psychosocial definition of employment. She is of the view that when one is employed with a paid income, it satisfies Jahoda's stand by the fact that there are five hidden satisfaction and unhidden functions derived from a paid job. First of all, the function is employment; although most people view it as a burden, one is psychologically affected when it ceases to exist, or it's not there in the first place. It offers a typical time structure where one has to abide by. Secondly, since a paid job involves a working area where adults spend most of their life outside of nuclear or extended families, it serves as a platform for social contacts. The following function of a paid job is the chance to work as a group to achieve collective goals. According to the author, the fourth function is that people have an inherent need to sign up for everyday activities. Last but not least, in today's society, where employment is seen as a must, a paid job provides prominence and prestige.

Jahoda's theory of paid employment has been widely castigated, the main one being the theory of an individual's well-being dependency on a paid job. This is because there are stressful, alienating and unsatisfactory jobs that negatively affect an individual's psychosocial situation. The loss of a good and stimulating job will also affect the unemployed. Another objection is that the theory is not precise. For instance, it does not establish the level of time structure and regular activity needed to satisfy the individual.

Peter Warr tries to solve some of the problems with Jahoda's theory with his vitamin theory (Warr, 1987) on employment and unemployment. Warr is of the stance that both work and unemployment affect the individual. This implies that a paid job may not necessarily be good for the individual. Warr's theory has nine criteria which are pivotal to the well-being of the individual. These criteria are considered vitamins vital for staying healthy. These vitamins can be utilised in both cases of employment and unemployment.

This goes to show that both employment and unemployment have their bad side because they both have varying levels of meaning to the different individuals. Warr's theory can explain the differences in the level of mental well-being of the individual in both categories. Because the theory assumes that employment, and the loss of employment, can have different meanings for different individuals, it can be used to explain differences between categories of unemployed concerning levels of mental well-being. But, perhaps the most severe critique against both

Jahoda's theory about the latent functions of employment and Warr's vitamin theory is the minimal role that the individual, or the actor, is assumed to play. The theories focus on how factors in the social environment affect the individuals rather than how they experience, interpret, and act towards their social structure. Further, they do not consider the possibility that people may have different psychosocial needs that can also be changeable over some time. All this means that the theories leave little room for personal and social change. This, which could be called a functionalistic perspective on the consequences of unemployment, has been criticised by David Fryer (Fryer & Payne, 1984; Fryer, 1986), who means that the perspective sees individuals as passive reactors which are dependent on, and determined by, psychosocial functions of social institutions. The agency theory assumes that individuals are influenced by their surroundings and peers. Hence, individuals' well-being is dependent on the degree of the social structure put in place to control their lives. The effects of unemployment on mental health depend on how the unemployment situation determines the agents from reaching what they see as desirable goals. In addition to being useful when analysing various consequences of unemployment, agency theory can also explain individual changes in mental well-being during, for instance, a more extended unemployment period.

A country with a young age structure will provide available and enough labour force that can contribute and exploit the resources of a country to generate income. A skilled labour force enhances economic growth (Duval, Eris, & Furceri, 2010). To improve economic growth, a country has to sustain a stable and positive gross domestic product (the value of final goods and services produced within the boundaries of countries during a year). The active population is directly involved in the production of goods and services in fulfilment of the society; therefore, studies show a strong relationship between economic growth and labour force participation rate. Mujahid and Zafar (2012) investigated economic growth-

Female labour force participation Nexus: empirical evidence of Pakistan. They used time series data spanning 1980 to 2010 and applied the ARDL technique to determine the nexus between economic growth and female labour participation. The result shows that there is a long-run relationship between female labour force participation and economic development.

Government Revenue Accumulation

Any country that wants to develop relies heavily on revenue. The expansion of government expenditure on infrastructure and services is dependent on the projections made on income to be mobilised. The majority of the countries deemed as developing build up their revenue by direct and indirect taxation, aids from developed countries, local production, amongst others. Tax is meant to generate the required funds to cater for the public expenses (Stijns, 2006). A country with a dense young population provides the labour force that can invariably provide the government with more tax revenue and, in the same vein, contribute to gross domestic gains.

Future Ecological Exploitations and Destructions

Youthful population growth necessitates the exploitation of available natural resources to meet the expectations and demands of the period. This menace puts pressure on the untapped resource that may be reserved for future generations. Young population in the long-run requires infrastructure advancement, in the form of road networks, residences, land enlargement through acclamations among others. In an attempt to resolve these challenges and meet the situational demand of the population, new dwellings will be provided, the forest needs to be depleted. Sophisticated machinery is required for the execution of this project resulting in the destruction of the ecosystem. The population environment literature concerns the relationship between increasing population density in subsistence agricultural areas and land degradation. The result of widely differing estimates regarding the extent of land degradation, with global estimates ranging from 20 to 51 million kilometre per square. However, another proponent believes that increasing population density in the contest of high poverty almost inevitably leads to land degradation (Sherbinin, Carr, Cassels, & Jiang, 2007).

Possibility of Social, Economic and Political Insecurity

The issue of insecurity is high in a situation where a country's population is high, especially when most of the population is within the workforce. However, insecurity can be viewed in terms of job, safeness, stable governance, among others. Globally especially in Africa, a population with a greater number in the youthful stage with a creeping economy are characterized by unemployment, high level of dependency, armed robbery, economic hardship, pressure on existing amenities and infrastructure, low per capita income, among others. Melcher (2021) defines economic insecurity as a psychological response to the possibility of hardship-causing financial loss. Insecurity requires real risk that threatens real hardship. In the same vein, according to Erlinghagen (2008), age is correlated with personal insecurities. With evidence, countries with these characteristics face political impunity through impeachment, coup d'etat and other forms of rebel attacks and others. Most developing countries such as Ghana, Nigeria, Ivory Coast suffer these challenges due to not satisfying the expectations of the young growing populations.

2.3 Population Policies

Population policies are actions taken to tackle the expected outcomes of birth and death, leading to internal or external relocation, which affects the population's size, the varying ages, and how the population is distributed in specific locations (Babiarz, Ma, Miller, Song, et al., 2018). The demographic transition, which involves first a decline in age-specific mortality rates and then a general decline in fertility rates, is one underlying development that motivates population policy research, either directly or indirectly. Due to the decrease in mortality and the sustained and substantial reduction in fertility, the natural rate of population growth tends to increase and the age composition of the population changes (Ge et al., 2018).

2.3.1 Population Policies used to Control Young Population

Population policies used to control the young population are anti-natalist policies. The anti-natalist policy is a population policy that aims at discouraging births (Shen, Wang, & Cai, 2020). This can be done through taxes on larger families, promotion of sterilisation, legalisation of abortion, and provision of contraception.

Birth Control/Contraceptives Policies

To control the existing young population explosion, governments and stakeholders around the world use various devices, drugs, agents, sexual practices, or surgical procedures to prevent conception (Zenz, 2020). They concentrate on preventative measures that will reduce the viability of producing offspring. As a result, the controlling department intensifies public education on the use and importance of contraceptives such as oral contraceptives, intrauterine devices (IUD), contraceptive implants, contraceptive injection, emergency contraception pill (The morning after pill), contraceptive ring, and diaphragms for childbirth. Furthermore, another school of thought believes that the coitus method, which involves pulling the man's penis out of the vagina before ejaculation, has been used as a birth control method over the years. When compared to traditional methods such as withdrawal and periodic abstinence, modern contraceptive methods such as male and female sterilisation, intrauterine devices (IUDs), implants, injectables, pills, male and female condoms, and spermicides are highly effective in preventing pregnancy (McCann, 2019). However, global contraceptive use has increased, rising from 54% in 1990 to 63% in 2007 (World Health Organization 2011). Given the availability of modern contraception, it is the most dependable method of preventing unintended pregnancy. As a result, the government and other non-governmental organisations should increase the supply and availability of these contraceptives for the policy to be effective because one of the most frequently cited reasons for women not using contraception is a lack of supply of methods and services (Rott, 2019).

Family Planning Policy and Reproductive Educations

The World Health Organization (WHO) defines family planning as "the ability of individuals and couples to anticipate and achieve their desired number of children, as well as the spacing and timing of their births." It is accomplished through the use of contraception and the treatment of involuntary infertility. This policy is critical for all countries, particularly developing countries because it effectively helps both married and unmarried people avoid unintended pregnancies. However, the United States has set a national goal of reducing unintended pregnancies from 50% in 2001 to 30% by 2010 (HHS, 2010). The goal of this policy is to increase responsible sexual behaviour and the proportion of adolescents who abstain from sexual intercourse or use condoms if they are currently sexually active, and the proportion of all sexually active people who use condoms. Family planning can increase partners' involvement in decisions about when to have children, in addition to its maternal and infant health benefits. One of the most critical aspects of assisting people in making pregnancy plans is avoiding unintended pregnancy. According to Alemayehu et al. (2016) family planning can be divided into two categories: natural family planning and artificial family planning. The former is a method of birth control that does not use pills or devices but instead uses the rhythm method, cervical mucus method and basal body temperature method, whereas the latter uses contraceptives and other sophisticated device applications. Other factors influencing the success of this family planning programme include education, effective execution, and implementation, among others. Mass media communication campaigns, for example, can raise awareness of the benefits of family planning and responsible parenting. Family planning programmes should also reach out to a wider range of people, such as religious and community leaders (Aliyu, 2018).

There is the need to incorporate reproductive and sex education into schools to prepare students for healthy and responsible living. Reproductive health programme designers should prioritise young people's negotiation skills. The establishment of youth-friendly service centres in convenient locations and the provision of necessary materials would encourage young people to use sexual health services (Keller, 2020). Studies revealed that most women, particularly adolescents, were unaware of reproductive organ physiology and contraceptive methods. Disseminating sexual health information through mass media could be one way to reach out to urban youths. Sexual health education delivered by trained peer educators can also be an effective method of increasing young people's knowledge of sexual and reproductive health issues (Mazur, Brindis, & Decker, 2018). Women's education is also an essential factor influencing contraceptive use and the effectiveness of family planning policies (Farahani, 2020). A Nepalese study found that older women (35 and up), who were educated, lived in cities, and worked in the business or service sectors, were more likely to use modern contraception methods (Sharma, Pratap, & Ghimire, 2011). Moreover, the ability to implement new educational reforms and create jobs and opportunities for women, in particular, has a significant impact on population growth. This can be attributed to the fact that urbanisation drifting and brain drain will occur due to education. According to Crankshaw and Borel-Saladin (2019), as development progresses, socioeconomic development and rapid urbanisation lead to a decrease in fertility rates. Furthermore, wealthy cities such as Shanghai had seen a drop in birth rates below the population replacement rate. This can be explained by the fact that women in urban areas have more education and labour opportunities than women in rural areas. As a result, employed urban women tend to marry later, resulting in fewer children. This is further supported by the fact that women who work in cities devote the majority of their time to education (Farahani, 2020).

Promoting Sterilization and Legalising Abortion

In general, the young population is more likely to reproduce, increasing the future population. Most decision-makers worldwide are promoting sterilisation and legalising abortion. Sterilisation is a permanent method of birth control. This procedure, however, is tubal ligation for women and vasectomy for men. In general, men use this method by tying the cut, clipped, or sealed to prevent a female egg from being fertilised by the man's sperm. In other words, surgical sterilisation is a procedure that permanently impairs a person's ability to bear children. Some countries, for example, have adopted this policy to control births. India enacted mandatory sterilisation laws and incentivised government employees who become sterilised by increasing their salaries and wages. The country also mandated the sterilisation of couples with more than two children. Legalising abortion, in contrast to sterilisation, is a positive step. Globally, abortion is one of the commonest gynaecological procedures (Rowlands & Thomas, 2020). The widespread practice of abortion, however, masks significant diversity in the social, political, and ethical meanings of terminating a pregnancy, as well as the practises surrounding abortion. All of these differ from one historical period to the next and among social groups within specific periods and confinement. That is, abortion is widely regarded as an unexceptional method of terminating an unwanted or untenable pregnancy (Abrams, 2015). Most countries, to keep their young population from exploding, legalise abortion. According to Macleod, Beynon-Jones, and Toerien (2017) and Sheldon (2016), British legislation has made abortion more accessible. This method, however, is either surgical or involves the use of contraceptives and pills such as misoprostol, Cytotec, Oxytocin, and Pitocin, among others. In some countries, abortion and child per parent policies have been implemented. For example, in Vietnam, abortion was introduced as part of the family planning process, which has resulted in a drastic reduction in births and population fertility growth (Hoang, Nguyen, & Duong, 2018). Furthermore, the implementation of the resistance to forced abortion was used as a tool to address the issue of overpopulation (Eklund & Purewal, 2017). Implementing mechanisms, governmental interventions, and the effectiveness of health practitioners and health delivery systems all contribute to the policy's effectiveness.

The use of Fiscal and Other Government Policies

The government and other policymakers use fiscal mechanisms such as taxation to discourage individuals from having large families or having multiple children to control population growth. Taxation is defined as a mandatory payment to a government based on the holdings of a tax base, but it is a significant political and economic issue (Saad & Al Foori, 2020). Currently, taxes are imposed not only for those purposes but also to further social and economic policy (Drazen, 2018). It is a mandatory contribution made by citizens to the state to carry out its political and financial responsibilities. According to Bongaarts (2016), the state and taxation are not mutually exclusive, and one cannot exist without the other. This policy is implemented so that families with new children or large family sizes face higher taxes. In the same vein, the government can encourage and enforce favourable regulations and laws on birth, marriage, and fertility, among other things, to reduce the young population exodus.

2.4 Definitions and Concept of Branching Process

This section discussed and reviewed the literature on the branching process and its applications.

The Branching Process is a simple but elegant model of population growth. It is also called the Galton Watson Process because some of the early theoretical results are derived from correspondence between Sir Francis Galton and the Reverend Henry William Watson in 1873.

Beznea, Lupaşcu-Stamate, and Vrabie (2020) defined branching process as a stochastic process, which consists of a set of random variables indexed by natural numbers in which each individual in the n^{th} generation produces a random number of individuals in $n + 1$ generation. A branching process models the reproduction of organisms such as human beings, cells or neutrons (Jagers, 1989; Kimmel & Axelrod, 2002). In the simplest branching processes, time is measured discretely in generations, and organisms are of only one type. Each organism is viewed as living one generation; during this period, it produces offspring contributing to the next generation. It is assumed that organisms reproduce independently according to the same probabilistic law. Interaction between organisms is forbidden.

Figure 2.1 presents an illustration of branching processes. In branching process, the population size (X_n) for each generation is of interest. It is assumed that the population size of the initial population is one ($X_0 = 1$). From Figure 2.1, the population size for first generation is 3 ($X_1 = 3$) and population size for second generation is 6 ($X_2 = 6$).

2.5 History of Branching Process

The study of branching processes began in the 1840s with a statistician called J. Bienaym, a probabilistic and later was advance by a clergy mathematician named Reverend. H. W. Watson in the 1870s and Francis Galton, a biometrician (Grosjean & Huillet, 2018). In the mid-19th century, most families in Victorian England realised that their family names could become extinct. Then in 1873, Sir

Francis Galton posed the question, "How many male children (on average) must each generation of a family have for the family name to continue in perpetuity?" (Galton, 1873). To answer the question, Reverend Henry William Watson and Sir Francis Galton wrote a paper on the probability of extinction of families in 1874. They proposed a solution using the theory of generating functions and functional iteration; they concluded that there is a possibility or probability that every family can go to extinction regardless of the family size and the average number of children each member reproduce.

The simple branching process model has independently arisen several times since 1845 to determine the influence of chance on the fluctuations in sizes of successive generations of family lines. The most thorough summary of this history is given in a paper by K. Albertsen, translated from the Danish and annotated by Albertsen (1995). The article points out that families realise that their family names are disappearing. However, the reason for the names disappearing was attributed to low fertility. During the 19th century, some scholars suggested that chance events influence the sizes of generations, and hence that a mathematical approach to the extinction of names ought to be feasible. Bienayme first did this in 1845, in a brief descriptive paper that was lost to the scholarly world until 1972. Heyde and Seneta (1977), wrote articles

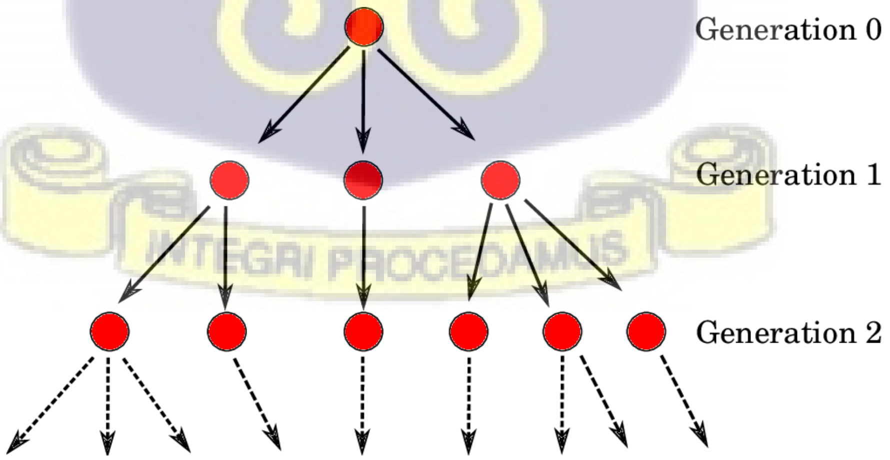

Figure 2.1: Branching Process

to explain the thoughts of Bienaymes articles when Seneta (1998) re-discovered Bienaymes paper (Béjot & Yaffe, 2019). The French text of this paper was reproduced in the end by Kendall (1975b). In relation to the probability of extinction of the male family line that descended from a single ancestor, Bienayme writes that if the mean of the number of male children who replace the number of males of the preceding generation were less than unity, it would be easily realised that the families are dying out due to the disappearance of the members of which they are composed. However, the analysis shows further that when the mean is equal to unity, families tend to disappear, although less rapidly. The analysis also shows that if the mean ratio is greater than unity, the probability of extinction of families with time no longer reduces to certainty. It only approaches a finite limit, which is fairly simple to calculate and which has the singular characteristic of being given by one of the roots of the equation (in which the number of generations is made infinite) that is not relevant to the question when the mean ratio is less than unity. This is a clear and correct statement of the fundamental criticality theorem from the probability generating function (PGF) of the offspring distribution $f(t) = p_0 + p_1(t) + p_2t^2 + p_3t^3 + \cdots$.

In 1873, Candolle drew attention to the desirability of a mathematical approach to family extinction. F. Galton had for many years been interested in quantitative aspects of heredity and hence was receptive to de Candolle's challenge. Galton rendered the problem in precise mathematical terms where he considered only adult males and published it in the Educational Times (Galton, 1873). Their distribution of the number of adult males is such that in each generation, p_0 is the percentage of the adult males who have no male children in their adult life; p_1 have one such; p_2 have two, and so on up to p_5, who has five. In his paper, he estimated the probability of the surname that will become extinct after n generations and also evaluated the possibilities of surnames being held by j number of persons.

As no correct solution was submitted, Galton asked his friend H. W. Watson (mathematician, rector, and mountaineer) for help (Kendall, 1975a). Watson made considerable progress, publishing his work in the Educational Times later in 1873. Along with preliminary remarks by Galton, this was formally published as Watson and Galton (1875). Watson fully answers Galton's specific questions (1) and (2) in the quotation. It is worthwhile summarising his contribution. His notation differs from ours. Watson assumes $f\left(.\right)$ is a polynomial, and he realises that the solution of Galton's problem is determined by the behaviour of its functional iterates, defined by

$$f_0\left(t\right)=t, \qquad f_n\left(t\right)=f\left(f_{n-1}\left(t\right)\right) \qquad \left(n\geq 1\right) \tag{2.1}$$

He states that the probability $p_{ij}^{(n)}$ that surname with i representatives in a generation will have j representatives n generations later is the coefficient of t^j in $\left(f_n\left(s\right)\right)^i$. Haldane (1927) clearly asserts the supercritical part of it, and with no restriction on the form of $f\left(.\right)$. His motivation for considering the problem is an interest in the survival of a rare mutation with a small selective advantage over the wild type genotype. He refers only to an earlier discussion of Fisher (1922), and he appeals to results about functional iteration.

The first general statement and proof of the criticality theorem arose in circumstances similar to, but independently of, the Galton-Watson collaboration. The Danish pioneer of queueing and bank traffic theory, A. K. Erlang, himself the last scion of one line of the distinguished Krarup family, posed the following problem in a 1929 issue of the Matematisk Tidsskrift.

Erlang obtained a partial and unpublished solution, having died soon after publishing the problem. J. F. Steffensen used his manuscript to arrive at a complete treatment of the problem, published in 1930 (Guttorp, Albertsen, Steffensen, & Kristensen, 1995). Steffensen (1932) highlights assumptions made to reach his solution, such as homogeneity of individual reproduction laws and independence of family lines. Steffensen's complete formulation and proof of the criticality theorem marks the end of the early development of branching process theory. Steffensen (1930), noticed that Galton and Watson's solution was right if the mean number of children reproduce is less than or equal to one, but if the mean number is more significant than one, the extinction probability of the family name is less than unity. Consequently, we shall not further pursue its historical development, except to remark that C. M. Christensen submitted a complete solution to the Matematisk Tidsskrift early in 1930. It was not published because it arrived a little after Steffensen's article had been sent to the printer. Christensen's manuscript is reproduced in Guttorp et al. (1995). In parallel with Stefiensen's studies, Fahlbeck (1898) also investigated the problem of extinction of Swedish noble families with extensive mathematical analysis. Fahlbeck found out that the ratio between male and female births of the Swedish noble families decreased and was called a degenerative phenomenon. Westergaard (1900), after using Fahlbeck's (1903) models, realised that without the assumptions of degeneration, the phenomenon noted by Fahlbeck must occur in all families whose male line becomes extinct.

Joffe and Spitzer (1967) in their research found out that individuals in the branching process may be independent but not identical. For example, type A males, incredibly fertile, maybe type B males, who are less productive. Furthermore, each type will have its offspring distribution; e.g., especially fertile males may be more likely to produce fertile offspring. Therefore, assuming common distribution to all of them will be problematic and proposed a multi-type branching process that allows for different kinds of objects in each generation. In multi-type branching processes, individuals are not identical but can be classified into m types of individuals. For instance, in population genetic, when considering the inheritance of alleles, a 3-type branching process, with types corresponding to the genotypes AA, Aa, and aa can be used as a model (CSERNICA, 2015).

Many extensions of Galson-Watson branching processes have been proposed, such as random walk and large deviation. The complexity of these processes continues to increase, allowing to describe more and more realistic population dynamics, biology (gene amplification, clonal resistance theory of cancer cells, polymerase chain reactions), epidemiology, genetics, and cell kinetics (the evolution of infectious diseases, sex-linked genes, stem cells), etc.

2.6 Applications of Branching Processes

There are some applications of branching processes in cells biology and epidemiology (Devroye, 1998; Mahmoud, 1992), Lagrange distributions, random walks and random forests (Pitman, 2006; Addario-Berry & Reed, 2009). This section discussed application of branching process in population dynamics, cell biology, epidemiology and infectious diseases.

2.6.1 Application of Branching Processes in Population Dynamics

In an attempt to solve the imbalance problems associated with population dynamics, many models have been proposed, such as branching process, birth and death models, life table, Leslie matrix, geometric population, etc.
Branching processes were once born out of a question from (human) population dynamics. Bienaymé (1845) formulated what later became known as the Galton-Watson process to study the extinction of surnames. Galton's formulation for estimating the possible survival of the family names of a specified number of adult males was famous, Kolmogorov (1959) came out with formulas to answer some questions on population dynamics half a century later. Kolmogorov came

out with two works on population dynamics which were titled ‘The solution of a biological problem’ and ‘The transition of branching processes into diffusion processes and related genetic problems’ (Kolmogorov, 1938, 1959).

The determination of offspring laws that minimise the probability of extinction also arises in connection with optimal life histories (Holgate, 1967; Daley, 1969). Branching process models take direct account of independent life histories, and the size of a population affect its extinction. Lebreton and Clobert (1991) made some interesting general observations about the utility and limitations of branching process models in assessing extinction probabilities of small populations. He did the reserch to answer questions such as why reproduction once in a lifetime (constant) is not universally favoured over reproduction on several occasions (varying). Given that reproduction increases the chance of mortality, some have argued that it is better to forgo the opportunity of several reproductive events in favour of just one when the individual has reached full maturity. Holgate (1967) investigates this question in a model system of biennial animals which can reproduce in each year of their lives, but where the probability of survival into a second-year involves production of a specified number of offspring in the first year. For some geometric and Poisson offspring laws, he finds that the probability that the population survives is an increasing function of the probability that an individual who is childless in the first year will survive to a second year.

Hairston, Olds and Munns (1985) discusses two different explanations of observed clutch-sizes of bird species whether they have evolved to maximise growth rates or minimise extinction probabilities. Arguments based on bet-hedging favour the latter as a way of coping with environmental uncertainty and avoiding direct competition for food. He computes clutch-size probabilities, which minimise extinction probabilities for models where environmental variation affects nestling survival through food abundance. Mountford finds that clutches of 2 or 3 eggs are optimal.

Grey (1980) pursues further aspects of this problem. His general set-up incorporates independent stationary environments and PGF's $f(s, N,)$ for the number of eggs in a clutch of size N, producing a laying bird when the environmental variable is constant. The aim is to find a clutch-size law that maximises survival probabilities for differing ancestral numbers. The optimal choice depends on the life-history strategy used by the birds. For example, the clutch-size law could vary from one generation to the next or be chosen and fixed for all generations. Numerical examples illustrate Grey's theoretical development.

Mountford (1988) uses a model of univoltine insects reproducing and dispersing leaves to discuss whether the detection of density dependence from empirical data is obscured by spatial and stochastic heterogeneity. His model is a Markov chain for the sizes of successive generations of insects. It allows for random dispersal of adult insects among many leaves, a Poisson larval production law for each adult, and size-dependent survival to adulthood on each leaf's larva. Although not modelled, infinite divisibility arguments show that this model is a size-dependent branching process.

The interaction of fecundity and sex ratio is investigated by Gabriel and Bürger (1992) using size-dependent branching processes. Individual offspring laws are Poisson with size-dependent means $m_{(i)}$, which take one of three ecologically motivated forms. The effects of sex are modelled as a two-type process in which the number of females and males born to an individual is independent with the same Poisson law, whose meaning depends on the number of females y and males k in the parent generation. Allowing sex in this way typically reduces the meantime to extinction. Some of the analytical properties of these models are inferred from simpler ones valid for extreme parameter configurations and which reduce the generation-size process to a stationary independent one. Ludwig (1996) conducts a numerical study of a size-dependent model, which also allows independent random environments and catastrophic mortality. Ludwig (1996) plots the probability of surviving n or more generations as functions of n for different population. He concluded that survival probabilities for small n (10 for his parameter values) are insensitive to changes in environment, but quite large differences are seen when n is large. This suggests that the survival of small populations is little affected by environmental changes unless the population size is simultaneously increased. By informal reasoning, Ludwig (2013) approximates his branching process model under appropriate conditions for large population with small changes over successive generations.

According to Mettle et al. (2019) probability of ultimate extinction can be used as a proxy for measuring the ageing population. Using parity of women in the terminal reproductive age instead of census data which is costly and time-consuming Mettle et al. (2019) analysed population dynamics of some selected countries in branching processes in a constant environment. They found out that Colombia was ageing than Honduras and Burkina Faso. These results were confirmed with World Development Indicators (2018), which state that the percentage of the population aged 65 and above to total population in Colombia (8.48%) was higher than Honduras (4.69%) and Burkina Faso (2.41%). According to Jagers (1997) the independent restriction of the branching process limits the application of the branching process in real-life situations since life is full of interaction. As a result, Jagers (1997) analysed the branching processes in the population dynamics where there is the interaction between individuals in the population. Lambert (2005), did not consider the independence of individuals in analysing the intraspecific competition pressure (density-dependence) in ecology population dynamics.

2.6.2 Application of Branching Processes Mutation/ Cells Biology

Branching theory was first applied in a substantial way by Fisher (1922) to some problems about the survival of genetic mutations. The genetic consequences of the branching process approximation are explored more fully in his influential monograph (Fisher, 1958). Within the limitations of applicability of the Wright-Fisher model, Fisher's branching process calculations show that for a small initial number of mutants, survival probabilities fall rather quickly. They give a rough idea of the number of generations required for effective loss of the mutant and the effects of a small selective advantage. His results show that tiny rates of recurrent mutation can ensure the survival of the mutant in the population. A related conclusion is that the fate of a selectively advantageous mutation is decided within the first few generations. It can become established if chance favours its early growth. Finally, calculation from current genotype frequencies permits a rough estimation of how far back a mutation arose.

Fisher's work has been extended in several directions. One is considering similar questions for genetic configurations giving different offspring laws, perhaps with other variations of the simple branching process structure. For example, Fisher himself argued that offspring laws for humans ought to have a variance to mean ratio exceeding unity. Kojima and Kelleher (1962) show that family size data from the U.S. census of 1950 are fitted better by a negative binomial law than the Poisson. Holgate (1966) uses the critical law to investigate the extent of increased survival of a neutral gene in a population that grows from a small size. Kojima and Schaffer (1964) consider the survival of two linked mutant genes. Here adults reproduce according to a general offspring law with mean M. A child acquires the mutant chromosome with probability $\frac{1-p}{2}$, where p is the recombination fraction. A fraction of affected children survive to adulthood. Hence the offspring mean for the mutation is $\frac{M(1-p)}{2}$. This shows, among other things, that tight linkage, meaning low recombination, enhances the accumulation of epistatically favourable mutations (Schaffer, 1970).

Bertoin (2010) gives an advanced simulation study to assess the meantime for loss of a neutral mutation in a human population. Recent theoretical work is based on diffusion methods, although branching processes have been used for deleterious mutations. All this is based on very simplifying assumptions about the structure and demographic characteristics of the model population. Bertoin (2010) try to inject greater realism. The simulation is designed to Fait Mully model the attributes of a Yanomami Amerindian community comprising four interacting villages. The tribulations of a new neutral mutant over 400 years are modelled as a branching process whose offspring law again is a zero-modified geometric. The authors appeal to Adami and Chu (2002) for this choice. A meantime to extinction of generations is found, slightly more than for an earlier and cruder simulation, but much less than the generations computed from the diffusion approximation.

According to Slavtchova-Bojkova, Trayanov, and Dimitrov (2017), branching process models of mutant survival in large populations give reliable results when the selective advantage is not too small, i.e., when evolution proceeds reasonably quickly. They are thus a helpful complement to diffusion methods which are most accurate when the selection advantage is small. In addition, the above justification for the characteristic independence of lineages requires that the mutation numbers remain relatively small. This may impose limitations on the time horizon for which the approximation is reliable.

These considerations are well illustrated by Sundd, Gladwin, and Novelli (2019). They considered the frequency of electrophoretically recognisable human haemoglobin mutations which arose between generations in the past. Sundd et al. (2019) make the point that estimates of the variance of the frequency of electrophoretically recognisable human haemoglobin mutations are necessary to assess the extent of agreement. A branching process model is used to compute this variance for the recent generations, and diffusion methods are used for the more remote past.

Lange (2010) computes median times to extinction for several critical offspring laws. They are concerned with the loss of genetic variability under a specific breeding policy for crop plants. The median time to extinction is interpreted as a segregation 'half-life'. These medians are typically relatively small, and hence it is essential to obtain accurate estimates for deciding whether to embark on a breeding programme for given plant species. Their calculations show that the simple branching models give more reliable results than diffusion approximations.

According to Henry-Labordere, Oudjane, Tan, Touzi, and Warin (2019), the theory of diffusion approximations for branching processes is valid when the expected number of offspring is near unity, and the number of ancestors is large. Jacob and Penisson (2011) studied the probability for which a gene becomes extinct before a mutant form appears. Jacob and Penisson (2011) numerically explores this using Poisson offspring law. Fijarczyk and Babik (2015) explored the above model in connection with detecting the appearance of a deleterious genotype in a family line. Allen (2015) established the relationship with the total number of births and treated the case separately. More complicated genetic configurations can be handled using multi-type branching processes. For example, modelling the numbers of normal and mutant genes typically will be a two-type process. A significant consideration is whether mutants can mutate back to the normal type. In either case, the mutation probabilities are very small, and if the models are appropriately parameterized, it is possible to derive

Poisson approximations. In the supercritical case, time can be used as the parameter, and the approximation then holds on to the set of non-extinction. Slavtchova-Bojkova et al. (2017) derive such results for a binary splitting model of mutation of the Q-galactosidase gene carried by a bacterial virus. McDonald and Kimmel (2017) explore the case where each type has the same general offspring law. Particular continuous-time mutation models comprise the quantitative basis of fluctuation tests for bacterial mutation. This subject has a quite extensive literature of its own, and which recently has been comprehensively reviewed (Slavtchova-Bojkova & Vitanov, 2019).

Another multi-type example is where the viability of a gene is spatially dependent, a situation which can be modelled by supposing the total population evolves within a set of discrete niches (Makarova, Kutsenko, & Yarovaya, 2020). Makarova et al. (2020) consider a large population of fixed size comprising N males and N_z females. A favourable mutation arises for which the offspring of mutant males occur in the ratio are less than unity. Burden and Wei (2018) extends Bartlett's approximation to the multi-type case. Burden and Soewongsono (2019) tighten Eshel's arguments by giving precise conditions under which the asymptotic equivalence is valid.

Rouzine (2020) examines the fate of a selectively neutral mutation growing in a multi-locus background that has attained a stable equilibrium at which there exists linkage disequilibrium. They assume Poisson offspring laws, and they prescribe genotypic viability coefficients of linkage disequilibrium. Nerman and Jagers (2020) ask how to define a notion of gene fixation meaningfully within the neutral theory of evolution. They envisage neutral mutations arising simultaneously and independently replicating according to a critical branching process, a linear birth and death process.

Baake, Cordero, and Hummel (2018) propose semi-stochastic models of mutation-selection balance to account for the increasing temporal pace of mutation induced by an increasing background population of normal types. As well as autosomal dominant and X-linked mutations, they model haplotype information on linked markers, as used in the linkage disequilibrium strategy of positional cloning. Their models have elements in common with the Lea—Coulson model of bacterial mutation used for Luria—Delbriick fluctuation tests (Zheng, 2021).

2.6.3 Application of Branching Processes in Epidemiology and Infectious Diseases

A basic model for the spread of a disease through a closed population is the S-I-R model. It assumes individuals progress partly or all through the irreversible sequence of states of health, susceptible infected, removed. Removal can mean physical isolation from susceptible populations, death, or recovery with lasting immunity. A fundamental simplifying assumption is that the population is fixed in size and that infectious and susceptible individuals mix homogeneously. This interaction between the two sub-populations is mirrored in non-linearities.

Branching process approximations to epidemic models have two principal uses. The first is to evaluate the likelihood of a major outbreak of the disease, which means that a large proportion of the susceptible have been infected by the time all infectives have been removed. The second use is concerned with the number of original susceptible ever infected, called the final size of the epidemic. A simple characterization of its law is not possible for even the simplest epidemic models. Under quite general conditions, this law can be approximated by the total number of births in an approximating branching process. Up until 1980, branching process approximations were used intuitively. A principal contribution since then has been the clarification of the precise nature of this approximation using modern techniques from the limit theory for stochastic processes.

Bailey et al. (1975) is the classical reference for epidemic modelling, with Gani and Daley (1999) giving a more modern account. Ball, Mollison, and Scalia-Tomba (1997) also conducted a survey on the role of branching processes in epidemic modelling. O'neill (1995) studies the modified epidemic where infections are classified into high- and low-risk groups, distinguished by a different constant rate. High-risk infections eventually switch to the low-risk group. O'neill (1995) allows a contact rate in the modified epidemic to depend on the current number of removals. These models are motivated by problems of sexually transmitted diseases. Ball et al. (1997) considers a population comprising M groups, or households, each with N individuals. Initially, one of the MN individuals is infectious. The contact mechanism for infections is defined by removal time and two-point processes, accounting for within-group and between-group contacts. The model is otherwise similar to the general epidemic. Strong convergence to an approximating Crump-Mode-Jagers process is obtained by allowing Lashari and Trapman (2018) to modify this arrangement so that infections become susceptible when their period of infectiousness ends. Mettle, Osei Affi, Aidoo, and Benn (2020), used a stochastic version of the deterministic epidemiological compartment (Susceptible – Infectious - Recovery) model through the branching process approximation, with continuous Time Markov Chains where the time variable is continuous and the state variable is discrete.

Two applications of multi-type branching processes to epidemic models are presented. The first application is to an SEIR epidemic model and the second application is to the same epidemic model but with dispersal. The SEIR epidemic is modelled as a two-type branching process. The occurrence of an outbreak depends on the number of exposed and infectious individuals. It is shown that the offspring pgfs for the exposed and infectious populations lead to an explicit formula for the probability of an outbreak. In the SEIR model with dispersal, the case of two regions with different healthcare situations is considered. One area has poor healthcare versus another area with excellent healthcare. It is shown that the rate and the direction of movement have a large impact on the occurrence of an outbreak. Branching process theory is used to investigate the probability of an epidemic when the movement rates differ between two regions.

Although the SIR and SEIR epidemic models are simple, they are often used as a first approximation during or after disease outbreaks to provide estimates of the potential spread of the disease or to understand the pattern of spread. For example, SIR and SEIR epidemic models in conjunction with data provided helpful information about the spread of the 2002–2003 SARS (severe acute respiratory syndrome) pandemic which began in China, the 2009–2010 H1N1 influenza pandemic which began in Mexico, and the 2014 Ebola outbreak in Africa.

2.6.4 Summary

This chapter had reviewed all the necessary literature with regards to the branching process and population dynamics. The theoretical literature was reviewed as well as the application of the branching processes in various fields. The study also reviewed the consequences of ageing and the young population and the various ways to control population dynamics. From the literature review, most branching processes applications are made in a constant environment and hence there is limited information on the application of branching processes in varying and random environments.

Chapter 3

Mathematical Description of Branching Processes

3.1 Introduction

This section presents the mathematical description of single-type branching process in a constant, varying and random environment as well as multi-type branching processes.

3.2 Branching Process in Constant Environment

Branching process in constant environmrnt is a branching process where the offspring distribution for all generation are the same and each individual produces a random number Z of offspring with probability distribution, $P(Z = r) = p_r, r = 0, 1, 2, ...,$ where $p_r \geq 0,\ r = 0, 1, 2, ...$ and $\sum_{r=0}^{\infty} p_r = 1$ (Palau & Pardo, 2018). All offspring act independently of each other and at the end of their lifetime, individually have random number of progenies in accordance with probability distribution. Further, the number of offspring born to an individual does not depend on how many other individuals are present.

To begin, we create a sequence that specifies the number of individuals in each generation branching process.

Definition 3.1.

For each generation n, we define X_n as the number of individuals in the generation. We write (X_n) for the sequence of the X_n. The initial population in the zero-th generation is $X_0 = 1$.

Definition 3.2.

Let $P(Z_1 = r) = p_r$. As we assumed $X_0 = 1$, this means that the probability that a single individual has r children is equal to p_r. Note that $\sum_{r=0}^{\infty} p_r = 1$. Given the population size of the n^{th} generation, the number of individuals in the $(n+1)^{th}$ generation is,

$$X_{n+1} = \sum_{r=1}^{X_n} Z_r \tag{3.1}$$

where $Z_r,\ r = 1, 2, 3, 4, ...,$ are iid random variables with probability distribution

$$P\left(Z = r\right) = p_r,\ r = 0, 1, 2, ..., \tag{3.2}$$

such that $p_r \geq 0,\ r = 0, 1, 2, \ldots,$ and $\sum_{r=0}^{\infty} p_r = 1$.

Define the probability generating function (pgf) of the offspring as

$$f\left(s\right) = \sum_{r=0}^{\infty} p_r s^r, 0 \leq s \leq 1 \tag{3.3}$$

and $f_n\left(s\right) = \sum_{r=0}^{\infty} P\left(X_{n+1} = r\right) s^r$ for $n = 0, 1, 2, 3, ..., 0 \leq s \leq 1$. It is clear that $f_0\left(s\right) = s$ and $f_1\left(s\right) = f\left(s\right)$.

We start with

$$\begin{aligned} f_{n+1}\left(s\right) &= E\left(s^{X_{n+1}}\right) = \sum_{r=0}^{\infty} P\left(X_{n+1} = r\right) s^r \\ &= \sum_{r=0}^{\infty} \sum_{j=0}^{\infty} P\left(X_{n+1} = r, X_n = j\right) s^r \\ &= \sum_{r=0}^{\infty} \sum_{j=0}^{\infty} P\left(X_{n+1} = r | X_n = j\right) P\left(X_n = j\right) s^r \end{aligned} \tag{3.4}$$

Therefore,

$$\begin{aligned} f_{n+1}(s) &= \sum_{r=0}^{\infty}\sum_{j=0}^{\infty} P(X_n = j)\, P(Z_1 + Z_2 + Z_3 + \ldots + Z_j = r)\, s^r \\ &= \sum_{j=0}^{\infty} P(X_n = j) \sum_{r=0}^{\infty} P(Z_1 + Z_2 + Z_3 + \ldots + Z_j = r)\, s^r \qquad (3.5) \end{aligned}$$

As Z_r, $r = 1, 2, 3, 4, \ldots, j$ are i.i.d random variables with common probability generating function $f(s)$, the sum $Z_1 + Z_2 + Z_3 + \ldots Z_j$ has the probability generating function $[f(s)]^j$. Thus,

$$f_{n+1}(s) = \sum_{j=0}^{\infty} P(X_n = j)\, [f(s)]^j \qquad (3.6)$$

which implies

$$f_{n+1}(s) = f_n(f(s)). \qquad (3.7)$$

3.2.1 Mean of Population Size of the n^{th} Generation

To find the expected size of the n^{th} generation, it can be found in Bhat and Miller (2002) that the initial population size is assumed to be one $(X_0 = 1)$ and the mean of the number of offspring to a single individual are finite, i.e., $m = E(X_1) < \infty$. Clearly, $E(X_n) = f_n'(1) = \left[\frac{df_n(s)}{ds}\right]_{s=1}$ is the mean number of individuals of the n^{th} generation. Differentiate 3.7 with respect to s and setting $s = 1$, gives

$$f_{n+1}'(1) = f_n'(1)\, f'(1). \qquad (3.8)$$

Solving it iteratively,

$$\begin{aligned} f_{n+1}'(1) &= f'(1)\, f_n'(1) \\ &= [f_1'(1)]^2\, f_{n-1}'(1) \\ &= [f'(1)]^3\, f_{n-2}'(1), \qquad (3.9) \end{aligned}$$

and by mathematical induction,

$$\begin{aligned} f'_{n+1}(1) &= \left[f'_n(1)\right]^n f'(1) \\ &= \left[f'(1)\right]^{n+1}. \end{aligned} \tag{3.10}$$

Given

$$E(X_1) = f'_1(1) = m \tag{3.11}$$

and for

$$E(X_{n+1}) = f'_{n+1}(1) = m^{n+1}. \tag{3.12}$$

Hence

$$E(X_n) = f'_n(1) = m^n. \tag{3.13}$$

3.2.2 Variance of Population Size of the n^{th} Generation

According to Bhat and Miller (2002) in derriving for the variance of the population size for the n^{th} generation, it is assume that the initial population size $X_0 = 1$ and variance of the number of offspring to a single individual is finite, i.e., $\sigma^2 = var(X_1) = E\left(X_1^2\right) - E\left(X_1\right)^2 < \infty$. The corresponding pgf is

$$f(s) = \sum_{r=0}^{\infty} P(X_n = r)\, s^r. \tag{3.14}$$

Obtain the second derivative of the p.g.f in equation (3.14) with respect to s and setting $s = 1$ to get

$$\begin{aligned} f''_n(1) &= \sum_{r=1}^{\infty} r(r-1) P(X_n = r) \\ &= E\left(X_n^2\right) - E(X_n) \\ &= E\left(X_n^2\right) - f'_n(1). \end{aligned} \tag{3.15}$$

Thus,

$$Var(X_n) = E\left(X_n^2\right) - \left[E(X_n)\right]^2. \tag{3.16}$$

but from equation (3.13),

$$\begin{aligned} f''_n(1) &= E\left(X_n^2\right) - f'_n(1) \\ E\left(X_n^2\right) &= f''_n(1) + f'_n(1) \end{aligned} \tag{3.17}$$

substituting equation (3.17) into equation (4.20), we have

$$\begin{aligned} Var\left(X_n\right) &= f''_n(1) + f'_n(1) - \left[E\left(X_n^2\right)\right]^2 \\ &= f''_n(1) + f'_n(1) - \left[f'_n(1)\right]^2 \end{aligned} \tag{3.18}$$

3.2.3 Extinction Probabilities in Constant Environment

Probability of extinction is the probability that the population will eventually die out. It can be found in Bhat and Miller (2002) that the probability of ultimate extinction q is then given by

$$q = \lim_{n\to\infty} f_n(o) = \lim_{n\to\infty} P(X_n = 0|X_0 = 1) = \lim_{n\to\infty} f_{n-r}\left\{f_r(0)\right\} \tag{3.19}$$

Lemma 3.1: Let $f(s)$ be the probability generating function of a branching process. The probability of extinction is equal to the smallest root of the fixed-point equation, $s = f(s)$.

