

# UNIVERSITY OF GHANA



## **Large Deviations of the Throughput in Multi-Channel Medium-Access Protocols**

A thesis Submitted to the  
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in

**Statistics**



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## Declaration

I declare that the entirety of the work contained therein is my own, original work towards the award of the PhD degree and that, to the best of my knowledge, it contains no material previously published by another person nor material which had been accepted for the award of any other degree of the university, except where due acknowledgement had been made in the text.

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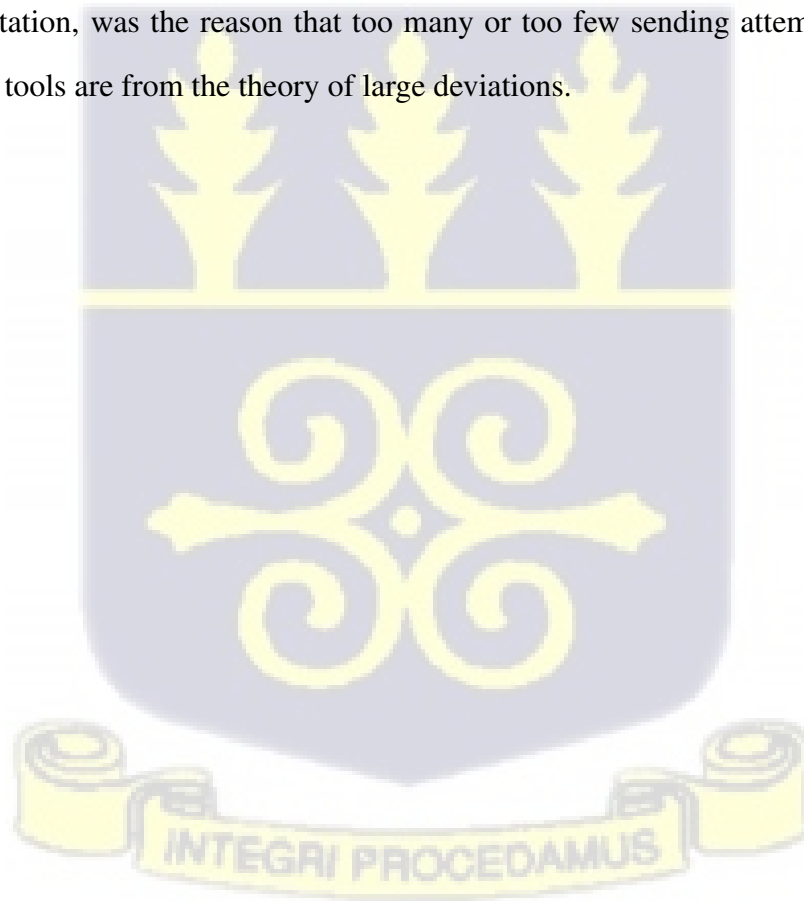
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## ABSTRACT

This thesis considers Aloha and slotted Aloha protocols as medium access rules for a multi-channel message delivery system. Users decide randomly and independently with a minimal amount of knowledge about the system at random times to make a sending attempt. The system has a fixed number of available channels; equivalently, interference constraints make the delivery of too many messages at a time impossible. We derive probabilistic formulas for the most important quantities like the number of successfully delivered messages and the number of sending attempts, and we derive large-deviation principles for these quantities in the limit of many participants and many sending attempts. We analyse the rate functions and their minimizers and derive laws of large numbers. In particular, we are interested in questions like “if the number of successfully delivered messages is significantly lower than the expectation, was the reason that too many or too few sending attempts were made?”. The main tools are from the theory of large deviations.