Proof

We will proceed by finding an expression for probability of ultimate extinction in the n^{th} generation q_n. Let r equal the number of offspring in generation 1. We can consider each of these r offspring as the zero-th generation of a new branching process. We then have that q_n is the probability that the processes started by each of these r offspring are now extinct. Thus, we have that

$$q_n = \sum_{r=0}^{\infty} p_r (q_{n-1})^r. \tag{3.20}$$

This is precisely the generating function $f(q_{n-1})$, so $q_n = f(q_{n-1})$. Taking the limit yields that $q = f(q)$. It remains to show that q is the smallest root. Because $P(Z_n = 0) = f_n(0)$,

$$\begin{aligned} q &= \lim_{n\to\infty} q_n & (3.21)\\ &= \lim_{n\to\infty} f_0. & (3.22)\end{aligned}$$

Probability generating functions are nondecreasing; thus, if there were another root, $q*$, of the fixed point equation, we would have

$$q_n = \lim_{n\to\infty} f_0 \leq \lim_{n\to\infty} f_n(q*) = q*. \qquad (3.23)$$

Thus, q is the smallest root of the fixed-point equation.

Lemma 3.2: Let $s \in (0;1]$. Both $f'(s)$ and $f''(s)$ are positive.

Proof.

We proceed first by differentiating f termwise. Recall that

$$\begin{aligned} f(s) &= \sum_{r=0}^{\infty} p_r s^r & (3.24)\\ &= p_0 + p_1 s + p_2 s^2 + \ldots \end{aligned}$$

we have

$$\begin{aligned} f'(s) &= \sum_{r=0}^{\infty} r p_r s^{r-1} & (3.25)\\ &= p_1 + 2p_2 s + 3p_3 s^2 + \ldots \end{aligned}$$

and

$$\begin{aligned} f''(s) &= \sum_{r=0}^{\infty} r(r-1) p_r s^{r-2} & (3.26)\\ &= 2p_2 + 3(2)p_3 s + 4(3)p_4 s^2 + \ldots \end{aligned}$$

Since 0<s≤ 1 and p_i are non-negative ($p_i > 0$), $f'(s)$ and $f''(s)$ are positive.

Theorem 3.1 : If the mean number of offspring $m \leq 1$, then the process becomes extinct with probability 1.

If $m > 1$, then the process becomes extinct with probability < 1.

Proof.

We must find the smallest root of the fixed-point equation. It is easy to verify that $s = 1$ is a solution, because if $s = 1$:

$$\begin{aligned} f(s) = f(1) &= \sum_{r=0}^{\infty} p_r (1)^r && (3.27) \\ &= \sum_{r=0}^{\infty} p_r = 1 && (3.28) \end{aligned}$$

as f is a probability generating function and that $m = f'(1)$. We will consider both the case $m \leq 1$ and $m >1$.

Case 1: $m \leq 1$

First, note that if $s = 0$ is a root, then $f(0) = p_0 = 0$. However, if $m \leq 1$, then p_0 cannot equal 0, as this would violate the assumptions $p_0>0$. Thus, $s = 0$ is not a root. Now, assume for the sake of contradiction that some $s \in (0; 1)$ is a root, with $s = f(s)$. Because we know that $f''(s)$ is positive for all $s \in (0; 1]$, we have that for all $s \in (0; 1)$, the inequality $f'(s) < f''(1) = m \leq 1$ holds. By the Mean Value Theorem, we have that

$$f'(c) = \frac{f(1) - f(s)}{1 - s} = 1 \tag{3.29}$$

for some $c \in (s, 1)$. However, as $c \in (s, 1)$, we know that $f'(c) < 1$, so this is a contradiction. Thus, 1 is the smallest nonnegative root, so the probability of extinction is 1.

Case 2: $m > 1$

Now, consider the case where mean number of offspring $m > 1$. If $p_0 = 0$, we have that $p_0 = f(0) = 0$, hence the smallest root is $s = 0$. Suppose now that $p_0 > 0$, and note that $f(0) = p_0 > 0$. Also, as $f'(1) = m > 1$, by Taylor's Theorem, we have that $f(s) < s$ when s is sufficiently close to 1. Thus, by the Intermediate Value Theorem, there exists some $x \in (0, 1)$ such that $f(x) = x$. Therefore, whenever $m > 1$, there is a root on the interval $[0, 1)$, so the probability of extinction is less than 1.

3.3 Branching Processes in Varying Environment

A branching process in varying environment is a modification of the constant environment branching process in which the reproduction laws may vary over the generations (González et al., 2019). This flexibility makes the process more appropriate to model real populations due to the fact that the stability in the reproductive capacity and in the immigration laws are not usually fulfilled.

Let's define $P(N_0)$ to be the space of all probability measures on the natural numbers $N_0 = \{0,\ 1,\ 2, \ldots\}$. For $f \in P(N_0)$, we denote its weights by $f\,[r]\,,\ r = 0,\ 1, \ldots$. We also define

$$f\,(s) = \sum_{r=0}^{\infty} f\,(r)\,s^r, \qquad 0 \le s \le 1$$

The resulting function on the interval $[0,\ 1]$ is the generating function of the measure f. f' represents the derivative of the function f. Let the mean and the normalized second factorial moment be giving as,

$$\bar{f} = \sum_{r=1}^{\infty} r f\,(r) \tag{3.30}$$

and

$$\bar{f} = \frac{1}{\bar{f}^2} \sum_{r=1}^{\infty} r\,(r-1)\,f\,(r) \tag{3.31}$$

respectively and can be obtained from the generating functions as

$$I = f'(1)\,,\ \bar{f} = \frac{f''(1)}{f'(1)^2}. \tag{3.32}$$

Definition 3.3

A sequence $\tau = (f_1, f_2, \ \ldots)$ of probability measures on N_0 (space of all probability measures on the natural numbers) is called a varying environment (Kersting & Vatutin, 2017).

Definition 3.4

Let $\tau = (f_n, \ n \geq 1)$ be a varying environment. Then a stochastic process $X = \{X_n, \ n \ \in N_0\}$ is called a branching process with environment τ, if for any non-negative integers, $x \geq 0, \ n \ \geq 1$,

$$P\left(X_n = x | X_0, \ldots, X_{n-1}\right) = \left(f_n^{x_{n-1}}\right)(x) \tag{3.33}$$

almost surely with respect to probability measure P (Kersting & Vatutin, 2017).

On the right-hand side, we have the $X_{n-1}th$ power of f_n. In particular, $X_n = 0$ almost surely with respect to probability measure P on the event that $X_{n-1} = 0$. If we want to emphasize that probabilities $P(.)$ are determined on the basis of the varying environment τ, we use the notation $P_\tau(.)$. In probabilistic terms, the definition says, for $n \geq 1$, that given $X_0, X_1, \ldots, X_{n-1}$ the random variable X_n may be realized as the sum of i.i.d. random variables $Z_{in},, \ i = 1, \ . \ . \ . \ , \ X_{n-1}$, with distribution f_n, where

$$X_n = \sum_{i=1}^{X_{n-1}} Z_{in}. \tag{3.34}$$

This corresponds to the following conception of the process X : X_n is the number of individuals of some population in generation n, where all individuals reproduce independently of each other and of X_0, and where f_n is the distribution of the number of offspring of an individual in generation $n-1$. The initial distribution of the population is mostly 1, may be arbitrary. Mostly we choose it to be $X_0 = 1$.

The distribution of X_n is conveniently expressed via composing generating functions. For probability measures $f_1, \ . \ . \ . \ f_n$ on N_0 and for natural numbers $0 \ \leq \ r \ < \ s$, we introduce the probability measures

$$f_{r,s} = \ f_{r+1} \circ \ldots \circ fs. \tag{3.35}$$

3.3.1 Extinction probabilities in Varying Environment

For a branching process X in varying environment, let

$$\theta = min \left\{ n \geq 1 : X_n = 0 \right\} \tag{3.36}$$

be the moment when the population dies out. Then $P\left(\theta \leq n\right) \ = \ P\left(X_n = 0\right)$, and the probability that the population becomes ultimately extinct is equal to

$$q = P\left(\theta < \infty\right) = \lim_{n \to \infty} P\left(X_n = 0\right) \tag{3.37}$$

3.4 Branching Processes in Random Environment

Branching Processes in a random environment is the process where individuals reproduce independently of each other and vary from one generation to another according to some random mechanism (Dyakonova, Vatutin, & Zhang, 2020).

Definition 3.5:

Let φ be the space of random probability measures on $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$. Let $\mathbf{F}$ be a random vector with independent components taking values in φ and let $F_n, \ n = 1, 2, \ldots$ be a sequence of independent copies of $\mathbf{F}$. The infinite sequence $\upsilon = \{F_1, F_2, \ldots\}$ is called a random environment .

A sequence of $\mathbb{N}_0$-valued random variables $X = \{X_n, n \in \mathbb{N}_0\}$ specified on the respective probability space (Ω, F, P) is called a branching process in the random environment (BPRE).

3.4.1 Extinction Probabilities in Random Environment

Recall that $\theta := min\{n \geq 1 : X_n = 0\}$ denotes the moment of extinction of a branching process. Now, we investigate when the extinction probability

$$q := \mathbb{P}(\theta < \infty) \tag{3.38}$$

is equal to one.

Definition 3.6

A BPRE $\mathbb{X}$ is said to be supercritical if the expected number of offspring $m < 1$, but when $m = 1$ and $m \gneq 1$, then the BPRE $\mathbb{X}$ is said to be critical and supercritical respectively (Mei, 2019).

Theorem 3.2

Let $\mathbb{X}$ be a subscritical or a critical BPRE. Then $q = 1$ (Kersting & Vatutin, 2017).

Proof

For subcritical or critical processes (up to strictly critical case), we have

$$\lim_{n\to\infty} \inf \mathbb{E}[X_n|X_0, \varphi] = \lim_{n\to\infty} \inf Z_0\bar{F}_1 \cdots \bar{F}_n = \lim_{n\to\infty} \inf X_0 e^{S_n} = 0 \quad a.s \tag{3.39}$$

It follows that

$$\mathbb{P}(\theta < \infty) = 1 \quad \mathbb{P} - a.s\ . \tag{3.40}$$

Taking expectation with respect to x, the theorem follows (Kersting & Vatutin, 2017).

For supercritical processes, $q < 1$ will typically be true. However, this requires an extra condition, which prevents catastrophes of the type where large populations die out all of a sudden in just one generation.

Theorem 3.3

For a supercritical BPRE $\mathcal{X}$, we have:

(i) If $\mathbb{E}[|\log(1 - F[0])|] < \infty$, then $q < 1$

(ii) If $q < 1$ and $0 < \mathbb{E}[X] < \infty$, then $\mathbb{E}[|\log(1 - F[0])|] < \infty$ (Kersting & Vatutin, 2017).

Proof

We assume that $X_0 = 1$ a.s. without loss of generality.

(i) First note that $1 - F[0] \leq \bar{F}$ or $\log(1 - F[0]) \leq X$. Therefore, from our assumption, we obtain $\mathbb{E}[X^-] < \infty$. Consequently, X possesses an expectation, where $\mathbb{E}[X] > 0$.

Now consider the representation $X_n = \sum_{i=1}^{X_{n-1}} Z_{in}$ and define recursively $X_0^* = X_0$,

$$X_0^* = \sum_{i=1}^{X_{n-1}^*} Z_{in}^c, \quad n \geq 1, \tag{3.41}$$

with some integer $c \geq 1$. Then $\mathcal{X}^* = \mathcal{X}^*{}_n$, $n \geq 0$ is a BPRE in a certain i.i.d. environment $\varphi^* = F_1^*, F_2^*, \ldots$, which are both defined on the same probability space as $\mathcal{X}$ and φ. Observe that $\mathbb{E}[X^*] > 0$ for c sufficiently large and also $F^*[0] = F[0]$, thus the BPRE $\mathcal{X}^*$ fulfills the same requirements as X. In addition, $X_n^* \leq X_n$, hence $q \leq q^*$ for the extinction probabilities. Therefore, it is sufficient to prove our claim for the process $\mathcal{X}^*$.

This consideration shows that, without loss of generality, we may assume for the BPRE $\mathcal{X}$ that there is an integer $c \geq 1$ such that $F[x] = 0$ a.s. for

$z > c$ and that $0 < \mu < \infty$ with $\mu = \mathbb{E}[X]$. From the strong Law of Large Numbers, it follows that as $n \to \infty$

$$\mathcal{S}_n = \mu_n + o(n), \quad X_n = o(n) \ \ a.s \tag{3.42}$$

Now using $\mathcal{X}$ conditionally on φ and X_0 and observing that $2\varphi F_k(1) = \tilde{F}_k \leq c^2/e^{2X_k}$, we obtain

$$\frac{1}{\mathbf{P}(\theta > n)} \leq \frac{1}{e^{S_n}} + \sum_{k=1}^{n} \frac{c^2}{e^{S_{k-1}} e^{2X_k}} \ \ a.s \tag{3.43}$$

where θ is the moment when the population dies out. Notice that as $n \to \infty$, it follows that

$$\mathbb{P}(\theta = \infty) > 0 \ \ a.s \ . \tag{3.44}$$

Taking expectation, we arrive at $1 - q = \mathcal{P}(\theta = \infty) > 0$. This is our claim.

(ii) Here too, we use the process $\mathcal{W} = W_n, \ n \geq 0$ given by

$$\mathcal{W} := e^{-S_n} X_n. \tag{3.45}$$

From independence of the components of the environment, we see that $\mathcal{W}$ is a non-negative martingale which consequently is $\mathbb{P}$ -a.s. convergent to a random variable $\mathcal{W}$. Therefore, since we assume $0 < \mathbb{E}[X] < \infty$, it follows that $\lim_{n\to\infty} sup e^{-2\mu n} Z_n = 0$ $\mathbb{P}$ $a.s$ and consequently,

$$\mathbb{P}(X_n = 0 | X_1, \ldots, X_{n-1}) = F_n[0]^{X_{n-1}} \geq F_n[0]^{e^{2\mu n}} \tag{3.46}$$

for all but finitely many n.

For a contradiction, we now suppose that $\mathbb{E}[|\log(1 - F[0])|] = \infty$. Then for any $a > 0$,

$$\sum_{n=1}^{\infty} \mathbb{P}(-\log(1 - F_n[0]) \geq an) = \infty \qquad (3.47)$$

and, from the Borel–Cantelli lemma, there are a.s. infinitely many moments $1 \leq n_1 < n_2 < \cdots$ such that $-\log(1 - F_{n_j}[0]) \geq 2\mu n_j$ and, consequently,

$$F_{n_j}[0]^{e^{2\mu n_j}} \geq (1 - e^{-2\mu})^{e^{2\mu}} > 0. \qquad (3.48)$$

We obtain $\sum_{x=1}^{\infty} \mathbf{P}(X_n = 0 | X_1, \ldots, X_{n-1}) = \infty \ \mathbb{P} - a.s.$

3.5 Multi-type Branching Processes

In many cases, the individuals involved in a branching process are not all alike Examples include:

(i) Population Genetics. When considering inheritance of alleles, a 3-type branching process, with types corresponding to the genotypes AA, Aa, and aa can be used as a model.

(ii) Physics. Cosmic-ray cascades involve both electrons and photons, with electrons producing photons and photons producing electrons. Such an example is modeled by a 2-type branching process.

We begin by describing the offspring distributions for all individuals. We will always be dealing with a branching process containing k type of individuals.

Definition 3.7.

For each generation n, we define $\mathbf{X}_n$ as an k-dimensional vector whose d^{th} entry gives the number of individuals of type d in the n^{th} generation of the branching process. We write $(\mathbf{X}_n)$ for the sequence of the $\mathbf{X}_n$.

Definition 3.8. (Offspring Vector)

Consider k-dimensional vectors $\mathbf{r} = (r_1, r_2, \ldots, r_k)$, where $r_1, \ldots, r_k$ are natural numbers. These vectors represent the offspring of each type created by an individual, with r_1 equalling the number of offspring of type 1, and so on. We call the vector $\mathbf{r}$ the Offspring Vector for the individual.

Definition 3.9. (Offspring Distribution)

For every type, let $d = 1, 2, \dots, k$, let $p_d(\mathbf{r})$ be the probability that an individual of type d has the offspring vector $\mathbf{r}$. Note that $\sum_r p_d(\mathbf{r}) = 1$. These probabilities define the offspring distribution for individuals of type d.

For single-type branching processes, we used the notation p_r to denote the probability that an individual had r children. We now use the notation $p_d(\mathbf{r})$ to denote the probability that an individual of type d has Offspring Vector $\mathbf{r}$.

Definition 3.10. (Probability Generating Function)

Let s denote an k-dimensional vector $(s_1, s_2, \dots, s_k)$ of complex numbers, and let $\max|s_d| = \|s\|$. For all $d = 1, 2, \dots, k$ and all such vectors with $\|s\| \leq 1$, let

$$f_d(\mathbf{s}) = \sum_{\mathbf{r}} p_d(\mathbf{r}) s_1^{r_1} s_2^{r_2} \dots s_k^{r_k} \tag{3.49}$$

We say f_d is the generating function for the offspring distribution of individuals of type d.

Let $\mathbf{N}_d$ with $d = 1, 2, \dots, k$ be a vector-valued random variable defined such that $P[\mathbf{N}_d = \mathbf{r}] = p_d(\mathbf{r})$. Fix d and let $\mathbf{Z}_{dj}$ be a sequence of vector-valued random variables with the same distribution as $\mathbf{N}_d$.

Consider $\mathbf{X}_n$. If $\mathbf{X}_{n-1} = \left(X_{1(n-1)}, X_{2(n-1)}, \dots, X_{k(n-1)}\right)$, then

$$\mathbf{X}_n = \sum_{d=1}^{k} \sum_{j=1}^{X_{d(n-1)}} \mathbf{Z}_{dj}, \tag{3.50}$$

where if $X_{d(n-1)} = 0$ for some d, the corresponding sum is interpreted as the zero vector. The above sum simply creates an appropriate offspring vector $\mathbf{r}$ for each individual in $\mathbf{X}_{(n-1)}$ and adds them to yield $\mathbf{X}_n$. We now construct a generating function for the sequence $(\mathbf{X}_n)$ by

$$F_n(d, \mathbf{s}) = \sum_{r} P\left[\mathbf{X}_n = \mathbf{r} | \mathbf{X}_0 = \mathbf{e}_d\right] s_1^{r_1} s_2^{r_2} \ldots s_k^{r_k}, \tag{3.51}$$

where $d = 1, 2, 3, \ldots, k$ and $\mathbf{s} = (s_1, s_2, \ldots s_k)$ with $\|s\| \leq 1$. Note that both

$$F_0(d, \mathbf{s}) = s_d, \tag{3.52}$$

$$F_1(d, \mathbf{s}) = f_d(\mathbf{s}), \tag{3.53}$$

where f_d is the generating function for the offspring distribution of individuals of type d.

We find the other iterates of F by defining an k-dimensional vector of generating functions f by setting

$$\mathbf{f}(1, \mathbf{s}) = (f_1(\mathbf{s}), \ldots f_k(\mathbf{s})), \tag{3.54}$$

and

$$\mathbf{f}(n, \mathbf{s}) = (f_1(\mathbf{f}(n-1, \mathbf{s})), \ldots f_k(\mathbf{f}(n-1, \mathbf{s})) \tag{3.55}$$

for all $n \geq 2$. It then follows from equation (3.53) that

$$F_n(d, \mathbf{s}) = f_d(\mathbf{f}(n-1, \mathbf{s})) \tag{3.56}$$

for $n \geq 2$ and $d = 1, 2, \ldots, k$.

Additionally, using equation (3.50) in the constant environment, we have that if $\mathbf{X}_{n-1} = \mathbf{y}$, then the generating function of $\mathbf{X}_n$ is

$$\prod_{d=1}^{k} f_d^{y_d}(\mathbf{s}) \tag{3.57}$$

Because all individuals reproduce independently, and offspring distributions do not change between generations in the constant environment, we know that $\mathbf{X}_n$

depends only on $\mathbf{X}_{n-1}$. Thus, $(\mathbf{X}_n)$ is a Markov chain with stationary transition probabilities. For this chain, we define the state space W as the set of all k-dimensional vectors of natural numbers. For any pair of vectors $\mathbf{x}, \mathbf{y} \in W$, let $p(\mathbf{x}, \mathbf{y})$ be the coefficient of $s_1^{r_1} s_2^{r_2} \ldots s_k^{r_k}$ in the generating function given in equation (3.57). We have that

$$P\left[\mathbf{X}_n = \mathbf{y} | \mathbf{X}_{n-1} = \mathbf{x}\right] = p(\mathbf{x},\ \mathbf{y}) \tag{3.58}$$

for all $n \geq 1$, and all vectors $\mathbf{x},\ \mathbf{y} \in W$, meaning that $(p(\mathbf{x},\ \mathbf{y}))$ is the matrix of transition probabilities. We can now use the fact that $(\mathbf{X}_n)$ is a Markov chain to prove an important result about the recurrence of states.

3.5.1 Positively Regular Processes and Irreducible States

The nonreccurence of states in Multi-type Branching processes is necessary to prove extinction theorems. We begin by what it means for a process to be positively regular:

Definition 3.11. (Positively Regular)

A multi-type process is irreducible if and only if, for every pair of types i, j, there exists some natural number n such that

$$P\left[X_{jn} \geq 1 | \mathbf{X}_0 = \mathbf{e}_i\right] > 0 \tag{3.59}$$

Here, X_{jn} gives the number of individuals of the jth type in generation n. If, for some n, the statement holds for all d, j; then the process is *positively regular*.

Associated Matrix:

For any multi-type branching process, there is an associated $k \times k$ matrix $\mathbf{K}$ defined by

$$k_{ij} = E[X_{j1}|\mathbf{X}_0 = \mathbf{e}_i] \tag{3.60}$$

for $i, j = 1, 2, \ldots k$. We can see the relationship between $\mathbf{K}$ and $\mathbf{X}_n$ by the following construction: Let

$$\mathbf{u}_n = E[\mathbf{X}_n] = (u_{1n}, \ldots, u_{kn}) \tag{3.61}$$

Taking the expected value of 3.61 yields that

$$\mathbf{u}_n = \mathbf{u}_{n-1}\mathbf{K} \tag{3.62}$$

and therefore,

$$\mathbf{u}_n = \mathbf{u}_0\mathbf{K}^n \tag{3.63}$$

If a multi-type branching process is positively regular, there exists a natural number n such that all elements in $\mathbf{K}^n$ are strictly positive. We now define what it means for the matrix $\mathbf{K}$ to be positively regular, which is stricter than the process being positively regular.

Definition 3.12. (Positively regular matrix)
The matrix $\mathbf{K}$ is positively regular if all elements of $\mathbf{K}$ are finite and there exists a natural number n such that all elements of $\mathbf{K}^n$ are strictly positive. We will need a few properties of positively regular matrices to deal with extinction.

Definition 3.13. (Perron-Frobenius Theorem)
If $\mathbf{K}$ is a positively regular matrix, then $\mathbf{K}$ has a positive eigenvalue h (the PerronFrobenius root of $\mathbf{K}$) of multiplicity 1, such that for all other eigenvalues λ of $\mathbf{K}$, we have that $|\lambda| < h$. The eigenvalue h has associated right and left eigenvectors $\mathbf{v}$ and $\mathbf{w}$ with strictly positive elements such that $h\mathbf{v} = \mathbf{v}\mathbf{K}$ and $\mathbf{K}\mathbf{w} = h\mathbf{w}$, and $\mathbf{v}\mathbf{w} = 1$. Let $\mathbf{K}_1 = \mathbf{w}\mathbf{v} = (\mathbf{w}_i v j)$ and $\mathbf{K}_2 = \mathbf{K} - h\mathbf{K}_1 = (k_{2dj})$. It

follows that $\mathbf{K}_1\mathbf{K}_1 = \mathbf{K}_1,\ \mathbf{K}_1\mathbf{K}_2 = \mathbf{K}_2\mathbf{K}_1 = \mathbf{0}$ and

$$\mathbf{K}^n = h^n\mathbf{K}_1 + \mathbf{K}_2^n \tag{3.64}$$

for all $n \in N$. Furthermore, there exists a positive number α with $0 < \alpha < r$ such that

$$k_{2dj}^{(n)} = O(\alpha^n) \tag{3.65}$$

where $\mathbf{K}_2^n = k_{2dj}^{(n)}$. The term $O(\alpha^n)$ states that some constant times α^n is an upper bound for $k_{2dj}^{(n)}$ as n approaches infinity. This bound is independent of d, j.

Definition 3.14. (Nonreccurent State)

A state $\mathbf{r} \in W$ is nonreccurent if and only if, for some $t \geq 1$,

$$P\left[\mathbf{X}_{n+t} = \mathbf{r} | \mathbf{X}_n = \mathbf{r}\right] < 1. \tag{3.66}$$

If a Markov Chain has stationary transition probabilities, the following definition is equivalent: A state is nonreccurent if and only if

$$P[\mathbf{X}_{n+t} = \mathbf{r} \text{ infinitely often } |\mathbf{X}_n = \mathbf{r}] = 0. \tag{3.67}$$

The following definition allows us to avoid the case where each individual produces one offspring with probability one.

Definition 3.15.

A multi-type branching process is singular if the generating functions $f_1(\mathbf{s}), \ldots, f_k(\mathbf{s})$ are all linear in $\mathbf{s}$ with no constant terms; i.e., each object has exactly one child.

3.5.2 Extinction Probability for Multi-type Branching Processes

Definition 3.16.

A multi-type branching process becomes extinct if there exists some $N \in \mathbb{N}$ for which, if $n \geq N$, then $\mathbf{X}_n = \mathbf{0}$, where $\mathbf{0}$ is the k-dimensional zero vector. Recall that we assumed that $\mathbf{X}_0 = \mathbf{e}_d$ for $d = 1, 2, \ldots, k$, where $\mathbf{e}_d$ is the dth m-dimensional standard basis vector. We write q_d for the probability that the process goes extinct when $\mathbf{X}_0 = \mathbf{e}_d$. We then use q_{dn} as the probability that the process beginning with $\mathbf{X}_0 = \mathbf{e}_d$ goes extinct by generation n. We have that

$$q_{d0} \leq q_{d1} \leq \cdots \leq q_{dn}$$

and, using some elementary measure theory,

$$q_d = \lim_{n \to \infty} q_{dn} \tag{3.68}$$

for $d = 1, 2, \ldots, k$. We let $\mathbf{q}_n = (q_{1n}, \ldots, q_{kn})$ be the k-dimensional vector of the extinction probabilities, and write

$$\mathbf{q} = \lim_{n \to \infty} \mathbf{q}_n \tag{3.69}$$

Theorem 3.4.

Let $\mathbf{K}$ be a positively regular matrix and let h be the Perron-Frobenius root of $\mathbf{K}$. If $h \leq 1$, then $\mathbf{q} = \mathbf{1}$, the k-dimensional vector $(1, 1, \ldots, 1)$. If $h > 1$, then $q_d < 1$ for all $d = 1, 2, \ldots, k$. In either case, $\mathbf{q}$ is such that, for any other nonnegative solution $\mathbf{q}*$ of the equation $\mathbf{q} = \mathbf{f}(\mathbf{q})$, we have that $q_d \leq q_d*$ for all $d = 1, 2, \ldots, k$.

Theorem 3.5.

Let the conditions of the above theorem hold.

(i) For every k-dimensional nonnegative vector $\mathbf{s} = (s_1, \ldots, s_k)$ such that $\|\mathbf{s}\| \leq 1$ but $\mathbf{s} \neq \mathbf{1}$, we have

$$\lim_{n\to\infty} \mathbf{f}(n, \mathbf{s}) = \mathbf{q}. \tag{3.70}$$

(ii) Additionally, the only nonnegative solutions of the equation $\mathbf{f(s)} = \mathbf{s}$ with $\|\mathbf{s}\| \leq 1$ are $\mathbf{1}$ and $\mathbf{q}$.

Proof of Theorem 3.4

First, we will prove that if $h \leq 1$, then $\mathbf{q} = \mathbf{1}$. As $\mathbf{K}$ is a positively regular matrix, all states are nonrecurrent. Thus, for all $d = 1, 2, \ldots, k$ and $\mathbf{r} \in W$, we have

$$\lim_{n\to\infty} P\left[X_n = \mathbf{r} | \mathbf{X}_0 = \mathbf{e}_d\right] = 0. \tag{3.71}$$

It follows that

$$P[\mathbf{X}_n \longrightarrow 0 | \mathbf{X}_0 = \mathbf{e}_d] + P[X_n \longrightarrow \infty | \mathbf{X}_0 = \mathbf{e}_d] = 1. \tag{3.72}$$

Using equations (3.63) and (3.64) yields that

$$E[\mathbf{X}_n] = \mathbf{u}_n = \mathbf{u}_0\mathbf{K}^n = \mathbf{u}_0 h^n \mathbf{K}_1 + \mathbf{K}_2^n. \tag{3.73}$$

As $h \leq 1$, the first term of this equation is bounded and $\mathbf{K}_2^n$ is bounded as well. Therefore, $E[\mathbf{X}_{jn} | \mathrm{X}_0 = \mathbf{e}_d]$ is bounded for all $n \in N$ and $d, j = 1, 2, \ldots k$. Thus, we have that

$$P[\mathbf{X}_n \longrightarrow 0 | \mathbf{X}_0 = \mathbf{e}_d] = 1, \tag{3.74}$$

so we know all states are nonrecurrent.

Chapter 4

METHODOLOGY

4.1 Introduction

This chapter presents the methodology that was used to achieve the study objectives. The chapter discusses the main theorem, data source, data description, the procedure for estimating ultimate extinction in random environment, proposed index of closeness and spectral analyses models.

4.2 Theorem

The application of branching processes are mostly done in constant (time-homogeneous) environment (Beznea, Deaconu, & Lupaşcu, 2015; Durrett, 2015; Mettle et al., 2019). This is because it is difficult to apply branching processes in the varying and random environment to obtain future probabilities of zero offspring. Hence the study proposed a more general theorem, based on the ideas of Bhat and Miller (2002) and Mettle et al. (2019), for computing generational probabilities of zero offspring in branching processes in a varying environment with the similar assumption in Bhat and Miller (2002) that the offspring distribution does not depend on previous population size.

Suppose in a branching process, the maximum number of offspring an organism in her reproductive lifetime in the n^{th} generation is m and $p_{(n)r}(r = 0, 1, 2, \ldots, m)$ is the probability distribution of the number of offspring the organism produces with a corresponding probability generation function $f_n(s)$, $|s| \leq 1$. Then the probability of producing no offspring $q_n(n = 1, 2, \ldots)$ at the n^{th} generation is

$$\begin{aligned} q_1 &= p_{(0)0} \\ q_n &= f_{n-1}\left(q_{n-1}\right) \end{aligned} \tag{4.1}$$

Proof

$$\begin{aligned} f_n\left(s\right) &= \sum_{i=0}^{m} p_{(0)i}s^i \qquad , \; |s| \leq 1 \\ f_1\left(s\right) &= p_{(0)0} + \sum_{i=0}^{m} p_{(0)i} = q_1 + Y_1\left(s\right) \end{aligned} \tag{4.2}$$

where $q_1 = p_{(0)0}$ and $Y\left(s\right) = \sum_{i=1}^{m} p_{(0)}s^i$.

Next,

$$\begin{aligned} f_2\left(s\right) &= f_1\left[f_1\left(s\right)\right] \\ &= \sum_{i=0}^{m} p_{(1)i}\left[q_1\right]^i \\ &= p_{10} + p_{(1)1}q_1 + y_{11}\left(s\right) + p_{(1)2}q_1^2 + y_{12}\left(s\right) + \ldots + p_{(1)m}q_1^m + y\left(s\right) \\ &= \sum_{i=0}^{m} p_{(1)i}\left(q_1\right)^i + Y_1\left(s\right) \\ &= f_1\left(q_1\right) + Y_1\left(s\right) \\ &= q_2 + Y_1\left(s\right) \end{aligned} \tag{4.3}$$

where $q_2 = f_1(q_1), Y_1(s) = \sum_{i=1}^{m} y_{1i}\left(s\right), y_{1i}(s)$, is a polynomial of order im in s with $y_{1i(0)} = 0$ for $i = 1, 2, 3, \ldots, m$ and $Y_1\left(s\right)$ is a polynomial of under m^2

in s. For $n = 3$, we have

$$\begin{aligned} f_3\left(s\right) &= f_2\left[f_1\left(s\right)\right] \\ &= \sum_{i=0}^{m} p_{(1)i}\left[f_1\left(s\right)\right]^i \\ &= p_{(2)0} + p_{(2)1}q_1 + y_{21}\left(s\right) + p_{(2)2}q_1^2 + y_{12}\left(s\right) + \ldots + p_{(2)m}q_1^m + y_{2m}\left(s\right) \\ &= \sum_{i=0}^{m} p_{(2)i}\left(q_2\right)^i + Y_2\left(s\right) \\ &= f_2\left(q_2\right) + Y_2\left(s\right) \\ &= q_3 + Y_2\left(s\right) \end{aligned} \tag{4.4}$$

where $q_3 = f_2(q_2), Y_2(s) = \sum_{i=1}^{m} y_{2i}\left(s\right), y_{2i}(s)$, is a polynomial of order im^2 in s with $y_{2i(0)} = 0$ for $i = 1, 2,\ 3, \ldots, m$ and $Y_2\left(s\right)$ is a polynomial of order m^3 in s.

Hence in general we have,

$$f_n\left(s\right) = q_n + Y_{n-1}\left(s\right) \tag{4.5}$$

where $q_n = \sum_{i=1}^{m} p_{(n-1)i}\left(q_{n-1}\right)^i, Y_{n-1}(s) = \sum_{i=1}^{m} y_{(n-1)i}\left(s\right), y_{(n-1)i}(s)$, is a polynomial of order im^{n-1} in t with $y_{(n-1)i}(0) = 0$ for $i = 1, 2,\ 3, \ldots, m$ and $Y_{n-1}\left(s\right)$ is a polynomial of order m^n in s.

Now, for $n + 1$, we have,

$$\begin{aligned} f_{n+1}\left(s\right) &= f_n\left[f_{n-1}\left(s\right)\right] \\ &= \sum_{i=0}^{m} p_{(n)i}\left[f_{n-1}\left(s\right)\right]^i \\ &= p_{(n)0} + p_{(n)1}q_1 + y_{n1}\left(s\right) + p_{(n)2}q_1^2 + y_{n2}\left(s\right) + \ldots + p_{(n)m}q_1^m + y_{nm}\left(s\right) \\ &= \sum_{i=0}^{m} p_{(n)i}\left(q_n\right)^i + Y_n\left(s\right) \\ &= f_n\left(q_n\right) + Y_n\left(s\right) \\ &= q_{n+1} + Y_n\left(s\right) \end{aligned} \tag{4.6}$$

where $q_{n+1} = \sum_{i=1}^{m} p_{(n)i} (q_n)^i, Y_n(s) = \sum_{i=1}^{m} y_{(n)i} (s), y_{(n)i}(s)$, is a polynomial of order im^n in s with $y_{(n)i}(0) = 0$ for $i = 1, 2, 3, \ldots, m$ and $Y_n(s)$ is a polynomial of under m^{n+1} in s.

Hence by induction equation (4.5) is true for all $n = 1, 2, 3, \ldots$. Clearly by putting $s = 0$ equation (4.5), we have

$$f_n(0) = f_{n-1}(q_{n-1}) \tag{4.7}$$

which proves the theorem.

Corollary

If the conditions in the theorem are true and in addition $p_{(n)r} = p_r(r = 0, 1, 2, 3, \ldots, m)$ with a corresponding pgf $f(t), |t| \leq 1$ for all $n = 1, 2, 3, \ldots$ then it can be seen in Mettle et al. (2019) that

$$\begin{aligned} &q_1 = p_0 \\ &q_n = f(q_{n-1}), n = 2, 3, \ldots \end{aligned} \tag{4.8}$$

4.3 Source of Data and Data description

The study used both empirical data and hypothetical data to achieve the study objectives.

4.3.1 Empirical Data

The empirical data for analysis of this study were secondary data obtained from the Demographic and Health Surveys (DHS). The DHS are nationally representative household surveys that provide data for a wide range of monitoring and impact evaluation indicators in population, health, and nutrition. The variables obtained from the study for each country are the age of respondents, the number of males ever born, the number of females ever born and the total number of children ever born. Below are the selected countries and the years used for the study:

Ghana	1993, 1998, 2003, 2008, 2013, 2018
Burkina Faso	1993, 1998, 2003, 2008, 2013, 2018
Mali	1992, 1997, 2002, 2007, 2012, 2017
Bangladesh	1993, 1998, 2003, 2008, 2013, 2018
Philippines	1993, 1998, 2003, 2008, 2013, 2018
Indonesia	1993, 1998, 2003, 2008, 2013, 2018
Colombia	1990, 1995, 2000, 2005, 2010, 2015

4.3.2 Hypothetical Data

Suppose the distribution of the number of offspring that an organism in the n^{th} generation of a branching process, follows a Poisson distribution with parameter θ

$$p(Z_n = r) = \frac{e^{-\theta}\theta^r}{r!}, \quad r = 0, 1, 2, \dots \ . \tag{4.9}$$

However, assuming the population of interest is heterogeneous at each generation, θ is itself a random variable according to the distribution

$$h(\theta) = \begin{cases} \beta_n e^{-\beta_n \theta}, & \theta > 0 \\ 0 & elsewhere \end{cases} , \beta_n > 0, \qquad n = 1, 2, 3 \dots \ . \tag{4.10}$$

The pgf of the number of offspring p_n is given by

$$f(t) = E_\theta E_{P_n}(t^{Z_n}|\theta) = E_\theta(e^{\theta(t-1)}) = \frac{\beta_n}{(\beta_n + 1 - t)} \ , \quad |t| < 1 \ . \tag{4.11}$$

Now differentiating with respect to t and evaluating at $t = 1$, the expected number of offsprings is given by

$$f'(t) = \mu = \beta_n^{-1}. \tag{4.12}$$

Hence by the theorem, equation (4.1) for such a population becomes

$$q_1 = f_1(0) = \frac{\beta_0}{(\beta_0 + 1)} \tag{4.13}$$

$$q_n = f_n(0) = \frac{\beta_{n-1}}{(\beta_{n-1} + 1 - q_{n-1})} \ , \quad n \geq 2 \ . \tag{4.14}$$

For the constant environment case where $\beta_n = \beta, \quad n = 1, 2, 3, \ldots$

$$\begin{aligned} q_1 &= \frac{\beta}{(\beta + 1)}, \\ q_n &= \frac{\beta}{(\beta + 1 - q_{n-1})}, \quad n = 2, 3, \ldots \ . \end{aligned} \tag{4.15}$$

4.4 Method of Data Analysis

The analysis of the data was done as follows, Firstly, after obtaining the parity data from women in their terminal reproductive age, the probability of the number of offspring produced by an individual in each generation was estimated using the MLE method. Then based on the estimated probabilities of offspring, we then apply the theorem to compute the generational probabilities of zero offspring for the constant and the two random environments. After which, the limiting probability of zero offspring is obtained, these were done for each country. Hypothetical data were obtained and some indexes of closeness were proposed to enable us to develop the conditions in which the estimated generational probabilities of zero offspring in random environments will behave similarly to a constant environment. Afterwards, three-way repeated-measures ANOVA and Bonferroni post hoc tests were used to test for the significant differences among the generational probabilities of zero offspring obtained from the constant and the two random environments. Multi-type branching processes for gender were also performed to obtain the generational probabilities for each gender. A paired t-test was employed to test the significance of the difference between the generational probabilities of zero offspring for males and females. Spectral analyses were then used to compute the expected time to revise population policy for each country.

4.5 Maximum Likelihood Estimation of Class Probabilities

The estimates of the probabilities of the number of offspring produced by an individual in each generation were obtained from the maximum likelihood estimation method.

Suppose B bootstrap samples of size n_t $(t = 1, 2, 3, \ldots, \varpi)$ are available from a multinomial population with $m + 1$ class probabilities $p_{(t)r}$ at time t such that x_{tr} individuals in the sample belong to class $r(r = 0, 1, 2, \ldots, m)$; and that $\sum_{r=0}^{m} x_{tr} = n_t$. In this study, n_t is the total number of females at the terminal reproductive age-group at generation t and x_{tr} is the number of females in the terminal age-group who have r children in generation t. The joint probability mass function of the sample at time t given n_t is

$$f\left(x_t|n_t\right) = \frac{n_t!}{x_{(t)0}!x_{(t)1}!\ldots x_{(t)m-1}!\left(n_t - \sum_{r=0}^{m-1} x_{(t)r}\right)}\left(\prod_{r=0}^{m-1} p_{(t)r}^{x_{(t)r}}\right)$$

$$\times\left(1 - \sum_{r=0}^{m-1} p_{(t)r}\right)^{n_t - \sum_{r=0}^{m-1} x_{tr}} \tag{4.16}$$

.

The log likelihood function is given by;

$$l\left(x_t|n_t\right) = ln\left[f\left(x_t|n_t\right)\right] = Q + \sum_{r=0}^{m-1} x_{(t)r} ln\left(p_{(t)r}\right) + \left(n_t - \sum_{r=0}^{m-1} x_{tr}\right) ln\left(1 - \sum_{r=0}^{m-1} p_{(t)r}\right), \tag{4.17}$$

where Q is independent of $p_{(t)r}$ $(r = 0, 1, 2, \ldots, m-1; t = 1, 2, .., \varpi)$.

Now, differentiating partially with respect to $p_{(t)r}\,(r = 0, 1, 2, \ldots, m-1)$ and equating to zero, the simultaneous solution of the resulting system of equations, we obtain the maximum likelihood estimators (MLE) (Bhat & Miller, 2002). Hence, the offspring probability distribution at time t from the B bootstrap samples is given as

$$\hat{p}^*_{(b)tr} = \frac{x^*_{(b)tr}}{n_{(b)t}}, \qquad r = 0, 1, 2, \ldots, m-1; \quad b = 1, 2, \ldots, B, \tag{4.18}$$

and

$$\hat{p}^*_{(b)tm} = 1 - \sum_{r=0}^{m-1} \hat{p}^*_{(b)tr}. \tag{4.19}$$

Now the estimated probabilities of the offspring distribution at time t is given as the averages of the B boostrap samples

$$\bar{p}^*_{(t)r} = \sum_{b=1}^{B} \hat{p}^*_{(b)tr}, \qquad r = 0, 1, 2, \ldots, m. \tag{4.20}$$

In the case of constant environment, information from all the ϖ samples are pooled, and the maximum likelihood estimates of the class probabilities $p_r(r = 0, 1, 2, ..., m)$ are given by the averages

$$\hat{p}_r = \frac{\sum_{t=1}^{\varpi} x_{tr}}{\sum_{t=1}^{\varpi} n_t}. \tag{4.21}$$

The probability of offspring distribution is obtained by finding the averages of the B bootstrap samples probabilties in (4.21) $p_{br}(r = 0, 1, 2, ..., m; \; b = 1, 2, \ldots, B)$

$$\bar{p}^*_r = \sum_{b=1}^{B} \hat{p}^*_{(b)r} \quad r = 0, 1, 2, ..., m. \tag{4.22}$$

4.6 Expected Number of Offspring

The expected number of offspring of the n^{th} generation is derived from the probability generation function f as the first derivative $f'(1)$.

$$\begin{aligned} f(s) &= \sum_{j=0}^{\infty} p_j s^j \\ &= p_0 + p_1 s + p_2 s^2 + \dots \\ f'(s) &= \sum_{j=0}^{\infty} j p_j s^{j-1} \\ &= p_1 + 2p_2 s + 3p_3 s^2 + \dots \\ f'(1) &= p_1 + 2p_2 + 3p_3 + \dots \\ E(X_n) &= p_1 + 2p_2 + 3p_3 + \dots \qquad (4.23) \end{aligned}$$

4.7 Estimation of Generational Probabilities of Zero Offspring in Random Branching Process for Single-type Branching Processes

Suppose estimates of the probability vectors $\{\mathbf{P}_{(1)}, \mathbf{P}_{(2)}, \mathbf{P}_{(3)}, \dots, \mathbf{P}_{(\varpi)}\}$ where ϖ may be finite or infinite and $\mathbf{P}_{(i)} = \left(p_{(i)1}, \dots, p_{(i)m}\right)$ are subject to α future generation. Then the generational probabilities of zero offspring are computed using the theorem in equation (4.1) for B sample paths resulting in $\mathbf{q}_i^* = (q_{i1}^*, q_{i2}^*, \dots, q_{i\alpha}^*)$, $i = 1, 2, \dots B$. The generational probabilities of zero offspring are obtained from the averages of all B sample paths for each generation,

$$\hat{q}_{(n)}^* = \frac{1}{B} \sum_{i=1}^{B} \mathbf{q}_i^* = (\hat{q}_1^*, \hat{q}_2^*, \dots, \hat{q}_\alpha^*) \ . \qquad (4.24)$$

The algorithm for estimating the generational Probabilities of Zero Offspring in Random Environment and constant environment can be found in Algorithm 1 and 2 respectively.

Algorithm 1 Generational Probabilities of Zero Offspring in Random Environment

---

(1) Given ϖ ($\varpi = 6$) different years of parity data .

(2) Generate B ($B = 1000$) bootstrap replications of the parity data for each year.

(3) For each year use MLE to estimate the offspring distribution parameters, from the bootstrap samples generated in (2) i.e $(\hat{p}^*_{0ij}, \hat{p}^*_{1ij}, \ldots, \hat{p}^*_{mij}), \quad i = 1, 2, \ldots, \varpi; \quad j = 1, 2, \ldots, B$.

(4) Find the average of the offspring distribution parameters (probability vectors) obtained from the bootstrap across the parameters for each year. $(\bar{p}^*_{0i}, \bar{p}^*_{1i}, \ldots, \bar{p}^*_{mi}), \quad i = 1, 2, \ldots, \varpi$.

(5) We now assumed that any of the probability vectors would be subjected to the future generation. In a situation where every probability vector has an equal chance of occurring in the future then it is called a **uniform random environment** (R1). But when more preferences are given to the most current probability vectors then it is called a **weighted random environment** (R2).

(6) Then compute the n_0 generational probabilities for each B bootstrap samples in (4) resulting in $q^*_{i1}, q^*_{i2}, \ldots, q^*_{in_0}, \quad i = 1, 2, \ldots, B$.

(7) Compute the averages of the generational probabilities from the boostrap estimates in (5) accross the generations, $\hat{q}^*_1, \hat{q}^*_2, \ldots, \hat{q}^*_{n_0}$ and the confidence interval.

Algorithm 2 Generational Probabilities of Zero Offspring in Constant Environment

---

(1) Given ϖ ($\varpi = 6$) different years of parity data .

(2) Pool the ϖ samples into one data set.

(2) Generate B ($B = 1000$) bootstrap samples of the parity data .

(3) Use MLE to estimate the offspring distribution parameters, from the bootstrap samples generated in (2) $(\hat{p}^*_{0j}, \hat{p}^*_{1j}, \ldots, \hat{p}^*_{mj}), \quad j = 1, 2, \ldots, B$.

(4) Find the average of the offspring distribution parameters obtained from the bootstrap across the parameters. $(\bar{p}^*_0, \bar{p}^*_1, \ldots, \bar{p}^*_m)$.

(5) With the assumption that offspring distribution will the same in future generations. Compute the n_0 generational probabilities from the corollary in equation (4.8).

4.8 Limiting Probability of Giving Birth to Zero Offspring for Single-type Branching Processes

The probability of ultimate extinction or the long-run probability of giving birth to zero offspring under a random environment exists. That is, there exists an n^{th} generation where $n = n*$ such that for all $n > n*$ one can find a real number ε (very small and greater than zero) such that

$$|q_n - q| < \varepsilon \tag{4.25}$$

for all $n > n*$ (Natanson, 2016).

The limiting probability of giving birth to zero offspring can be easily computed for the constant environment case. This is because the sequence of $\{q_n\}_{n\geq1}$ is monotone, and it is bounded between zero and one and hence the long-run probability of zero offspring exists

$$\lim_{n\to\infty} q_n = q \ . \tag{4.26}$$

4.9 Multi-type Branching Process

Suppose there are $d(d = 1, 2, 3, \ldots, k)$ types of individuals in a population. Each individual independently reproduces new offspring of any of the types. Let Z_n represent the total population size in the nth generation. Let $Z_{(d),i}$ represent the number of d types of offspring the ith individual in generation n reproduces. Then, the number of individuals in the n^{th} generation is

$$\mathbf{X}_n = \sum_{d=1}^{k} \sum_{i=1}^{X_{d(n-1)}} Z_{(d)i} \; . \tag{4.27}$$

Suppose in a two-type branching process, the maximum number of offspring of types $d(d = 1, 2)$ an individual will produce at the terminal reproductive age are m_1 and m_2 respectively. Let $p_{ij}(i = 0, 1, 2, \ldots, m_1; j = 0, 1, 2, \ldots, m_2)$ be the probability that an individual at the terminal reproductive age will produce i offspring of type 1 and j offspring of type 2. Then the joint pgf of the process is given as

$$f(s_1, s_2) = \sum_{i=0}^{m_1} \sum_{j=0}^{m_2} p_{ij} s_1^i s_2^j, \tag{4.28}$$

the marginal pgfs of type 1 and 2 are respectively

$$f(s_1) = f(s_1, 1) = \sum_{i=0}^{m_1} v_{1i} s_1^i \tag{4.29}$$

and

$$f(s_2) = f(1, s_2) = \sum_{j=0}^{m_2} v_{2j} s_2^j \tag{4.30}$$

where

$$v_{1i} = \sum_{j=0}^{m_2} p_{ij}, \quad i = 0, 1, 2, \ldots, m_1 \tag{4.31}$$

and

$$v_{2j} = \sum_{i=0}^{m_1} p_{ij}, \quad j = 0, 1, 2, \ldots, m_2 \tag{4.32}$$

The pgf of single-type branching process for the given joint distribution assuming the total number of offspring an individual will produce at the terminal reproductive age irrespective of the type is given as

$$f(s) = f(s,s) = \sum_{r=0}^{m_1+m_2} v_r s^r \tag{4.33}$$

where v_r = the sum of all joint probability p_{ij} such that $i + j = r, \quad r = 0, 1, 2, \ldots m_1 + m_2$.

4.9.1 Estimation of Probabilities of Offspring for Multi-type Branching Processes

The estimates of the joint probabilities of the number of two types of offspring produced by an individual in each generation is obtain from the maximum likelihood estimation method.

Suppose the maximum number of type $d(d = 1, 2)$ offspring an individual will produce at the terminal reproductive age is m_d. Suppose B bootstrap samples of size n_t $(t = 1, 2, 3, \ldots, \varpi)$ are available from a multinomial population of $m_1 + m_2 + 1$ class probabilities $p_{ij(t)}$ at time t such that $x_{ij(t)}$ $(i = 0, 1, 2, \ldots, m_1; j = 0, 1, 2, \ldots, m_2)$ represent the number of individuals at the terminal reproductive age who produced r offspring of type i and j at time t and $n_{(t)}$ is the total number of females at the terminal reproductive age-group at generation t.

Then the maximum likelihood estimate of the joint probabilities from the B bootstrap samples is given as

$$\hat{p}^*_{ij(b)t} = \frac{x^*_{ij(b)t}}{n_{(b)t}}, \ i = 0, 1, 2, \ldots, m_1; \ j = 0, 1, 2, \ldots, m_2; \ b = 1, 2, \ldots, B \tag{4.34}$$

Then the joint probabilities of the offspring distribution at time t is given as the averages of the B boostrap samples

$$\bar{p}^*_{ij(t)} = \sum_{b=1}^{B} \hat{p}^*_{ij(b)t}, \quad i = 0, 1, 2, \ldots, m_1; \ j = 0, 1, 2, \ldots, m_2, \tag{4.35}$$

and let $v_0, v_1, v_2, \dots$ represent the joint probability of giving birth to 0, 1, 2, ... offspring respectively. Therefore, v_0 is the probability of giving birth to 0 type i and type j offspring and v_1 is the probability of giving birth to only 1 type i offspring or only 1 type j offspring. Given that p_{ij} represent the probability of giving birth to a number of type i offspring and type j offspring. The joint probability of giving birth to 0, 1, 2, ... number of offfspring is $v_0 = p_{00}$; $v_1 = p_{01} + p_{10}$; $v_2 = p_{02} + p_{11} + p_{20}; \dots$.

4.9.2 Estimation of Generational Probabilities of Zero Offspring in Random Branching Process for Multi-type Branching Processes

Suppose estimates of the $d \times m$ probability matrice $\mathbf{P}_{(i)} = \left(\mathbf{p}^d_{(i)1}, \mathbf{p}^d_{(i)2} \dots, \ \mathbf{p}^d_{(i)m}\right)$ where $d = 1, 2$ and the probability matrice are subject to future generations $i = 1, 2, 3, \dots \alpha$. Through a random selection (with repetition allowed) from the set of the probability matrice $\left\{\mathbf{P}_{(1)}, \mathbf{P}_{(2)}, \mathbf{P}_{(3)}, \dots, \mathbf{P}_{(\varpi)}\right\}$ where ϖ may be finite or infinite. The generational probabilities of zero offspring are computed using the theorem in equation (4.1) for each n_0 sample paths $\mathbf{q}_i = \left(\mathbf{q}^d_{i1}, \mathbf{q}^d_{i2}, \dots, \mathbf{q}^d_{i\alpha}\right), \ \ i = 1, 2, \dots n_0; \ d = 1, 2$. Then the generational probabilities of zero offspring are obtained from the averages of all sample paths for each generation, ,

$$\hat{\mathbf{q}}_{(n)} = \frac{1}{n_o} \sum_{i=1}^{n_o} \mathbf{q}_{in} \ , \quad n = 1, 2, \dots, \alpha \tag{4.36}$$

where $\hat{\mathbf{q}}_{(n)} = (q_{(1)n}, q_{(2)n})$ and $\mathbf{q}_{in} = (q_{(1)in}, q_{(2)in})$.

4.9.3 Limiting Probability of Giving Birth to Zero Offspring for Multi-type Branching Processes

The long-run probability of giving birth to zero offspring in a random environment for multi-type branching processes can be obtained similarly under the single-type. It is given as

$$|\mathbf{q}_n - \mathbf{q}| < \boldsymbol{\varepsilon} \tag{4.37}$$

where, $\mathbf{q}_n = (q_{1n}, q_{2n})$, $\mathbf{q} = (q_1, q_2)$, the limiting probability of zero offspring for type 1 and type 2, respectively and $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2)$ where $\varepsilon_i (i = 1, 2)$ is very small and $\varepsilon_i > 0$.

4.10 Index of Closeness

In the process of obtaining the conditions under which the generational probabilities of zero offspring for the random and constant environments are almost the same, this study proposed two indexes of closeness. One of the indexes of closeness measures the degree of closeness among the set of probability mass functions obtained at different times in the past. The other measure of closeness examines how close the generational probabilities of zero offspring between the constant environment and random environment branching processes are.

4.10.1 Definition: $\mathbf{I}_1$

Consider the finite set $\left\{P_{(1)}, P_{(2)}, P_{(3)}, \ldots, P_{(\alpha)}\right\}$ of probability distributions such that $P_{(i)} = P_{(i)1}, \ldots, P_{(i)k}$ $i = 1, 2, 3, \ldots, \alpha$ and $\sum_{j=0}^{k} P_{(j)k} = 1$. Define $I_{\ell m}$ as

$$I_{\ell m} = \frac{p_{(\ell)}}{p_{(m)}}, \quad m \neq \ell; m, l = 1, 2, \ldots, \alpha \tag{4.38}$$

An index of closeness, $\mathbf{I}_1$ among the α distributions is the geometric mean of the $I_{\ell m} (m \neq \ell)$ given by

$$\mathbf{I}_1 = \left(\prod_{\ell=1}^{\alpha} \prod_{m=\ell+1}^{\alpha} \mathbf{I}_{\ell m} \right)^{\frac{2}{\alpha(\alpha-1)}} \tag{4.39}$$

4.10.2 Definition: $\mathbf{I}_2$

Suppose $q_{in}(i = 1, 2; n = 1, 2, \ldots, L)$ is the probability of zero offspring in the n^{th} generation of the i^{th} branching process. Define $\mathbf{I}_2$ the index of closeness between the zero offspring probabilities of the two processes as

$$\mathbf{I}_2 = \left(\prod_{n=1}^{L} \frac{q_{1n}}{q_{2n}}\right)^{\frac{1}{L}} \tag{4.40}$$

values of $\mathbf{I}_1$ and $\mathbf{I}_2$ close to unity implies corresponding quantities being compared are very close to each other.

4.11 Repeated Measures Analysis of Variance (ANOVA)

The repeated measures ANOVA tests whether there are any differences between related population means. A repeated measures design often involves measuring subjects at different points in time or subjects measured under different experimental conditions (Gladwin, 2020). It can be viewed as an extension of the paired-samples t-test (but the groups are three or more). For this study, the dependent variable is the probability of zero offspring, and for each point in time (generation), the probability of zero offspring is computed in a constant, uniform random and weighted random environments. Hence, the repeated measures ANOVA will assess if the probabilities of zero offspring in a constant, uniform and weighted random environment are significantly different from each other.

4.11.1 Hypothesis

Null hypothesis: The null hypothesis (H_0) states that the probability of zero offspring in a constant, uniform and weighted random environment are not significantly different for each generation:

$$H_0 : q_1 = q_2 = q_3 \tag{4.41}$$

Alternative hypothesis: The alternative hypothesis (H_A) states that the probability of zero offspring in a constant, uniform and weighted random environment are significantly different for each generation (at least one).

$$H_A : \text{at least two conditions are significantly different}$$

4.11.2 Test Statistics

The test statistics of repeated ANOVA is similar to independent ANOVA and can be seen in Gladwin (2020) as a ratio of Means Squares between subject (MS_b) and Mean Square Error (MS_e).

$$\begin{aligned} MS_b &= \frac{SS_b}{(k-1)} && (4.42) \\ MS_e &= \frac{SS_e}{(n-1)(k-1)} && (4.43) \\ F &= \frac{MS_b}{MS_e} && (4.44) \end{aligned}$$

The computation of Sum of Squares between subject (SS_b) and Sum of Square Error (SS_e) are

$$\begin{aligned} SS_b &= \sum_{j=1}^{k} n_j \left(\bar{x_j} - \bar{x}\right) && (4.45) \\ SS_e &= \sum_{j=1}^{k} (x_{ij} - \bar{x_j})^2 - SS_b && (4.46) \end{aligned}$$

where $\bar{x_j}$ is group mean, $\bar{x}$ is grand mean and x_{ij} represents the i^{th} observation in the j^{th} group.

A repeated-measures ANOVA will not inform you where the differences between groups lie as it is an omnibus statistical test (Gladwin, 2020; Ugbolue et al., 2020). To highlight exactly where these differences occur Bonferroni post hoc test was employed.

4.12 Unit Root Tests

Unit root test plays a vital role in the time-series analysis as well as spectral analysis. The importance of unit root tests and their development has been emphasized in applied statistics over the past four decades (Diebold & Rudebusch, 2021). This is because many time-series models rely on the assumptions of the stationarity of the data before they can provide a precise estimate. Similarly, the spectral analysis also requires the data to be stationary. Stationarity means that the statistical properties of a time series, such as mean, variance, autocorrelation, etc., are all constant over time (Palachy, 2019). The study used the Augmented Dickey-Fuller (ADF) test and Phillips-Perron (PP) unit root test to determine the stationarity of the generational probabilities of zero offspring.

4.12.1 The Augmented Dickey-Fuller (ADF) Test

The standard Dickey-Fuller (DF) test is estimated by expressing the time series variable as autoregressive (AR) as shown below:

$$y_t = \rho y_{t-1} + x'\delta + \epsilon_t \tag{4.47}$$

where x', are optional exogenous regressors, which may consist of constant, or a constant and trend, ρ and δ are parameters to be estimated, and the $\in_t$ are assumed to be white noise. Then after subtracting y_{t-1} from both sides of the equation, the equation becomes:

$$\Delta y_t = \alpha y_{t-1} + x'\delta + \epsilon_t \tag{4.48}$$

where $\alpha = \rho - 1$. The null and alternative hypotheses may be written as, $H_0 : \alpha = 0$ and $H_1 : \alpha < 0$. Using the conventional t-statistic, the test statistic is:

$$t_\alpha = \widehat{\alpha}/(se(\widehat{\alpha})) \tag{4.49}$$

where $\widehat{\alpha}$ is the estimate of α, and se $(\widehat{\alpha})$ is the standard error of the coefficient $\alpha = \rho - 1$.

Dickey and Fuller (1979) show that, under the null hypothesis of a unit root, the critical value is generated by MacKinnon (1991, 1996). However, if the series is not an AR model and correlated at its lags then the Augmented Dickey-Fuller (ADF) test is preferred. The ADF correct the correlations of the series with its lags by adding p lagged difference terms of the variable y to the right-hand side of the test regression:

$$\Delta y_t = \alpha y_{t-1} + x_t'\delta + \beta_1 \Delta y_{t-1} + \beta_2 \Delta y_{t-2} + \cdots + \beta_p \Delta y_{t-p} + v_t \tag{4.50}$$

4.12.2 The Phillips-Perron (PP) Test

Phillips and Perron (1988) unit root test is a unit roots test that control series with higher correlation when testing for a unit root. The PP test is based on the statistic:

$$t_\alpha = t_\alpha \left(\frac{\gamma_0}{f_0}\right)^{1/2} - \frac{T(f_0 - \gamma_0)(se\,(\widehat{\alpha}))}{2 f_0^{1/2} s} \tag{4.51}$$

where $\widehat{\alpha}$ is the estimate, and t_α the t-ratio of α, $se(\widehat{\alpha})$ is the standard error of the coefficient α, and s is the standard error of the test regression. The term f_0 is an estimator of the residual spectrum at frequency zero. In addition, γ_0 is a consistent estimate of the error variance calculated as $(T - k)\, s^2/T$, where k is the number of regressors. The null hypothesis states that the series has a unit root against the alternative that the series has no unit root.

4.13 Spectral Analysis

Assuming that the probabilities of giving birth to zero offspring the generations (time) q_t are periodic (cyclic), then the series would be modelled as a harmonic regression shown below as

$$q_t = R cos\left(ft + d\right) + \varepsilon_t \ , \tag{4.52}$$

where R represent the amplitude of variation, f is the frequency of periodic variation, measured in number of radians per unit time, d represent the phase and ε_t is the white noise of the component.

The harmonic regression model can be further decomposed into

$$q_t = a\ cos\left(ft\right) + b\ sin\left(ft\right) + \varepsilon_t, \tag{4.53}$$

where $a = \ R\ cos(d)$ and $b\ =\ -R\ sin(d)$.

For k sum of frequencies the model is written as

$$q_t = \sum_{i=1}^{k} a_i\ cos\left(ft\right) + \sum_{i=1}^{k} b_i\ sin\left(ft\right) + \varepsilon_t \tag{4.54}$$

The selection of frequencies are given as

$$f_k = \frac{2\pi k}{N}, \qquad \left(k = 1, 2, .., \frac{N}{2}\right) \tag{4.55}$$

where N is the total number of observation, the k^{th} frequency is often referred to as the k^{th} harmonic.

According to Granger and Hatanaka (2015) the sum of squares SS_k for a particular frequency is given by

$$SS_k = \frac{N}{2}\left(a_k^2 + b_k^2\right) \tag{4.56}$$

If we regard SS_k as the portion of the total sum of squares accounted for by frequencies in the range

$$f_k \pm \frac{\pi}{N} \tag{4.57}$$

we can draw a histogram so that the area of each bar is proportional SS_k. The height of the histogram would be

$$I\left(f_k\right) = \frac{N}{4\pi}\left(a_k^2 + b_k^2\right) \tag{4.58}$$

The plot of $I(f)$ versus f is called the periodogram.

This definition of the periodogram equates the total sum of squares to the area under the periodogram. $I(f)$ may be calculated directly from the data as

$$I\left(f_k\right) = \frac{\left[\sum q_t cos\left(\frac{2\pi kt}{N}\right)\right]^2 + \left[\sum q_t sin\left(\frac{2\pi kt}{N}\right)\right]^2}{N\pi}. \tag{4.59}$$

4.13.1 Period

As mentioned before, waves consist of oscillations. The time taken to complete one oscillation is called the period. Period is computed as the inverse of frequency and often denoted as T (Dikpati & McIntosh, 2020);

$$T = \frac{1}{f} \tag{4.60}$$

where f is the frequency.

4.13.2 Wavelength

According to Chen et al. (2019), wavelength is defined as the distance between two sequential peaks or trough of a wave. Therefore, it is the length of one complete wave cycle. It is the distance between consecutive corresponding points of the same phase on the wave. Wavelength is commonly designated by the Greek letter lambda (λ),

$$\lambda = \frac{2\pi}{f}. \tag{4.61}$$

4.13.3 Amplitude

The amplitude of a wave refers to the maximum amount of displacement of a particle on the medium from its rest position. In a sense, the amplitude is the distance from rest to crest. Similarly, the amplitude can be measured from the rest position to the trough position. The formula is given as

$$Amplitude \; = \sqrt{a^2 + b^2} \;\; , \tag{4.62}$$

where a and b are estimates from the harmonic regression.

4.13.4 Phase

Phase specifies the location or timing of a point within a wave cycle of a repetitive waveform;

$$Phase = tan^{-1}\left(\frac{-b}{a}\right). \tag{4.63}$$

4.13.5 Proposed time

The proposed time is defined as the appropriate time to reverse population dynamics policies. This time was proposed based on the wavelength calculated with the idea that half of the wavelengths are always peak or trough, where extreme values are observed. Therefore, in revising a policy, it would be advisable not to wait for half of the wave cycle since waiting for that period will have a significant impact on the economy (Frias, Smolyakov, Kaganovich, & Raitses, 2013). Therefore, the study proposed that the required time frame for population policy revision should be less than half of the wavelength. Hence, the suggested time is calculated as

$$Proposed\ time = \frac{wavelength}{4}. \tag{4.64}$$

Chapter 5

DATA ANALYSIS

5.1 Introduction

This chapter presents the results from the proposed method of analysing the branching process in a varying environment and the existing method (time-homogeneous). This includes obtaining a situation where results from the proposed method will be close to the existing method. The chapter also presents the results from estimating the generational probability of zero offspring in two type branching processes and the expected time to revise population policy.

5.2 Estimation of the Long-run Probabilities of Zero Offspring for Population Subject to Oscillation and Varying Environment

This section examined the long-run behaviours of the probabilities of zero offspring for populations subject to oscillation and varying environments. This study focused on a particular case of the varying environment where the sequence of offspring distribution approaches a limit.

5.2.1 Oscillating Environment

The oscillating environment in the study focused on the situation in which the offspring distributions oscillate among two, three, four and five different offspring distributions. Using the propose method in equation (4.1) the results are presented in Table 5.1.

Table 5.1: Oscillating Environment of Offspring Distribution

Number of oscillating offspring distributions	The generation in which q starts to oscillate	Oscillating q
2	6	0.02354677, 0.02715530
3	8	0.02354289, 0.01489922, 0.02701496
4	8	0.01171598, 0.02696468, 0.02354150, 0.01489916
5	10	0.01171598, 0.01779376, 0.02706131, 0.02354417, 0.01489927

Data: Ghana parity data 1993, 1998, 2003, 2008, 2013

Source: Author's computation, (2021).

From Table 5.1, it can be seen that when the distribution of offspring oscillates among n number of distributions, the long-run probability of zero offspring also oscillates around the exact number of oscillating offspring distributions. For instance, when the offspring distribution was oscillating between two offspring distributions, the long-run probability of zero offspring was oscillating between two probabilities (0.02354677, 0.02354677); for three offspring distributions, the long-run probability of zero offspring also oscillated among three probabilities (0.02354289, 0.01489922, 0.02701496). However, the study revealed that the long-run probability of zero offspring oscillation starts after a specific number of generation. The number at which it starts to oscillates is determined by the number of offspring distributions the population is subjected to. Moreover, the fewer the number of distributions the population is subjected to, the faster the lof-run zero probability starts to oscillate. This implies that when a country is subjected to two offspring distributions, it is expected that the probability of zero offspring will oscillate among two values in the long-run. These results confirmed with Vandermeer (2006), who found out that established populations often exhibit oscillations in their sizes and the probability of extinction can also oscillate. He explained that even though the population size oscillates when the population is isolated (exposed to extreme or bad environments), the intrinsic stochasticity of fundamental processes can ultimately bring it to extinction.

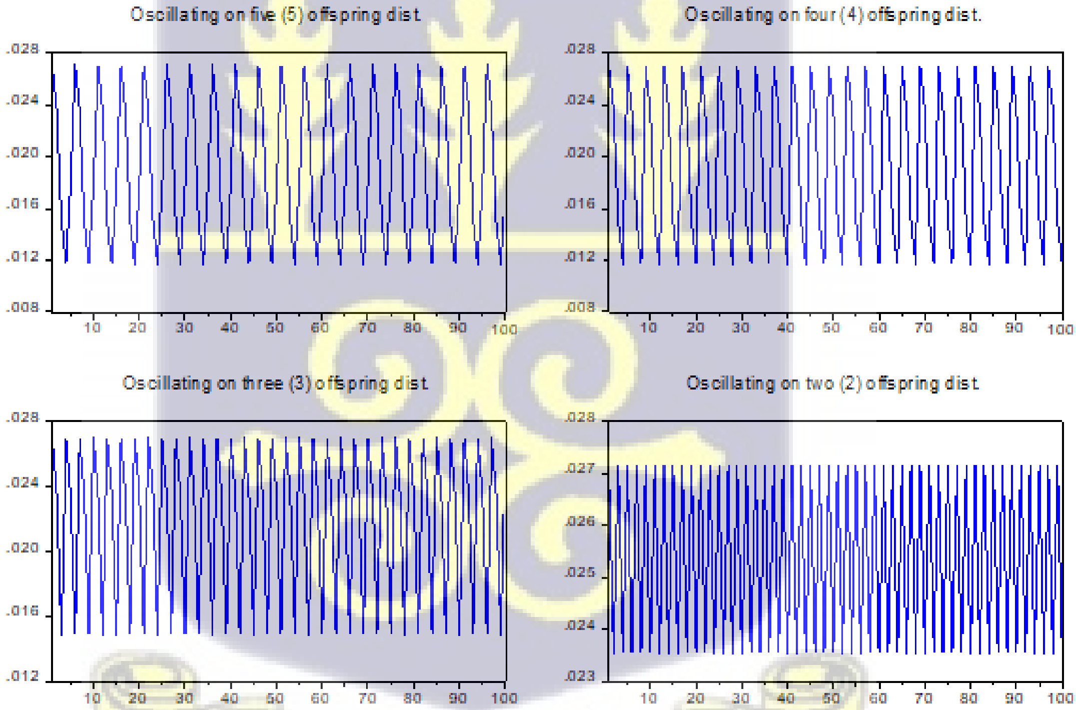

Figure 5.1: Oscillating Environment of Offspring Distribution

Figure 5.1 displays the graphical representation of the long-run probabilities of zero offspring from oscillating distribution of offspring. It was observed that when the distribution of offspring oscillates among fewer distributions of offspring, the oscillation of the long-run probabilities of zero offspring becomes faster and starts at an earlier generation.

5.2.2 Limiting Offspring Distribution

This section estimates two long-run probability of zero offspring. The first limiting probability is obtained from a sequence of offpring distributions which approach a limit. The second limiting probability is obatined from the limit of the offspring distribution.

The limit of the offspring distribution is given as

$$p_0 = 0.01953,\ p_1 = 0.02986,\ p_2 = 0.95061 \tag{5.1}$$

The sequence of offspring distributions that approach a limit is given as

$$p_0 = 0.01953 + \frac{1}{(n+1)},\ p_1 = 0.02986 - \frac{1}{2\,(n+1)},\ p_2 = 0.06416 - \frac{1}{2\,(n+1)} \tag{5.2}$$

The Long-run Probability of Zero Offspring from a Sequence of Offspring Distributions

Figure 5.2 presents the results of the generational probabilities of offspring from a sequence of offspring distributions that approach a limit using equation (4.24).

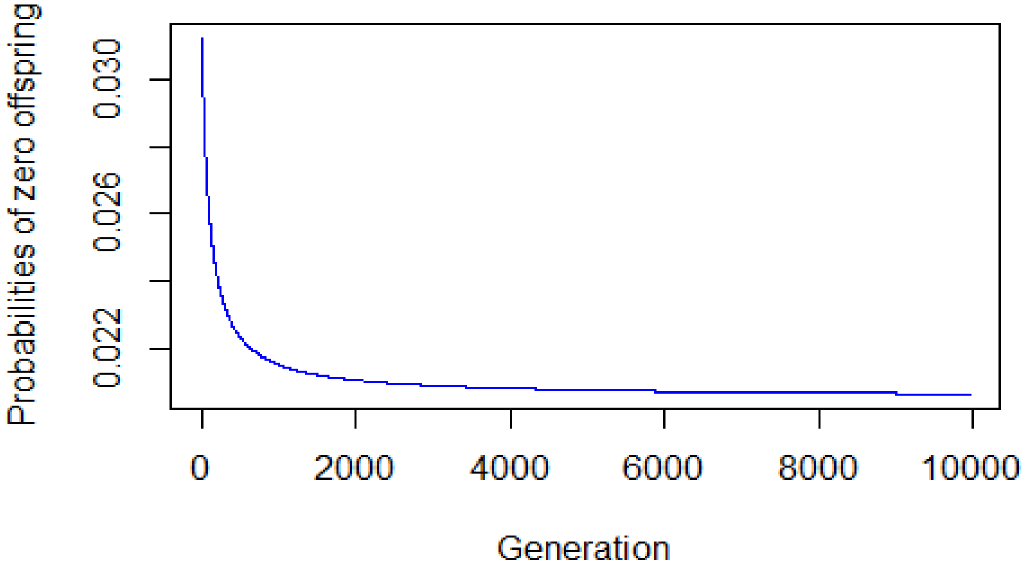

Figure 5.2: The Probability of Ultimate Extinction for Sequence of Offspring Distributions which Approach a Limit

From Figure 5.2, it can be seen that the generational probability of zero offspring is decreasing and gets to a generation where it is stable with a limiting probability of 0.020545.

The Long-run Probability of Zero Offspring of a Population Subject to the Offspring Distribution in Equation (5.1)

The long-run probability of zero offspring can be derrieved from generation function (pgf) using a point equation $q = f(q)$. The probability generation function (pgf) of the limit distribution (in equation 5.1) is given as

$$\begin{aligned} f(t) &= \sum_{r=0}^{\infty} p_r t^r \\ &= 0.01953 + 0.02986t + 0.95061t^2 \qquad (5.3) \end{aligned}$$

Using the pgf the probability of ultimate extinction q is the solution of $q = f(q)$. Thus

$$
\begin{aligned}
q &= 0.01953 + 0.02986q + 0.95061q^2 \\
0 &= 0.95061q^2 - 0.97014q + 0.01953 \\
q &= \frac{0.97014 - \sqrt{(0.97014)^2 - (4 \times 0.95061 \times 0.01953}}{2 \times 0.95061} \\
q &= 0.020545 \qquad (5.4)
\end{aligned}
$$

it can be seen that the long-run probability of zero offspring from equation (5.2) gives 0.020545, which is the same as that for the population, which is subject to a sequence of the probability distribution that approach a limit. Therefore it can be inferred that

$$q = \lim_{n \to \infty} f_{n-1}(q_{n-1}) \qquad (5.5)$$

d'Souza and Biggins (1992), obtained similar results with regards to the expected number of offspring. They explained that in supercritical monotype branching processes in varying environments (when the expected number of offspring is greater than one), the expectation of the offspring distribution Z_n converges almost surely to a finite random variable Z. This implies that when a country targets a certain long-run probability of giving birth, it can work towards the fertility rate to achieve the goal.

5.3 Conditions under which Estimations of Long-run Probability of Zero Offspring in a Random Environment Behaves Like a Constant Environment

This section analyses the long-run probability of zero offspring under a constant environment and random environment. Two random processes were considered:

uniform random process and weighted random process. The generational probabilities of zero offspring and index of closeness were computed for both hypothetical data and empirical data.

5.3.1 The Long-run Probability of Zero Offspring for Hypothetical Data

To demonstrate the conditions under which constant environment and random environment yield similar results, the proposed method in equation (4.1) was applied to 8 hypothetical populations. For each population, a random sample of six parameters β_n of pgf in (equation 4.11) was selected. These were used by means of equation (4.14) to estimate generational probabilities of zero offspring (q_n) under the random environment for each population. Table 5.2 presents the randomly selected values of the parameters (β_n) for each population together with their averages which were used to compute the generational probabilities of zero offspring under the constant environment based on equation (4.15). For each hypothetical population, the expected number of offspring were computed based on equation (4.12).

Table 5.2: Hypothetical Data

β_n	Pop1	Pop2	Pop3	Pop4	Pop5	Pop6	Pop7	Pop8
β_1	0.829362	0.312276	0.784718	0.674018	0.451921	1.433557	0.815467	0.448958
β_2	0.216146	0.376013	0.747908	0.084751	0.288907	2.326234	0.815615	0.449743
β_3	0.156823	0.366983	0.844737	0.131543	0.373664	2.620822	0.815823	0.451266
β_4	0.144930	0.396490	0.781333	0.094861	0.356545	1.681935	0.815249	0.454673
β_5	0.895739	0.190096	0.744832	0.265426	0.103555	3.681874	0.815257	0.458665
β_6	0.207025	0.169934	0.844548	0.057165	0.126145	3.052839	0.815202	0.461842
Average	0.408338	0.301965	0.791346	0.217961	0.283456	2.466210	0.815436	0.454191
μ	2.448955	3.311638	1.263670	4.587984	3.527882	0.405480	1.226339	2.201715

Source: Author's computation, (2021).

It can be inferred from Table 5.2 that in population one, the selected parameters were widely spread; the minimum value was 0.144930 and the maximum value is 0.895739, with the expected number of offspring being 2.4. Population 2 had the first four parameters close to each other, while the last two were very low. The parameters of populations 3 and 7 are very close, while the parameters in population 8 are exhibiting an increasing trend. The expected number of offspring in all the population was more than one except in population 6.

Probability of Ultimate Extinction of Hypothetical Data

The long-run probabilities of giving birth to zero offspring were computed in a constant environment and random environment. The results of the probability of ultimate extinctions and descriptive statistics of absolute difference of $|q_n - q|$ are displayed in Table 5.3.

Table 5.3: Probability of Ultimate Extinctions and Descriptive Statistics of Absolute Differences $|q_n - q|$

Pop.	Constant	Random		Mean	St Dev.	Max	Min
π_1	0.408338	R1	0.344513	5.09×10^{-04}	3.46×10^{-04}	1.31×10^{-03}	7.67×10^{-08}
		R2	0.322875	5.21×10^{-04}	4.04×10^{-04}	1.88×10^{-03}	1.32×10^{-07}
π_2	0.301965	R1	0.295242	2.25×10^{-04}	1.64×10^{-04}	5.45×10^{-04}	4.84×10^{-07}
		R2	0.271004	2.26×10^{-04}	1.65×10^{-04}	7.08×10^{-04}	1.72×10^{-07}
π_3	0.791346	R1	0.790441	3.57×10^{-05}	2.54×10^{-05}	9.13×10^{-05}	3.71×10^{-07}
		R2	0.794715	2.33×10^{-05}	1.71×10^{-05}	5.54×10^{-05}	4.26×10^{-07}
π_4	0.217961	R1	0.187225	3.41×10^{-04}	3.02×10^{-04}	1.11×10^{-03}	3.18×10^{-07}
		R2	0.148745	4.83×10^{-04}	2.97×10^{-04}	1.24×10^{-03}	4.81×10^{-08}
π_5	0.283456	R1	0.269826	2.22×10^{-04}	1.73×10^{-04}	5.80×10^{-04}	9.56×10^{-08}
		R2	0.226641	4.15×10^{-04}	3.04×10^{-04}	1.10×10^{-03}	3.95×10^{-08}
π_6	1	R1	1	0.0000	0.0000	0.0000	0.0000
		R2	1	0.0000	0.0000	0.0000	0.0000
π_7	0.815435	R1	0.815435	5.66×10^{-07}	7.63×10^{-07}	3.79×10^{-06}	3.58×10^{-09}
		R2	0.815375	1.06×10^{-06}	9.79×10^{-07}	4.94×10^{-06}	2.40×10^{-07}
π_8	0.454191	R1	0.454189	1.27×10^{-05}	8.36×10^{-06}	3.62×10^{-05}	4.33×10^{-09}
		R2	0.456096	7.79×10^{-06}	5.35×10^{-06}	2.37×10^{-05}	2.14×10^{-08}

R1: Uniform random, R2: Weighted random

Source: Author's computation, (2021).

For the uniform random process (R1) in population π_1, the probability of ultimate extinction was obtained to be $q = 0.344513$. The absolute differences $|q_n - 0.344513|$, $n \geq 50$; ranges from 7.67×10^{-08} to 1.31×10^{-03} with a mean of 5.09×10^{-04} and a standard deviation of 3.46×10^{-04}. These indicate that for $n \geq 50$ $|q_n - q| < \varepsilon = 0.0004$ confirming $q = 0.344513$, according to equation (4.25). Similar results were obtained for π_2, π_3, π_4, π_5, π_7, π_8, for instance in π_2 for $n \geq 50$ $|q_n - q| < \varepsilon$ were 0.0002 and 0.00004 respectively. In the cases of populations $\pi 6$, $q_n = 1$ for $n \geq 50$. Hence the probability of ultimate extinction is unity, indicating long-run extinction of the population. Under the constant environment case for π_1, $q = 0.408338$, whiles those of populations π_2 and π_3 are 0.301965 and 0.791346 respectively. Population π_4, π_5, and π_6 recorded the probability of ultimate extinction of 0.217961, 0.283456, 1 respectively.

From the expected number of offspring in Table 5.2, the study revealed that the probability of the ultimate extinction has some relation with the expected number of offspring. Regardless of the three approaches, when the expected number of offspring is less than one, as seen in population 6, the probability of ultimate extinction is one. It confirms with Bhat and Miller (2002) that when the expected number of offspring is less than one, the long-run probability is one. In addition, when the expected number of offspring is greater than one, the probability of ultimate extinction becomes less than one. However, when the expected number of offspring is greater than one but close to one, the long-run probability of giving birth to zero offspring becomes closer to one, as seen in populations 3 and 7. These results confirmed with Beznea et al. (2020) that the expected number of offspring has a relation to the long-run probability of zero offspring.

Index of Closeness of Hypothetical Data

From Table 5.3, it was found that in some hypothetical populations, the probability of ultimate extinction (long-run probability of zero offspring) under the constant environment was very close to the random environment. The study, therefore, proposed indices of closeness, based on equations (4.39) and (4.40), respectively, to ascertain the level of closeness that recorded a very close probability of ultimate extinction in each environment. As discussed in chapter three, the index of closeness I_1 measures how the offspring distributions are close to each other. I_{21} shows how close the values of the probabilities of zero offspring under a constant environment are close to the uniform random environment, and I_{22} measures the closeness between the constant environment and weighted random environment. The spearman correlation of the absolute difference of the indices of closeness was also computed. ρ_1 represent the spearman correlation between $|1 - I_1|$ and $|1 - I_{21}|$, and ρ_2 represent the spearman correlation between $|1 - I_1|$ and $|1 - I_{22}|$. The results are presented in Table 5.4.

Table 5.4: Index of Closeness of the Hypothetical Data

Population	I_1	$\|1 - I_1\|$	I_{21}	$\|1 - I_{21}\|$	I_{22}	$\|1 - I_{22}\|$
$\pi1$	1.20144	0.20144	1.05705	0.05705	1.29195	0.29195
$\pi2$	1.39667	0.39667	1.11234	0.11234	1.33939	0.33939
$\pi\mathbf{3}$	0.98170	0.01830	1.00518	0.00518	0.99614	0.00386
$\pi4$	1.85134	0.85134	1.24111	0.24111	1.44988	0.44988
$\pi5$	1.88454	0.88454	1.25051	0.25051	1.45794	0.45794
$\pi\mathbf{6}$	0.73035	0.26965	1.00000	0.00000	1.00001	0.00001
$\pi\mathbf{7}$	1.00024	0.00024	1.00000	0.00000	1.00007	0.00007
$\pi8$	0.98623	0.01377	1.00003	0.00003	0.88058	0.11942
ρ_1	0.814					
ρ_2	0.738					

Source: Author's computation, (2021).

From Table 5.4, it is found that hypothetical populations 3, 7 and 8 had $|1 - I_1|$

less than one and exhibited a high degree of measure of closeness. For instance, in population 3, the value of $|1 - I_1|$ was 0.01830 and portray a very high measure of closeness of I_{21} and I_{22}. ($|1 - I_{21}| = 0.00024; |1 - I_{22}| = 0.03298$). Population 7 recorded the lowest measure of closeness of I_1, I_{21} and I_{22}. It can be seen that when the $|1 - I_1|$ is low the index of closeness with regards to $|1 - I_{21}|$ and $|1 - I_{22}|$ were also low. This informs us that when offspring distributions for the generation are very close I_1, the probabilities of zero offspring under the constant environment will be similar to the random environment I_{21} and I_{22}. To confirm this, a Spearman correlation analysis was done, and strong positive correlations were recorded. This implies that at a certain level closeness $|1 - I_1| < 0.1$, the probability of zero offspring under a random environment is similar to the constant environment. Hence analysing the branching processes in the contant environment will be a step in the right direction. However, if the distribution of offspring is increasing or decreasing from generation to generation, as in the case of population 8. The long-run probability of zero offspring under the weighted random environment will not be close to the constant environment. This happens because the weighted random procedure emphasises on the most recent values; hence more of the current parameters will be selected. Therefore, weighted random is more appropriate in estimating the probability of ultimate extinction when such decreasing or increasing trends are seen in the offspring distribution.

To corroborate our position, repeated ANOVA and multiple t-tests were used to assess if the probabilities of zero offspring in each generation statistically differ among the three processes. A multivariate normality test was performed before the Repeated Measures ANOVA was done. The results of the multivariate normality test are displayed in Appendix D, whilst the Repeated Measures ANOVA are presented in Table 5.5.

Table 5.5: Repeated ANOVA and Multiple T-Test

Population	Repeated ANOVA		t-test	
	Test stat.	p-value	pair	test stat.
π_1	1850.576	0.000	Const. & Rand1	40.5356***
			Const. & Rand2	55.9448***
			Rand1 & Rand2	16.9549***
π_2	579.442	0.000	Const. & Rand1	6.4746***
			Const. & Rand2	32.9059***
			Rand1 & Rand2	26.5369***
π_3	0.240	0.7869		
π_4	12699.83	0.000	Const. & Rand1	60.9211***
			Const. & Rand2	156.6851***
			Rand1 & Rand2	109.2843***
π_5	2678.323	0.000	Const. & Rand1	15.4762***
			Const. & Rand2	71.9995***
			Rand1 & Rand2	57.85***
π_6	0.010	0.990		
π_7	0.0005	0.999		
π_8	3484.793	0.000	Const. & Rand1	1.07
			Const. & Rand2	-155***
			Rand1 & Rand2	-156***

**t-test is significant at the 0.01 level (2-tailed)

Source: Author's computation, (2021).