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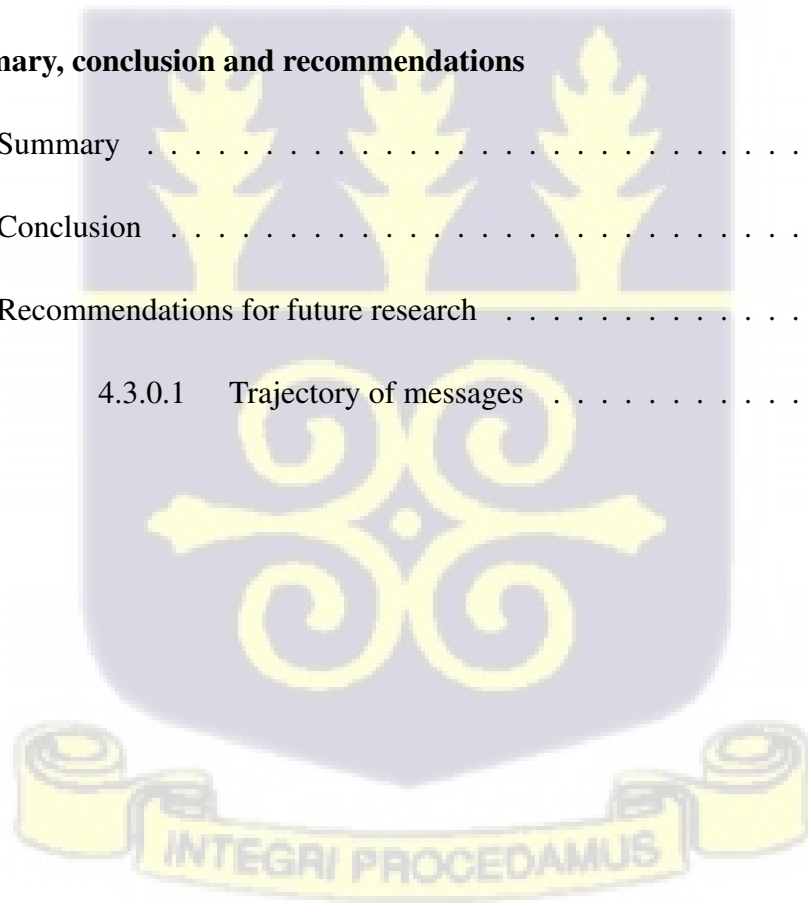


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# Chapter 1

## Introduction

### 1.1 Overview

The probabilistic analysis of large telecommunication systems is one of the cornerstones of research towards developing new technologies in the telecommunication industry. In this thesis, we consider capacity and connectivity of network protocols, where we consider users located in space who intend to transmit messages to some intended receivers. The ubiquitous questions are about connectivity and capacity, the two most important characteristic quantities of telecommunication systems.

For many years, probabilistic tools with the help of random point processes and stochastic geometry have been used successfully for the theoretical analysis of spatially distributed wireless networks. However, in spite of these ideas, which obviously has a great potential for technical progress, such kind of networks have hardly been established in reality yet. It seems as though the precise set-up that makes the idea successful has not yet been identified. A number of new questions arise for such kind of systems. The most obvious questions are about connectivity, interference and capacity which are much more prominent problems. One has to admit that the functionality of an ad-hoc multi-hop network has not yet been

sufficiently understood in terms of its consequences and properties.

There is not yet a sufficient understanding of the influence of the decisive parameters on the performance qualities of telecommunication systems. In particular, sufficient reliable rigorous research and resulting rules of thumbs are currently missing or have not yet been developed. Questions about the most important quantities like connectivity under the influence of interference, network capacity under restrictions of storage size of the user equipments, reach of message trajectories, algorithms for finding optimal routings, dependence on the environment and more, still gives ample room for rigorous scientific research, notably for mathematical research. In particular, research in the field of large deviations and stochastic geometry are necessary. The seminal paper Gupta and Kumar (2000) (from the field of information science), which is the first (to our best of knowledge) rigorous investigation of capacity in a multi-hop system with unbounded number of steps and under the influence of interference; provoked a lot of successive works in the sequel. In the use of large deviations for analysing connectivity, not much has been done. Engineers who work in this area rely mostly on simulations for drawing conclusions.

## 1.2 Problem statement

In telecommunication networks, specifically in the newly introduced 5G networks where mobile and other devices are used as transmitters instead of base stations, many advantages together with some disadvantages arise. There is the advantage of reducing cost when using device to device (D2D) transmissions as it is not heavily dependent on costly infrastructure installations. The question of connectivity however comes into play. This is due to the fact that network connection depends on the availability of enough active devices in a vicinity.