From Table 5.5, observing hypothetical population one, the repeated ANOVA shows that the probabilities of zero offspring in each generation on the average are significantly different among the three methods ($F(2, 198) = 1850.58; p = 0.000$). The multiple paired t-test also confirmed that all the methods were significantly different from each other. Similar results can be found in population 4 (π_4) and population 5 (π_5), where on average, the probabilities of zero offspring for each generation were statistically different in the various environments under consideration. Regarding populations 3 and 7, which exibit a close index ($|1 - I_1| < 0.1$) the generational probabilities of zero offspring for the constant and the random environments were not significantly different

$(F(2,198) = 0.24; p = 0.7869)$ and $(F(2,198) = 0.0005; p = 0.999)$ respectively. This confirms that under certain closeness conditions, one can use the method under the constant environment instead of that random environment.

5.3.2 Empirical Data

This section discussed the application of the proposed method on empirical data. The selected countries were Ghana, Bangladesh, Burkina Faso, the Philippines, Colombia, Mali and Indonesia.

Offspring Distribution and Expected Number of Offsprings for Ghana

The offspring distribution and the expected number of offspring for six years with five years interval (1993, 1998, 2003, 2008, 2013, 2018) were computed based on equation (4.18), (4.19), (4.21) and (4.23). The results are presented in Table 5.6.

Table 5.6: Probability Distribution and Expected Number of Offsprings for Ghana by Year of Survey

probability	1993	1998	2003	2008	2013	2018	Average
$\bar{p}_0^*$	0.0271	0.0227	0.0139	0.0113	0.0173	0.0248	0.0195
$\bar{p}_1^*$	0.0151	0.0249	0.038	0.0138	0.0464	0.0416	0.0300
$\bar{p}_2^*$	0.0323	0.0435	0.0596	0.072	0.0985	0.0785	0.0641
$\bar{p}_3^*$	0.0775	0.0838	0.0842	0.1194	0.0918	0.1348	0.0986
$\bar{p}_4^*$	0.0739	0.1096	0.1075	0.1445	0.1223	0.1475	0.1176
$\bar{p}_5^*$	0.0986	0.1509	0.1061	0.1212	0.1457	0.1223	0.1241
$\bar{p}_6^*$	0.1128	0.1388	0.1537	0.131	0.1781	0.1369	0.1419
$\bar{p}_7^*$	0.1555	0.1379	0.1358	0.1426	0.1228	0.0994	0.1323
$\bar{p}_8^*$	0.1335	0.1069	0.1393	0.1202	0.0736	0.0796	0.1089
$\bar{p}_9^*$	0.1252	0.0804	0.0877	0.0554	0.0607	0.0661	0.0793
$\bar{p}_{10+}^*$	0.1484	0.1007	0.0741	0.0686	0.043	0.0685	0.0839
μ	6.5449	5.9401	5.8878	5.6304	5.2298	5.2384	5.7463

Source: Author's computation, (2021).

From Table 5.6, it can be seen that the expected number of offspring in 1993 was 6.5 and decreased to 5.94 in 1998, then 5.89 in 2003, it finally reduced to 5.24 in 2019. The expected number of offspring for the entire period was 5.7463. The mean number of offspring decreases as the years go by. This result confirms with Asumadu-Sarkodie and Owusu (2016) that there has been a reduction in Ghana fertility rate. According to them, due to economic situations, people tend to give birth to few children compared to previous years when children were used as labour force in farming. Abdul-Salam, Baba, and Jabir (2018) in their study explained that the pursuit of high education and the social empowerment of women had been shown to decrease fertility rates as a result of better childhood health and women's participation in the workforce. According to Blanc and Grey (2000) the frequent use of contraceptives has led to a reduction in the birth rate in Ghana. They concluded that for every increase of 15 percentage points in contraception among married women, a decline of one child is expected.

Probability Distribution and Expected Number of Offsprings for Bangladesh

The offspring distribution of Bangladesh were computed for 1993,1998, 2003, 2008, 2013, and 2018. As well as the expected number of offspring. The results are presented in Table 5.7.

Table 5.7: Probability Distribution and Expected Number of Offsprings for Bangladesh by Year of Survey

probability	1993	1998	2003	2008	2013	2018	Average
$\bar{p}_0^*$	0.00621	0.01894	0.01674	0.02489	0.01874	0.01449	0.01667
$\bar{p}_1^*$	0.01553	0.02840	0.01954	0.04491	0.04961	0.04646	0.03408
$\bar{p}_2^*$	0.03416	0.04142	0.06419	0.12879	0.14057	0.17349	0.09710
$\bar{p}_3^*$	0.04814	0.07219	0.10791	0.18344	0.20948	0.22549	0.14111
p_4	0.08385	0.12426	0.15628	0.17587	0.19956	0.22506	0.16081
$\bar{p}_5^*$	0.11491	0.14201	0.15256	0.15747	0.16979	0.13299	0.14496
$\bar{p}_6^*$	0.13509	0.15621	0.14977	0.11039	0.10088	0.08781	0.12336
$\bar{p}_7^*$	0.16304	0.14911	0.11535	0.08496	0.05788	0.05414	0.10408
$\bar{p}_8^*$	0.13199	0.10769	0.09395	0.04437	0.02426	0.02387	0.07102
$\bar{p}_9^*$	0.11180	0.07219	0.05581	0.02327	0.01599	0.00895	0.04800
$\bar{p}_{10+}^*$	0.15528	0.08757	0.06791	0.02165	0.01323	0.00725	0.05881
μ	6.70497	5.90296	5.49861	4.38149	4.08710	3.88491	5.07667

Source: Author's computation, (2021)

From Table 5.7, there is a decrease in the expected number of children an individual will have from 1993 to 2018. In 1993, the mean number of offspring was 6.70497, and in 2017, the mean number of offspring drastically reduced to 3.88491. This implies that the birth rate of Bangladesh has reduced. This reduction confirms with Schoen (2018), who explained that Bangladesh has come out with some population policies in 2012, which has greatly had an impact on the fertility rate of the country.

Probability Distribution and Expected Number of Offsprings for Colombia

The expected number of offspring and the offspring distribution of Colombia were computed for the periods namely 1990, 1995, 2000, 2005, 2010 and 2015 and presented in Table 5.8.

Table 5.8: Probability Distribution and Expected Number of Offsprings for Colombia by Year of Survey

Probability	1990	1995	2000	2005	2010	2015	Average
$\bar{p}_0^*$	0.06496	0.07829	0.07814	0.06427	0.06381	0.05191	0.06690
$\bar{p}_1^*$	0.06838	0.07620	0.08093	0.09411	0.11462	0.11472	0.09149
$\bar{p}_2^*$	0.09744	0.14092	0.15442	0.18943	0.23340	0.26141	0.17950
$\bar{p}_3^*$	0.10940	0.17015	0.20372	0.23066	0.23374	0.24519	0.19881
$\bar{p}_4^*$	0.13162	0.15658	0.15721	0.16105	0.14080	0.13882	0.14768
$\bar{p}_5^*$	0.12821	0.10647	0.12093	0.10866	0.08393	0.07416	0.10373
$\bar{p}_6^*$	0.09402	0.08768	0.07907	0.05845	0.05375	0.04519	0.06970
$\bar{p}_7^*$	0.08718	0.06159	0.04000	0.03396	0.02826	0.02480	0.04596
$\bar{p}_8^*$	0.06838	0.03549	0.03163	0.02522	0.02063	0.01784	0.03320
$\bar{p}_9^*$	0.05128	0.03340	0.02326	0.01504	0.01179	0.01205	0.02447
$\bar{p}_{10+}^*$	0.09915	0.05324	0.03070	0.01916	0.01526	0.01391	0.03857
μ	4.93333	4.10125	3.75814	3.46956	3.20964	3.13418	3.76769

Source: Author's computation, (2021)

The study revealed that in 1990 the probability of giving birth to a child, two and three children were higher than the probability of giving birth to no child. Similar results were seen in all the years. This implies that it is more likely to give birth to children in Colombian than having no child. It was observed that in 1990, the expected number of children was 4.9333 and in 1995, it decreased to 4.10125 and 3.13418 in 2015. The overall expected number of offspring over the period was 3.76769 and was greater than the values obtained in the current years (2000 to 2015). This implies that the expected offspring for the most recent years were low. This result is in line with Batyra (2016) who found out that the fertility trend in Colombia since 1990 has been declining. They attributed the reason to changes in the timing of childbearing, especially first birth. Other factors were increases in women's education and career opportunities in Colombia. Davalos and Morales (2017) in their study on the economic crisis and childbearing, concluded that periods of economic recession are associated with the low birth rate in poor areas of Colombia with a reduction of 0.0002 children per woman.

Probability Distribution and Expected Number of Offsprings for Burkina Faso

Table 5.9 presents the results of expected number of offspring and the offspring distribution of Burkina Faso in 1993, 1998, 2003, 2008, 2013 and 2018.

Table 5.9: Probability Distribution and Expected Number of Offsprings for Burkina Faso by Year of Survey

Probability	1993	1998	2003	2008	2013	2018	Average
$\bar{p}_0^*$	0.02452	0.01046	0.00929	0.00905	0.01055	0.00555	0.01157
$\bar{p}_1^*$	0.01090	0.01046	0.01207	0.01584	0.01055	0.00739	0.01120
$\bar{p}_2^*$	0.01090	0.02092	0.02507	0.01584	0.01582	0.01294	0.01691
$\bar{p}_3^*$	0.03270	0.03975	0.04178	0.03469	0.03691	0.05360	0.03991
$\bar{p}_4^*$	0.06267	0.05230	0.05293	0.06184	0.05800	0.07579	0.06059
$\bar{p}_5^*$	0.04087	0.05858	0.07707	0.10030	0.08963	0.16266	0.08819
$\bar{p}_6^*$	0.09264	0.09414	0.11699	0.15158	0.17223	0.15712	0.13079
$\bar{p}_7^*$	0.12262	0.15900	0.17177	0.17119	0.16520	0.19039	0.16336
$\bar{p}_8^*$	0.17166	0.15272	0.16713	0.18326	0.14236	0.12569	0.15714
$\bar{p}_9^*$	0.16076	0.17573	0.13463	0.11312	0.11072	0.10351	0.13308
$\bar{p}_{10+}^*$	0.26976	0.22594	0.19127	0.14329	0.18805	0.10536	0.18728
μ	7.51771	7.41423	7.15042	6.92534	7.03866	6.57671	7.10385

Source: Author's computation, (2021)

From Table 5.9, the probability of giving birth to more than 6 offspring is very high over the study period. For instance, the probability of giving birth to 10 or more children in 1993 was 0.269756, and that of 1998 was 0.22594. Burkina Faso, recorded an average of 7.10385 children per woman with the highest recorded in 1993 ($\mu = 7.51771$). On the average the probability of giving birth to 10 or more children was the highest ($p_{10+} = 0.18728$). According to Bloom (2016) Burkina Faso is among the few countries that have a high birth rate, with the yearly population growth rate more than double the world average, of which more than two-fifths of the population is younger than age 15. According to Jahn, Kynast-Wolf, Kouyaté and Becher (2006), the expected number of children in Burkina Faso is very high in most rural areas due to multiple pregnancies (twins birth).

Probability Distribution and Expected Number of Offsprings for Philippines

The offspring distribution of Philippines and the expected number of offspring were computed for 1993, 1998, 2003, 2008, 2013 and 2018. The results are displayed in Table 5.10.

Table 5.10: Probability Distribution and Expected Number of Offsprings for The Philippines

Probability	1993	1998	2003	2008	2013	2018	Average
$\bar{p}_0^*$	0.08042	0.07901	0.06031	0.07266	0.08031	0.08388	0.07610
$\bar{p}_1^*$	0.03671	0.05021	0.05436	0.06644	0.07536	0.09154	0.06244
$\bar{p}_2^*$	0.06818	0.09218	0.10648	0.12941	0.15127	0.15489	0.11707
$\bar{p}_3^*$	0.11364	0.11770	0.15339	0.18685	0.15237	0.18482	0.15146
$\bar{p}_4^*$	0.12762	0.12757	0.15860	0.13426	0.15842	0.15802	0.14408
$\bar{p}_5^*$	0.12500	0.10617	0.12584	0.11211	0.11551	0.11069	0.11589
$\bar{p}_6^*$	0.09528	0.11193	0.10350	0.08720	0.08581	0.07936	0.09385
$\bar{p}_7^*$	0.08829	0.10947	0.07595	0.06713	0.05666	0.05360	0.07518
$\bar{p}_8^*$	0.08566	0.05926	0.05808	0.04914	0.05006	0.03133	0.05559
$\bar{p}_9^*$	0.06119	0.05268	0.03574	0.03391	0.02970	0.02645	0.03995
$\bar{p}_{10}^*$	0.11801	0.09383	0.06776	0.06090	0.04455	0.02541	0.06841
μ	5.25525	4.95309	4.60760	4.28374	4.07096	3.73547	4.48435

Source: Author's computation, (2021).

The expected number of offspring in 1993 was 5.25525 and reduced slightly to 4.95309 in 1998 and continued to reduce to 3.73547 in 2018. This implies that there was a decline in the birth rate of the Philippines over the years. These results are confirmed with Hackenberg and Magalit (2019), who attributes the causes of declining fertility in the Philippines as an increase in the use of family planning.

Probability Distribution and Expected Number of Offsprings for Mali

The offspring distribution and the expected number of offspring for six years with five years interval (1992, 1997, 2002, 2007, 2012 and 2017) were computed. The results were presented in Table 5.11.

Table 5.11: Probability Distribution and Expected Number of Offsprings for Mali

Probability	1992	1997	2002	2007	2012	2017	Average
$\bar{p}_0^*$	0.03376	0.03644	0.02604	0.02586	0.02115	0.03639	0.02994
$\bar{p}_1^*$	0.03798	0.02699	0.03333	0.02931	0.04381	0.04589	0.03622
$\bar{p}_2^*$	0.03376	0.03374	0.02604	0.03190	0.06042	0.04430	0.03836
$\bar{p}_3^*$	0.05907	0.04858	0.03646	0.05086	0.07553	0.08228	0.05880
$\bar{p}_4^*$	0.08017	0.05128	0.04583	0.05776	0.09819	0.09019	0.07057
$\bar{p}_5^*$	0.05907	0.05398	0.06979	0.07414	0.14351	0.10760	0.08468
$\bar{p}_6^*$	0.11392	0.08772	0.06667	0.09914	0.13746	0.10443	0.10156
$\bar{p}_7^*$	0.08439	0.11066	0.08646	0.10776	0.09668	0.10760	0.09892
$\bar{p}_8^*$	0.06751	0.10661	0.15625	0.12931	0.10725	0.10760	0.11242
$\bar{p}_9^*$	0.14346	0.12416	0.13542	0.12500	0.08761	0.09810	0.11896
$\bar{p}_{10+}^*$	0.28692	0.31984	0.31771	0.26897	0.12840	0.17563	0.24958
μ	6.87342	7.18489	7.37813	7.04569	5.93354	6.15981	6.76258

Source: Author's computation, (2021)

Results from Table 5.11 show that the probability of zero offspring varied over the years and was in the range of 0.02115 to 0.03639, with an average of 0.029941. It was observed that the expected number of offspring in 2001 was very high ($\mu = 7.37813$) and the lowest expected number of children was in 2012 ($\mu = 5.93354$). One can see that the expected number of offspring fluctuated over the years, and the overall average was 6.76258. Philibert, Tourigny, Coulibaly, and Fournier (2013), assess the seasonality of the birth rate in Mali and their results confirmed that the average birth rate of Mali varies in months and years due to rainfall, food supply and economic conditions.

Probability Distribution and Expected Number of Offsprings for Indonesia

Table 5.12, represents the offspring distribution of gender as well as the expected number of offspring for six consecutive years with five years interval (1993, 1998, 2003, 2008, 2013, 2018) for Indonesia.

Table 5.12: Probability Distribution and Expected Number of Offsprings for Indonesia by Year of Survey

Probability	1993	1998	2003	2008	2013	2018	Average
$\bar{p}_0^*$	0.03205	0.02603	0.02784	0.03019	0.05572	0.06097	0.03880
$\bar{p}_1^*$	0.06374	0.05437	0.05934	0.06650	0.06508	0.08859	0.06627
$\bar{p}_2^*$	0.07490	0.09572	0.11526	0.15774	0.19403	0.24565	0.14722
$\bar{p}_3^*$	0.10371	0.1341	0.17265	0.20222	0.23682	0.25523	0.18414
$\bar{p}_4^*$	0.13036	0.15182	0.18901	0.17635	0.17433	0.16042	0.16371
$\bar{p}_5^*$	0.12856	0.13765	0.14750	0.13141	0.10826	0.08428	0.12294
$\bar{p}_6^*$	0.11559	0.12493	0.10867	0.08761	0.07005	0.04757	0.09240
$\bar{p}_7^*$	0.11559	0.10295	0.07204	0.05788	0.04219	0.02650	0.06952
$\bar{p}_8^*$	0.09471	0.06767	0.05299	0.03677	0.02309	0.01804	0.04888
$\bar{p}_9^*$	0.06158	0.04598	0.02466	0.02156	0.01652	0.00686	0.02953
$\bar{p}_{10+}^*$	0.07922	0.05870	0.03004	0.03178	0.01393	0.00591	0.03660
Mean	5.29564	4.95633	4.40391	4.08784	3.59045	3.14461	4.24646

Source: Author's computation, (2021).

The results in Table 5.12 show that in 1993 the probability of zero offspring was 0.03205 but reduced in 1998 to 0.02603 and later experienced an increment from 2008 to 2018. On average, the probability of giving birth to three offspring was the highest. This implies that it is more likely to give birth to three offspring in Indonesia. The overall mean for the number of children was 4.24646. The expected number of offspring declined from 5.29564 in 1993 to 3.1446 in 2018. This result is in line with Hatton, Sparrow, Suryadarma, and Eng (2018), they found out that there is a decline in the family size of the people in Indonesia. The family size was influenced from the supply side, as a family planning programme, and demand side, such as modern media (social media, internet).

Index of Closeness and Probability of Ultimate Extinction per Country

The indexes of closeness for each country were computed using equations (4.39) and (4.40). The probability of ultimate extinction were computed using equation (4.24) and (4.25) from 10000 sample path. The results are presented in Table 5.13.

Table 5.13: Index of Closeness and Probability of Ultimate Extinction q per Country

Country	I_1	$\|1 - I_1\|$	q
Ghana	0.974427	0.025573	0.019385
Bangladesh	1.143058	0.143058	0.018652
Philippines	1.071941	0.071941	0.08343
Burkina Faso	1.009437	0.009437	0.009717
Colombia	1.254567	0.254567	0.072851
Mali	0.927010	0.072990	0.03026
Indonesia	1.265860	0.265860	0.047128

Source: Author's computation, (2021).

Table 5.13 shows that Ghana, Burkina Faso, and Mali's exibit a close offspring distribution from generation to generation. From the hypothetical data, it was found that when the closeness of index $|1 - I_1| < 0.01$ the generational probabilities of zero offspring in a random environment behaves like the constant environment. Nevertheless, when the offspring distribution shows an increasing or decreasing trend from generation to generation, then the weighted random environment is prefered. It was found earlier from Table 5.6 to 5.12 that the expected number of offspring in all the countries decreased from generation to generation. As a result of the decreasing trend, the study computed the long-run limiting probability of zero offspring of the empirical data using the weighted random environment.

The results of the long-run probability of zero offspring show that the country with the highest ultimate extinction was the Philippines, followed by Colombia and Indonesia. Burkina Faso recorded the lowest ultimate extinction, followed by Bangladesh and Ghana. This implies that the Philippines has the aged population among the selected countries and Burkin Faso has the youngest population. These results confirmed with world Bank Data on World Development Indicator (2020) that Burkina Faso, Mali and Ghana display the lowest percentage of their population above 65 years (Burkina Faso = 2.4% and Ghana=3.5%).

5.4 Estimation of Probability of Zero Offspring in Multi-type Branching Processes with respect to Gender

This section presents the estimation of the long-run probability of zero offspring in a multi-type branching process concerning gender. The multi-type branching processes were analysed in a random environment. The study performed a paired t-test to determine if the generational probabilities of zero offspring per gender are significantly different.

5.4.1 Offspring Distribution of Ghana with respect to Gender by Year of Survey

The marginal probability generation function (pgf) of offspring of Ghana for 1993, 1998, 2003, 2008, 2013, 2018 were computed from the joint probability table in the Appendix B using equations (4.29) to (4.32). Let f_1 denote the pgf of male offspring and f_2 denote the pgf of female offspring.

Year: 1993

In order to obtain the marginal probability generation function (p.g.f), the marginal probabilities v_{1i} and v_{2j} were computed from equation (4.31). For example in 1993 for Ghana, the marginal probabilities of males were computed as;

$v_{10} = \sum_{j=0}^{m_2} \pi_{0j}$

$v_{10} = \pi_{00} + \pi_{01} + \pi_{02} + \pi_{03} + \pi_{04} + \pi_{05} + \pi_{06} + \pi_{07} + \pi_{08} + \pi_{09} + \pi_{0(10)}$

$v_{10} = 0.0271 + 0.0031 + 0.0058 + 0.0090 + 0.0058 + 0.0057 + 0.0028 + 0.0000 + 0.0000 + 0.0000 + 0.0000 = 0.0593$

$v_{11} = 0.0120 + 0.0206 + 0.0241 + 0.0088 + 0.0241 + 0.0149 + 0.0058 + 0.0090 +$

$0.0057 + 0.0000 + 0.0000 = 0.1250$

$v_{12} = 0.0059 + 0.0353 + 0.0206 + 0.0391 + 0.0240 + 0.0326 + 0.0087 + 0.0060 +$
$0.0028 + 0.0031 + 0.0000 = 0.1781$

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$v_{1,10} = 0.0000 + 0.0030 + 0.0000 + 0.0000 + 0.0000 + 0.0000 + 0.0000 + 0.0000 +$
$0.0000 + 0.0000 + 0.0000 = 0.0030.$

Hence;

$f_1(s) = 0.0593 + 0.1250s + 0.1781s^2 + 0.1815s^3 + 0.1603s^4 + 0.1160s^5 + 0.1201s^6 +$
$0.0387s^7 + 0.0121s^8 + 0.0059s^9 + 0.0030s^{10}$

$f_2(s) = 0.0632 + 0.1482s + 0.1644s^2 + 0.2089s^3 + 0.1557s^4 + 0.1471s^5 + 0.0558s^6 +$
$0.0394s^7 + 0.0113s^8 + 0.0060s^9$

Year: 1998

$f_1(s) = 0.0704 + 0.1162s + 0.1960s^2 + 0.2437s^3 + 0.1414s^4 + 0.1125s^5 + 0.0645s^6 +$
$0.0369s^7 + 0.0139s^8 + 0.0022s^9 + 0.0023^{10}$

$f_2(s) = 0.0872 + 0.1771s + 0.1854s^2 + 0.1806s^3 + 0.1781s^4 + 0.1273s^5 + 0.0366s^6 +$
$0.0207s^8 + 0.0070s^9$

Year: 2003

$f_1(s) = 0.0638 + 0.1527s + 0.1897s^2 + 0.1955s^3 + 0.1768s^4 + 0.1277s^5 + 0.0602s^6 +$
$0.0255s^7 + 0.0081s^8$

$f_2(s) = 0.0780 + 0.1618s + 0.2268s^2 + 0.1772s^3 + 0.1747s^4 + 0.0950s^5 + 0.0624s^6 +$
$0.0142s^7 + 0.0099s^8$

Year: 2008

$f_1(s) = 0.0624 + 0.1544s + 0.2391s^2 + 0.2150s^3 + 0.1500s^4 + 0.1173s^5 + 0.0412s^6 +$
$0.0116s^7 + 0.0068s^8 + 0.0024s^9$

$f_2(s) = 0.0711 + 0.1705s + 0.2409s^2 + 0.2023s^3 + 0.1295s^4 + 0.1237s^5 + 0.0324s^6 + 0.0137s^7 + 0.0117s^8 + 0.0023s^9 + 0.0021s^{10}$

Year: 2013

$f_1(s) = 0.0836 + 0.2018s + 0.2134s^2 + 0.2032s^3 + 0.1526s^4 + 0.0775s^5 + 0.0425s^6 + 0.0129s^7 + 0.0080s^8 + 0.0047s^9$

$f_2(s) = 0.0923 + 0.1969s + 0.2168s^2 + 0.2366s^3 + 0.1265s^4 + 0.0863s^5 + 0.0273s^6 + 0.0116s^7 + 0.0059s^8$

Year: 2018

$f_1(s) = 0.1039 + 0.1757s + 0.1922s^2 + 0.1977s^3 + 0.1468s^4 + 0.1025s^5 + 0.0499s^6 + 0.0185s^7 + 0.0103s^8 + +0.0000s^9 + 0.0021s^{10}$

$f_2(s) = 0.0788 + 0.2107s + 0.2654s^2 + 0.2005s^3 + 0.1341s^4 + 0.0605s^5 + 0.0226s^6 + 0.0208s^7 + 0.0020s^8 + 0.0000s^9 + 0.0042s^{10}$

The overall pgf

$f_1(s) = 0.0739 + 0.1543s + 0.2014s^2 + 0.2061s^3 + 0.1547s^4 + 0.1089s^5 + 0.0631s^6 + 0.0240s^7 + 0.0099 + s^8 + 0.0025s^9 + 0.0012s^{10}$

$f_2(s) = 0.0784 + 0.1775s + 0.2166s^2 + 0.2010s^3 + 0.1498s^4 + 0.1067s^5 + 0.0395s^6 + 0.0201s^7 + 0.0080s^8 + 0.0014s^9 + 0.0011s^{10}$

From the pgfs, the probability of giving birth to no female is higher than zero probability of male offspring in all years except for 2018. The probability of giving birth to one to three female offspring was higher than one to three males. This implies that it is more likely to give birth to three females than males. Nevertheless, it is less likely to give birth to zero males as compared to females. For instance, in 1993, the probability of giving birth to no male was 0.0593, and that of females was 0.0632. In 2018, the probability of giving birth to no male was 0.1039, which was higher than that of giving birth to no female (0.0788).

5.4.2 Offspring Distribution of Bangladesh with respect to Gender by Year of Survey

The probability generation function of offspring distribution for Bangladesh for each gender for the years (1993, 1998, 2003, 2008, 2013, 2018) were obtained as follows;

Year: 1993

$f_1(s) = 0.0264 + 0.104s + 0.1972s^2 + 0.1848s^3 + 0.1879s^4 + 0.1491s^5 + 0.0745s^6 + 0.045s^7 + 0.0155s^8 + 0.0124s^9 + 0.0031s^{10}$

$f_2(s) = 0.0435 + 0.1335s + 0.1957s^2 + 0.2189s^3 + 0.1537s^4 + 0.146s^5 + 0.0683s^6 + 0.0248s^7 + 0.0093s^8 + 0.0016s^9 + 0.0047s^{10}$

Year: 1998

$f_1(s) = 0.0556 + 0.1491s + 0.2095s^2 + 0.2379s^3 + 0.168s^4 + 0.0899s^5 + 0.0592s^6 + 0.0189s^7 + 0.0095s^8 + 0.0024s^9$

$f_2(s) = 0.0556 + 0.1834s + 0.1976s^2 + 0.2083s^3 + 0.1586s^4 + 0.1077s^5 + 0.0521s^6 + 0.0189s^7 + 0.0095s^8 + 0.0047s^9 + 0.0036s^{10}$

Year: 2003

$f_1(s) = 0.0567 + 0.1563s + 0.2493s^2 + 0.2251s^3 + 0.1609s^4 + 0.0865s^5 + 0.0437s^6 + 0.0149s^7 + 0.0047s^8 + 0.0009s^9 + 0.0009s^{10}$

$f_2(s) = 0.0735 + 0.1991s + 0.2298s^2 + 0.2186s^3 + 0.1247s^4 + 0.0698s^5 + 0.0512s^6 + 0.0233s^7 + 0.0047s^8 + 0.0028s^9 + 0.0028s^{10}$

Year: 2008

$f_1(s) = 0.092 + 0.2338s + 0.2895s^2 + 0.1921s^3 + 0.1028s^4 + 0.0552s^5 + 0.0216s^6 + 0.0097s^7 + 0.0022s^8 + 00.0011s^9$

$f_2(s) = 0.1369 + 0.2738s + 0.2462s^2 + 0.1677s^3 + 0.0974s^4 + 0.0449s^5 + 0.0254s^6 +$

$0.0054s^7 + 0.0016s^8 + 0.0005s^9$

Year: 2013

$f_1(s) = 0.1009 + 0.2778s + 0.2966s^2 + 0.1836s^2 + 0.0888s^2 + 0.0331s^2 + 0.0138s^2 + 0.0039s^2 + 0.0011s^2 + 0.0006s^2$

$f_2(s) = 0.1373 + 0.2806s + 0.247s^2 + 0.1808s^2 + 0.0932s^2 + 0.038s^2 + 0.0165s^2 + 0.005s^2 + 0.0006s^2 + 0.0006s^2 + 0.0006s^2$

Year: 2018

$f_1(s) = 0.1083 + 0.283s + 0.3005s^2 + 0.1812s^3 + 0.0891s^4 + 0.0256s^5 + 0.0085s^6 + 0.003s^7 + 0.0004s^8 + 0.0004s^9$

$f_2(s) = 0.1236 + 0.3248s + 0.2818s^2 + 0.1445s^3 + 0.0759s^4 + 0.0298s^5 + 0.0149s^6 + 0.0038s^7 + 0.0009s^8$

The overall pgf

$f_1(s) = 0.0733 + 0.2007s + 0.2571s^2 + 0.2008s^3 + 0.1329s^4 + 0.0732s^5 + 0.0369s^6 + 0.0159s^7 + 0.0056s^8 + 0.0028s^9 + 0.0009s^{10}$

$f_2(s) = 0.0951 + 0.2325s + 0.233s^2 + 0.1898s^3 + 0.1172s^4 + 0.0727s^5 + 0.0381s^6 + 0.0135s^7 + 0.0044s^8 + 0.0017s^9 + 0.0019s^{10}$

From the pgfs, the probability of zero male offspring was lower than the zero female offspring. In all the years, the probabilities of zero male children were less than the probability of zero female offspring. In the year 1993, the probability of zero male offspring was 0.026398 and 0.043478 was recorded for females .

It can also be seen that in the last year 2018, the probability of giving birth to two or three males offspring, were more than the probability of giving birth to two or three females. This implies that in Bangladesh it is more likely to give birth to males than females.

5.4.3 Offspring Distribution of Colombia with respect to Gender by Year of Survey

The probability generation function (pgf) of the offspring distributions of Colombia for each gender in 1990, 1995, 2000, 2005, 2010, 2015 were computed. They pgf are;

Year: 1990

$f_1(s) = 0.1453 + 0.1863s + 0.2188s^2 + 0.159s^3 + 0.1162s^4 + 0.0786s^5 + 0.0547s^6 + 0.0205s^7 + 0.0137s^8 + 0.0034s^9 + 0.0034s^{10}$

$f_2(s) = 0.1641 + 0.1897s + 0.1966s^2 + 0.1761s^3 + 0.1333s^4 + 0.0718s^5 + 0.0274s^6 + 0.0239s^7 + 0.0171s^8$

Year: 1995

$f_1(s) = 0.1733 + 0.2349s + 0.2474s^2 + 0.1441s^3 + 0.0919s^4 + 0.0511s^5 + 0.0292s^6 + 0.0167s^7 + 0.0084s^8 + 0.0021s^9 + 0.001s^{10}$

$f_2(s) = 0.19 + 0.2589s + 0.2328s^2 + 0.1555s^3 + 0.0793s^4 + 0.0418s^5 + 0.0251s^6 + 0.0115s^7 + 0.0031s^8 + 0.0021s^{10}$

Year: 2000

$f_1(s) = 0.1963 + 0.2567s + 0.2279s^2 + 0.1712s^3 + 0.0781s^4 + 0.0353s^5 + 0.0149s^6 + 0.013s^7 + 0.0047s^8 + 0.0019s^9$

$f_2(s) = 0.2112 + 0.2586s + 0.2447s^2 + 0.1526s^3 + 0.0679s^4 + 0.04s^5 + 0.0149s^6 + 0.0065s^7 + 0.0028s^8 + 0.0009s^9$

Year: 2005

$f_1(s) = 0.1989 + 0.2959s + 0.2535s^2 + 0.1446s^2 + 0.0602s^2 + 0.0274s^2 + 0.0102s^2 + 0.0063s^2 + 0.0012s^2 + 0.001s^2 + 0.001s^2$

$f_2(s) = 0.1957 + 0.3124s + 0.2479s^2 + 0.1276s^2 + 0.0635s^2 + 0.033s^2 + 0.0119s^2 +$

$0.0061s^2 + 0.0012s^2 + 0.0007s^2$

Year: 2010

$f_1(s) = 0.2081 + 0.3282s + 0.24s^2 + 0.1307s^2 + 0.0506s^2 + 0.0243s^2 + 0.0118s^2 + 0.0036s^2 + 0.0019s^2 + 0.0005s^2 + 0.0002s^2$

$f_2(s) = 0.2216 + 0.3321s + 0.2383s^2 + 0.1176s^2 + 0.0529s^2 + 0.022s^2 + 0.0113s^2 + 0.0029s^2 + 0.0009s^2 + 0.0005s^2$

Year: 2015

$f_1(s) = 0.2028 + 0.3418s + 0.2487s^2 + 0.1233s^2 + 0.0468s^2 + 0.0222s^2 + 0.009s^2 + 0.0035s^2 + 0.0012s^2 + 0.0007s^2$

$f_2(s) = 0.222{+}0.3418s{+}0.2452s^2 0.1105s^2 0.047s^2 0.0176s^2 0.0104s^2 0.0028s^2 0.0016s^2 0.0009s^2$

The overall pfg

$f_1(s) = 0.1874 + 0.274s + 0.2394s^2 + 0.1455s^2 + 0.074s^2 + 0.0398s^2 + 0.0216s^2 + 0.0106s^2 + 0.0052s^2 + 0.0016s^2 + 0.0009s^2$

$f_2(s) = 0.2008 + 0.2823s + 0.2342s^2 + 0.14s^3 + 0.074s^4 + 0.0377s^5 + 0.0168s^6 + 0.009s^7 + 0.0045s^2 + 0.0005s^2 + 0.0003s^2$

From the pgfs, all over the years, the probability of giving birth to zero female offspring was more than the probability of giving birth to male offspring. From the overall pgf the probability of giving birth to zero female offspring (0.2008) was higher than zero males (0.1874). This implies that it is more likely to have no female child in Columbia than a male child.

5.4.4 Offspring Distribution of Burkina Faso with respect to Gender by Year of Survey

The following probability generating function (pgf) represents the pgf of offspring of Burkina Faso for each gender in 1993, 1998, 2003, 2008, 2013 and 2018.

Year: 1993

$f_1(s) = 0.0409 + 0.079s + 0.1335s^2 + 0.1608s^3 + 0.218s^4 + 0.1635s^5 + 0.1063s^6 + 0.0545s^7 + 0.0272s^8 + 0.0136s^9 + 0.0027s^{10}$

$f_2(s) = 0.0545 + 0.0763s + 0.1008s^2 + 0.188s^3 + 0.2398s^4 + 0.1362s^5 + 0.1226s^6 + 0.0599s^7 + 0.0163s^8 + 0.0027s^9 + 0.0027s^{10}$

Year: 1998

$f_1(s) = 0.0523 + 0.0669s + 0.1402s^2 + 0.1611s^3 + 0.2259s^4 + 0.1527s^5 + 0.1213s^6 + 0.0418s^7 + 0.0251s^8 + 0.0063s^9 + 0.0063s^{10}$

$f_2(s) = 0.0356 + 0.0649s + 0.159s^2 + 0.2071s^3 + 0.2176s^4 + 0.1172s^5 + 0.1255s^6 + 0.0481s^7 + 0.0209s^8 + 0.0356s^9 + 0.0649s^{10}$

Year: 2003

$f_1(s) = 0.0344 + 0.0947s + 0.1216s^2 + 0.2043s^3 + 0.2061s^4 + 0.1532s^5 + 0.1123s^6 + 0.0436s^7 + 0.0186s^8 + 0.0074s^9 + 0.0037s^{10}$

$f_2(s) = 0.0381 + 0.1003s + 0.1746s^2 + 0.1903s^3 + 0.2015s^4 + 0.1569s^5 + 0.0817s^6 + 0.0334s^7 + 0.0149s^8 + 0.0074s^9 + 0.0009s^{10}$

Year: 2008

$f_1(s) = 0.0294 + 0.0875s + 0.184s^2 + 0.2217s^3 + 0.1916s^4 + 0.1463s^5 + 0.0769s^6 + 0.037s^7 + 0.0181s^8 + 0.0045s^9 + 0.003s^{10}$

$f_2(s) = 0.0415 + 0.0928s + 0.1659s^2 + 0.2353s^3 + 0.19s^4 + 0.1448s^5 + 0.0784s^6 + 0.0317s^7 + 0.0136s^8 + 0.0038s^9 + 0.0023s^{10}$

Year: 2013

$f_1(s) = 0.0246 + 0.0685s + 0.1634s^2 + 0.2408s^3 + 0.2091s^4 + 0.1336s^5 + 0.0791s^6 + 0.0351s^7 + 0.0299s^8 + 0.0105s^9 + 0.0053s^{10}$

$f_2(s) = 0.0475 + 0.1019s + 0.188s^2 + 0.1968s^3 + 0.1705s^4 + 0.1424s^5 + 0.0826s^6 + 0.0387s^7 + 0.0211s^8 + 0.0053s^9 + 0.0053s^{10}$

Year: 2018

$f_1(s) = 0.024 + 0.0795s + 0.2015s^2 + 0.2421s^3 + 0.1996s^4 + 0.1368s^5 + 0.0832s^6 + 0.024s^7 + 0.0074s^8 + 0.0018s^9$

$f_2(s) = 0.037 + 0.1109s + 0.2181s^2 + 0.2384s^3 + 0.1756s^4 + 0.1257s^5 + 0.0518s^6 + 0.0259s^7 + 0.0129s^8 + 0.0018s^9 + 0.0018s^{10}$

The overall pgf

$f_1(s) = 0.0343 + 0.0794s + 0.1574s^2 + 0.2051s^3 + 0.2084s^4 + 0.1477s^5 + 0.0965s^6 + 0.0394s^7 + 0.021s^8 + 0.0074s^9 + 0.0035s^{10}$

$f_2(s) = 0.0423 + 0.0912s + 0.1677s^2 + 0.2093s^3 + 0.1992s^4 + 0.1372s^5 + 0.0904s^6 + 0.0396s^7 + 0.0166s^8 + 0.0039s^9 + 0.0025s^{10}$

From the pgfs, for each year the probability of zero offspring of males were more than the females. Also, from the overall pgf, the probability of giving birth to zero females offspring (0.042338) was higher than the probability of giving birth to zero males (0.034262). This implies that in Burkina Faso it is more likely to have no female child than a male child.

5.4.5 Offspring Distribution of Philippines with respect to Gender by Year of Survey

The offspring distribution of Phillippines for each gender (male and female) are display for 1993, 1998, 2003, 2008, 2013 and 2018 as;

Year: 1993

$f_1(s) = 0.1364 + 0.1573s + 0.2045s^2 + 0.1809s^2 + 0.1285s^2 + 0.0909s^2 + 0.0446s^2 + 0.0367s^2 + 0.0131s^2 + 0.0035s^2 + 0.0035s^2$

$f_2(s) = 0.139 + 0.1888s + 0.2098s^2 + 0.1661s^2 + 0.1311s^2 + 0.0726s^2 + 0.0516s^2 + 0.0271s^2 + 0.0079s^2 + 0.0044s^2 + 0.0017s^2$

Year: 1998

$f_1(s) = 0.1506 + 0.1811s^2 + 0.1984s^2 + 0.1663s^2 + 0.1325s^2 + 0.0864s^2 + 0.0436s^2 + 0.023s^2 + 0.0148s^2 + 0.0016s^2 + 0.0016s^2$

$f_2(s) = 0.1613 + 0.1942s + 0.2173s^2 + 0.1712s^2 + 0.1185s^2 + 0.0675s^2 + 0.037s^2 + 0.0156s^2 + 0.0123s^2 + 0.0033s^2 + 0.0016s^2$

Year: 2003

$f_1(s) = 0.1236 + 0.2085s + 0.2353s^2 + 0.1713s^3 + 0.1273s^4 + 0.0752s^5 + 0.0276s^6 + 0.0186s^7 + 0.0074s^8 + 0.0037s^9 + 0.0015s^{10}$

$f_2(s) = 0.1549 + 0.2487s + 0.2226s^2 + 0.1705s^3 + 0.1028s^4 + 0.0529s^5 + 0.0261s^6 + 0.0104s^7 + 0.006s^8 + 0.0022s^9 + 0.003s^{10}$

Year: 2008

$f_1(s) = 0.1522 + 0.2311s + 0.2567s^2 + 0.1419s^3 + 0.092s^4 + 0.0644s^5 + 0.0332s^6 + 0.0173s^7 + 0.0062s^8 + 0.0042s^9 + 0.0007s^{10}$

$f_2(s) = 0.1744 + 0.2651s + 0.2298s^2 + 0.1419s^3 + 0.09s^4 + 0.0491s^5 + 0.0353s^6 + 0.0104s^7 + 0.0028s^8 + 0.0007s^9 + 0.0007s^{10}$

Year: 2013

$f_1(s) = 0.187 + 0.2387s + 0.231s^2 + 0.1397s^3 + 0.1001s^4 + 0.0578s^5 + 0.027s^6 + 0.0132s^7 + 0.0028s^8 + 0.0011s^9 + 0.0017s^{10}$

$f_2(s) = 0.1793 + 0.2745s + 0.2211s^2 + 0.1584s^3 + 0.0809s^4 + 0.0479s^5 + 0.0209s^6 + 0.0105s^7 + 0.005s^8 + 0.0006s^9 + 0.0011s^{10}$

Year: 2018

$f_1(s) = 0.1897 + 0.2541s + 0.2447s^2 + 0.15s^3 + 0.0877s^4 0.0397s^5 + 0.0184s^6 + 0.008s^7 + 0.0056s^8 + 0.001s^9 + 0.001s^{10}$

$f_2(s) = 0.2189 + 0.2833s + 0.2238s^2 + 0.1368s^3 + 0.0811s^4 + 0.0324s^5 + 0.0125s^6 + 0.0063s^7 + 0.0035s^8 + 0.001s^8 + 0.0003s^9$

The overall pgf

$f_1(s) = 0.1566 + 0.2118s + 0.2284s^2 + 0.1583s^3 + 0.1114s^4 + 0.0691s^5 + 0.0324s^5 + 0.0195s^6 + 0.0083s^7 + 0.0025s^8 + 0.0017s^9$

$f_2(s) = 0.1713 + 0.2424s + 0.2207s^2 + 0.1575s^3 + 0.1007s^4 + 0.0537s^5 + 0.0306s^6 + 0.0134s^7 + 0.0062s^8 + 0.002s^9 + 0.0014s^{10}$

In 1993, the probability of zero offspring from mle and femalee were 0.1613, 0.1506 respectively. In addition to that, it was found that over the years, the probability of zero male offspring was lower than the female zero offspring. This means that it is more likely to have no female than a male child.

5.4.6 Offspring Distribution of Mali with respect to Gender by Year of Survey

The probability generation function (pgf) of the offspring distributions of Mali for each gender are display for 1992, 1997, 2002, 2007, 2012 and 2017.

Year: 1992

$$f_1(s) = 0.0675 + 0.1139s + 0.1139s^2 + 0.173s^3 + 0.1983s^4 + 0.1435s^5 + 0.0759s^6 + 0.0675s^7 + 0.0169s^8 + 0.0169s^9 + 0.0127s^{10}$$

$$f_2(s) = 0.0844 + 0.1139s + 0.1857s^2 + 0.1308s^3 + 0.1688s^4 + 0.1435s^5 + 0.0759s^6 + 0.0633s^7 + 0.0295s^8 + 0.0042s^9$$

Year: 1997

$$f_1(s) = 0.0796 + 0.0729s + 0.1215s^2 + 0.1714s^3 + 0.1592s^4 + 0.1592s^5 + 0.1161s^6 + 0.0702s^7 + 0.0351s^8 + 0.0121s^9 + 0.0027s^{10}$$

$$f_2(s) = 0.0675 + 0.1066s + 0.1363s^2 + 0.1714s^3 + 0.1768s^4 + 0.1484s^5 + 0.0904s^6 + 0.0607s^7 + 0.0337s^8 + 0.0067s^9 + 0.0014s^{10}$$

Year: 2002

$$f_1(s) = 0.0615 + 0.0802s + 0.1208s^2 + 0.1625s^3 + 0.1781s^4 + 0.1448s^5 + 0.1323s^6 + 0.0646s^7 + 0.0333s^8 + 0.0125s^9 + 0.0094s^{10}$$

$$f_2(s) = 0.0583 + 0.0969s + 0.1385s^2 + 0.1708s^3 + 0.1813s^4 + 0.1469s^5 + 0.1073s^6 + 0.0625s^7 + 0.0292s^8 + 0.0052s^9 + 0.0031s^{10}$$

Year: 2007

$$f_1(s) = 0.0509 + 0.1017s + 0.1422s^2 + 0.1457s^3 + 0.206s^4 + 0.1621s^5 + 0.0862s^6 + 0.0681s^7 + 0.0241s^8 + 0.0095s^9 + 0.0034s^{10}$$

$$f_2(s) = 0.0716 + 0.0974s + 0.1569s^2 + 0.1793s^3 + 0.1879s^4 + 0.1388s^5 + 0.0914s^6 + 0.0422s^7 + 0.0241s^8 + 0.0078s^9 + 0.0026s^{10}$$

Year: 2012

$$f_1(s) = 0.068 + 0.1269s + 0.1798s^2 + 0.2145s^3 + 0.2039s^4 + 0.1027s^5 + 0.0619s^6 + 0.0242s^7 + 0.0106s^8 + 0.006s^9 + 0.0015s^{10}$$

$$f_2(s) = 0.0906 + 0.1722s + 0.21s^2 + 0.1798s^3 + 0.1435s^4 + 0.1073s^5 + 0.065s^6 +$$

$0.0181s^7 + 0.0091s^8 + 0.0015s^9 + 0.003s^{10}$

Year: 2017

$f_1(s) = 0.0823 + 0.1535s + 0.1725s^2 + 0.1646s^3 + 0.1867s^4 + 0.1203s^5 + 0.0506s^6 + 0.0396s^7 + 0.0158s^8 + 0.0127s^9 + 0.0016s^{10}$

$f_2(s) = 0.0981 + 0.1313s + 0.1725s^2 + 0.2057s^3 + 0.1614s^4 + 0.0949s^5 + 0.0775s^6 + 0.0348s^7 + 0.0174s^8 + 0.0016s^9 + 0.0048s^{10}$

The overall pgf

$f_1(s) = 0.0683 + 0.1082s + 0.1418s^2 + 0.1719s^3 + 0.1887s^4 + 0.1388s^5 + 0.0872s^6 + 0.0557s^7 + 0.0226s^8 + 0.0116s^9 + 0.0052s^{10}$

$f_2(s) = 0.0784 + 0.1197s + 0.1666s^2 + 0.173s^3 + 0.1699s^4 + 0.13s^5 + 0.0846s^6 + 0.047s^7 + 0.0238s^8 + 0.0045s^9 + 0.0025s^{10}$

On average, the probability of having no female offspring is higher than having no male offspring. For each year similar result was recorded where the female zero offspring were higher than the male probability of zero offspring. For example, in the year 2017, the probability of zero offspring for males (0.0823) was lower than the probability of zero probability for female (0.0981) offspring. However, in 2018, the probability of having three or four female offspring was higher than the probability of having three or four males. This means that even though it is more likely to have no female children than males, it is also likely to have more female children than males.

5.4.7 Offspring Distribution of Indonesia with respect to Gender by Year of Survey

The pgf distribution of offspring for Indonesia, for each gender and were computed for 1993, 1998, 2003, 2008, 2013 and 2018.

Year: 1993

$$f_1(s) = 0.0969 + 0.1819s + 0.2085s^2 + 0.1941s^3 + 0.1484s^4 + 0.0886s^5 + 0.049s^6 + 0.0209s^7 + 0.0072s^8 + 0.0032s^9 + 0.0014s^{10}$$

$$f_2(s) = 0.1221 + 0.1894s + 0.2042s^2 + 0.193s^3 + 0.139s^4 + 0.0835s^5 + 0.0468s^6 + 0.0151s^7 + 0.005s^8 + 0.0014s^9 + 0.0004s^{10}$$

Year: 1998

$$f_1(s) = 0.1003 + 0.2027s + 0.2366s^2 + 0.1784s^3 + 0.1353s^4 + 0.0752s^5 + 0.044s^6 + 0.0191s^7 + 0.0061s^8 + 0.0012s^9 + 0.0012s^{10}$$

$$f_2(s) = 0.1131 + 0.2183s + 0.2412s^2 + 0.1955s^3 + 0.1194s^4 + 0.0645s^5 + 0.0275s^6 + 0.0127s^7 + 0.0064s^8 + 0.0014s^9$$

Year: 2003

$$f_1(s) = 0.1131 + 0.2269s + 0.2608s^2 + 0.1905s^3 + 0.1128s^4 + 0.0527s^5 + 0.0252s^6 + 0.0142s^7 + 0.0032s^8 + 0.0007s^9$$

$$f_2(s) = 0.1324 + 0.2647s + 0.2571s^2 + 0.1746s^3 + 0.0938s^4 + 0.0447s^5 + 0.0217s^6 + 0.0073s^7 + 0.0027s^8 + 0.0005s^8 + 0.0005s^9$$

Year: 2008

$$f_1(s) = 0.1287 + 0.2592s + 0.2619s^2 + 0.1718s^3 + 0.0974s^4 + 0.0454s^5 + 0.0229s^6 + 0.0093s^7 + 0.0023s^8 + 0.0007s^9 + 0.0005s^{10}$$

$$f_2(s) = 0.1562 + 0.2842s + 0.2578s^2 + 0.1557s^3 + 0.0797s^4 + 0.0384s^5 + 0.0166s^6 + 0.0075s^7 + 0.0027s^8 + 0.0009s^9 + 0.0005s^{10}$$

Year: 2013

$$f_1(s) = 0.1612 + 0.2913s + 0.2706s^2 + 0.1516s^3 + 0.0732s^4 + 0.0308s^5 + 0.0139s^6 + 0.0044s^7 + 0.0022s^8 + 0.0004s^9 + 0.0002s^{10}$$

$$f_2(s) = 0.1857 + 0.3166s + 0.2565s^2 + 0.1351s^3 + 0.0615s^4 + 0.0295s^5 + 0.009s^6 + 0.0038s^7 + 0.0022s^8 + 0.0002s^9$$

Year: 2018

$f_1(s) = 0.1994 + 0.3242s + 0.2621s^2 + 0.1269s^2 + 0.0543s^2 + 0.0227s^2 + 0.007s^2 + 0.0024s^2 + 0.0006s^2 + 0.0003s^2 + 0.0002s^2$

$f_2(s) = 0.2147 + 0.3413s + 0.2587s^2 + 0.1133s^2 + 0.048s^2 + 0.016s^2 + 0.0056s^2 + 0.0021s^2 + 0.0002s^2 + 0.0002s^2$

The overall

$f_1(s) = 0.1333 + 0.2477s + 0.2501s^2 + 0.1689s^2 + 0.1036s^2 + 0.0526s^2 + 0.027s^2 + 0.0117s^2 + 0.0036s^2 + 0.0011s^2 + 0.0006s^2$

$f_2(s) = 0.154 + 0.2691s + 0.2459s^2 + 0.1612s^2 + 0.0902s^2 + 0.0461s^2 + 0.0212s^2 + 0.0081s^2 + 0.0032s^2 + 0.0008s^2 + 0.0002s^2$

It can be seen that in all the years, the probability of zero male offspring was less than the probability of giving birth to no female. The study revealed that at some specific number of children, there is a high probability of giving birth to more male children than females and vice versa. For instance, in 1993, the probability of birth to two males was higher than that of two females. However, in the same year, giving birth to three females was higher than the probability of giving birth to three males.

5.4.8 Estimation of Long-run Probability of Zero Offsprings and Expected Number of Offspring per Gender

The long-run probability of zero offspring of the selected countries was computed in a random environment for each gender. The first row of the vectors represent results for males and the second row represent females.

Ghana $\mu = \begin{pmatrix} 2.9968 \\ 2.8116 \end{pmatrix}$ $\mathbf{q} = \begin{pmatrix} 0.0964 \\ 0.1009 \end{pmatrix}$

Bangladesh $\mu = \begin{pmatrix} 2.6257 \\ 2.4935 \end{pmatrix}$ $\mathbf{q} = \begin{pmatrix} 0.1151 \\ 0.1584 \end{pmatrix}$

Philippines $\mu = \begin{pmatrix} 2.3712 \\ 2.1871 \end{pmatrix}$ $\mathbf{q} = \begin{pmatrix} 0.2337 \\ 0.2690 \end{pmatrix}$

Burkina Faso $\mu = \begin{pmatrix} 3.7058 \\ 3.5501 \end{pmatrix}$ $\mathbf{q} = \begin{pmatrix} 0.0338 \\ 0.0472 \end{pmatrix}$

Colombia $\mu = \begin{pmatrix} 1.9533 \\ 1.8624 \end{pmatrix}$ $\mathbf{q} = \begin{pmatrix} 0.3182 \\ 0.3475 \end{pmatrix}$

Mali $\mu = \begin{pmatrix} 3.6069 \\ 3.3936 \end{pmatrix}$ $\mathbf{q} = \begin{pmatrix} 0.0790 \\ 0.0964 \end{pmatrix}$

Indonesia $\mu = \begin{pmatrix} 2.2197 \\ 2.0543 \end{pmatrix}$ $\mathbf{q} = \begin{pmatrix} 0.2211 \\ 0.2680 \end{pmatrix}$

From the expected unmber of offspring and ultimate extinctions, it can be seen that in Ghana, the long-run probability of zero male offspring is less than the long-run probability of zero offspring for females. Similar results can be seen in the rest of the countries where the probability of ultimate extinction of males was less than the extinction of females. This implies that the females are ageing as compare to the males. The study also revealed that, the expected number of offspring for males were more than that of females in all the countries. This is also in line with the findings of Pillay and Maharaj (2013) and Warner and Corley (2017), that females constitute the highest percentage in the world aged population. According to Sen (2017), more males are born than females, but since males have low life expectancy than females, females are found more in the aged group than the males.

5.4.9 Assessing the Significant Difference of Probabilities of Zero Offspring between Gender

A paired t-test was later used to assess if the generational probabilities of zero offspring were significantly different with respect to gender. The results are displayed in Table 5.14.

Table 5.14: Assessing the Significant Difference of Probabilities of Zero Offspring between Gender

Country	t-test	mean diff.	p-value
Ghana	-545.123*	-0.0042	0.000
Bangladesh	-1622.83*	-0.0431	0.000
Philippines	-1237.29*	-0.035 3	0.000
Burkina Faso	-3313.8*	-0.0135	0.000
Colombia	-1057.49*	-0.0293	0.000
Mali	-1221.31*	-0.0462	0.000
Indonesia	-1221.31*	-0.0462	0.000

Source: Author's computation, (2021).

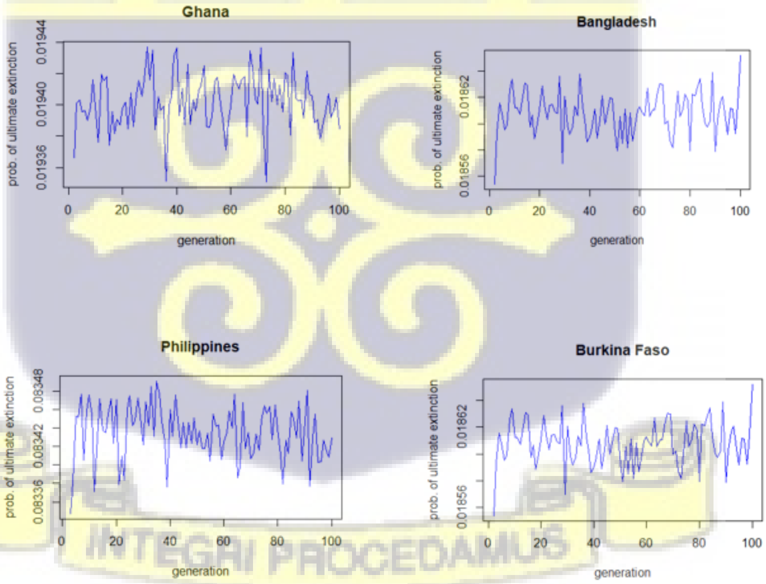
From Table 5.14, the t-test statistics were negative, which means that the probability of ultimate extinction of males was less than the females for all the countries. There was a significant difference between male and female long-run probabilities of zero offspring in all the countries. In all the counties, it was clear that Ghana recorded the lowest mean difference (close to zero), followed by Burkina Faso. Apart from Ghana, all the countries have mean difference greater than 0.01 in absolute terms.

5.5 Estimation of Expected Revision Time for Population Policy

This section presents the results and discussions of the branching process in a random environment using spectral analyses. These include the plot of generational probabilities of zero offspring for each selected country, unit root test, plot of the periodogram, Harmonic regression and wavelength analyses.

5.5.1 Plot of Generational Probability of Zero Offspring of Selected Countries

Figure 5.3 represents the plots of the points $(n, \hat{q}_n)$, of the estimates of $q_n (n = 1, 2, 3, \ldots, 100)$ based on the proposed method in equation (4.36) for each country. The plots indicate that in all the countries, the probability of zero offspring looks stationary.



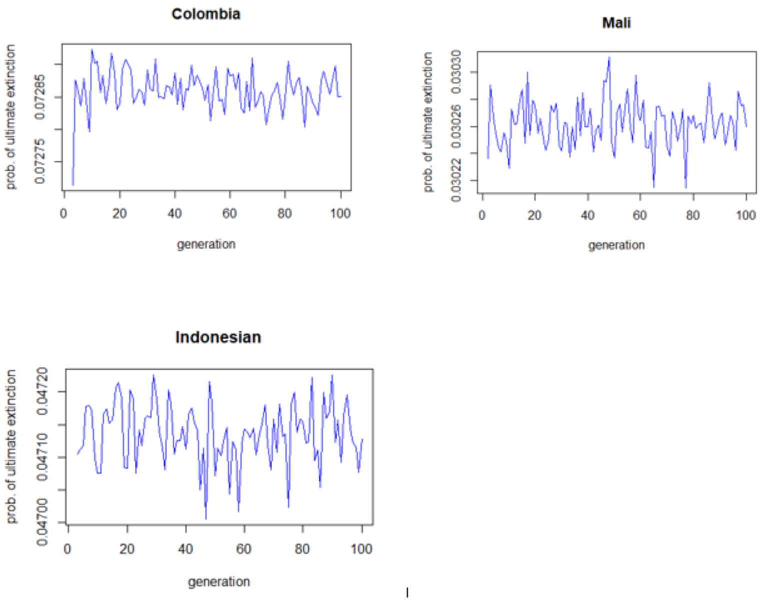


Figure 5.3: Plot of Probability of Zero Offspring of Selected Countries

5.5.2 Unit Root Test

One of the assumptions for time series analyses is that the data should be stationary (no unit root). This is because non-stationary data are unpredictable and cannot be forecasted (Shi et al., 2021). The study used Augmented Dickey-Fuller (ADF) and Phillip Perron unit root tests to test for the unit root or stationarity of the series. The results are presented in Table 5.15.

Table 5.15: Unit Root Test

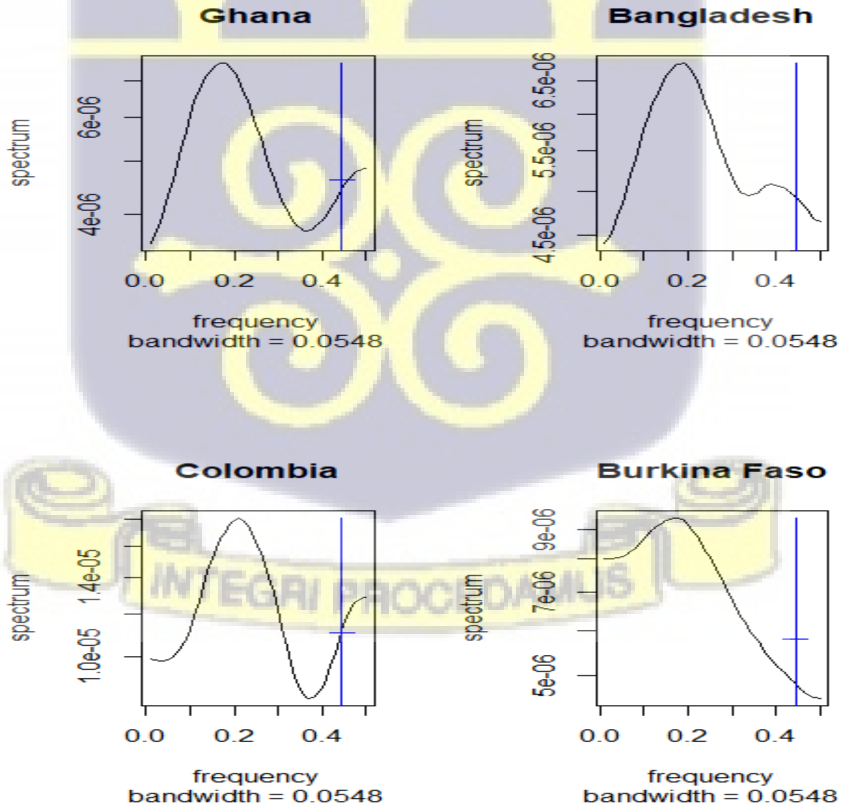
	ADF		Phillip Perron	
Country	Test statistic	p-value	Test statistic	p-value
Ghana	-37.9896	0.0000	-37.4527	0.0000
Bangladesh	-47.8552	0.0000	-46.9447	0.0000
Philippines	-222.2324	0.0000	-233.5383	0.0000
Burkina Faso	-9.4714	0.0001	-18.3632	0.0001
Colombia	-268.3727	0.0000	-313.5747	0.0000
Indonesia	-74.7584	0.0000	-81.0878	0.0000
Mali	-67.3049	0.0000	-67.3046	0.0000

Source: Author's computation, (2021).

Results generated by the Augmented Dickey-Fuller (ADF) and Phillip Perron unit root test show that the probabilities of zero offspring were stationary in all the selected countries (p-value < 0.05). This implies that there is no need for any transformation (differencing) before the spectral analysis is carried out.

5.5.3 Identifying Appropriate Set of Frequencies Using Periodogram

The most difficult task in spectral analyses is to determine the appropriate set of frequencies to fit in the harmonic regression model. By inspecting the periodogram, one can determine those frequencies that should be represented in the harmonic regression model (Delisle, Hara, & Ségransan, 2020). Figure 5.4, presents the periodograms of the probability of zero offspring for each of the selected countries to select the appropriate set of frequencies to calculate for wavelength and harmonic regression.



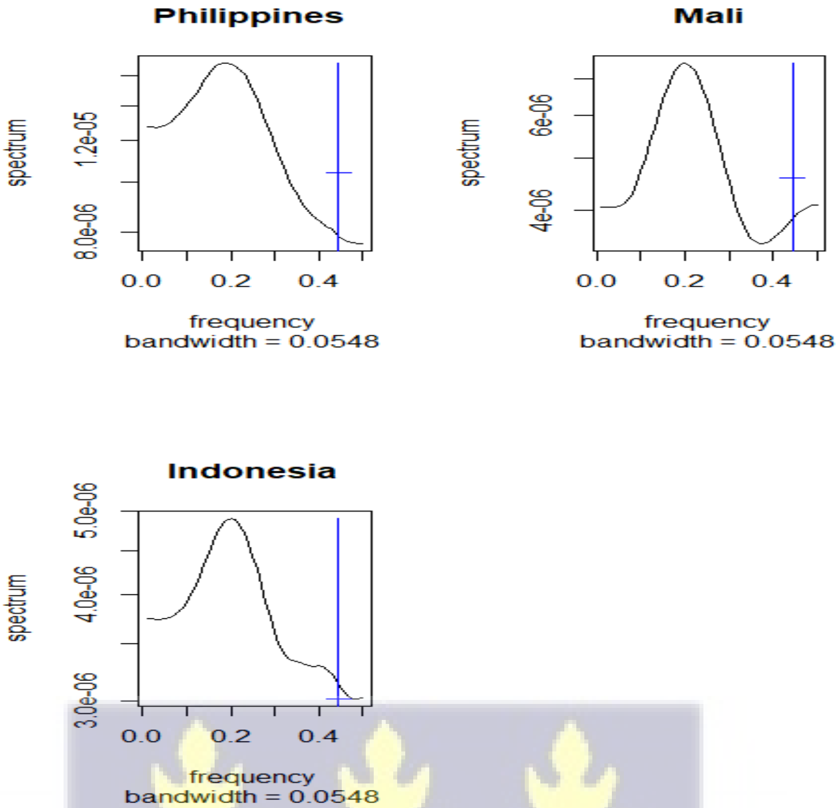


Figure 5.4: Periodogram of Selected Countries

From Figure 5.4, the bandwidth for measuring the width of the frequency interval(s) used for smoothing the periodogram is 0.0548. In all the countries the periodogram, spike within 0.15 to 0.25. The significant frequency identified for Burkina Faso, Mali, Ghana, Colombia, Bangladesh, the Philippines and Indonesia were 0.19961, 0.20845, 0.20953, 0.22946, 0.22655, 0.22860 and 0.21865, respectively.

5.5.4 Estimation of Period, Wavelength, Proposed Policy Revision Time

After obtaining the frequency from the periodogram, the period, wavelength and proposed population policy revision time were estimated using equations (4.61) to (4.64). Since the time intervals were five years, the results obtained were multiplied by five and are presented in Table 5.16. The spectral plots of the various countries are shown in Appendix E.

Table 5.16: Estimation of Period, Wavelength, Proposed time

Country	Frequency	Period	Wavelength	Proposed time
Burkina Faso	0.19961	25.04869	157.38559	39.34640
Mali	0.20845	23.98620	150.70974	37.67743
Ghana	0.20953	23.86248	149.93236	37.48309
Colombia	0.22946	21.79058	136.91422	34.22856
Bangladesh	0.22655	22.07038	138.67227	34.66807
Philippines	0.22860	21.87255	137.42930	34.35733
Indonesia	0.21865	22.86781	143.68266	35.92067

Source: Author's computation, (2021).

The wavelengths of the countries range from 136 to 158 years. Burkina Faso had a maximum length of 157.38559 years with Colombia having the lowest wavelength of 136.91422 years. This implies that it will be much longer for Burkina Faso to have a complete population cycle than Colombia and the other countries. The proposed times estimated based on equation (4.64) for the countries were close and range from 33 to 40 years. Again Burkina Faso recorded the highest wavelength of 39.34640 years. This result is in line with the one-child policy of China. According to Zeng and Hesketh (2016) the one-child policy was gradually resulting in a threat to the country and hence there was the need to revise the policy within 35 years.

5.5.5 Harmonic Regression for Selected Countries

Table 5.17 presents the results of the harmonic regression for each country. Based on equations (4.62) and (4.55), the values of amplitude and phase were also estimated and presented in Table 5.17. The amplitude measures the height of the wave, and the phase measure how far the function is shifted horizontally from the usual position.

Table 5.17: Results of Harmonic Regression by Selected Country

Country		coeff.	st. error	t-value	p-value	Amplitude (R)	Phase(d)
Ghana	Cons.	0.02038	0.00021	95.475	0.0000	0.0011	-2.2902
	A	-0.00073	0.00030	-2.402	0.0182		
	B	0.00083	0.00030	2.804	0.0061		
Bangladesh	Cons.	0.01753	0.00023	77.247	0.0000	0.0013	2.1851
	A	-0.00073	0.00032	-2.308	0.0231		
	B	0.00104	0.00032	3.202	0.0019		
Philippines	Cons.	0.08197	0.00038	214.218	0.0000	0.0015	2.2292
	A	-0.00094	0.00054	-1.753	0.0827		
	B	-0.00121	0.00054	-2.231	0.0280		
Burkina Faso	Cons.	0.01145	0.00022	51.532	0.0000	0.0012	-0.7769
	A	0.00084	0.00031	2.698	0.0082		
	B	0.00083	0.00032	2.623	0.0101		
Colombia	Cons.	0.07478	0.00039	191.57	0.0000	0.0018	0.6924
	A	0.00140	0.00055	2.557	0.0121		
	B	-0.00116	0.00056	-2.087	0.0395		
Mali	Cons.	0.03122	0.00023	134.384	0.0000	0.0012	2.3298
	A	-0.00080	0.00033	-2.464	0.0155		
	B	-0.00085	0.00033	-2.569	0.0117		

Table 5.17 – *Continued from previous page*

Country		coeff.	st. error	t-value	p-value	Amplitude (R)	Phase(d)
Indonesia	Cons.	0.18978	0.00214	88.529	0.0000	0.0095	2.4519
	A	-0.00733	0.00302	-2.431	0.0169		
	B	-0.00605	0.00302	-2.001	0.0482		

Source: Author's computation, (2021)

The amplitude values were small and were between 0.0011 to 0.0095. Indonesia recorded the highest amplitude, meaning that Indonesia's wave depth to the equilibrium was high among the countries. This implies that the generational probabilities of zero offspring in Indonesia shoot higher from the equilibrium than in other countries. Also, some countries recorded a positive phase and others a negative phase, which implies that for some countries, the wave function shifted to the left from its normal position and others shifted to the right from its normal position. And Colombia recorded the lowest phase value of 0.6924; hence the wave function of Mali moved horizontally to the left by 0.6924.

Chapter 6

SUMMARY, CONCLUSION AND RECOMMENDATIONS

6.1 Introduction

This chapter presents the summary, contribution to literature, discussion of findings, conclusions, recommendations and areas for further research in estimating the long-run probability of zero offspring in varying environments.

6.2 Summary

This thesis is premised on the proposition that population dynamics (young and old) can be monitored regularly from the offspring distribution of a population using branching processes rather than relying on a complete census, which is costly. The study aims to develop a method to monitor population dynamics using a branching process in varying environments. The specific objectives are to determine the long-run probability of zero offspring for a population subject to an oscillating and varying environment where the offspring distribution goes to a limit; to determine the condition under which estimations of the long-run probability of zero offspring in a random environment behaves like that of a constant environment; to analyse the probability of zero offspring in multi-type branching processes concerning gender; and to determine the expected time to revise population dynamics policies based on parity data from selected countries. Literature on population dynamics as well as branching processes and their applications were reviewed. From Mettle et al. (2019), based on parity of women in the terminal reproductive age, the dynamics of a population can be monitored. This implies that the distribution of population offspring can be used to monitor its dynamics.

Hypothetical population data were simulated from Poisson distribution, and seven empirical data of women in their terminal age reproductive were used. The countries include Ghana, Burkina Faso, Bangladesh, Philippines, Mali, Indonesia and Colombia from 1990 to 2019 with five years intervals. The study stated and proved a theorem for analysing the branching processes in varying environments. The study also proposed an estimated period for population policy to be revised using spectral analysis.

6.3 Contribution to Literature

The applications of branching processes are mostly made in the constant environment assumption. This is because getting information about the future is problematic (Mettle et al., 2019; Abrams, 2015; Beznea et al., 2020). This assumption is not true in real-life situations since environmental circumstances change (Albano & Giorno, 2020). This study develops a method to analyse the branching process in varying environments. The proposed method was applied to both hypothetical and empirical populations. This study contributes to the literature by proposing a method for applying branching processes in varying environments.

This study also contributes to the literature by proposing a methodology to estimate population policy revisions time using spectral analysis in branching processes.

6.4 Discussion of Findings

The study found that when the distributions of offspring oscillate, the long-run probabilities of zero offspring also oscillate after a specific number of generations. In addition, the fewer the number of oscillating distributions the population is subject to, the quicker the limiting probabilities start to oscillate. This study is in line with Vandermeer (2006) that when the population oscillates in terms of sizes, the probability of ultimate extinction also oscillates.

The study also found that when the distribution of offspring approaches a limit, the long-run probability of offspring also approaches a finite value similar to the long-run probability obtained from the limiting offspring distribution. This result is confirmed by d'Souza and Biggins (1992), that in supercritical monotype branching processes in which the expectation of the offspring distribution converges almost surely to a finite random variable Z, the expectations is the same as the expectation obtained from finite random variable Z.

The study proposed a closeness of index to ascertain the conditions for which estimations of the long-run probability of zero offspring in a random environment behave like that of a constant environment. The study found that when the offspring distributions from generation to generation are very close, the probability of ultimate extinction results obtained in random and constant environments are close.

From the empirical analyses, This study found out that Burkina Faso exhibited the youngest population whilst the Philippines exhibits an ageing population. According to World development indicators (2020) , the percentage of aged people in the total population of Burkina Faso (2.41%) is lower than that of the Philippines (5.51%). The result also confirmed with Mettle et al. (2019), who used the probability of ultimate extinction in a constant environment, that Burkina Faso has a young population. From the multi-type analysis, the male population is more younger than the female population. The results were in line with Li, Zhou and Jia (2019) that females live longer than males in many species, including humans, due to differences in chromosomes and hormones between men and women. From GSS (2010), the percentage of females aged 65 years and above to the total population was more (5.3%) than the percentage recorded for males(4.1%).

From the spectral analyses, the proposed period for policy revision was between 34 to 40 years. This implies that when a country presents a population policy, the duration should range from 34 to 40 years. Beyond 40 years, the government will experience a server impact on the policy before it is revised. The average population policy revision time for West African countries is 37 years and 34 years for Asian countries. These results confirm with Gietel-Basten, Han, and Cheng (2019), who noticed that the impact of the China intervention was similar in the province that shares common social and economic conditions.

6.5 Conclusion

Based on the findings, the following conclusions were made;

- The long-run probability of zero offspring from oscillating offspring distribution also oscillates. The oscillation starts after some generations. The fewer the number of oscillating offspring distributions, the quicker the limiting probability starts to oscillate.

  The long-run probability of zero offspring from the distribution of offspring that approaches a limit also approaches a finite value. This finite value is the same as the long-run probability of zero offspring obtained from the limiting offspring distribution.

- Under a certain closeness of the offspring distribution for each generation, $|1 - I_1| < 0.1$, the estimations of the long-run probability of zero offspring in a random environment behaves like that of a constant environment. However, if there is an increasing or decreasing pattern in the offspring distribution from generation to generation, only the long-run probability of zero offspring in a uniform random environment behaves like that of a constant environment. Under such conditions, the type of randomisation (uniform or weighted) influenced the limiting probability.

- Using the long-run probability of zero offspring as a proxy, based on the empirical data in all the selected countries, the male population are younger than the female population.
- The expected time to revise population dynamics policies based on parity data from selected countries ranges 34 to 40 years. Any effective population policy which is not revised within this period will lead to consequences for the nation.

6.6 Recommendation

Based on the conclusion and findings, the following recommendations were made;

- The application of the branching processes must be carried out in a random environment based on the proposed method. Under certain closeness conditions, as defined earlier, applying the branching process in a constant environment is a step in the right direction. However, if there is an increasing or decreasing trend in offspring distribution; the weighted random environment should be prefered (i.e., preferences should be given to most current offspring distributions).
- Parity data from women in their terminal age can be used to monitor population dynamics regularly to inform the government for effective planning. The government should also put measures to ensure that the economic growth is in line with the population dynamics. This will ensure that the increasing demand for services arising from the young or ageing population is met. For example, Ghana and Burkina Faso exhibit a young

population. Hence government should work towards creating more jobs and a well-educated workforce. Flexible policies should be given to investors to encourage them to invest more in the country. In addition to that well-educated labour force will provide investors with the confidence to invest and create more jobs for the citizens.

- Effective population policies should be guided by the appropriate time frame so that the policy will serve its purpose rather than creating problems in future. A well-managed population policy will ensure that both the population and the country's economic status complement each other and call for adjustment when the need arises.
- From the empirical results, all the countries were gradually ageing as their fertility rates decreased. Therefore, population ageing awareness should be intensified by the World Health Organisation (WHO).
  In addition to that, health systems need to be well organised around older people's needs and preferences to enhance their aged intrinsic capacity. Moreover, an aged friendly environment should be created across the globe. Society can give special assistance and extra care for the elderly poor. The government can also provide homes with recreational facilities for elderly people.

6.7 Areas for Future Research

Areas for further studies include developing theorems involving generational probabilities of parities other than zero. This will help estimate the long-run probability of giving birth to a specific number of offspring other than zero. Another assumption of the branching processes is that offspring gives birth to its kind independently; applying this assumption does not work in most cases. Hence methods should be provided in applying branching processes when the production of offspring is dependent on each other.

References

Abdul-Salam, S., Baba, S. S., & Jabir, H. (2018). The impact of mother's education on fertility in ghana. *Int J Probab Stat*, *7*(2), 31–43.

Abrams, P. (2015). The bad mother: stigma, abortion and surrogacy. *Journal of Law, Medicine & Ethics*, *43*(2), 179–191.

Acemoglu, D., & Restrepo, P. (2017). Secular stagnation? the effect of aging on economic growth in the age of automation. *American Economic Review*, *107*(5), 174–79.

Adami, C., & Chu, J. (2002). Critical and near-critical branching processes. *Physical Review E*, *66*(1), 011907.

Addario-Berry, L., & Reed, B. (2009). Minima in branching random walks. *The Annals of Probability*, *37*(3), 1044–1079.

Akanle, O., & Nwaobiala, U. R. (2020). Changing but fragile: Female breadwinning and family stability in nigeria. *Journal of Asian and African Studies*, *55*(3), 398–411.

Albano, G., & Giorno, V. (2020). Inferring time non-homogeneous ornstein uhlenbeck type stochastic process. *Computational Statistics & Data Analysis*, *150*, 107008.

Albertsen, K. (1995). The extinction of families. *International Statistical Review/Revue Internationale de Statistique*, 234–239.

Alemayehu, M., Lemma, H., Abrha, K., Adama, Y., Fisseha, G., Yebyo, H., et al. (2016). Family planning use and associated factors among pastoralist community of afar region, eastern ethiopia. *BMC women's health*, *16*(1), 1–9.

Alferink, L. J., Jong, J. C. Kiefte-de, Erler, N. S., Veldt, B. J., Schoufour, J. D., De Knegt, R. J., et al. (2019). Association of dietary macronutrient composition and non-alcoholic fatty liver disease in an ageing population:

the rotterdam study. *Gut*, *68*(6), 1088–1098.

Aliyu, A. A. (2018). Family planning services in africa: The successes and challenges. *Family Planning*, *69*.

Allen, L. J. (2015). Stochastic population and epidemic models. *Mathematical biosciences lecture series, stochastics in biological systems*.

Andrade-Ines, E., Beaugé, C., Michtchenko, T., & Robutel, P. (2016). Secular dynamics of s-type planetary orbits in binary star systems: applicability domains of first-and second-order theories. *Celestial Mechanics and Dynamical Astronomy*, *124*(4), 405–432.

Asumadu-Sarkodie, S., & Owusu, P. A. (2016). The casual nexus between child mortality rate, fertility rate, gdp, household final consumption expenditure, and food production index. *Cogent Economics & Finance*, *4*(1), 1191985.

Baake, E., Cordero, F., & Hummel, S. (2018). A probabilistic view on the deterministic mutation–selection equation: dynamics, equilibria, and ancestry via individual lines of descent. *Journal of mathematical biology*, *77*(3), 795–820.

Babiarz, K. S., Ma, P., Miller, G., Song, S., et al. (2018). *The limits (and human costs) of population policy: Fertility decline and sex selection in china under mao*. National Bureau of Economic Research.

Bailey, N. T., et al. (1975). *The mathematical theory of infectious diseases and its applications* (No. 2nd ediition). Charles Griffin & Company Ltd 5a Crendon Street, High Wycombe, Bucks HP13 6LE.

Baldwin, R., & Teulings, C. (2014). Secular stagnation: facts, causes and cures. *London: Centre for Economic Policy Research-CEPR*.

Ball, F., Mollison, D., & Scalia-Tomba, G. (1997). Epidemics with two levels of mixing. *The Annals of Applied Probability*, 46–89.

Bank, W. (2020). *World development indicators 2020*. The World Bank.

Bansaye, V., Millan, J. C. P., & Smadi, C. (2013). On the extinction of continuous state branching processes with catastrophes. *Electronic Journal* *of Probability*, *18*, 1–31.

Batyra, E. (2016). Fertility and the changing pattern of the timing of childbearing in colombia. *Demographic Research*, *35*, 1343–1372.

Beard, J. R., Officer, A., De Carvalho, I. A., Sadana, R., Pot, A. M., Michel, J.-P., et al. (2016). The world report on ageing and health: a policy framework for healthy ageing. *The lancet*, *387*(10033), 2145–2154.

Béjot, Y., & Yaffe, K. (2019). Ageing population: A neurological challenge. *Neuroepidemiology*, *52*(1-2), 76–78.

Bellenbaum, R. (1995). Reinsurance of environmental risk pricing and risk assessment. *The Geneva Papers on Risk and Insurance-Issues and Practice*, *20*(3), 393–401.

Bertoin, J. (2010). A limit theorem for trees of alleles in branching processes with rare neutral mutations. *Stochastic Processes and their Applications*, *120*(5), 678–697.

Beznea, L., Deaconu, M., & Lupaşcu, O. (2015). Branching processes for the fragmentation equation. *Stochastic Processes and their Applications*, *125*(5), 1861–1885.

Beznea, L., Lupaşcu-Stamate, O., & Vrabie, C. I. (2020). Stochastic solutions to evolution equations of non-local branching processes. *Nonlinear Analysis*, *200*, 112021.

Bhagat, R. (2018). Development impacts of migration and urbanisation. *Economic and political weekly*, *53*(48), 15–19.

Bhandari, H., & Mishra, A. K. (2018). Impact of demographic transformation on future rice farming in asia. *Outlook on Agriculture*, *47*(2), 125–132.

Bhat, U. N., & Miller, G. K. (2002). *Elements of applied stochastic processes* (Vol. 3). Wiley-Interscience Hoboken^ eN. JNJ.

Bienaymé, I.-J. (1845). De la loi de multiplication et de la durée des familles. *Soc. Philomat. Paris Extraits, Sér*, *5*(37-39), 4.

Blanc, A. K., & Grey, S. (2000). *Greater than expected fertility decline in ghana:*

An examination of the evidence. Macro International Incorporated.

Bloom, D. E. (2016). Demographic upheaval: the world will struggle with population growth, aging, migration, and urbanization. *Finance & Development*, *53*(001).

Bloom, D. E., Canning, D., & Fink, G. (2010). Implications of population ageing for economic growth. *Oxford review of economic policy*, *26*(4), 583–612.

Bloom, D. E., Canning, D., Kotschy, R., Prettner, K., & Schünemann, J. J. (2019). *Health and economic growth: reconciling the micro and macro evidence* (Tech. Rep.). National Bureau of Economic Research.

Bloom, D. E., & Luca, D. L. (2016). The global demography of aging: facts, explanations, future. In *Handbook of the economics of population aging* (Vol. 1, pp. 3–56). Elsevier.

Bongaarts, J. (2016). Development: Slow down population growth. *Nature News*, *530*(7591), 409.

Bradshaw, J., & Finch, N. (2002). A comparison of child benefit packages in 22 countries.

Burden, C. J., & Soewongsono, A. C. (2019). Coalescence in the diffusion limit of a bienaymé–galton–watson branching process. *Theoretical population biology*, *130*, 50–59.

Burden, C. J., & Wei, Y. (2018). Mutation in populations governed by a galton–watson branching process. *Theoretical population biology*, *120*, 52–61.

Chaloupka, M., & Musick, J. A. (2017). Age, growth, and population dynamics. *The biology of sea turtles*, 233–276.

Chang, A. Y., Skirbekk, V. F., Tyrovolas, S., Kassebaum, N. J., & Dieleman, J. L. (2019). Measuring population ageing: an analysis of the global burden of disease study 2017. *The Lancet Public Health*, *4*(3), e159–e167.

Charlebois, D. A., & Balázsi, G. (2019). Modeling cell population dynamics. *In silico biology*, *13*(1-2), 21–39.

Chen, L., Zheng, Q., Xiong, X., Yuan, Y., Xie, H., Guo, Y., et al. (2019). Dynamic and statistical features of internal solitary waves on the continental slope in the northern south china sea derived from mooring observations. *Journal of Geophysical Research: Oceans*, *124*(6), 4078–4097.

Coale, A. J. (2017). How a population ages or grows younger. In *Population growth* (pp. 47–58). Routledge.

Crankshaw, O., & Borel-Saladin, J. (2019). Causes of urbanisation and counter-urbanisation in zambia: natural population increase or migration? *Urban Studies*, *56*(10), 2005–2020.

CSERNICA, T. (2015). Extinction in single and multi-type branching processes.

Daley, D. (1969). Quasi-stationary behaviour of a left-continuous random walk. *The Annals of Mathematical Statistics*, *40*(2), 532–539.

Davalos, E., & Morales, L. F. (2017). Economic crisis promotes fertility decline in poor areas: Evidence from colombia. *Demographic Research*, *37*, 867–888.

Delisle, J.-B., Hara, N., & Ségransan, D. (2020). Efficient modeling of correlated noise-i. statistical significance of periodogram peaks. *Astronomy & Astrophysics*, *635*, A83.

Devroye, L. (1998). Branching processes and their applications in the analysis of tree structures and tree algorithms. In *Probabilistic methods for algorithmic discrete mathematics* (pp. 249–314). Springer.

Dickey, D. A., & Fuller, W. A. (1979). Distribution of the estimators for autoregressive time series with a unit root. *Journal of the American statistical association*, *74*(366a), 427–431.

Diebold, F., & Rudebusch, G. (2021). *13. on the power of dickey-fuller tests against fractional alternatives.* Princeton University Press.

Dikpati, M., & McIntosh, S. W. (2020). Space weather challenge and forecasting implications of rossby waves. *Space Weather*, *18*(3), e2018SW002109.

Dingle, H. (2014). *Migration: the biology of life on the move.* Oxford University Press, USA.

Drazen, A. (2018). *Political economy in macroeconomics.* Princeton University

Press.

d'Souza, J., & Biggins, J. (1992). The supercritical galton-watson process in varying environments. *Stochastic processes and their applications*, *42*(1), 39–47.

Durrett, R. (2015). *Branching process models of cancer.* Springer.

Duval, R., Eris, M., & Furceri, D. (2010). Labour force participation hysteresis in industrial countries: Evidence and causes.

Dyakonova, E., Li, D., Vatutin, V., & Zhang, M. (2020). Branching processes in a random environment with immigration stopped at zero. *Journal of Applied Probability*, *57*(1), 237–249.

Eklund, L., & Purewal, N. (2017). The bio-politics of population control and sex-selective abortion in china and india. *Feminism & Psychology*, *27*(1), 34–55.

Erlinghagen, M. (2008). Self-perceived job insecurity and social context: A multi-level analysis of 17 european countries. *European Sociological Review*, *24*(2), 183–197.

Fahlbeck, P. E. (1898). *Sveriges adel. statistisk undersökning öfver de å riddarhuset introducerade ätterna. i ätternas demografi. ii den lefvande adeln i sverige och finland.* CWK Kleerup.

Farahani, F. K. (2020). Adolescents and young people's sexual and reproductive health in iran: A conceptual review. *The Journal of Sex Research*, *57*(6), 743–780.

Fijarczyk, A., & Babik, W. (2015). Detecting balancing selection in genomes: limits and prospects. *Molecular ecology*, *24*(14), 3529–3545.

Fisher, R. A. (1922). On the mathematical foundations of theoretical statistics. *Philosophical Transactions of the Royal Society of London. Series A, Containing Papers of a Mathematical or Physical Character*, *222*(594-604), 309–368.

Fisher, R. A. (1958). *The genetical theory of natural selection.* New York: Dover

Publications.

Frias, W., Smolyakov, A. I., Kaganovich, I. D., & Raitses, Y. (2013). Long wavelength gradient drift instability in hall plasma devices. ii. applications. *Physics of Plasmas*, *20*(5), 052108.

Fryer, D. (1986). Employment deprivation and personal agency during unemployment: A critical discussion of jahoda's explanation of the psychological effects of unemployment.

Fryer, D., & Payne, R. (1984). Proactive behaviour in unemployment: Findings and implications. *Leisure studies*, *3*(3), 273–295.