There is also the question of how many hops should a typical trajectory make from a transmitter to an intended receiver in comparison to Euclidean distance. There is also the problem

of optimal sending probability that network devices should allocate to individual devices in order to achieve maximum throughput.

In addressing these issues, several researchers have looked at medium access protocols. Some of these protocols are Aloha and its slotted versions, Carrier sense medium access (CSMA) and other protocols that control how messages are sent out. In Ganti and Haenggi (2009), it is assumed that network nodes form a Poisson Point Process (PPP) on a plane. Time is divided (slotted) and at every time slot, each node transmits with probability  $p$ . Further it is assumed that a node  $x$  can transmit to a node  $y$  only if  $B(y, \beta|x - y|)$  for  $\beta > 0$  contain no other transmitting node. Here  $B(y, r)$  represents a ball centered around  $y$  with a radius of  $r$ . Like many other research, Aloha models considered allow messages to be transmitted independently with probability  $p$  without considering interference and capacity. However, most messages fail to be transmitted due to interference or capacity constraints. It is therefore important to consider the introduction of interference or capacity constraints in these models hence computing the network connectivity metrics using appropriate mathematical tools in order to assess the quality of the protocol/strategy. It is clear that, due to interference constraints or restricted availability of communication channels, the task can be successfully managed only if a suitable rule is enforced that will regulate which of the many messages is gaining access to the medium at a given time. Clearly, this rule should be as simple as possible, and it should not require much infrastructure.

To this end, this research seeks to introduce capacity constraint into the ALOHA and slotted ALOHA models and study their connectivity metrics using large deviations. A similar constraint was introduced by König and Tóbiás (2019) in the Gibbsian model with the simplicity that the interference/capacity constraint is only present at the first hop of message transmission (Also see Munari et al., 2013). This research however, looks at a more realistic fact that at every hop there is an interference/capacity constraint which should be managed.

### 1.3 Objectives of the study

The main objective of the thesis is to study probabilistic random protocols for message trajectories in telecommunication networks with capacity constraints. The specific objectives are;

- Propose random protocols/strategies for describing message hops in wireless networks under capacity constraint.
- Derive large deviation principles for the proposed random protocols/strategies.
- Derive minimisers for the large deviation rate functions.
- Describe and analyse the conditions for attaining maximum throughput and the uniqueness of the throughput probability parameter.

### 1.4 Research questions

The following are the research questions;

- What probabilistic models could provide us strategies to describe message hops in wireless networks under capacity constraint?
- To what rates and rate functions would the probabilistic models be satisfying large deviation principle (LDP)?
- What are the minimisers for the rate functions?
- Is there an optimal throughput probability parameter? If any, is it unique?

## 1.5 Literature review

### 1.5.1 Interference constraint in telecommunication networks

The study of interference using large deviations was championed by Torisi and co-authors under different network settings (see for example, Ganesh and Torrisi, 2008; Torrisi and Leonardi, 2012; Leonardi and Torrisi, 2013).

Interference limits the capacity of networks thereby resulting in many frustrated users. Capacity of wireless networks which is the focus of this thesis is a form of interference which limits the number of messages that can be transmitted (throughput) at any point in time. There has been efforts in literature to study the behavior of interference with respect to different point process which determines how nodes of a network are distributed. Throughput probability in networks is influenced by the mutual interference among simultaneous transmissions over the same physical channel (See for example, Hunter et al., 2008; Torrisi and Leonardi, 2012)

Leonardi and Torrisi (2013) studied the tail behaviour of interference in networks whose nodes are distributed according to a Ginibre process over a bounded region in a plane. They assumed different distributional assumptions for the fading variable. Their results shows that, the dependence of interference on network nodes depends heavily on the distribution of the fading variable. Ganesh and Torrisi (2008) studied the behaviour of interference in wireless networks using large deviations. They described the network nodes by homogenous Poisson point process with random transmission powers (a combination of fading and path loss function).