Fuller, R., Carleton, K., Fadool, J., Spady, T., & Travis, J. (2004). Population variation in opsin expression in the bluefin killifish, lucania goodei: a real-time pcr study. *Journal of Comparative Physiology A*, *190*(2), 147–154.

Gabriel, W., & Bürger, R. (1992). Survival of small populations under demographic stochasticity. *Theoretical Population Biology*, *41*(1), 44–71.

Galton, F. (1873). Problem 4001. *Educational Times*, *1*(17), 4.

Gani, D. D. J., & Daley, D. (1999). Epidemic modeling: an introduction. *Cambridge U. Press, Cambridge*.

Ge, S., Yang, D. T., & Zhang, J. (2018). Population policies, demographic structural changes, and the chinese household saving puzzle. *European Economic Review*, *101*, 181–209.

Gietel-Basten, S., Han, X., & Cheng, Y. (2019). Assessing the impact of the "one-child policy" in china: a synthetic control approach. *PloS one*, *14*(11), e0220170.

Gladwin, T. E. (2020). An implementation of n-way repeated measures anova: effect coding, automated unpacking of interactions, and randomization testing. *MethodsX*, *7*, 100947.

González, M., Kersting, G., Minuesa, C., & Puerto, I. del. (2019). Branching processes in varying environment with generation-dependent immigration. *Stochastic Models*, *35*(2), 148–166.

González, M., & Puerto, I. M. del. (2010). Branching processes: genealogy and evolution. *Boletín de Estadística e Investigación operativa*, *26*(2), 107–123.

Granger, C. W. J., & Hatanaka, M. (2015). *Spectral analysis of economic time series.(psme-1)*. Princeton university press.

Grey, D. (1980). Minimisation of extinction probabilities in reproducing populations. *Theoretical Population Biology*, *18*(3), 430–443.

Grosjean, N., & Huillet, T. (2018). On the genealogy and coalescence times of bienaymé–galton–watson branching processes. *Stochastic Models*, *34*(1), 1–24.

Grzenda, W. (2019). Socioeconomic aspects of long-term unemployment in the context of the ageing population of europe: the case of poland. *Economic research-Ekonomska istraživanja*, *32*(1), 1561–1582.

GSS, G. (2010). Population and housing census: Summary report of final results. *Accra: Ghana Statistical Service*.

Guttorp, P., Albertsen, K., Steffensen, J., & Kristensen, E. (1995). Three papers on the history of branching processes. *International statistical review*, *63*(2), 233–245.

Hackenberg, R. A., & Magalit, H. F. (2019). *Demographic responses to development: sources of declining fertility in the philippines*. Routledge.

Hairston Jr, N. G., Olds, E. J., & Munns Jr, W. R. (1985). Bet-hedging and environmentally cued diapause strategies of diaptomid copepods: with 3 figures in the text. *Internationale Vereinigung für theoretische und angewandte Limnologie: Verhandlungen*, *22*(5), 3170–3177.

Haldane, J. B. S. (1927). A mathematical theory of natural and artificial selection, part v: selection and mutation. In *Mathematical proceedings of the cambridge philosophical society* (Vol. 23, pp. 838–844).

Hatton, T. J., Sparrow, R., Suryadarma, D., & Eng, P. van der. (2018). Fertility and the health of children in indonesia. *Economics & Human Biology*, *28*, 67–78.

Henry-Labordere, P., Oudjane, N., Tan, X., Touzi, N., & Warin, X. (2019). Branching diffusion representation of semilinear pdes and monte carlo approximation. In *Annales de l'institut henri poincaré, probabilités et statistiques* (Vol. 55, pp. 184–210).

Heyde, C., & Seneta, E. (1977). Miscellaneous writings. In *Ij bienaymé* (pp. 129–137). Springer.

Hoang, A., Nguyen, C. Q., & Duong, C. D. (2018). Youth experiences in accessing sexual healthcare services in vietnam. *Culture, health & sexuality*, *20*(5), 545–559.

Hoffmann, E. M., Konerding, V., Nautiyal, S., & Buerkert, A. (2019). Is the push-pull paradigm useful to explain rural-urban migration? a case study in uttarakhand, india. *PloS one*, *14*(4), e0214511.

Holgate, P. (1966). Genetic algebras associated with polyploidy. *Proceedings of the Edinburgh Mathematical Society*, *15*(1), 1–9.

Holgate, P. (1967). Population survival and life history phenomena. *Journal of theoretical Biology*, *14*(1), 1–10.

Jacob, C., & Penisson, S. (2011). Limit models for a new general class of multitype branching processes with memory and population dependence. In *Workshop branching processes and derived processes cirm.*

Jagers, P. (1989). General branching processes as markov fields. *Stochastic Processes and their Applications*, *32*(2), 183–212.

Jagers, P. (1995). Branching processes as population dynamics. *Bernoulli*, 191–200.

Jagers, P. (1997). Towards dependence in general branching processes. In *Classical and modern branching processes* (pp. 127–139). Springer.

Jahn, A., Kynast-Wolf, G., Kouyaté, B., & Becher, H. (2006). Multiple pregnancy in rural burkina faso: frequency, survival, and use of health services. *Acta obstetricia et gynecologica Scandinavica*, *85*(1), 26–32.

Jahoda, M. (1982). Employment and unemployment. *Cambridge Books*.

Joffe, A., & Spitzer, F. (1967). On multitype branching processes with q≤ 1. *Journal of Mathematical Analysis and Applications*, *19*(3), 409–430.

Jowit, J. (2013). Ageing population will have huge impact on social services. *The Guardian*.

Juni, M. H. (2015). Ageing population: a public health implications. *International Journal of Public Health and Clinical Sciences*, *2*(3).

Keller, L. (2020). Reducing sti cases: young people deserve better sexual health information and services. *age*, *79*(65), 65.

Kendall, D. G. (1975a). The genealogy of genealogy branching processes before (and after) 1873. *Bulletin of the London Mathematical Society*, *7*(3), 225–253.

Kendall, D. G. (1975b). Review lecture, the recovery of structure from fragmentary information. *Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences*, *279*(1291), 547–582.

Kersting, G., & Vatutin, V. (2017). *Discrete time branching processes in random environment*. John Wiley & Sons.

Kimmel, M., & Axelrod, D. E. (2002). The age-dependent process: The markov case. In *Branching processes in biology* (pp. 65–86). Springer.

King, R., Lulle, A., Sampaio, D., & Vullnetari, J. (2017). Unpacking the ageing–migration nexus and challenging the vulnerability trope. *Journal of Ethnic and Migration Studies*, *43*(2), 182–198.

Kojima, K., & Schaffer, H. (1964). Accumulation of epistatic gene complexes. *Evolution*, *18*(1), 127–129.

Kojima, K.-I., & Kelleher, T. M. (1962). Survival of mutant genes. *The American Naturalist*, *96*(891), 329–346.

Kolmogorov, A. N. (1938). On the analytic methods of probability theory. *Uspekhi matematicheskikh nauk*(5), 5–41.

Kolmogorov, A. N. (1959). Entropy per unit time as a metric invariant of automorphisms. In *Dokl. akad. nauk sssr* (Vol. 124, pp. 754–755).

Lambert, A. (2005). The branching process with logistic growth. *The Annals of Applied Probability*, *15*(2), 1506–1535.

Lange, K. (2010). Branching processes. In *Applied probability* (pp. 217–245). Springer.

Lashari, A. A., & Trapman, P. (2018). Branching process approach for epidemics in dynamic partnership network. *Journal of mathematical biology*, *76*(1), 265–294.

Lebreton, J., & Clobert, J. (1991). Bird population dynamics, management, and conservation: the role of mathematical modelling. *Bird population studies*, 105–125.

Lee, R. D., & Mason, A. (2011). *Population aging and the generational economy: A global perspective*. Edward Elgar Publishing.

Lee-Six, H., Øbro, N. F., Shepherd, M. S., Grossmann, S., Dawson, K., Belmonte, M., et al. (2018). Population dynamics of normal human blood inferred from somatic mutations. *Nature*, *561*(7724), 473–478.

Le Page, E., Peigné, M., & Pham, C. (2018). The survival probability of a critical multi-type branching process in iid random environment. *The Annals of Probability*, *46*(5), 2946–2972.

Li, D., Vatutin, V., & Zhang, M. (2021). Subcritical branching processes in random environment with immigration stopped at zero. *Journal of Theoretical Probability*, *34*(2), 874–896.

Li, H., Zhou, T., & Jia, C. (2019). The influence of the universal two-child policy on china's future population and ageing. *Journal of Population Research*, *36*(3), 183–203.

Lonnie, M., Hooker, E., Brunstrom, J. M., Corfe, B. M., Green, M. A., Watson, A. W., et al. (2018). Protein for life: Review of optimal protein intake, sustainable dietary sources and the effect on appetite in ageing adults. *Nutrients*, *10*(3), 360.

Ludwig, D. (1996). The distribution of population survival times. *The American Naturalist*, *147*(4), 506–526.

Ludwig, D. (2013). *Stochastic population theories* (Vol. 3). Springer Science & Business Media.

Macleod, C. I., Beynon-Jones, S., & Toerien, M. (2017). Articulating reproductive justice through reparative justice: case studies of abortion in great britain and south africa. *Culture, health & sexuality*, *19*(5), 601–615.

Madsen, E. L., Daumerie, B., & Hardee, K. (2010). The effects of age structure on development. *Policy and Issue Brief, Population Action International*, 1–4.

Mahmoud, H. M. (1992). Distances in random plane-oriented recursive trees. *Journal of Computational and Applied Mathematics*, *41*(1-2), 237–245.

Maity, S., & Sinha, A. (2020). Linkages between economic growth and population ageing with a knowledge spillover effect. *Journal of the Knowledge Economy*, 1–20.

Makarova, Y., Kutsenko, V., & Yarovaya, E. (2020). On two-type branching random walks and their applications for genetic modelling. In *International conference on stochastic methods* (pp. 255–268).

Mason, E., McDougall, L., Lawn, J. E., Gupta, A., Claeson, M., Pillay, Y., et al. (2014). From evidence to action to deliver a healthy start for the next generation. *The Lancet*, *384*(9941), 455–467.

Mazur, A., Brindis, C. D., & Decker, M. J. (2018). Assessing youth-friendly sexual and reproductive health services: a systematic review. *BMC health services research*, *18*(1), 1–12.

McCann, C. R. (2019). *Birth-control politics in the united states, 1916–1945*. Cornell University Press.

McDonald, T. O., & Kimmel, M. (2017). Multitype infinite-allele branching processes in continuous time. *Journal of Applied Probability*, *54*(2), 550–568.

Mei, Z. (2019). Limit theorems of branching processes with immigration. *SCIENTIA SINICA Mathematica*, *49*(3), 581.

Melcher, C. R. (2021). Economic self-interest and americans' redistributive, class, and racial attitudes: The case of economic insecurity. *Political Behavior*, 1–22.

Merchant, R. A., Chen, M. Z., Tan, L. W. L., Lim, M. Y., Ho, H. K., & Dam, R. M. van. (2017). Singapore healthy older people everyday (hope) study: prevalence of frailty and associated factors in older adults. *Journal of the American Medical Directors Association*, *18*(8), 734–e9.

Mettle, F. O., Osei Affi, P., Aidoo, E. K., & Benn, S. (2020). Stochastic modeling approach of infectious disease with sir epidemiological compartment model. *Commun. Math. Biol. Neurosci.*, *2020*, Article–ID.

Mettle, F. O., Quaye, E. N. B., Appiah, M., & Aidoo, E. K. (2019). Analysis of generational probability of zero offspring in a branching process. *Far East Journal of Mathematical Science*, *111*(1), 89-106.

Michoń, P. (2019). The baltic miracle? the economic crisis and its consequences for young people in the labor market of the baltic states, 2007–2017. *Journal of Baltic Studies*, *50*(1), 7–20.

Mordechay, K., Gándara, P., & Orfield, G. (2019). Demographic change. *Educational Leadership*.

Mountford, M. (1988). Population regulation, density dependence, and heterogeneity. *The Journal of Animal Ecology*, 845–858.

Mujahid, N., & Zafar, N. uz. (2012). Economic growth-female labour force participation nexus: an empirical evidence for pakistan. *The Pakistan Development Review*, 565–585.

Mutunga, C., Zulu, E. M., & De Souza, R.-M. (2012). Population dynamics, climate change and sustainable development in africa.

Natanson, I. P. (2016). *Theory of functions of a real variable*. Courier Dover Publications.

Nerman, O., & Jagers, P. (2020). Branching processes and neutral mutations. In *Mathematical statistics theory and applications* (pp. 683–692). De Gruyter.

O'neill, P. (1995). Epidemic models featuring behaviour change. *Advances in applied probability*, *27*(4), 960–979.

Organization, W. H. (2013). *Global tuberculosis report 2013.* Author.

Palachy, S. (2019). Stationarity in time series analysis. *Towards Data Science. Saatavissa: https://towardsdatascience. com/stationarity-in-time-seriesanalysis-90c94f27322. Hakupäivä*, *31*, 2019.

Palau, S., & Pardo, J. (2018). Branching processes in a lévy random environment. *Acta Applicandae Mathematicae*, *153*(1), 55–79.

Peng, X. (2011). China's demographic history and future challenges. *science*, *333*(6042), 581–587.

Pettinger, R. (2013). *Organizational behaviour: Performance management in practice.* Routledge.

Philibert, A., Tourigny, C., Coulibaly, A., & Fournier, P. (2013). Birth seasonality as a response to a changing rural environment (kayes region, mali). *Journal of biosocial science*, *45*(4), 547–565.

Phillips, P. C., & Perron, P. (1988). Testing for a unit root in time series regression. *Biometrika*, *75*(2), 335–346.

Pillay, N. K., & Maharaj, P. (2013). Population ageing in africa. , 11–51.

Pitman, J. (2006). Random walks and random forests. In *Combinatorial stochastic processes* (pp. 121–141). Springer.

Reynaud, C., & Miccoli, S. (2019). Population ageing in italy after the 2008 economic crisis: A demographic approach. *Futures*, *105*, 17–26.

Rott, H. (2019). Birth control pills and thrombotic risks: differences of contraception methods with and without estrogen. *Hämostaseologie*, *39*(01), 042–048.

Rouzine, I. M. (2020). Multi-locus theory of asexual populations. In *Mathematical modeling of evolution* (pp. 61–100). De Gruyter.

Rowlands, S., & Thomas, K. (2020). Mandatory waiting periods before abortion and sterilization: Theory and practice. *International Journal of Women's Health*, *12*, 577.

Saad, A., & Al Foori, A. (2020). Zakat and tax: A comparative study in malaysia. *International Journal of Innovation, Creativity and Change*, *10*(12), 140–151.

Schaffer, H. (1970). Survival of mutant genes as a branching process. In *Mathematical topics in population genetics* (pp. 317–336). Springer.

Schoen, R. F. (2018). Shifting the burden to daughters: A qualitative examination of population policy, labor migration, and filial responsibility in rural bangladesh. *Qualitative Sociology Review*, *14*(3), 106–124.

Seale, L., Awosika, O., & Lim, H. W. (2021). Trends in sessions in diversity at the american academy of dermatology annual meetings: 2013–2019. *International journal of women's dermatology*, *7*(2), 197.

Sen, A. (2017). More than 100 million women are missing. In *Gender and justice* (pp. 219–222). Routledge.

Seneta, E. (1998). Early influences on probability and statistics in the russian empire. *Archive for history of exact sciences*, *53*(3), 201–213.

Sharifi, A., & Hosseingholizadeh, M. (2019). The effect of rapid population growth on urban expansion and destruction of green space in tehran from 1972 to 2017. *Journal of the Indian Society of Remote Sensing*, *47*(6), 1063–1071.

Sharma, S. K., Pratap, N., & Ghimire, D. R. (2011). Ethnic differentials of the impact of family planning program on contraceptive use in nepal. *Demographic Research*, *25*, 837–868.

Sheldon, S. (2016). The decriminalisation of abortion: An argument for modernisation. *Oxford Journal of Legal Studies*, *36*(2), 334–365.

Shen, K., Wang, F., & Cai, Y. (2020). Government policy and global fertility change: A reappraisal. *Asian Population Studies*, *16*(2), 145–166.

Sherbinin, A. d., Carr, D., Cassels, S., & Jiang, L. (2007). Population and environment. *Annu. Rev. Environ. Resour.*, *32*, 345–373.

Shi, Z., Bai, Y., Jin, X., Wang, X., Su, T., & Kong, J. (2021). Parallel deep prediction with covariance intersection fusion on non-stationary time series. *Knowledge-Based Systems*, *211*, 106523.

Sidorenko, A. (2007). World policies on aging and the united nations.

Slavtchova-Bojkova, M., Trayanov, P., & Dimitrov, S. (2017). Branching processes in continuous time as models of mutations: Computational approaches and algorithms. *Computational Statistics & Data Analysis*, *113*, 111–124.

Slavtchova-Bojkova, M., & Vitanov, K. (2019). Multi-type age-dependent branching processes as models of metastasis evolution. *Stochastic Models*, *35*(3), 284–299.

Steffensen, J. (1930). Infantile mortality from an actuarial point of view. *Scandinavian Actuarial Journal*, *1930*(2), 272–286.

Steffensen, J. (1932). On stieltjes' integral and its applications to actuarial questions. *Journal of the Institute of Actuaries*, *63*(3), 443–483.

Stiglbauer, B., & Batinic, B. (2012). The role of jahoda's latent and financial benefits for work involvement: A longitudinal study. *Journal of Vocational Behavior*, *81*(2), 259–268.

Stijns, J.-P. (2006). Natural resource abundance and human capital accumulation. *World development*, *34*(6), 1060–1083.

Sundd, P., Gladwin, M. T., & Novelli, E. M. (2019). Pathophysiology of sickle cell disease. *Annual review of pathology: mechanisms of disease*, *14*, 263–292.

Taylor, I. (2016). Dependency redux: Why africa is not rising. *Review of African Political Economy*, *43*(147), 8–25.

Tran, J., Norton, R., Conrad, N., Rahimian, F., Canoy, D., Nazarzadeh, M., et al. (2018). Patterns and temporal trends of comorbidity among adult patients with incident cardiovascular disease in the uk between 2000 and 2014: a population-based cohort study. *PLoS medicine*, *15*(3), e1002513.

Ugbolue, U. C., Yates, E. L., Wearing, S. C., Gu, Y., Lam, W.-K., Valentin, S., et al. (2020). Sex differences in heel pad stiffness during in vivo loading and unloading. *Journal of Anatomy*, *237*(3), 520–528.

Ugochukwu, U. S., & Chinyere, U. P. (2013). The impact of capital formation on the growth of nigerian economy. *Research journal of finance and accounting*, *4*(9), 36–42.

Vandermeer, J. (2006). Oscillating populations and biodiversity maintenance. *Bioscience*, *56*(12), 967–975.

Vincent, G. K. (2010). *The next four decades: The older population in the united states: 2010 to 2050* (No. 1138). US Department of Commerce, Economics and Statistics Administration, US . . . .

Vyas, S., Golub, M. D., Sussillo, D., & Shenoy, K. V. (2020). Computation through neural population dynamics. *Annual Review of Neuroscience*, *43*, 249–275.

Warner, J., & Corley, D. (2017). The women's leadership gap. *Center for American Progress*, *21*.

Warr, P. (1987). *Work, unemployment, and mental health.* Oxford University Press.

Watson, H. W., & Galton, F. (1875). On the probability of the extinction of families. *The Journal of the Anthropological Institute of Great Britain and Ireland*, *4*, 138–144.

Westergaard, H. (1900). Anmeldelse af pe fahlbeck: Sveriges adel, 1. *Nationallumn. Tidsskr*, *8*(3), 284–291.

Yenilmez, I. M. (2015). Economic and social consequences of population aging the dilemmas and opportunities in the twenty-first century. *Applied Research in Quality of Life*, *10*, 735–752.

Zeng, Y., & Hesketh, T. (2016). The effects of china's universal two-child policy. *The Lancet*, *388*(10054), 1930–1938.

Zenz, A. (2020). *Sterilizations, iuds, and mandatory birth control: The ccp's* *campaign to suppress uyghur birthrates in xinjiang*. Jamestown Foundation Washington.

Zhang, J. (2017). The evolution of china's one-child policy and its effects on family outcomes. *Journal of Economic Perspectives*, *31*(1), 141–60.

Zheng, Q. (2021). New approaches to mutation rate fold change in luria–delbrück fluctuation experiments. *Mathematical Biosciences*, *335*, 108572.

Zhu, H., Chen, B., Cheng, Y., Zhou, Y., Yan, Y.-S., Luo, Q., et al. (2019). Insulin therapy for gestational diabetes mellitus does not fully protect offspring from diet-induced metabolic disorders. *Diabetes*, *68*(4), 696–708.

Zubiashvili, T., & Zubiashvili, N. (2021). Population aging–a global challenge. *Ecoforum Journal*, *10*(2).

APPENDIX

Appendix A: Probability of Ultimate Extinction of Empirical Data

Probability of Ultimate Extinction of Empirical data

Country		prob. Extin.	Repeated ANOVA	p-value		paired t-tests
Ghana	Constant	0.020131	4480	0.0000	constant Random1	-1.79765
	Random1	0.020176			constant Random2	439.6569***
	Random2	0.019385			Random1 Random2	299.8701***
Bangladesh	Constant	0.017264	12223.43	0.0000	constant Random1	-3.2607***
	Random1	0.017305			constant Random2	-591.584***
	Random2	0.018652			Random1 Random2	-456.786***
Philippines	Constant	0.082107	143.328	0.0000	constant Random1	-1.1816
	Random1	0.082134			constant Random2	-146.294***

Table 6.1 – *Continued from previous page*

Country		prob. Extin.	Repeated ANOVA	p-value		paired t-tests
	Random2	0.08343			Random1 Random2	-145.447
Burkina Faso	Constant	0.011704	3481.89	0.0000	constant Random1	-0.65619
	Random1	0.011729			constant Random2	1631.926***
	Random2	0.009717			Random1 Random2	792.4049***
Colombia	Constant	0.074837	199.577	0.0000	constant Random1	-5.74075***
	Random1	0.07489			constant Random2	417.6901***
	Random2	0.072851			Random1 Random2	384.522***
Mali	Constant	0.031165	1734.981	0.0000	constant Random1	-1.59679
	Random1	0.031111			constant Random2	487.0524***
	Random2	0.03026			Random1 Random2	336.3917***
Indonesian	Constant	0.041830	8240.651	0.0000	constant Random1	-7.44142***

Country	prob. Extin.	Repeated ANOVA	p-value		paired t-tests
Random1	0.041861			constant Random2	-598.342***
Random2	0.047128			Random1 Random2	-556.374***

Source: Author's computation, (2021)

Appendix B: Joint Probability Mass Function (PMF) and Marginal PMF

Ghana

1993	Female											
Male	0	1	2	3	4	5	6	7	8	9	10	v_{1i}
0	0.0271	0.0031	0.0058	0.0090	0.0058	0.0057	0.0028	0.0000	0.0000	0.0000	0.0000	0.0593
1	0.0120	0.0206	0.0241	0.0088	0.0241	0.0149	0.0058	0.0090	0.0057	0.0000	0.0000	0.1250
2	0.0059	0.0353	0.0206	0.0391	0.0240	0.0326	0.0087	0.0060	0.0028	0.0031	0.0000	0.1781
3	0.0091	0.0387	0.0176	0.0266	0.0301	0.0293	0.0179	0.0122	0.0000	0.0000	0.0000	0.1815
4	0.0000	0.0090	0.0328	0.0420	0.0237	0.0352	0.0116	0.0031	0.0000	0.0029	0.0000	0.1603
5	0.0030	0.0117	0.0181	0.0299	0.0210	0.0205	0.0090	0.0000	0.0028	0.0000	0.0000	0.1160
6	0.0000	0.0208	0.0270	0.0333	0.0270	0.0059	0.0000	0.0061	0.0000	0.0000	0.0000	0.1201
7	0.0061	0.0060	0.0061	0.0145	0.0000	0.0030	0.0000	0.0030	0.0000	0.0000	0.0000	0.0387
8	0.0000	0.0000	0.0092	0.0029	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0121
9	0.0000	0.0000	0.0031	0.0028	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0059
10	0.0000	0.0030	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0030
v_{2i}	0.0632	0.1482	0.1644	0.2089	0.1557	0.1471	0.0558	0.0394	0.0113	0.0060	0.0000	$n = 336$

1998	Female											
Male	0	1	2	3	4	5	6	7	8	9	10	v_{1i}
0	0.0227	0.0110	0.0139	0.0091	0.0068	0.0046	0.0000	0.0023	0.0000	0.0000	0.0000	0.0704
1	0.0139	0.0183	0.0201	0.0228	0.0136	0.0159	0.0046	0.0024	0.0046	0.0000	0.0000	0.1162
2	0.0113	0.0385	0.0322	0.0409	0.0364	0.0251	0.0068	0.0024	0.0024	0.0000	0.0000	0.1960
3	0.0160	0.0432	0.0501	0.0412	0.0389	0.0340	0.0113	0.0090	0.0000	0.0000	0.0000	0.2437
4	0.0046	0.0322	0.0272	0.0322	0.0295	0.0157	0.0000	0.0000	0.0000	0.0000	0.0000	0.1414
5	0.0093	0.0110	0.0187	0.0137	0.0254	0.0206	0.0116	0.0022	0.0000	0.0000	0.0000	0.1125
6	0.0070	0.0161	0.0114	0.0092	0.0162	0.0022	0.0000	0.0024	0.0000	0.0000	0.0000	0.0645
7	0.0000	0.0068	0.0118	0.0047	0.0066	0.0070	0.0000	0.0000	0.0000	0.0000	0.0000	0.0369
8	0.0024	0.0000	0.0000	0.0046	0.0024	0.0022	0.0023	0.0000	0.0000	0.0000	0.0000	0.0139
9	0.0000	0.0000	0.0000	0.0022	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0022
10	0.0000	0.0000	0.0000	0.0000	0.0023	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0023
v_{2i}	0.0872	0.1771	0.1854	0.1806	0.1781	0.1273	0.0366	0.0207	0.0070	0.0000	0.0000	$n = 438$

2003	Female											
Male	0	1	2	3	4	5	6	7	8	9	10	v_{1i}
0	0.0139	0.0161	0.0158	0.0101	0.0039	0.0040	0.0000	0.0000	0.0000	0.0000	0.0000	0.0638
1	0.0219	0.0297	0.0398	0.0238	0.0138	0.0117	0.0080	0.0019	0.0021	0.0000	0.0000	0.1527
2	0.0141	0.0283	0.0358	0.0240	0.0470	0.0142	0.0181	0.0062	0.0020	0.0000	0.0000	0.1897
3	0.0060	0.0341	0.0402	0.0297	0.0442	0.0274	0.0079	0.0041	0.0019	0.0000	0.0000	0.1955
4	0.0099	0.0179	0.0455	0.0298	0.0338	0.0199	0.0141	0.0020	0.0039	0.0000	0.0000	0.1768
5	0.0062	0.0159	0.0257	0.0381	0.0258	0.0099	0.0061	0.0000	0.0000	0.0000	0.0000	0.1277
6	0.0040	0.0119	0.0161	0.0158	0.0041	0.0020	0.0063	0.0000	0.0000	0.0000	0.0000	0.0602
7	0.0020	0.0039	0.0059	0.0059	0.0021	0.0038	0.0019	0.0000	0.0000	0.0000	0.0000	0.0255
8	0.0000	0.0040	0.0020	0.0000	0.0000	0.0021	0.0000	0.0000	0.0000	0.0000	0.0000	0.0081
9	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
10	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
v_{2i}	0.0780	0.1618	0.2268	0.1772	0.1747	0.0950	0.0624	0.0142	0.0099	0.0000	0.0000	$n = 501$

2008	Female											
Male	0	1	2	3	4	5	6	7	8	9	10	v_{1i}
0	0.0113	0.0092	0.0118	0.0115	0.0023	0.0116	0.0024	0.0000	0.0023	0.0000	0.0000	0.0624
1	0.0046	0.0417	0.0435	0.0324	0.0090	0.0069	0.0046	0.0046	0.0048	0.0023	0.0000	0.1544
2	0.0186	0.0482	0.0641	0.0343	0.0347	0.0253	0.0093	0.0023	0.0023	0.0000	0.0000	0.2391
3	0.0161	0.0343	0.0458	0.0410	0.0322	0.0365	0.0023	0.0045	0.0023	0.0000	0.0000	0.2150
4	0.0115	0.0161	0.0321	0.0392	0.0305	0.0137	0.0048	0.0000	0.0000	0.0000	0.0021	0.1500
5	0.0045	0.0115	0.0320	0.0301	0.0162	0.0162	0.0068	0.0000	0.0000	0.0000	0.0000	0.1173
6	0.0023	0.0071	0.0069	0.0115	0.0023	0.0066	0.0022	0.0023	0.0000	0.0000	0.0000	0.0412
7	0.0022	0.0000	0.0047	0.0000	0.0000	0.0047	0.0000	0.0000	0.0000	0.0000	0.0000	0.0116
8	0.0000	0.0000	0.0000	0.0023	0.0023	0.0022	0.0000	0.0000	0.0000	0.0000	0.0000	0.0068
9	0.0000	0.0024	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0024
10	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
v2i	0.0711	0.1705	0.2409	0.2023	0.1295	0.1237	0.0324	0.0137	0.0117	0.0023	0.0021	$n = 435$

2013	Female										
Male	0	1	2	3	4	5	6	7	8	9	10	v_{1i}
0	0.0173	0.0177	0.0218	0.0162	0.0059	0.0035	0.0012	0.0000	0.0000	0.0000	0.0000	0.0836
1	0.0287	0.0515	0.0378	0.0311	0.0218	0.0219	0.0056	0.0011	0.0023	0.0000	0.0000	0.2018
2	0.0252	0.0308	0.0403	0.0504	0.0322	0.0229	0.0058	0.0046	0.0012	0.0000	0.0000	0.2134
3	0.0070	0.0380	0.0378	0.0640	0.0288	0.0161	0.0079	0.0012	0.0024	0.0000	0.0000	0.2032
4	0.0071	0.0310	0.0423	0.0356	0.0204	0.0103	0.0046	0.0013	0.0000	0.0000	0.0000	0.1526
5	0.0012	0.0130	0.0229	0.0187	0.0104	0.0057	0.0022	0.0034	0.0000	0.0000	0.0000	0.0775
6	0.0034	0.0069	0.0068	0.0148	0.0059	0.0047	0.0000	0.0000	0.0000	0.0000	0.0000	0.0425
7	0.0000	0.0035	0.0047	0.0035	0.0000	0.0012	0.0000	0.0000	0.0000	0.0000	0.0000	0.0129
8	0.0012	0.0045	0.0012	0.0011	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0080
9	0.0012	0.0000	0.0012	0.0012	0.0011	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0047
10	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
v_{2i}	0.0923	0.1969	0.2168	0.2366	0.1265	0.0863	0.0273	0.0116	0.0059	0.0000	0.0000	$n = 870$

2018	Female										
Male	0	1	2	3	4	5	6	7	8	9	10	v_{1i}
0	0.0250	0.0297	0.0246	0.0164	0.0061	0.0021	0.0000	0.0000	0.0000	0.0000	0.0000	0.1039
1	0.0125	0.0435	0.0559	0.0348	0.0165	0.0125	0.0000	0.0000	0.0000	0.0000	0.0000	0.1757
2	0.0103	0.0436	0.0622	0.0351	0.0143	0.0145	0.0082	0.0020	0.0000	0.0000	0.0020	0.1922
3	0.0188	0.0355	0.0479	0.0480	0.0228	0.0164	0.0041	0.0042	0.0000	0.0000	0.0000	0.1977
4	0.0082	0.0208	0.0352	0.0249	0.0348	0.0106	0.0041	0.0062	0.0020	0.0000	0.0000	0.1468
5	0.0000	0.0252	0.0272	0.0145	0.0207	0.0022	0.0062	0.0043	0.0000	0.0000	0.0022	0.1025
6	0.0020	0.0082	0.0043	0.0206	0.0127	0.0000	0.0000	0.0021	0.0000	0.0000	0.0000	0.0499
7	0.0020	0.0000	0.0061	0.0041	0.0021	0.0022	0.0000	0.0020	0.0000	0.0000	0.0000	0.0185
8	0.0000	0.0021	0.0020	0.0021	0.0041	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0103
9	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
10	0.0000	0.0021	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0021
v_{2i}	0.0788	0.2107	0.2654	0.2005	0.1341	0.0605	0.0226	0.0208	0.0020	0.0000	0.0042	$n = 482$

Bangladesh

1993	Female											
Male	0	1	2	3	4	5	6	7	8	9	10	v_{1i}
0	0.0062	0.0078	0.0047	0.0031	0.0016	0.0016	0.0000	0.0016	0.0000	0.0000	0.0000	0.0264
1	0.0078	0.0155	0.0186	0.0202	0.0155	0.0109	0.0047	0.0062	0.0016	0.0016	0.0016	0.1040
2	0.0140	0.0217	0.0342	0.0388	0.0217	0.0419	0.0140	0.0031	0.0047	0.0000	0.0031	0.1972
3	0.0047	0.0233	0.0280	0.0450	0.0357	0.0311	0.0093	0.0078	0.0000	0.0000	0.0000	0.1848
4	0.0047	0.0280	0.0419	0.0404	0.0264	0.0295	0.0140	0.0016	0.0016	0.0000	0.0000	0.1879
5	0.0031	0.0140	0.0311	0.0280	0.0326	0.0217	0.0140	0.0031	0.0016	0.0000	0.0000	0.1491
6	0.0016	0.0078	0.0155	0.0217	0.0140	0.0047	0.0093	0.0000	0.0000	0.0000	0.0000	0.0745
7	0.0000	0.0093	0.0124	0.0124	0.0047	0.0031	0.0031	0.0000	0.0000	0.0000	0.0000	0.0450
8	0.0016	0.0016	0.0047	0.0031	0.0016	0.0016	0.0000	0.0016	0.0000	0.0000	0.0000	0.0155
9	0.0000	0.0031	0.0031	0.0062	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0124
10	0.0000	0.0016	0.0016	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0031
v_{2i}	0.0435	0.1335	0.1957	0.2189	0.1537	0.1460	0.0683	0.0248	0.0093	0.0016	0.0047	$n = 644$

1998	Female											
Male	0	1	2	3	4	5	6	7	8	9	10	v_{1i}
0	0.0189	0.0166	0.0036	0.0036	0.0024	0.0036	0.0047	0.0024	0.0000	0.0000	0.0000	0.0556
1	0.0118	0.0272	0.0260	0.0237	0.0237	0.0225	0.0059	0.0036	0.0012	0.0012	0.0024	0.1491
2	0.0107	0.0343	0.0462	0.0391	0.0355	0.0249	0.0107	0.0047	0.0024	0.0012	0.0000	0.2095
3	0.0083	0.0485	0.0497	0.0556	0.0391	0.0213	0.0071	0.0036	0.0036	0.0012	0.0000	0.2379
4	0.0036	0.0249	0.0237	0.0402	0.0308	0.0237	0.0154	0.0036	0.0000	0.0012	0.0012	0.1681
5	0.0012	0.0130	0.0260	0.0225	0.0118	0.0059	0.0071	0.0012	0.0012	0.0000	0.0000	0.0899
6	0.0012	0.0107	0.0142	0.0178	0.0083	0.0047	0.0012	0.0000	0.0012	0.0000	0.0000	0.0592
7	0.0000	0.0047	0.0036	0.0036	0.0059	0.0012	0.0000	0.0000	0.0000	0.0000	0.0000	0.0189
8	0.0000	0.0024	0.0047	0.0012	0.0012	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0095
9	0.0000	0.0012	0.0000	0.0012	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0024
10	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
v_{2i}	0.0556	0.1834	0.1976	0.2083	0.1586	0.1077	0.0521	0.0189	0.0095	0.0047	0.0036	$n = 845$

2003						Female						
Male	0	1	2	3	4	5	6	7	8	9	10	v_{1i}
0	0.0167	0.0084	0.0112	0.0093	0.0056	0.0019	0.0028	0.0009	0.0000	0.0000	0.0000	0.0567
1	0.0112	0.0326	0.0344	0.0354	0.0205	0.0102	0.0074	0.0037	0.0009	0.0000	0.0000	0.1563
2	0.0205	0.0540	0.0549	0.0521	0.0326	0.0102	0.0149	0.0065	0.0009	0.0009	0.0019	0.2493
3	0.0102	0.0521	0.0521	0.0512	0.0214	0.0158	0.0121	0.0074	0.0028	0.0000	0.0000	0.2251
4	0.0084	0.0223	0.0372	0.0437	0.0242	0.0140	0.0084	0.0019	0.0000	0.0009	0.0000	0.1609
5	0.0037	0.0149	0.0233	0.0158	0.0130	0.0121	0.0019	0.0009	0.0000	0.0000	0.0009	0.0865
6	0.0009	0.0084	0.0130	0.0065	0.0065	0.0019	0.0037	0.0019	0.0000	0.0009	0.0000	0.0437
7	0.0000	0.0047	0.0019	0.0047	0.0000	0.0037	0.0000	0.0000	0.0000	0.0000	0.0000	0.0149
8	0.0019	0.0009	0.0009	0.0000	0.0009	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0047
9	0.0000	0.0009	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0009
10	0.0000	0.0000	0.0009	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0009
v_{2i}	0.0735	0.1991	0.2298	0.2186	0.1247	0.0698	0.0512	0.0233	0.0047	0.0028	0.0028	$n = 1075$

2008						Female						
Male	0	1	2	3	4	5	6	7	8	9	10	v_{1i}
0	0.0249	0.0200	0.0200	0.0135	0.0060	0.0038	0.0022	0.0016	0.0000	0.0000	0.0000	0.0920
1	0.0249	0.0660	0.0557	0.0417	0.0238	0.0119	0.0081	0.0011	0.0005	0.0000	0.0000	0.2338
2	0.0428	0.0909	0.0639	0.0471	0.0244	0.0114	0.0070	0.0011	0.0005	0.0005	0.0000	0.2895
3	0.0233	0.0498	0.0536	0.0336	0.0206	0.0076	0.0022	0.0011	0.0005	0.0000	0.0000	0.1921
4	0.0146	0.0238	0.0254	0.0189	0.0092	0.0060	0.0043	0.0005	0.0000	0.0000	0.0000	0.1028
5	0.0054	0.0125	0.0162	0.0103	0.0065	0.0027	0.0016	0.0000	0.0000	0.0000	0.0000	0.0552
6	0.0005	0.0076	0.0070	0.0016	0.0038	0.0011	0.0000	0.0000	0.0000	0.0000	0.0000	0.0217
7	0.0005	0.0022	0.0043	0.0005	0.0022	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0097
8	0.0000	0.0011	0.0000	0.0000	0.0011	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0022
9	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
10	0.0000	0.0000	0.0000	0.0005	0.0000	0.0005	0.0000	0.0000	0.0000	0.0000	0.0000	0.0011
v_{2i}	0.1369	0.2738	0.2462	0.1678	0.0974	0.0449	0.0254	0.0054	0.0016	0.0005	0.0000	$n = 1848$

2013						Female						
Male	0	1	2	3	4	5	6	7	8	9	10	v_{1i}
0	0.0187	0.0187	0.0187	0.0210	0.0116	0.0066	0.0039	0.0011	0.0000	0.0000	0.0006	0.1009
1	0.0309	0.0832	0.0706	0.0513	0.0276	0.0094	0.0033	0.0011	0.0006	0.0000	0.0000	0.2778
2	0.0386	0.0888	0.0783	0.0518	0.0232	0.0099	0.0050	0.0011	0.0000	0.0000	0.0000	0.2966
3	0.0292	0.0474	0.0507	0.0320	0.0165	0.0039	0.0028	0.0011	0.0000	0.0000	0.0000	0.1836
4	0.0110	0.0287	0.0199	0.0165	0.0066	0.0050	0.0006	0.0006	0.0000	0.0000	0.0000	0.0888
5	0.0044	0.0094	0.0055	0.0050	0.0050	0.0022	0.0011	0.0000	0.0000	0.0006	0.0000	0.0331
6	0.0033	0.0039	0.0028	0.0011	0.0022	0.0006	0.0000	0.0000	0.0000	0.0000	0.0000	0.0138
7	0.0011	0.0000	0.0006	0.0022	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0039
8	0.0000	0.0000	0.0000	0.0000	0.0006	0.0006	0.0000	0.0000	0.0000	0.0000	0.0000	0.0011
9	0.0000	0.0006	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0006
10	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
v_{2i}	0.1373	0.2806	0.2470	0.1808	0.0932	0.0380	0.0165	0.0050	0.0006	0.0006	0.0006	$n = 1814$

2018						Female						
Male	0	1	2	3	4	5	6	7	8	9	10	v_{1i}
0	0.0145	0.0243	0.0286	0.0209	0.0102	0.0051	0.0034	0.0013	0.0000	0.0000	0.0000	0.1083
1	0.0222	0.1002	0.0806	0.0469	0.0196	0.0085	0.0043	0.0004	0.0004	0.0000	0.0000	0.2830
2	0.0448	0.0993	0.0895	0.0324	0.0217	0.0077	0.0043	0.0009	0.0000	0.0000	0.0000	0.3005
3	0.0247	0.0639	0.0477	0.0230	0.0145	0.0051	0.0009	0.0009	0.0004	0.0000	0.0000	0.1812
4	0.0145	0.0260	0.0239	0.0141	0.0060	0.0030	0.0017	0.0000	0.0000	0.0000	0.0000	0.0891
5	0.0021	0.0068	0.0085	0.0051	0.0021	0.0004	0.0000	0.0004	0.0000	0.0000	0.0000	0.0256
6	0.0004	0.0034	0.0021	0.0013	0.0013	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0085
7	0.0004	0.0009	0.0004	0.0004	0.0004	0.0000	0.0004	0.0000	0.0000	0.0000	0.0000	0.0030
8	0.0000	0.0000	0.0000	0.0004	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0004
9	0.0000	0.0000	0.0004	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0004
10	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
v_{2i}	0.1236	0.3248	0.2818	0.1445	0.0759	0.0298	0.0149	0.0038	0.0009	0.0000	0.0000	$n = 2346$

Colombia

1990	Female											
Male	0	1	2	3	4	5	6	7	8	9	10	v_{1i}
0	0.0650	0.0291	0.0256	0.0171	0.0034	0.0034	0.0000	0.0017	0.0000	0.0000	0.0000	0.1453
1	0.0393	0.0427	0.0359	0.0342	0.0205	0.0068	0.0017	0.0017	0.0034	0.0000	0.0000	0.1863
2	0.0291	0.0479	0.0479	0.0393	0.0308	0.0188	0.0017	0.0034	0.0000	0.0000	0.0000	0.2188
3	0.0086	0.0359	0.0342	0.0274	0.0222	0.0154	0.0068	0.0051	0.0034	0.0000	0.0000	0.1590
4	0.0103	0.0222	0.0222	0.0239	0.0154	0.0120	0.0051	0.0000	0.0051	0.0000	0.0000	0.1162
5	0.0086	0.0034	0.0137	0.0188	0.0154	0.0051	0.0068	0.0051	0.0017	0.0000	0.0000	0.0786
6	0.0034	0.0051	0.0154	0.0068	0.0068	0.0051	0.0017	0.0068	0.0034	0.0000	0.0000	0.0547
7	0.0000	0.0000	0.0000	0.0051	0.0120	0.0034	0.0000	0.0000	0.0000	0.0000	0.0000	0.0205
8	0.0000	0.0034	0.0000	0.0034	0.0051	0.0000	0.0017	0.0000	0.0000	0.0000	0.0000	0.0137
9	0.0000	0.0000	0.0017	0.0000	0.0000	0.0000	0.0017	0.0000	0.0000	0.0000	0.0000	0.0034
10	0.0000	0.0000	0.0000	0.0000	0.0017	0.0017	0.0000	0.0000	0.0000	0.0000	0.0000	0.0034
v_{2i}	0.1641	0.1897	0.1966	0.1761	0.1333	0.0718	0.0274	0.0239	0.0171	0.0000	0.0000	$n = 585$

1995	Female											
Male	0	1	2	3	4	5	6	7	8	9	10	v_{1i}
0	0.0783	0.0376	0.0292	0.0146	0.0094	0.0021	0.0010	0.0000	0.0010	0.0000	0.0000	0.1733
1	0.0386	0.0710	0.0605	0.0365	0.0146	0.0094	0.0042	0.0000	0.0000	0.0000	0.0000	0.2349
2	0.0407	0.0793	0.0605	0.0355	0.0157	0.0073	0.0042	0.0031	0.0010	0.0000	0.0000	0.2474
3	0.0157	0.0386	0.0386	0.0292	0.0115	0.0021	0.0031	0.0042	0.0000	0.0000	0.0010	0.1441
4	0.0115	0.0136	0.0230	0.0230	0.0094	0.0042	0.0031	0.0031	0.0010	0.0000	0.0000	0.0919
5	0.0021	0.0094	0.0104	0.0073	0.0115	0.0063	0.0042	0.0000	0.0000	0.0000	0.0000	0.0512
6	0.0000	0.0052	0.0063	0.0063	0.0021	0.0063	0.0021	0.0010	0.0000	0.0000	0.0000	0.0292
7	0.0000	0.0031	0.0031	0.0031	0.0031	0.0010	0.0021	0.0000	0.0000	0.0000	0.0010	0.0167
8	0.0021	0.0010	0.0000	0.0000	0.0021	0.0021	0.0010	0.0000	0.0000	0.0000	0.0000	0.0084
9	0.0010	0.0000	0.0000	0.0000	0.0000	0.0010	0.0000	0.0000	0.0000	0.0000	0.0000	0.0021
10	0.0000	0.0000	0.0010	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0010
v_{2i}	0.1900	0.2589	0.2328	0.1555	0.0793	0.0418	0.0251	0.0115	0.0031	0.0000	0.0021	$n = 958$

2000						Female						
Male	0	1	2	3	4	5	6	7	8	9	10	v_{1i}
0	0.0781	0.0391	0.0409	0.0223	0.0102	0.0047	0.0000	0.0009	0.0000	0.0000	0.0000	0.1963
1	0.0419	0.0763	0.0800	0.0354	0.0167	0.0037	0.0009	0.0009	0.0009	0.0000	0.0000	0.2567
2	0.0372	0.0744	0.0530	0.0372	0.0140	0.0084	0.0019	0.0009	0.0009	0.0000	0.0000	0.2279
3	0.0270	0.0428	0.0400	0.0354	0.0093	0.0121	0.0037	0.0009	0.0000	0.0000	0.0000	0.1712
4	0.0158	0.0158	0.0186	0.0093	0.0093	0.0037	0.0028	0.0019	0.0000	0.0009	0.0000	0.0781
5	0.0065	0.0065	0.0056	0.0037	0.0047	0.0047	0.0019	0.0009	0.0009	0.0000	0.0000	0.0354
6	0.0009	0.0019	0.0028	0.0056	0.0000	0.0028	0.0009	0.0000	0.0000	0.0000	0.0000	0.0149
7	0.0037	0.0009	0.0028	0.0000	0.0028	0.0000	0.0028	0.0000	0.0000	0.0000	0.0000	0.0130
8	0.0000	0.0009	0.0009	0.0028	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0047
9	0.0000	0.0000	0.0000	0.0009	0.0009	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0019
10	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
v_{2i}	0.2112	0.2586	0.2447	0.1526	0.0679	0.0400	0.0149	0.0065	0.0028	0.0009	0.0000	$n = 1075$

2005						Female						
Male	0	1	2	3	4	5	6	7	8	9	10	v_{1i}
0	0.0643	0.0500	0.0429	0.0267	0.0090	0.0032	0.0015	0.0012	0.0000	0.0002	0.0000	0.1989
1	0.0441	0.1019	0.0854	0.0371	0.0167	0.0070	0.0024	0.0010	0.0002	0.0000	0.0000	0.2959
2	0.0446	0.0897	0.0636	0.0320	0.0129	0.0075	0.0017	0.0007	0.0005	0.0002	0.0000	0.2535
3	0.0289	0.0427	0.0357	0.0163	0.0095	0.0070	0.0032	0.0012	0.0000	0.0002	0.0000	0.1446
4	0.0087	0.0180	0.0121	0.0075	0.0068	0.0041	0.0015	0.0015	0.0000	0.0000	0.0000	0.0602
5	0.0032	0.0078	0.0046	0.0051	0.0039	0.0015	0.0007	0.0002	0.0005	0.0000	0.0000	0.0274
6	0.0010	0.0007	0.0024	0.0015	0.0029	0.0010	0.0007	0.0000	0.0000	0.0000	0.0000	0.0102
7	0.0005	0.0010	0.0007	0.0007	0.0015	0.0015	0.0002	0.0002	0.0000	0.0000	0.0000	0.0063
8	0.0002	0.0002	0.0002	0.0005	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0012
9	0.0002	0.0002	0.0000	0.0000	0.0002	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000	0.0010
10	0.0000	0.0002	0.0002	0.0002	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0010
v_{2i}	0.1957	0.3124	0.2479	0.1276	0.0636	0.0330	0.0119	0.0061	0.0012	0.0007	0.0000	$n = 4123$

2010	Female											
Male	0	1	2	3	4	5	6	7	8	9	10	v_{1i}
0	0.0638	0.0565	0.0493	0.0248	0.0097	0.0023	0.0014	0.0004	0.0000	0.0000	0.0000	0.2081
1	0.0581	0.1287	0.0895	0.0347	0.0108	0.0045	0.0012	0.0005	0.0002	0.0002	0.0000	0.3283
2	0.0555	0.0888	0.0498	0.0246	0.0132	0.0050	0.0028	0.0002	0.0002	0.0000	0.0000	0.2400
3	0.0307	0.0369	0.0307	0.0173	0.0082	0.0040	0.0021	0.0005	0.0002	0.0002	0.0000	0.1307
4	0.0097	0.0132	0.0111	0.0075	0.0050	0.0026	0.0009	0.0005	0.0002	0.0000	0.0000	0.0506
5	0.0024	0.0050	0.0042	0.0052	0.0031	0.0026	0.0012	0.0004	0.0002	0.0000	0.0000	0.0243
6	0.0012	0.0019	0.0028	0.0019	0.0023	0.0009	0.0005	0.0004	0.0000	0.0000	0.0000	0.0118
7	0.0000	0.0004	0.0010	0.0012	0.0004	0.0000	0.0005	0.0000	0.0000	0.0002	0.0000	0.0036
8	0.0000	0.0005	0.0000	0.0002	0.0004	0.0002	0.0005	0.0002	0.0000	0.0000	0.0000	0.0019
9	0.0002	0.0002	0.0000	0.0000	0.0000	0.0000	0.0002	0.0000	0.0000	0.0000	0.0000	0.0005
10	0.0000	0.0000	0.0000	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002
v_{2i}	0.2216	0.3321	0.2383	0.1176	0.0529	0.0220	0.0113	0.0030	0.0009	0.0005	0.0000	$n = 5767$

2015	Female											
Male	0	1	2	3	4	5	6	7	8	9	10	v_{1i}
0	0.0519	0.0554	0.0607	0.0241	0.0074	0.0016	0.0012	0.0002	0.0002	0.0000	0.0000	0.2028
1	0.0593	0.1372	0.0962	0.0331	0.0104	0.0037	0.0012	0.0005	0.0002	0.0000	0.0000	0.3418
2	0.0635	0.0915	0.0487	0.0255	0.0125	0.0030	0.0021	0.0012	0.0005	0.0002	0.0000	0.2487
3	0.0334	0.0403	0.0223	0.0148	0.0060	0.0032	0.0016	0.0007	0.0002	0.0007	0.0000	0.1233
4	0.0093	0.0111	0.0088	0.0058	0.0058	0.0037	0.0021	0.0000	0.0002	0.0000	0.0000	0.0468
5	0.0032	0.0032	0.0058	0.0049	0.0030	0.0007	0.0012	0.0002	0.0000	0.0000	0.0000	0.0223
6	0.0009	0.0023	0.0007	0.0012	0.0012	0.0014	0.0012	0.0000	0.0002	0.0000	0.0000	0.0090
7	0.0005	0.0005	0.0012	0.0009	0.0002	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000	0.0035
8	0.0000	0.0000	0.0005	0.0002	0.0005	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0012
9	0.0000	0.0002	0.0005	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0007
10	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
v_{2i}	0.2220	0.3418	0.2452	0.1105	0.0471	0.0176	0.0104	0.0028	0.0016	0.0009	0.0000	$n = 4315$

Burkina Faso

1993							Female					
Male	0	1	2	3	4	5	6	7	8	9	10	v_{1i}
0	0.0245	0.0000	0.0027	0.0055	0.0027	0.0027	0.0027	0.0000	0.0000	0.0000	0.0000	0.0409
1	0.0109	0.0055	0.0082	0.0191	0.0055	0.0109	0.0109	0.0027	0.0027	0.0000	0.0027	0.0790
2	0.0027	0.0136	0.0164	0.0082	0.0273	0.0136	0.0164	0.0273	0.0055	0.0027	0.0000	0.1335
3	0.0055	0.0164	0.0109	0.0191	0.0327	0.0354	0.0218	0.0136	0.0055	0.0000	0.0000	0.1608
4	0.0082	0.0109	0.0164	0.0436	0.0600	0.0327	0.0300	0.0164	0.0000	0.0000	0.0000	0.2180
5	0.0027	0.0164	0.0109	0.0436	0.0463	0.0218	0.0191	0.0000	0.0027	0.0000	0.0000	0.1635
6	0.0000	0.0109	0.0136	0.0164	0.0354	0.0136	0.0164	0.0000	0.0000	0.0000	0.0000	0.1063
7	0.0000	0.0000	0.0109	0.0164	0.0218	0.0027	0.0027	0.0000	0.0000	0.0000	0.0000	0.0545
8	0.0000	0.0027	0.0027	0.0136	0.0055	0.0027	0.0000	0.0000	0.0000	0.0000	0.0000	0.0273
9	0.0000	0.0000	0.0055	0.0027	0.0027	0.0000	0.0027	0.0000	0.0000	0.0000	0.0000	0.0136
10	0.0000	0.0000	0.0027	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0027
v_{2i}	0.0545	0.0763	0.1008	0.1880	0.2398	0.1362	0.1226	0.0600	0.0164	0.0027	0.0027	$n = 367$

1998							Female					
Male	0	1	2	3	4	5	6	7	8	9	10	v_{1i}
0	0.0105	0.0042	0.0084	0.0146	0.0042	0.0042	0.0021	0.0021	0.0021	0.0000	0.0000	0.0523
1	0.0063	0.0042	0.0105	0.0146	0.0084	0.0000	0.0105	0.0084	0.0042	0.0000	0.0000	0.0670
2	0.0084	0.0126	0.0146	0.0188	0.0251	0.0251	0.0167	0.0105	0.0063	0.0000	0.0021	0.1402
3	0.0021	0.0146	0.0146	0.0293	0.0314	0.0209	0.0293	0.0146	0.0042	0.0000	0.0000	0.1611
4	0.0042	0.0105	0.0356	0.0502	0.0460	0.0335	0.0398	0.0063	0.0000	0.0000	0.0000	0.2259
5	0.0021	0.0021	0.0251	0.0335	0.0544	0.0188	0.0084	0.0042	0.0021	0.0021	0.0000	0.1527
6	0.0000	0.0126	0.0251	0.0272	0.0314	0.0084	0.0167	0.0000	0.0000	0.0000	0.0000	0.1213
7	0.0021	0.0000	0.0126	0.0126	0.0084	0.0021	0.0000	0.0021	0.0021	0.0000	0.0000	0.0418
8	0.0000	0.0042	0.0063	0.0042	0.0063	0.0021	0.0021	0.0000	0.0000	0.0000	0.0000	0.0251
9	0.0000	0.0000	0.0042	0.0000	0.0000	0.0021	0.0000	0.0000	0.0000	0.0000	0.0000	0.0063
10	0.0000	0.0000	0.0021	0.0021	0.0021	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0063
v_{2i}	0.0356	0.0649	0.1590	0.2071	0.2176	0.1172	0.1255	0.0481	0.0209	0.0021	0.0021	$n = 478$

2003						Female						
Male	0	1	2	3	4	5	6	7	8	9	10	v_{1i}
0	0.0093	0.0046	0.0037	0.0037	0.0056	0.0037	0.0028	0.0000	0.0009	0.0000	0.0000	0.0344
1	0.0074	0.0167	0.0158	0.0130	0.0130	0.0139	0.0102	0.0019	0.0019	0.0009	0.0000	0.0947
2	0.0046	0.0167	0.0186	0.0149	0.0204	0.0223	0.0158	0.0028	0.0028	0.0019	0.0009	0.1216
3	0.0056	0.0121	0.0325	0.0297	0.0539	0.0390	0.0167	0.0102	0.0028	0.0019	0.0000	0.2043
4	0.0037	0.0121	0.0325	0.0436	0.0381	0.0371	0.0241	0.0102	0.0037	0.0009	0.0000	0.2061
5	0.0009	0.0149	0.0269	0.0381	0.0316	0.0223	0.0074	0.0065	0.0028	0.0019	0.0000	0.1532
6	0.0028	0.0139	0.0260	0.0288	0.0223	0.0130	0.0037	0.0019	0.0000	0.0000	0.0000	0.1124
7	0.0009	0.0046	0.0130	0.0121	0.0084	0.0037	0.0009	0.0000	0.0000	0.0000	0.0000	0.0436
8	0.0028	0.0028	0.0037	0.0028	0.0065	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0186
9	0.0000	0.0009	0.0009	0.0028	0.0009	0.0019	0.0000	0.0000	0.0000	0.0000	0.0000	0.0074
10	0.0000	0.0009	0.0009	0.0009	0.0009	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0037
v_{2i}	0.0381	0.1003	0.1746	0.1903	0.2015	0.1569	0.0817	0.0334	0.0149	0.0074	0.0009	$n = 1077$

2008						Female						
Male	0	1	2	3	4	5	6	7	8	9	10	v_{1i}
0	0.0091	0.0060	0.0030	0.0038	0.0030	0.0015	0.0000	0.0023	0.0008	0.0000	0.0000	0.0294
1	0.0098	0.0091	0.0106	0.0166	0.0181	0.0098	0.0098	0.0030	0.0008	0.0000	0.0000	0.0875
2	0.0038	0.0136	0.0272	0.0332	0.0385	0.0294	0.0272	0.0060	0.0030	0.0015	0.0008	0.1840
3	0.0068	0.0106	0.0279	0.0581	0.0430	0.0354	0.0211	0.0098	0.0060	0.0015	0.0015	0.2217
4	0.0045	0.0181	0.0309	0.0490	0.0437	0.0279	0.0083	0.0060	0.0023	0.0008	0.0000	0.1916
5	0.0015	0.0121	0.0287	0.0400	0.0241	0.0272	0.0083	0.0038	0.0008	0.0000	0.0000	0.1463
6	0.0023	0.0075	0.0241	0.0189	0.0106	0.0106	0.0023	0.0008	0.0000	0.0000	0.0000	0.0769
7	0.0015	0.0075	0.0083	0.0083	0.0075	0.0030	0.0008	0.0000	0.0000	0.0000	0.0000	0.0370
8	0.0015	0.0060	0.0030	0.0060	0.0008	0.0000	0.0008	0.0000	0.0000	0.0000	0.0000	0.0181
9	0.0000	0.0015	0.0015	0.0008	0.0008	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0045
10	0.0008	0.0008	0.0008	0.0008	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0030
v_{2i}	0.0415	0.0928	0.1659	0.2353	0.1901	0.1448	0.0784	0.0317	0.0136	0.0038	0.0023	$n = 1326$

2013						Female						
Male	0	1	2	3	4	5	6	7	8	9	10	v_{1i}
0	0.0105	0.0035	0.0018	0.0018	0.0035	0.0035	0.0000	0.0000	0.0000	0.0000	0.0000	0.0246
1	0.0070	0.0105	0.0070	0.0088	0.0070	0.0105	0.0123	0.0018	0.0035	0.0000	0.0000	0.0685
2	0.0035	0.0264	0.0211	0.0229	0.0229	0.0264	0.0176	0.0158	0.0070	0.0000	0.0000	0.1634
3	0.0018	0.0158	0.0439	0.0615	0.0475	0.0369	0.0193	0.0053	0.0053	0.0018	0.0018	0.2408
4	0.0088	0.0088	0.0527	0.0457	0.0299	0.0299	0.0158	0.0105	0.0018	0.0035	0.0018	0.2091
5	0.0035	0.0193	0.0246	0.0264	0.0211	0.0211	0.0105	0.0035	0.0018	0.0000	0.0018	0.1336
6	0.0053	0.0053	0.0211	0.0123	0.0211	0.0088	0.0053	0.0000	0.0000	0.0000	0.0000	0.0791
7	0.0035	0.0053	0.0053	0.0070	0.0088	0.0035	0.0018	0.0000	0.0000	0.0000	0.0000	0.0352
8	0.0035	0.0035	0.0053	0.0088	0.0053	0.0018	0.0000	0.0000	0.0018	0.0000	0.0000	0.0299
9	0.0000	0.0035	0.0035	0.0018	0.0018	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0105
10	0.0000	0.0000	0.0018	0.0000	0.0018	0.0000	0.0000	0.0018	0.0000	0.0000	0.0000	0.0053
v_{2i}	0.0475	0.1019	0.1881	0.1968	0.1705	0.1424	0.0826	0.0387	0.0211	0.0053	0.0053	$n = 569$

2018	Female											
Male	0	1	2	3	4	5	6	7	8	9	10	v_{1i}
0	0.0056	0.0037	0.0037	0.0000	0.0019	0.0074	0.0019	0.0000	0.0000	0.0000	0.0000	0.0240
1	0.0037	0.0037	0.0222	0.0074	0.0166	0.0074	0.0092	0.0074	0.0019	0.0000	0.0000	0.0795
2	0.0056	0.0259	0.0351	0.0499	0.0277	0.0314	0.0129	0.0056	0.0074	0.0000	0.0000	0.2015
3	0.0056	0.0240	0.0592	0.0629	0.0444	0.0240	0.0129	0.0056	0.0037	0.0000	0.0000	0.2421
4	0.0074	0.0259	0.0388	0.0592	0.0407	0.0185	0.0037	0.0019	0.0000	0.0019	0.0019	0.1996
5	0.0037	0.0166	0.0333	0.0240	0.0314	0.0185	0.0092	0.0000	0.0000	0.0000	0.0000	0.1368
6	0.0019	0.0111	0.0148	0.0259	0.0111	0.0129	0.0019	0.0037	0.0000	0.0000	0.0000	0.0832
7	0.0019	0.0000	0.0074	0.0092	0.0000	0.0056	0.0000	0.0000	0.0000	0.0000	0.0000	0.0240
8	0.0019	0.0000	0.0019	0.0000	0.0019	0.0000	0.0000	0.0019	0.0000	0.0000	0.0000	0.0074
9	0.0000	0.0000	0.0019	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0019
10	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
v_{2i}	0.0370	0.1109	0.2181	0.2385	0.1756	0.1257	0.0518	0.0259	0.0129	0.0019	0.0019	$n = 541$

Philippines

1993	Female											
Male	0	1	2	3	4	5	6	7	8	9	10	v_{1i}
0	0.0804	0.0201	0.0149	0.0114	0.0052	0.0026	0.0009	0.0000	0.0009	0.0000	0.0000	0.1364
1	0.0166	0.0376	0.0446	0.0192	0.0201	0.0087	0.0052	0.0035	0.0000	0.0018	0.0000	0.1573
2	0.0157	0.0490	0.0559	0.0341	0.0201	0.0122	0.0105	0.0052	0.0009	0.0000	0.0009	0.2046
3	0.0087	0.0385	0.0420	0.0332	0.0245	0.0175	0.0079	0.0044	0.0026	0.0009	0.0009	0.1809
4	0.0087	0.0210	0.0192	0.0227	0.0254	0.0096	0.0131	0.0052	0.0026	0.0009	0.0000	0.1285
5	0.0052	0.0131	0.0157	0.0192	0.0166	0.0087	0.0070	0.0044	0.0009	0.0000	0.0000	0.0909
6	0.0000	0.0061	0.0052	0.0114	0.0105	0.0035	0.0052	0.0018	0.0000	0.0009	0.0000	0.0446
7	0.0018	0.0018	0.0087	0.0096	0.0061	0.0070	0.0009	0.0009	0.0000	0.0000	0.0000	0.0367
8	0.0018	0.0018	0.0018	0.0044	0.0009	0.0009	0.0009	0.0009	0.0000	0.0000	0.0000	0.0131
9	0.0000	0.0000	0.0009	0.0000	0.0009	0.0009	0.0000	0.0009	0.0000	0.0000	0.0000	0.0035
10	0.0000	0.0000	0.0009	0.0009	0.0009	0.0009	0.0000	0.0000	0.0000	0.0000	0.0000	0.0035
v_{2i}	0.1390	0.1888	0.2098	0.1661	0.1311	0.0726	0.0516	0.0271	0.0079	0.0044	0.0018	$n = 1144$

1998	Female											
Male	0	1	2	3	4	5	6	7	8	9	10	v_{1i}
0	0.0790	0.0239	0.0198	0.0124	0.0066	0.0033	0.0049	0.0000	0.0008	0.0000	0.0000	0.1506
1	0.0263	0.0461	0.0436	0.0346	0.0165	0.0082	0.0033	0.0008	0.0008	0.0000	0.0008	0.1811
2	0.0263	0.0469	0.0494	0.0305	0.0140	0.0173	0.0049	0.0066	0.0008	0.0017	0.0000	0.1984
3	0.0148	0.0296	0.0288	0.0379	0.0263	0.0124	0.0082	0.0041	0.0033	0.0008	0.0000	0.1663
4	0.0074	0.0255	0.0329	0.0255	0.0173	0.0107	0.0066	0.0008	0.0041	0.0008	0.0008	0.1325
5	0.0017	0.0124	0.0247	0.0148	0.0148	0.0099	0.0066	0.0008	0.0008	0.0000	0.0000	0.0864
6	0.0017	0.0082	0.0074	0.0066	0.0140	0.0025	0.0000	0.0017	0.0017	0.0000	0.0000	0.0436
7	0.0041	0.0008	0.0041	0.0049	0.0041	0.0025	0.0017	0.0008	0.0000	0.0000	0.0000	0.0231
8	0.0000	0.0008	0.0041	0.0041	0.0041	0.0008	0.0008	0.0000	0.0000	0.0000	0.0000	0.0148
9	0.0000	0.0000	0.0017	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0017
10	0.0000	0.0000	0.0008	0.0000	0.0008	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0017
v_{2i}	0.1613	0.1942	0.2173	0.1712	0.1185	0.0675	0.0370	0.0156	0.0124	0.0033	0.0017	$n = 1215$

2003	Female											
Male	0	1	2	3	4	5	6	7	8	9	10	v_{1i}
0	0.0603	0.0238	0.0171	0.0127	0.0052	0.0037	0.0007	0.0000	0.0000	0.0000	0.0000	0.1236
1	0.0305	0.0618	0.0588	0.0313	0.0149	0.0045	0.0030	0.0022	0.0015	0.0000	0.0000	0.2085
2	0.0276	0.0685	0.0603	0.0365	0.0238	0.0075	0.0075	0.0022	0.0007	0.0000	0.0007	0.2353
3	0.0134	0.0469	0.0395	0.0357	0.0231	0.0082	0.0022	0.0000	0.0015	0.0000	0.0007	0.1713
4	0.0149	0.0268	0.0216	0.0223	0.0209	0.0127	0.0037	0.0015	0.0015	0.0007	0.0007	0.1273
5	0.0045	0.0142	0.0142	0.0164	0.0052	0.0104	0.0060	0.0030	0.0000	0.0007	0.0007	0.0752
6	0.0030	0.0052	0.0030	0.0075	0.0037	0.0045	0.0007	0.0000	0.0000	0.0000	0.0000	0.0276
7	0.0007	0.0000	0.0037	0.0060	0.0037	0.0015	0.0007	0.0015	0.0007	0.0000	0.0000	0.0186
8	0.0000	0.0007	0.0030	0.0022	0.0007	0.0000	0.0007	0.0000	0.0000	0.0000	0.0000	0.0075
9	0.0000	0.0007	0.0007	0.0000	0.0015	0.0000	0.0000	0.0000	0.0000	0.0007	0.0000	0.0037
10	0.0000	0.0000	0.0007	0.0000	0.0000	0.0000	0.0007	0.0000	0.0000	0.0000	0.0000	0.0015
v_{2i}	0.1549	0.2487	0.2226	0.1705	0.1028	0.0529	0.0261	0.0104	0.0060	0.0022	0.0030	$n = 1343$

2008	Female											
Male	0	1	2	3	4	5	6	7	8	9	10	v_{1i}
0	0.0727	0.0305	0.0249	0.0166	0.0028	0.0035	0.0007	0.0007	0.0000	0.0000	0.0000	0.1523
1	0.0360	0.0692	0.0706	0.0263	0.0194	0.0055	0.0028	0.0014	0.0000	0.0000	0.0000	0.2311
2	0.0353	0.0865	0.0574	0.0318	0.0242	0.0132	0.0062	0.0021	0.0000	0.0000	0.0000	0.2568
3	0.0132	0.0408	0.0311	0.0270	0.0132	0.0069	0.0076	0.0014	0.0000	0.0007	0.0000	0.1419
4	0.0069	0.0194	0.0201	0.0187	0.0118	0.0062	0.0062	0.0014	0.0014	0.0000	0.0000	0.0920
5	0.0069	0.0090	0.0111	0.0111	0.0090	0.0090	0.0048	0.0028	0.0007	0.0000	0.0000	0.0644
6	0.0007	0.0062	0.0083	0.0042	0.0055	0.0035	0.0035	0.0000	0.0007	0.0000	0.0007	0.0332
7	0.0014	0.0028	0.0042	0.0035	0.0028	0.0000	0.0021	0.0007	0.0000	0.0000	0.0000	0.0173
8	0.0007	0.0000	0.0014	0.0014	0.0000	0.0014	0.0014	0.0000	0.0000	0.0000	0.0000	0.0062
9	0.0007	0.0007	0.0007	0.0007	0.0014	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0042
10	0.0000	0.0000	0.0000	0.0007	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0007
v_{2i}	0.1744	0.2651	0.2298	0.1419	0.0900	0.0491	0.0353	0.0104	0.0028	0.0007	0.0007	$n = 1445$

2013	Female											
Male	0	1	2	3	4	5	6	7	8	9	10	v_{1i}
0	0.0803	0.0418	0.0303	0.0176	0.0088	0.0044	0.0028	0.0011	0.0000	0.0000	0.0000	0.1870
1	0.0336	0.0853	0.0594	0.0374	0.0116	0.0061	0.0033	0.0017	0.0006	0.0000	0.0000	0.2387
2	0.0358	0.0627	0.0605	0.0391	0.0165	0.0094	0.0050	0.0022	0.0000	0.0000	0.0000	0.2310
3	0.0127	0.0446	0.0292	0.0259	0.0105	0.0099	0.0033	0.0011	0.0022	0.0000	0.0006	0.1397
4	0.0072	0.0242	0.0220	0.0187	0.0138	0.0066	0.0044	0.0011	0.0011	0.0006	0.0006	0.1001
5	0.0072	0.0116	0.0105	0.0110	0.0094	0.0050	0.0011	0.0017	0.0006	0.0000	0.0000	0.0578
6	0.0011	0.0017	0.0061	0.0055	0.0083	0.0028	0.0006	0.0011	0.0000	0.0000	0.0000	0.0270
7	0.0017	0.0028	0.0022	0.0028	0.0011	0.0028	0.0000	0.0000	0.0000	0.0000	0.0000	0.0132
8	0.0000	0.0000	0.0006	0.0000	0.0006	0.0011	0.0000	0.0006	0.0000	0.0000	0.0000	0.0028
9	0.0000	0.0000	0.0006	0.0006	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0011
10	0.0000	0.0000	0.0000	0.0000	0.0006	0.0000	0.0006	0.0000	0.0006	0.0000	0.0000	0.0017
v_{2i}	0.1793	0.2745	0.2211	0.1584	0.0809	0.0479	0.0209	0.0105	0.0050	0.0006	0.0011	n = 1818

2018	Female											
Male	0	1	2	3	4	5	6	7	8	9	10	v_{1i}
0	0.0839	0.0432	0.0320	0.0171	0.0098	0.0007	0.0021	0.0007	0.0004	0.0000	0.0000	0.1897
1	0.0484	0.0790	0.0654	0.0366	0.0153	0.0059	0.0021	0.0004	0.0007	0.0004	0.0000	0.2541
2	0.0439	0.0787	0.0564	0.0324	0.0198	0.0087	0.0021	0.0021	0.0004	0.0004	0.0000	0.2447
3	0.0237	0.0442	0.0348	0.0209	0.0153	0.0063	0.0031	0.0007	0.0010	0.0000	0.0000	0.1500
4	0.0111	0.0223	0.0212	0.0136	0.0094	0.0059	0.0021	0.0007	0.0010	0.0000	0.0004	0.0877
5	0.0052	0.0087	0.0080	0.0073	0.0066	0.0021	0.0004	0.0010	0.0000	0.0004	0.0000	0.0397
6	0.0007	0.0049	0.0031	0.0045	0.0028	0.0014	0.0007	0.0004	0.0000	0.0000	0.0000	0.0185
7	0.0004	0.0010	0.0017	0.0031	0.0010	0.0007	0.0000	0.0000	0.0000	0.0000	0.0000	0.0080
8	0.0014	0.0014	0.0010	0.0004	0.0004	0.0007	0.0000	0.0004	0.0000	0.0000	0.0000	0.0056
9	0.0004	0.0000	0.0000	0.0004	0.0004	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0010
10	0.0000	0.0000	0.0000	0.0007	0.0004	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0010
v_{2i}	0.2189	0.2833	0.2238	0.1368	0.0811	0.0324	0.0125	0.0063	0.0035	0.0010	0.0004	n = 2873

Mali

1992							Female					
Male	0	1	2	3	4	5	6	7	8	9	10	v_{1i}
0	0.0338	0.0169	0.0127	0.0000	0.0000	0.0000	0.0000	0.0042	0.0000	0.0000	0.0000	0.0675
1	0.0211	0.0127	0.0295	0.0169	0.0127	0.0084	0.0084	0.0000	0.0042	0.0000	0.0000	0.1139
2	0.0084	0.0211	0.0253	0.0127	0.0253	0.0042	0.0000	0.0127	0.0000	0.0042	0.0000	0.1139
3	0.0084	0.0295	0.0127	0.0338	0.0253	0.0253	0.0042	0.0211	0.0127	0.0000	0.0000	0.1730
4	0.0084	0.0169	0.0295	0.0338	0.0253	0.0422	0.0169	0.0169	0.0084	0.0000	0.0000	0.1983
5	0.0042	0.0169	0.0084	0.0084	0.0380	0.0338	0.0211	0.0084	0.0042	0.0000	0.0000	0.1435
6	0.0000	0.0000	0.0084	0.0084	0.0295	0.0169	0.0127	0.0000	0.0000	0.0000	0.0000	0.0760
7	0.0000	0.0000	0.0338	0.0084	0.0084	0.0084	0.0084	0.0000	0.0000	0.0000	0.0000	0.0675
8	0.0000	0.0000	0.0042	0.0042	0.0042	0.0000	0.0042	0.0000	0.0000	0.0000	0.0000	0.0169
9	0.0000	0.0000	0.0127	0.0042	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0169
10	0.0000	0.0000	0.0084	0.0000	0.0000	0.0042	0.0000	0.0000	0.0000	0.0000	0.0000	0.0127
v_{2i}	0.0844	0.1139	0.1857	0.1308	0.1688	0.1435	0.0760	0.0633	0.0295	0.0042	0.0000	$n = 237$

1997	Female											
Male	0	1	2	3	4	5	6	7	8	9	10	v_{1i}
0	0.0364	0.0243	0.0108	0.0041	0.0027	0.0000	0.0000	0.0000	0.0014	0.0000	0.0000	0.0796
1	0.0027	0.0135	0.0189	0.0135	0.0027	0.0095	0.0041	0.0041	0.0041	0.0000	0.0000	0.0729
2	0.0095	0.0189	0.0148	0.0202	0.0216	0.0148	0.0081	0.0068	0.0041	0.0027	0.0000	0.1215
3	0.0068	0.0148	0.0135	0.0324	0.0297	0.0229	0.0189	0.0175	0.0148	0.0000	0.0000	0.1714
4	0.0054	0.0148	0.0122	0.0243	0.0324	0.0229	0.0256	0.0162	0.0027	0.0014	0.0014	0.1592
5	0.0027	0.0108	0.0297	0.0256	0.0324	0.0256	0.0216	0.0068	0.0027	0.0014	0.0000	0.1592
6	0.0014	0.0054	0.0108	0.0256	0.0243	0.0283	0.0095	0.0068	0.0027	0.0014	0.0000	0.1161
7	0.0027	0.0014	0.0135	0.0122	0.0216	0.0148	0.0000	0.0027	0.0014	0.0000	0.0000	0.0702
8	0.0000	0.0000	0.0068	0.0108	0.0068	0.0081	0.0027	0.0000	0.0000	0.0000	0.0000	0.0351
9	0.0000	0.0027	0.0041	0.0027	0.0014	0.0014	0.0000	0.0000	0.0000	0.0000	0.0000	0.0122
10	0.0000	0.0000	0.0014	0.0000	0.0014	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0027
v_{2i}	0.0675	0.1066	0.1363	0.1714	0.1768	0.1485	0.0904	0.0607	0.0337	0.0068	0.0014	$n = 741$

2002	Female											
Male	0	1	2	3	4	5	6	7	8	9	10	v_{1i}
0	0.0260	0.0177	0.0063	0.0063	0.0031	0.0010	0.0000	0.0000	0.0010	0.0000	0.0000	0.0615
1	0.0156	0.0156	0.0104	0.0094	0.0125	0.0063	0.0021	0.0031	0.0031	0.0000	0.0021	0.0802
2	0.0042	0.0167	0.0146	0.0177	0.0135	0.0125	0.0188	0.0104	0.0104	0.0021	0.0000	0.1208
3	0.0031	0.0125	0.0240	0.0198	0.0302	0.0302	0.0167	0.0177	0.0063	0.0021	0.0000	0.1625
4	0.0063	0.0146	0.0188	0.0229	0.0427	0.0344	0.0188	0.0156	0.0042	0.0000	0.0000	0.1781
5	0.0000	0.0083	0.0156	0.0323	0.0323	0.0250	0.0188	0.0094	0.0031	0.0000	0.0000	0.1448
6	0.0000	0.0031	0.0198	0.0260	0.0281	0.0250	0.0240	0.0042	0.0000	0.0010	0.0010	0.1323
7	0.0000	0.0063	0.0115	0.0208	0.0115	0.0063	0.0063	0.0010	0.0010	0.0000	0.0000	0.0646
8	0.0021	0.0010	0.0083	0.0115	0.0063	0.0042	0.0000	0.0000	0.0000	0.0000	0.0000	0.0333
9	0.0000	0.0000	0.0063	0.0021	0.0010	0.0021	0.0000	0.0010	0.0000	0.0000	0.0000	0.0125
10	0.0010	0.0010	0.0031	0.0021	0.0000	0.0000	0.0021	0.0000	0.0000	0.0000	0.0000	0.0094
v_{2i}	0.0583	0.0969	0.1385	0.1708	0.1813	0.1469	0.1073	0.0625	0.0292	0.0052	0.0031	$n = 960$

2007	Female											
Male	0	1	2	3	4	5	6	7	8	9	10	v_{1i}
0	0.0259	0.0095	0.0060	0.0043	0.0026	0.0009	0.0009	0.0009	0.0000	0.0000	0.0000	0.0509
1	0.0198	0.0164	0.0216	0.0172	0.0103	0.0069	0.0026	0.0026	0.0026	0.0017	0.0000	0.1017
2	0.0095	0.0190	0.0181	0.0241	0.0250	0.0121	0.0155	0.0086	0.0060	0.0017	0.0026	0.1422
3	0.0060	0.0147	0.0241	0.0198	0.0259	0.0267	0.0103	0.0095	0.0052	0.0035	0.0000	0.1457
4	0.0052	0.0129	0.0319	0.0405	0.0388	0.0336	0.0267	0.0095	0.0060	0.0009	0.0000	0.2060
5	0.0017	0.0129	0.0207	0.0319	0.0371	0.0267	0.0190	0.0078	0.0043	0.0000	0.0000	0.1621
6	0.0017	0.0035	0.0086	0.0181	0.0259	0.0138	0.0129	0.0017	0.0000	0.0000	0.0000	0.0862
7	0.0017	0.0052	0.0129	0.0181	0.0121	0.0138	0.0026	0.0017	0.0000	0.0000	0.0000	0.0681
8	0.0000	0.0017	0.0086	0.0026	0.0069	0.0035	0.0009	0.0000	0.0000	0.0000	0.0000	0.0241
9	0.0000	0.0009	0.0043	0.0026	0.0017	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0095
10	0.0000	0.0009	0.0000	0.0000	0.0017	0.0009	0.0000	0.0000	0.0000	0.0000	0.0000	0.0035
v_{2i}	0.0716	0.0974	0.1569	0.1793	0.1879	0.1388	0.0914	0.0422	0.0241	0.0078	0.0026	$n = 1160$

2012	Female											
Male	0	1	2	3	4	5	6	7	8	9	10	v_{1i}
0	0.0212	0.0227	0.0106	0.0030	0.0045	0.0000	0.0030	0.0015	0.0015	0.0000	0.0000	0.0680
1	0.0212	0.0332	0.0227	0.0181	0.0136	0.0106	0.0030	0.0030	0.0000	0.0015	0.0000	0.1269
2	0.0166	0.0332	0.0423	0.0423	0.0212	0.0106	0.0076	0.0045	0.0015	0.0000	0.0000	0.1798
3	0.0166	0.0317	0.0529	0.0378	0.0242	0.0332	0.0136	0.0015	0.0015	0.0000	0.0015	0.2145
4	0.0015	0.0302	0.0483	0.0378	0.0302	0.0287	0.0181	0.0030	0.0045	0.0000	0.0015	0.2039
5	0.0045	0.0106	0.0121	0.0151	0.0257	0.0181	0.0121	0.0045	0.0000	0.0000	0.0000	0.1027
6	0.0060	0.0060	0.0121	0.0121	0.0151	0.0045	0.0060	0.0000	0.0000	0.0000	0.0000	0.0619
7	0.0015	0.0030	0.0015	0.0076	0.0076	0.0015	0.0015	0.0000	0.0000	0.0000	0.0000	0.0242
8	0.0015	0.0015	0.0030	0.0030	0.0015	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0106
9	0.0000	0.0000	0.0030	0.0030	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0060
10	0.0000	0.0000	0.0015	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0015
v_{2i}	0.0906	0.1722	0.2100	0.1798	0.1435	0.1073	0.0650	0.0181	0.0091	0.0015	0.0030	$n = 662$

2017	Female											
Male	0	1	2	3	4	5	6	7	8	9	10	v_{1i}
0	0.0364	0.0190	0.0111	0.0095	0.0032	0.0032	0.0000	0.0000	0.0000	0.0000	0.0000	0.0823
1	0.0269	0.0237	0.0285	0.0364	0.0158	0.0095	0.0079	0.0000	0.0016	0.0016	0.0016	0.1535
2	0.0095	0.0301	0.0301	0.0348	0.0269	0.0174	0.0127	0.0079	0.0016	0.0000	0.0016	0.1725
3	0.0142	0.0158	0.0317	0.0269	0.0348	0.0158	0.0158	0.0063	0.0032	0.0000	0.0000	0.1646
4	0.0048	0.0206	0.0269	0.0269	0.0269	0.0237	0.0269	0.0174	0.0111	0.0000	0.0016	0.1867
5	0.0016	0.0127	0.0158	0.0332	0.0301	0.0158	0.0095	0.0016	0.0000	0.0000	0.0000	0.1203
6	0.0016	0.0016	0.0142	0.0111	0.0111	0.0079	0.0032	0.0000	0.0000	0.0000	0.0000	0.0506
7	0.0032	0.0048	0.0063	0.0142	0.0095	0.0000	0.0016	0.0000	0.0000	0.0000	0.0000	0.0396
8	0.0000	0.0016	0.0048	0.0063	0.0016	0.0016	0.0000	0.0000	0.0000	0.0000	0.0000	0.0158
9	0.0000	0.0016	0.0032	0.0048	0.0016	0.0000	0.0000	0.0016	0.0000	0.0000	0.0000	0.0127
10	0.0000	0.0000	0.0000	0.0016	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0016
v_{2i}	0.0981	0.1313	0.1725	0.2057	0.1614	0.0949	0.0775	0.0348	0.0174	0.0016	0.0048	$n = 632$

Indonesia

1993	Female											
Male	0	1	2	3	4	5	6	7	8	9	10	v_{1i}
0	0.0321	0.0277	0.0140	0.0112	0.0061	0.0032	0.0022	0.0000	0.0004	0.0000	0.0000	0.0969
1	0.0360	0.0396	0.0310	0.0339	0.0205	0.0108	0.0061	0.0025	0.0011	0.0000	0.0004	0.1819
2	0.0213	0.0465	0.0465	0.0364	0.0256	0.0173	0.0108	0.0029	0.0011	0.0004	0.0000	0.2085
3	0.0151	0.0328	0.0439	0.0346	0.0331	0.0180	0.0115	0.0029	0.0014	0.0007	0.0000	0.1941
4	0.0112	0.0209	0.0299	0.0339	0.0249	0.0148	0.0086	0.0032	0.0007	0.0004	0.0000	0.1484
5	0.0036	0.0104	0.0180	0.0227	0.0144	0.0133	0.0036	0.0022	0.0004	0.0000	0.0000	0.0886
6	0.0022	0.0065	0.0122	0.0101	0.0090	0.0043	0.0032	0.0014	0.0000	0.0000	0.0000	0.0490
7	0.0007	0.0032	0.0058	0.0079	0.0022	0.0004	0.0007	0.0000	0.0000	0.0000	0.0000	0.0209
8	0.0000	0.0011	0.0018	0.0011	0.0022	0.0011	0.0000	0.0000	0.0000	0.0000	0.0000	0.0072
9	0.0000	0.0007	0.0004	0.0011	0.0007	0.0004	0.0000	0.0000	0.0000	0.0000	0.0000	0.0032
10	0.0000	0.0000	0.0007	0.0004	0.0004	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0014
v_{2i}	0.1221	0.1894	0.2042	0.1930	0.1390	0.0835	0.0468	0.0151	0.0050	0.0014	0.0004	$n = 2777$

1998	Female											
Male	0	1	2	3	4	5	6	7	8	9	10	v_{1i}
0	0.0260	0.0243	0.0188	0.0191	0.0055	0.0041	0.0012	0.0006	0.0006	0.0003	0.0000	0.1004
1	0.0301	0.0541	0.0463	0.0359	0.0182	0.0116	0.0049	0.0017	0.0000	0.0000	0.0000	0.2027
2	0.0229	0.0550	0.0648	0.0390	0.0307	0.0139	0.0061	0.0017	0.0023	0.0003	0.0000	0.2366
3	0.0139	0.0344	0.0448	0.0382	0.0243	0.0130	0.0055	0.0026	0.0015	0.0003	0.0000	0.1784
4	0.0113	0.0275	0.0312	0.0301	0.0176	0.0090	0.0043	0.0026	0.0015	0.0003	0.0000	0.1353
5	0.0041	0.0095	0.0182	0.0162	0.0133	0.0075	0.0035	0.0023	0.0003	0.0003	0.0000	0.0752
6	0.0026	0.0090	0.0095	0.0104	0.0067	0.0032	0.0017	0.0006	0.0003	0.0000	0.0000	0.0440
7	0.0020	0.0029	0.0052	0.0049	0.0017	0.0017	0.0000	0.0006	0.0000	0.0000	0.0000	0.0191
8	0.0000	0.0006	0.0017	0.0015	0.0015	0.0006	0.0003	0.0000	0.0000	0.0000	0.0000	0.0061
9	0.0000	0.0009	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0012
10	0.0003	0.0003	0.0003	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0012
v_{2i}	0.1131	0.2183	0.2412	0.1955	0.1194	0.0645	0.0275	0.0127	0.0064	0.0015	0.0000	$n = 3458$

2003	Female											
Male	0	1	2	3	4	5	6	7	8	9	10	v_{1i}
0	0.0278	0.0305	0.0242	0.0171	0.0078	0.0034	0.0020	0.0000	0.0002	0.0000	0.0000	0.1131
1	0.0288	0.0586	0.0625	0.0440	0.0193	0.0093	0.0017	0.0020	0.0002	0.0005	0.0000	0.2269
2	0.0325	0.0730	0.0733	0.0413	0.0239	0.0093	0.0054	0.0015	0.0005	0.0000	0.0002	0.2608
3	0.0200	0.0503	0.0498	0.0366	0.0183	0.0095	0.0046	0.0007	0.0005	0.0000	0.0000	0.1905
4	0.0137	0.0291	0.0242	0.0186	0.0151	0.0059	0.0037	0.0020	0.0005	0.0000	0.0002	0.1128
5	0.0046	0.0105	0.0147	0.0112	0.0054	0.0039	0.0012	0.0010	0.0002	0.0000	0.0000	0.0528
6	0.0022	0.0071	0.0049	0.0037	0.0024	0.0022	0.0022	0.0002	0.0002	0.0000	0.0000	0.0252
7	0.0024	0.0044	0.0024	0.0017	0.0010	0.0012	0.0007	0.0000	0.0002	0.0000	0.0000	0.0142
8	0.0002	0.0010	0.0007	0.0005	0.0005	0.0000	0.0002	0.0000	0.0000	0.0000	0.0000	0.0032
9	0.0000	0.0002	0.0005	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0007
10	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
v_{2i}	0.1324	0.2647	0.2571	0.1746	0.0938	0.0447	0.0217	0.0073	0.0027	0.0005	0.0005	$n = 4095$

2008	Female											
Male	0	1	2	3	4	5	6	7	8	9	10	v_{1i}
0	0.0302	0.0318	0.0361	0.0159	0.0093	0.0043	0.0007	0.0005	0.0000	0.0000	0.0000	0.1287
1	0.0347	0.0828	0.0763	0.0384	0.0150	0.0079	0.0020	0.0016	0.0002	0.0000	0.0002	0.2592
2	0.0388	0.0806	0.0665	0.0422	0.0184	0.0084	0.0041	0.0014	0.0007	0.0007	0.0002	0.2619
3	0.0295	0.0481	0.0404	0.0250	0.0170	0.0057	0.0032	0.0018	0.0011	0.0000	0.0000	0.1718
4	0.0141	0.0243	0.0229	0.0163	0.0100	0.0048	0.0030	0.0014	0.0005	0.0002	0.0000	0.0974
5	0.0052	0.0104	0.0084	0.0082	0.0061	0.0050	0.0014	0.0005	0.0002	0.0000	0.0000	0.0454
6	0.0023	0.0043	0.0052	0.0048	0.0030	0.0009	0.0020	0.0005	0.0000	0.0000	0.0000	0.0229
7	0.0009	0.0016	0.0009	0.0039	0.0009	0.0009	0.0002	0.0000	0.0000	0.0000	0.0000	0.0093
8	0.0005	0.0002	0.0007	0.0007	0.0000	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000	0.0023
9	0.0000	0.0000	0.0002	0.0002	0.0000	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000	0.0007
10	0.0000	0.0000	0.0002	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0005
v_{2i}	0.1562	0.2842	0.2578	0.1557	0.0797	0.0384	0.0166	0.0075	0.0027	0.0009	0.0005	$n = 4406$

2013						Female					
Male	0	1	2	3	4	5	6	7	8	9	10	v_{1i}
0	0.0557	0.0301	0.0396	0.0215	0.0096	0.0040	0.0006	0.0002	0.0000	0.0000	0.0000	0.1612
1	0.0350	0.1101	0.0844	0.0360	0.0149	0.0068	0.0030	0.0008	0.0004	0.0000	0.0000	0.2913
2	0.0444	0.0987	0.0712	0.0310	0.0151	0.0070	0.0014	0.0010	0.0008	0.0000	0.0000	0.2707
3	0.0322	0.0462	0.0328	0.0205	0.0106	0.0052	0.0026	0.0012	0.0002	0.0002	0.0000	0.1516
4	0.0113	0.0205	0.0175	0.0135	0.0048	0.0040	0.0008	0.0004	0.0004	0.0000	0.0000	0.0732
5	0.0050	0.0078	0.0058	0.0068	0.0038	0.0010	0.0002	0.0002	0.0004	0.0000	0.0000	0.0309
6	0.0018	0.0022	0.0034	0.0032	0.0022	0.0012	0.0000	0.0000	0.0000	0.0000	0.0000	0.0139
7	0.0000	0.0006	0.0012	0.0020	0.0000	0.0004	0.0002	0.0000	0.0000	0.0000	0.0000	0.0044
8	0.0002	0.0004	0.0006	0.0004	0.0004	0.0000	0.0002	0.0000	0.0000	0.0000	0.0000	0.0022
9	0.0000	0.0000	0.0000	0.0002	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0004
10	0.0000	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002
v_{2i}	0.1857	0.3166	0.2565	0.1351	0.0615	0.0295	0.0090	0.0038	0.0022	0.0002	0.0000	$n = 5025$

2018						Female					
Male	0	1	2	3	4	5	6	7	8	9	10	v_{1i}
0	0.0610	0.0423	0.0584	0.0252	0.0094	0.0024	0.0005	0.0002	0.0000	0.0000	0.0000	0.1994
1	0.0463	0.1271	0.0953	0.0385	0.0112	0.0035	0.0011	0.0011	0.0002	0.0000	0.0000	0.3242
2	0.0602	0.1039	0.0595	0.0227	0.0102	0.0037	0.0013	0.0005	0.0000	0.0002	0.0000	0.2621
3	0.0308	0.0413	0.0262	0.0153	0.0080	0.0037	0.0016	0.0000	0.0000	0.0000	0.0000	0.1269
4	0.0117	0.0182	0.0110	0.0056	0.0059	0.0013	0.0005	0.0002	0.0000	0.0000	0.0000	0.0543
5	0.0037	0.0061	0.0062	0.0037	0.0021	0.0005	0.0003	0.0002	0.0000	0.0000	0.0000	0.0227
6	0.0010	0.0018	0.0019	0.0010	0.0005	0.0008	0.0002	0.0000	0.0000	0.0000	0.0000	0.0070
7	0.0000	0.0003	0.0002	0.0011	0.0005	0.0002	0.0002	0.0000	0.0000	0.0000	0.0000	0.0024
8	0.0002	0.0002	0.0000	0.0000	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0006
9	0.0000	0.0000	0.0000	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0003
10	0.0000	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002
v_{2i}	0.2147	0.3413	0.2587	0.1133	0.0480	0.0160	0.0056	0.0021	0.0002	0.0002	0.0000	$n = 6265$

Appendix C: Estimated Offspring Distribution, Standard Error and bias-corrected and accelerated (BCa) bootstrap interval

Ghana

Prob. **2008**	estimate	st. error	Bca Confidence Interval	
p0	0.0113	0.0051	0.0023	0.0230
p1	0.0138	0.0059	0.0046	0.0276
p2	0.0720	0.0127	0.0460	0.0966
p3	0.1194	0.0155	0.0897	0.1494
p4	0.1445	0.0164	0.1126	0.1816
p5	0.1212	0.0161	0.0920	0.1540
p6	0.1310	0.0160	0.1011	0.1609
p7	0.1426	0.0168	0.1103	0.1770
p8	0.1202	0.0156	0.0897	0.1494
p9	0.0554	0.0110	0.0345	0.0759
p10	0.0686	0.0118	0.0460	0.0920

Prob. **2013**	estimate	st. error	Bca Confidence Interval	
p0	0.0173	0.0044	0.0092	0.0264
p1	0.0464	0.0071	0.0345	0.0609
p2	0.0985	0.0100	0.0793	0.1184
p3	0.0918	0.0101	0.0724	0.1126
p4	0.1223	0.0105	0.1023	0.1414
p5	0.1457	0.0118	0.1241	0.1701
p6	0.1781	0.0130	0.1540	0.2023
p7	0.1228	0.0118	0.1011	0.1460
p8	0.0736	0.0089	0.0575	0.0920
p9	0.0607	0.0084	0.0448	0.0770
p10	0.0430	0.0066	0.0299	0.0552

Prob. **2018**	estimate	st. error	Bca Confidence Interval	
p0	0.0248	0.0071	0.0124	0.0394
p1	0.0416	0.0089	0.0228	0.0581
p2	0.0785	0.0124	0.0560	0.1058
p3	0.1348	0.0160	0.1037	0.1660
p4	0.1475	0.0164	0.1141	0.1805
p5	0.1223	0.0153	0.0934	0.1535
p6	0.1369	0.0154	0.1079	0.1639
p7	0.0994	0.0132	0.0726	0.1245
p8	0.0796	0.0122	0.0539	0.1017
p9	0.0661	0.0114	0.0456	0.0892
p10	0.0685	0.0116	0.0456	0.0913

Bangladesh

Prob. **1993**	estimate	st. error	Bca Confidence Interval	
p0	0.0062	0.0030	0.0016	0.0124
p1	0.0155	0.0049	0.0062	0.0248
p2	0.0342	0.0070	0.0217	0.0481
p3	0.0481	0.0086	0.0326	0.0652
p4	0.0839	0.0109	0.0637	0.1040
p5	0.1149	0.0125	0.0916	0.1413
p6	0.1351	0.0131	0.1087	0.1599
p7	0.1630	0.0142	0.1351	0.1925
p8	0.1320	0.0130	0.1056	0.1568
p9	0.1118	0.0127	0.0885	0.1351
p10	0.1553	0.0148	0.1289	0.1848

Prob. **1998**	estimate	st. error	Bca Confidence Interval	
p0	0.0189	0.0047	0.0107	0.0272
p1	0.0284	0.0056	0.0189	0.0402
p2	0.0414	0.0068	0.0284	0.0556
p3	0.0722	0.0092	0.0544	0.0899
p4	0.1243	0.0114	0.1030	0.1467
p5	0.1420	0.0121	0.1183	0.1669
p6	0.1562	0.0124	0.1314	0.1799
p7	0.1491	0.0123	0.1266	0.1751
p8	0.1077	0.0106	0.0876	0.1278
p9	0.0722	0.0090	0.0556	0.0899
p10	0.0876	0.0096	0.0698	0.1089

Prob. **2003**	estimate	st. error	Bca Confidence Interval	
p0	0.0167	0.0040	0.0102	0.0251
p1	0.0195	0.0042	0.0112	0.0270
p2	0.0642	0.0076	0.0493	0.0781
p3	0.1079	0.0096	0.0902	0.1256
p4	0.1563	0.0110	0.1358	0.1795
p5	0.1526	0.0112	0.1321	0.1749
p6	0.1498	0.0109	0.1284	0.1712
p7	0.1154	0.0098	0.0967	0.1358
p8	0.0940	0.0087	0.0781	0.1107
p9	0.0558	0.0068	0.0428	0.0707
p10	0.0679	0.0075	0.0521	0.0800

Prob. **2008**	estimate	st. error	Bca Confidence Interval	
p0	0.0249	0.0036	0.0179	0.0325
p1	0.0449	0.0048	0.0346	0.0536
p2	0.1288	0.0077	0.1136	0.1439
p3	0.1834	0.0090	0.1645	0.1991
p4	0.1759	0.0090	0.1580	0.1905
p5	0.1575	0.0087	0.1402	0.1732
p6	0.1104	0.0075	0.0974	0.1255
p7	0.0850	0.0067	0.0731	0.0979
p8	0.0444	0.0047	0.0352	0.0541
p9	0.0233	0.0036	0.0168	0.0303
p10	0.0217	0.0033	0.0162	0.0292

Prob. **2013**	estimate	st. error	Bca Confidence Interval	
p0	0.0187	0.0032	0.0121	0.0248
p1	0.0496	0.0051	0.0397	0.0595
p2	0.1406	0.0079	0.1240	0.1566
p3	0.2095	0.0094	0.1913	0.2266
p4	0.1996	0.0094	0.1803	0.2166
p5	0.1698	0.0085	0.1521	0.1863
p6	0.1009	0.0074	0.0882	0.1163
p7	0.0579	0.0057	0.0480	0.0689
p8	0.0243	0.0037	0.0171	0.0314
p9	0.0160	0.0030	0.0105	0.0215
p10	0.0132	0.0027	0.0083	0.0187

Prob. **2018**	estimate	st. error	Bca Confidence Interval	
p0	0.0145	0.0025	0.0098	0.0200
p1	0.0465	0.0044	0.0379	0.0546
p2	0.1735	0.0080	0.1573	0.1884
p3	0.2255	0.0088	0.2084	0.2430
p4	0.2251	0.0088	0.2080	0.2413
p5	0.1330	0.0071	0.1194	0.1466
p6	0.0878	0.0057	0.0754	0.0993
p7	0.0541	0.0046	0.0448	0.0622
p8	0.0239	0.0032	0.0179	0.0303
p9	0.0090	0.0019	0.0055	0.0132
p10	0.0073	0.0017	0.0043	0.0107

Colombia

Prob. **1990**	estimate	st. error	Bca Confidence Interval	
p0	0.0650	0.0102	0.0462	0.0838
p1	0.0684	0.0104	0.0496	0.0906
p2	0.0974	0.0126	0.0735	0.1231
p3	0.1094	0.0124	0.0855	0.1350
p4	0.1316	0.0143	0.1026	0.1590
p5	0.1282	0.0137	0.1026	0.1556
p6	0.0940	0.0123	0.0701	0.1197
p7	0.0872	0.0117	0.0650	0.1111
p8	0.0684	0.0106	0.0479	0.0889
p9	0.0513	0.0091	0.0325	0.0701
p10	0.0992	0.0118	0.0769	0.1214

Prob. **1995**	estimate	st. error	Bca Interval	Confidence
p0	0.0783	0.0084	0.0626	0.0950
p1	0.0762	0.0084	0.0626	0.0950
p2	0.1409	0.0109	0.1200	0.1628
p3	0.1702	0.0121	0.1441	0.1921
p4	0.1566	0.0117	0.1326	0.1795
p5	0.1065	0.0100	0.0846	0.1242
p6	0.0877	0.0090	0.0710	0.1054
p7	0.0616	0.0077	0.0470	0.0772
p8	0.0355	0.0059	0.0240	0.0470
p9	0.0334	0.0061	0.0230	0.0480
p10	0.0532	0.0071	0.0397	0.0689

Prob. **2000**	estimate	st. error	Bca Interval	Confidence
p0	0.0781	0.0084	0.0623	0.0958
p1	0.0809	0.0086	0.0623	0.0967
p2	0.1544	0.0110	0.1330	0.1758
p3	0.2037	0.0123	0.1795	0.2279
p4	0.1572	0.0111	0.1340	0.1777
p5	0.1209	0.0099	0.1014	0.1405
p6	0.0791	0.0085	0.0633	0.0977
p7	0.0400	0.0060	0.0279	0.0521
p8	0.0316	0.0054	0.0223	0.0428
p9	0.0233	0.0047	0.0149	0.0316
p10	0.0307	0.0053	0.0205	0.0409

Prob. **2005**	estimate	st. error	Bca Confidence Interval	
p0	0.0643	0.0038	0.0568	0.0715
p1	0.0941	0.0046	0.0851	0.1026
p2	0.1894	0.0060	0.1773	0.2008
p3	0.2307	0.0067	0.2173	0.2425
p4	0.1611	0.0057	0.1492	0.1720
p5	0.1087	0.0048	0.0994	0.1181
p6	0.0585	0.0037	0.0521	0.0657
p7	0.0340	0.0029	0.0281	0.0393
p8	0.0252	0.0024	0.0206	0.0303
p9	0.0150	0.0019	0.0114	0.0187
p10	0.0192	0.0021	0.0150	0.0230

Prob. **2010**	estimate	st. error	Bca Confidence Interval	
p0	0.0638	0.0032	0.0581	0.0706
p1	0.1146	0.0043	0.1066	0.1231
p2	0.2334	0.0056	0.2228	0.2445
p3	0.2337	0.0055	0.2228	0.2448
p4	0.1408	0.0046	0.1320	0.1490
p5	0.0839	0.0037	0.0766	0.0910
p6	0.0538	0.0030	0.0484	0.0598
p7	0.0283	0.0021	0.0241	0.0324
p8	0.0206	0.0019	0.0170	0.0244
p9	0.0118	0.0015	0.0088	0.0146
p10	0.0153	0.0016	0.0125	0.0187

Prob. **2015**	estimate	st. error	Bca Confidence Interval	
p0	0.0519	0.0033	0.0457	0.0584
p1	0.1147	0.0048	0.1050	0.1242
p2	0.2614	0.0070	0.2484	0.2758
p3	0.2452	0.0064	0.2320	0.2577
p4	0.1388	0.0054	0.1291	0.1504
p5	0.0742	0.0041	0.0663	0.0818
p6	0.0452	0.0031	0.0385	0.0508
p7	0.0248	0.0024	0.0204	0.0299
p8	0.0178	0.0020	0.0141	0.0216
p9	0.0121	0.0016	0.0088	0.0153
p10	0.0139	0.0018	0.0104	0.0176

Burkina Faso

Prob. **1993**	estimate	st. error	Bca Confidence Interval	
p0	0.0245	0.0082	0.0109	0.0409
p1	0.0109	0.0053	0.0027	0.0218
p2	0.0109	0.0054	0.0027	0.0218
p3	0.0327	0.0096	0.0163	0.0545
p4	0.0627	0.0126	0.0381	0.0872
p5	0.0409	0.0102	0.0245	0.0627
p6	0.0926	0.0150	0.0654	0.1226
p7	0.1226	0.0174	0.0899	0.1580
p8	0.1717	0.0198	0.1281	0.2071
p9	0.1608	0.0191	0.1226	0.1989
p10	0.2698	0.0238	0.2207	0.3134

Prob. **1998**	estimate	st. error	Bca Confidence Interval	
p0	0.0105	0.0047	0.0021	0.0188
p1	0.0105	0.0048	0.0021	0.0209
p2	0.0209	0.0065	0.0084	0.0335
p3	0.0398	0.0089	0.0230	0.0565
p4	0.0523	0.0098	0.0335	0.0732
p5	0.0586	0.0102	0.0397	0.0795
p6	0.0941	0.0131	0.0690	0.1213
p7	0.1590	0.0167	0.1234	0.1883
p8	0.1527	0.0169	0.1213	0.1862
p9	0.1757	0.0178	0.1444	0.2134
p10	0.2259	0.0193	0.1883	0.2615

Prob. **2003**	estimate	st. error	Bca Confidence Interval	
p0	0.0093	0.0030	0.0046	0.0158
p1	0.0121	0.0033	0.0065	0.0186
p2	0.0251	0.0049	0.0158	0.0353
p3	0.0418	0.0061	0.0306	0.0548
p4	0.0529	0.0071	0.0399	0.0669
p5	0.0771	0.0079	0.0613	0.0929
p6	0.1170	0.0100	0.0994	0.1393
p7	0.1718	0.0114	0.1476	0.1922
p8	0.1671	0.0115	0.1458	0.1903
p9	0.1346	0.0103	0.1151	0.1551
p10	0.1913	0.0122	0.1671	0.2145

Prob. **2008**	estimate	st. error	Bca Confidence Interval	
p0	0.0091	0.0025	0.0045	0.0143
p1	0.0158	0.0033	0.0098	0.0226
p2	0.0158	0.0035	0.0090	0.0234
p3	0.0347	0.0049	0.0264	0.0452
p4	0.0618	0.0064	0.0498	0.0754
p5	0.1003	0.0082	0.0852	0.1169
p6	0.1516	0.0098	0.1312	0.1689
p7	0.1712	0.0101	0.1508	0.1908
p8	0.1833	0.0106	0.1614	0.2044
p9	0.1131	0.0084	0.0965	0.1275
p10	0.1433	0.0095	0.1252	0.1621

Prob. **2013**	estimate	st. error	Bca Confidence Interval	
p0	0.0106	0.0043	0.0035	0.0193
p1	0.0106	0.0042	0.0035	0.0193
p2	0.0158	0.0052	0.0070	0.0281
p3	0.0369	0.0076	0.0228	0.0510
p4	0.0580	0.0099	0.0387	0.0773
p5	0.0896	0.0119	0.0668	0.1125
p6	0.1722	0.0159	0.1406	0.2021
p7	0.1652	0.0158	0.1336	0.1968
p8	0.1424	0.0144	0.1160	0.1722
p9	0.1107	0.0133	0.0844	0.1388
p10	0.1881	0.0163	0.1564	0.2197

Prob. **2018**	estimate	st. error	Bca Confidence Interval	
p0	0.0056	0.0033	0.0000	0.0148
p1	0.0074	0.0037	0.0018	0.0148
p2	0.0129	0.0049	0.0055	0.0240
p3	0.0536	0.0096	0.0351	0.0721
p4	0.0758	0.0114	0.0536	0.0998
p5	0.1627	0.0166	0.1331	0.1959
p6	0.1571	0.0158	0.1238	0.1885
p7	0.1904	0.0166	0.1590	0.2218
p8	0.1257	0.0144	0.0980	0.1534
p9	0.1035	0.0138	0.0776	0.1312
p10	0.1054	0.0130	0.0813	0.1312

Indonesia

Prob. **1993**	estimate	st. error	Bca Confidence Interval	
p0	0.0321	0.0033	0.0252	0.0385
p1	0.0637	0.0049	0.0551	0.0735
p2	0.0749	0.0050	0.0648	0.0846
p3	0.1037	0.0058	0.0929	0.1145
p4	0.1304	0.0063	0.1185	0.1430
p5	0.1286	0.0064	0.1160	0.1404
p6	0.1156	0.0060	0.1041	0.1275
p7	0.1156	0.0059	0.1037	0.1275
p8	0.0947	0.0056	0.0839	0.1062
p9	0.0616	0.0043	0.0537	0.0702
p10	0.0792	0.0054	0.0688	0.0897

Prob. **1998**	estimate	st. error	Bca Confidence Interval	
p0	0.0260	0.0027	0.0211	0.0312
p1	0.0544	0.0039	0.0460	0.0619
p2	0.0957	0.0051	0.0859	0.1056
p3	0.1342	0.0059	0.1229	0.1457
p4	0.1518	0.0063	0.1403	0.1645
p5	0.1377	0.0061	0.1264	0.1498
p6	0.1249	0.0056	0.1128	0.1353
p7	0.1030	0.0051	0.0925	0.1131
p8	0.0677	0.0042	0.0587	0.0752
p9	0.0460	0.0035	0.0393	0.0535
p10	0.0587	0.0040	0.0512	0.0665

Prob. **2003**	estimate	st. error	Bca Confidence Interval	
p0	0.0278	0.0026	0.0230	0.0327
p1	0.0593	0.0037	0.0520	0.0664
p2	0.1153	0.0051	0.1050	0.1250
p3	0.1727	0.0059	0.1609	0.1839
p4	0.1890	0.0059	0.1775	0.2007
p5	0.1475	0.0054	0.1365	0.1575
p6	0.1087	0.0047	0.0991	0.1177
p7	0.0720	0.0039	0.0645	0.0799
p8	0.0530	0.0036	0.0462	0.0598
p9	0.0247	0.0023	0.0198	0.0288
p10	0.0300	0.0027	0.0249	0.0354

Prob. **2008**	estimate	st. error	Bca Confidence Interval	
p0	0.0302	0.0027	0.0250	0.0359
p1	0.0665	0.0037	0.0592	0.0740
p2	0.1577	0.0055	0.1464	0.1680
p3	0.2022	0.0062	0.1904	0.2147
p4	0.1764	0.0059	0.1639	0.1870
p5	0.1314	0.0050	0.1221	0.1414
p6	0.0876	0.0043	0.0790	0.0953
p7	0.0579	0.0035	0.0515	0.0651
p8	0.0368	0.0028	0.0313	0.0420
p9	0.0216	0.0023	0.0172	0.0259
p10	0.0318	0.0026	0.0270	0.0372

Prob. **2013**	estimate	st. error	Bca Interval	Confidence
p0	0.0557	0.0033	0.0501	0.0627
p1	0.0651	0.0035	0.0587	0.0714
p2	0.1940	0.0057	0.1829	0.2052
p3	0.2368	0.0060	0.2259	0.2490
p4	0.1743	0.0054	0.1636	0.1847
p5	0.1083	0.0047	0.0989	0.1174
p6	0.0701	0.0036	0.0635	0.0774
p7	0.0422	0.0027	0.0368	0.0478
p8	0.0231	0.0021	0.0193	0.0271
p9	0.0165	0.0018	0.0131	0.0199
p10	0.0139	0.0017	0.0105	0.0173

Prob. **2018**	estimate	st. error	Bca Confidence Interval	
p0	0.0610	0.0028	0.0559	0.0672
p1	0.0886	0.0037	0.0817	0.0966
p2	0.2457	0.0054	0.2346	0.2565
p3	0.2552	0.0057	0.2439	0.2666
p4	0.1604	0.0048	0.1513	0.1705
p5	0.0843	0.0035	0.0774	0.0905
p6	0.0476	0.0027	0.0421	0.0528
p7	0.0265	0.0020	0.0227	0.0305
p8	0.0180	0.0017	0.0150	0.0214
p9	0.0069	0.0011	0.0049	0.0091
p10	0.0059	0.0010	0.0042	0.0078

Mali

Prob. **1992**	estimate	st. error	Bca Confidence Interval	
p0	0.0338	0.0117	0.0127	0.0591
p1	0.0380	0.0124	0.0169	0.0633
p2	0.0338	0.0116	0.0127	0.0591
p3	0.0591	0.0154	0.0295	0.0928
p4	0.0802	0.0176	0.0464	0.1181
p5	0.0591	0.0148	0.0295	0.0886
p6	0.1139	0.0204	0.0759	0.1561
p7	0.0844	0.0174	0.0506	0.1181
p8	0.0675	0.0158	0.0380	0.0970
p9	0.1435	0.0234	0.0928	0.1857
p10	0.2869	0.0284	0.2278	0.3376

Prob. **1997**	estimate	st. error	Bca Confidence Interval	
p0	0.0364	0.0069	0.0243	0.0499
p1	0.0270	0.0059	0.0148	0.0378
p2	0.0337	0.0066	0.0216	0.0486
p3	0.0486	0.0079	0.0337	0.0661
p4	0.0513	0.0081	0.0364	0.0661
p5	0.0540	0.0085	0.0378	0.0702
p6	0.0877	0.0104	0.0675	0.1066
p7	0.1107	0.0115	0.0891	0.1336
p8	0.1066	0.0112	0.0850	0.1282
p9	0.1242	0.0120	0.1012	0.1498
p10	0.3198	0.0173	0.2874	0.3549

Prob. **2002**	estimate	st. error	Bca Confidence Interval	
p0	0.0260	0.0051	0.0156	0.0354
p1	0.0333	0.0058	0.0229	0.0448
p2	0.0260	0.0051	0.0156	0.0354
p3	0.0365	0.0060	0.0250	0.0479
p4	0.0458	0.0068	0.0323	0.0594
p5	0.0698	0.0079	0.0552	0.0865
p6	0.0667	0.0083	0.0500	0.0823
p7	0.0865	0.0088	0.0688	0.1031
p8	0.1563	0.0120	0.1333	0.1813
p9	0.1354	0.0111	0.1135	0.1563
p10	0.3177	0.0153	0.2885	0.3458

Prob. **2007**	estimate	st. error	Bca Confidence Interval	
p0	0.0259	0.0047	0.0172	0.0353
p1	0.0293	0.0049	0.0198	0.0397
p2	0.0319	0.0051	0.0224	0.0422
p3	0.0509	0.0063	0.0388	0.0638
p4	0.0578	0.0068	0.0448	0.0707
p5	0.0741	0.0077	0.0603	0.0897
p6	0.0991	0.0088	0.0819	0.1155
p7	0.1078	0.0092	0.0905	0.1267
p8	0.1293	0.0099	0.1103	0.1483
p9	0.1250	0.0098	0.1069	0.1457
p10	0.2690	0.0129	0.2448	0.2948

Prob. **2012**	estimate	st. error	Bca Confidence Interval	
p0	0.0212	0.0057	0.0106	0.0317
p1	0.0438	0.0078	0.0287	0.0589
p2	0.0604	0.0090	0.0453	0.0785
p3	0.0755	0.0101	0.0559	0.0937
p4	0.0982	0.0121	0.0755	0.1239
p5	0.1435	0.0136	0.1148	0.1707
p6	0.1375	0.0135	0.1103	0.1616
p7	0.0967	0.0114	0.0740	0.1178
p8	0.1073	0.0121	0.0831	0.1299
p9	0.0876	0.0112	0.0665	0.1073
p10	0.1284	0.0134	0.1012	0.1541

Prob. **2017**	estimate	st. error	Bca Confidence Interval	
p0	0.0364	0.0073	0.0222	0.0506
p1	0.0459	0.0083	0.0301	0.0617
p2	0.0443	0.0082	0.0285	0.0601
p3	0.0823	0.0110	0.0601	0.1028
p4	0.0902	0.0113	0.0696	0.1139
p5	0.1076	0.0124	0.0823	0.1297
p6	0.1044	0.0121	0.0807	0.1282
p7	0.1076	0.0122	0.0823	0.1313
p8	0.1076	0.0121	0.0839	0.1313
p9	0.0981	0.0113	0.0759	0.1203
p10	0.1756	0.0150	0.1456	0.2057

Philippines

Prob. **1993**	estimate	st. error	Bca Confidence Interval	
p0	0.0804	0.0081	0.0656	0.0970
p1	0.0367	0.0056	0.0271	0.0490
p2	0.0682	0.0076	0.0542	0.0830
p3	0.1136	0.0093	0.0962	0.1329
p4	0.1276	0.0100	0.1093	0.1477
p5	0.1250	0.0099	0.1049	0.1434
p6	0.0953	0.0085	0.0795	0.1119
p7	0.0883	0.0086	0.0726	0.1049
p8	0.0857	0.0081	0.0708	0.1014
p9	0.0612	0.0070	0.0472	0.0752
p10	0.1180	0.0093	0.0988	0.1355

Prob. **1998**	estimate	st. error	Bca Confidence Interval	
p0	0.0790	0.0078	0.0642	0.0947
p1	0.0502	0.0063	0.0379	0.0617
p2	0.0922	0.0084	0.0765	0.1086
p3	0.1177	0.0095	0.0996	0.1358
p4	0.1276	0.0095	0.1095	0.1465
p5	0.1062	0.0088	0.0889	0.1235
p6	0.1119	0.0089	0.0955	0.1309
p7	0.1095	0.0092	0.0914	0.1267
p8	0.0593	0.0070	0.0461	0.0733
p9	0.0527	0.0061	0.0403	0.0650
p10	0.0938	0.0084	0.0774	0.1103

Prob. **2003**	estimate	st. error	Bca Interval	Confidence
p0	0.0603	0.0064	0.0484	0.0730
p1	0.0544	0.0061	0.0417	0.0655
p2	0.1065	0.0088	0.0901	0.1243
p3	0.1534	0.0098	0.1355	0.1735
p4	0.1586	0.0103	0.1392	0.1794
p5	0.1258	0.0094	0.1065	0.1445
p6	0.1035	0.0083	0.0871	0.1199
p7	0.0760	0.0073	0.0611	0.0901
p8	0.0581	0.0064	0.0454	0.0707
p9	0.0357	0.0049	0.0268	0.0454
p10	0.0678	0.0067	0.0536	0.0804

Prob. **2008**	estimate	st. error	Bca Confidence Interval	
p0	0.0727	0.0069	0.0595	0.0865
p1	0.0664	0.0068	0.0540	0.0803
p2	0.1294	0.0084	0.1128	0.1453
p3	0.1869	0.0106	0.1654	0.2069
p4	0.1343	0.0090	0.1156	0.1509
p5	0.1121	0.0080	0.0955	0.1273
p6	0.0872	0.0072	0.0740	0.1024
p7	0.0671	0.0063	0.0554	0.0796
p8	0.0491	0.0059	0.0374	0.0609
p9	0.0339	0.0047	0.0242	0.0422
p10	0.0609	0.0063	0.0484	0.0734

Prob. **2013**	estimate	st. error	Bca Confidence Interval	
p0	0.0803	0.0063	0.0677	0.0930
p1	0.0754	0.0062	0.0633	0.0875
p2	0.1513	0.0082	0.1342	0.1672
p3	0.1524	0.0083	0.1375	0.1689
p4	0.1584	0.0085	0.1408	0.1749
p5	0.1155	0.0074	0.1018	0.1315
p6	0.0858	0.0068	0.0737	0.1001
p7	0.0567	0.0055	0.0451	0.0671
p8	0.0501	0.0050	0.0402	0.0605
p9	0.0297	0.0039	0.0226	0.0374
p10	0.0446	0.0049	0.0347	0.0545

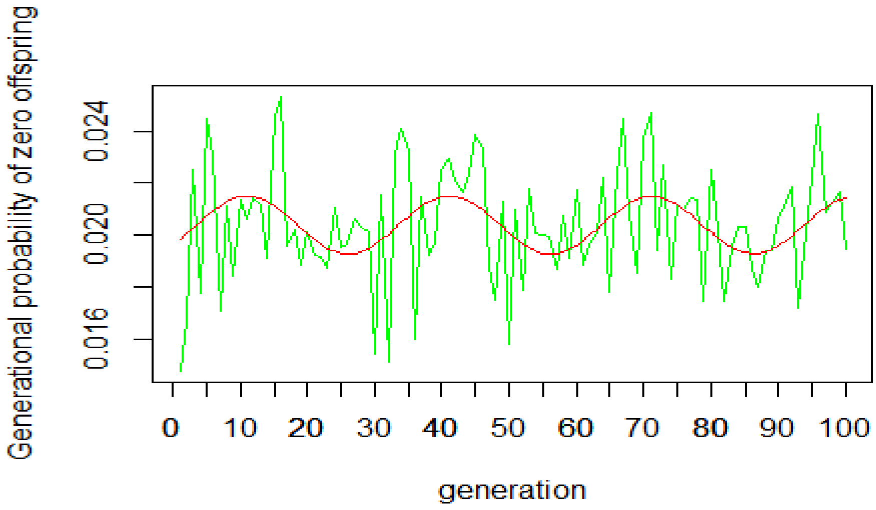
Prob. **2018**	estimate	st. error	Bca Confidence Interval	
p0	0.0839	0.0050	0.0738	0.0929
p1	0.0915	0.0056	0.0814	0.1023
p2	0.1549	0.0065	0.1420	0.1671
p3	0.1848	0.0074	0.1712	0.1994
p4	0.1580	0.0069	0.1448	0.1723
p5	0.1107	0.0057	0.0989	0.1215
p6	0.0794	0.0053	0.0689	0.0891
p7	0.0536	0.0042	0.0456	0.0620
p8	0.0313	0.0031	0.0254	0.0376
p9	0.0265	0.0029	0.0205	0.0327
p10	0.0254	0.0030	0.0198	0.0317

Appendix D: Multivariate Normality Test for the Hypothetical Data

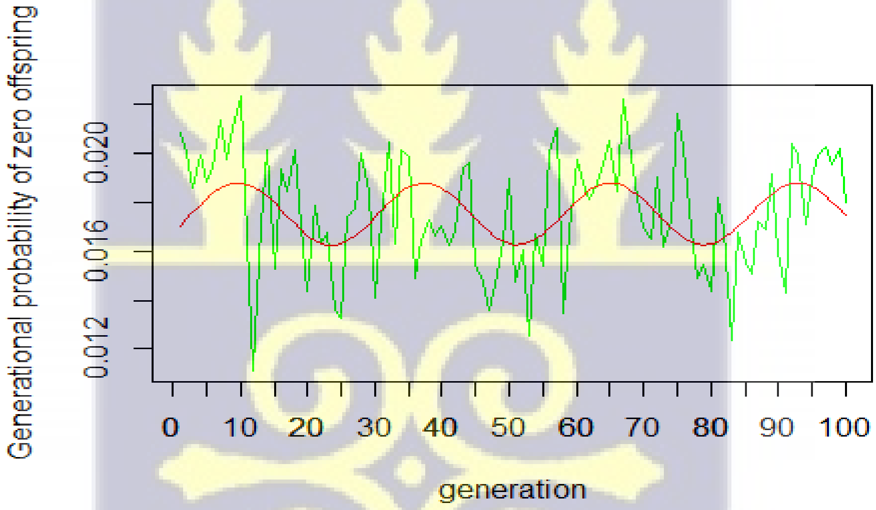
Henze-Zirkler Multivariate Normality Test

Population	Henze-Zirkler Test value	P-value
π_1	1.0913	0.1154
π_2	0.7954	0.6662
π_3	0.9868	0.2482
π_4	0.9564	0.3019
π_5	0.9026	0.2273
π_6	0.9753	0.2677
π_7	0.6441	0.9352
π_8	0.8620	0.5062

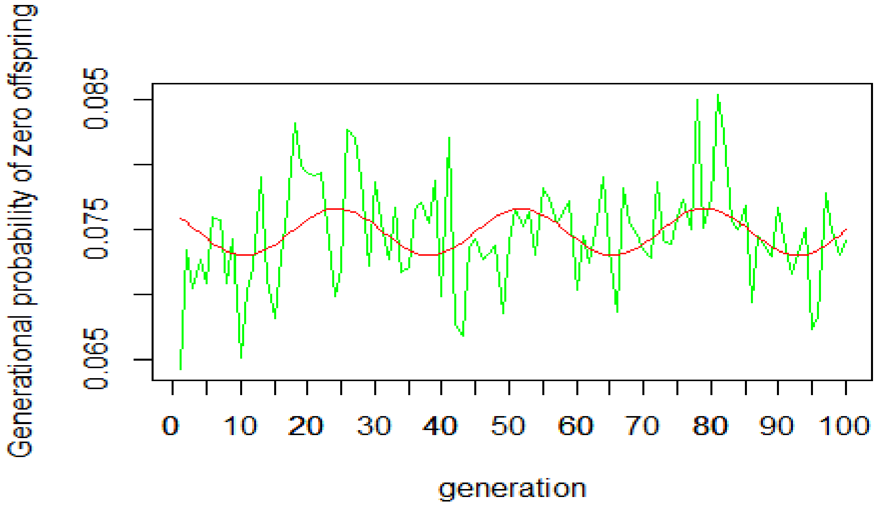
Appendix E: Spectral Plot



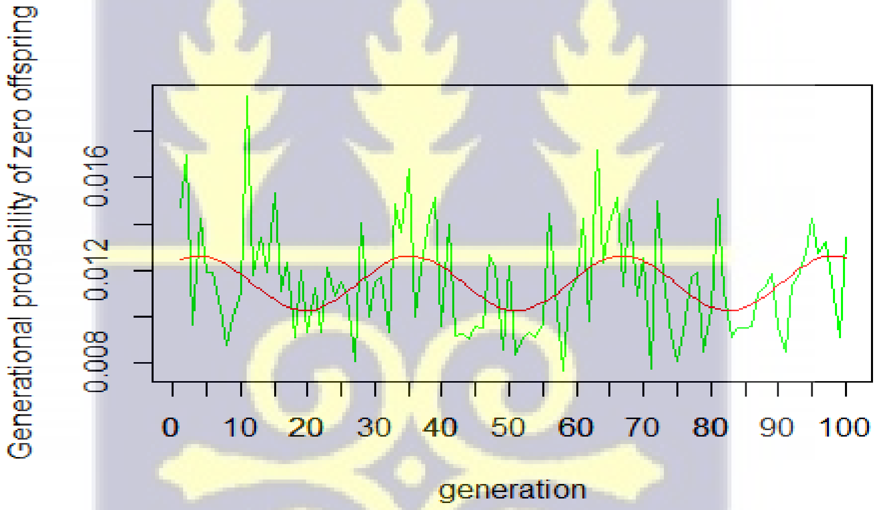
Spectral Plot of Ghana



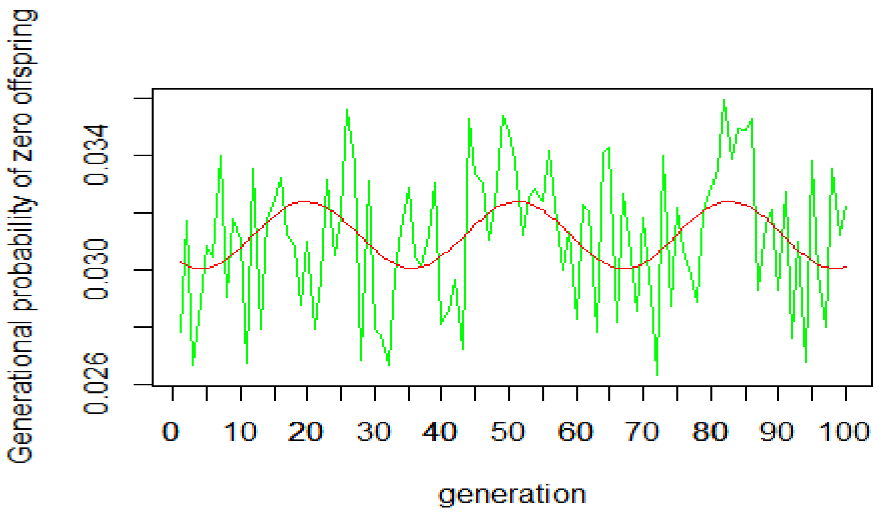
Spectral Plot of Bangladesh



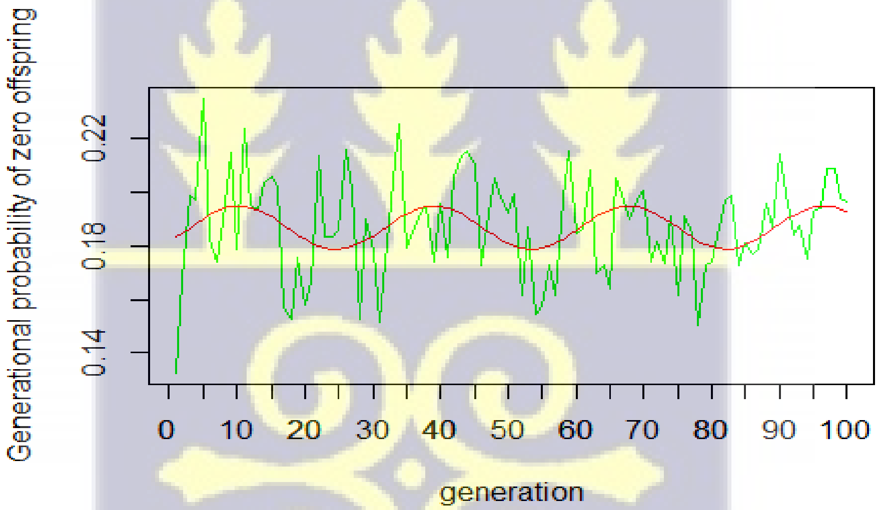
Spectral Plot of Colombia



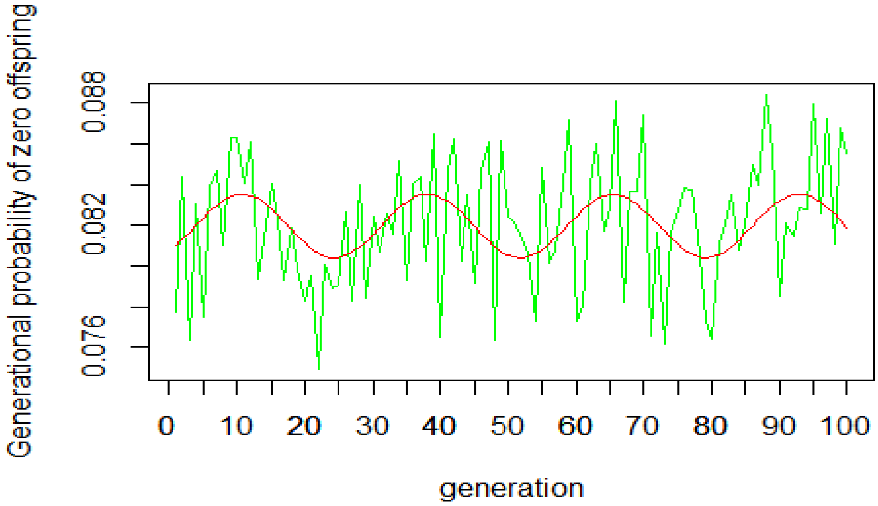
Spectral Plot of Burkina Faso



Spectral Plot of Mali



Spectral Plot of Indonesia



Spectral Plot of Philippines