These studies made different assumptions for the fading variable and leaves it open for a more general fading or power variable. Also, the distribution of network nodes can be extended to several other processes like the Matern process, the Cox point process and

others. It could also be studied for a more general point process. In this thesis, network nodes form a Poisson point process.

### **1.5.2 Capacity Constraints in Communication Networks**

With limited resources notwithstanding, the performance of communication networks are required to operate at optimum levels. Capacity constraints in communication networks has received maximum attention in relation to how to optimize resources and maximize the utility of transmission Li et al. (2017). Capacity crunch, which is associated with communication networks has shown to perform at a constrained capacity due to the exponentially increasing traffic demands. According to Ellis et al. (2016), communication networks need modest changes and optimisation solutions. In the study of Frank and El-Bardai (1969), they investigated the problem of finding best possible message routings that minimize network losses subject to network capacity constraints.

There is a general consensus on the need to find optimisation solutions to improve network capacity without necessarily expanding infrastructure due to budget constraints.

### **1.5.3 Computing the expected number of transmitters in a wireless network**

It is believed that interference/capacity constraint is a performance-limiting factor in wireless networks. One-shot scheduling of wireless network capacity has been a major problem in network protocols. In computing the expected number of transmitters that will maximize wireless network capacity, Dinitz (2010) developed a technique from machine learning and game theory. Their method saw some improvement in network protocols but still lacked a lot of randomness and could have some problems with implementation.

Multi-channel wireless Networks are known to have connectivity and capacity constraint

problems. In identifying and addressing performance of multi-channel network under channel constraints, Bhandari and Vaidya (2007) analysed capacity and connectivity metrics of a wireless network. Key was assessing the optimal number of transmitters for optimum throughput. However, the network structure they studied comprised of  $n$  randomly deployed nodes with only single channel or interface. This thesis, studies a slightly different regime with multiple channels each having an upper bound for number of allowable messages that can be transmitted at a given time.

#### **1.5.4 Throughput Properties of Telecommunication Networks**

Improving transmission efficiency within wireless telecommunication by avoiding transmission collision has been a research need (Dumas et al., 2021; Khan et al., 2021). Aloha protocol was developed to transmit data packets to nodes but usually results in collision. According to Khan et al. (2021), the efficiency of Aloha is twice less than slotted Aloha. According to (Dumas et al. (2021)), slotted Aloha was introduced because of the challenges identified with the Aloha protocol. Slotted Aloha defines slots of packets transmission to improve transmission efficiency .

#### **1.5.5 Medium access protocols**

Medium Access Protocols such as Aloha are very important in the successful deployment of wireless networks. In most protocols users manage resources in a decentralized manner (Haykin, 2005; Neel et al., 2002). There are several papers on designs of medium access protocols and how the existing ones can be improved to increase connectivity. The most basic protocol is the Aloha with its existing extensions such as the slotted Aloha. Different tools have been used to study connectivity properties such as expected number of successful transmissions, optimal transmission probability and number of collisions.

### 1.5.5.1 Aloha

Aloha is known to be the first random access protocol that was developed for communication networks (Médard et al., 2004). The quest to raise the performance of Aloha led to the design of slotted Aloha. In analyzing the performance metrics of Aloha and slotted Aloha, different flexible versions have been proposed and some attempts have been made to compute closed form expressions for their connectivity properties. However, in most cases engineers rely on simulations to study network metrics such as throughput.

Several papers have looked at different variants of the Aloha model under varying network assumptions. In Błaszczyszyn and Mühlethaler (2015), mobile ad-hoc network nodes comprising of transmitters and receivers are fixed and only the Medium access control (MAC) status evolves over time. Also, in Błaszczyszyn et al. (2012) receivers and transmitters are nearest neighbours forming a Poisson process with a key assumption that locations of the nodes are static.

Baccelli and Singh (2013) also studied spatial Aloha in which the medium access probability (MAP) of nodes is adapted to the local topology of their local environment. Achir et al. (2016) compared the performance of CSMA and spatial Aloha by computing their spatial throughput using stochastic geometry. In carrying out the comparison, both schemes are optimized by adjusting their parameters. The transmission probability for spatial Aloha is said to be adapted, whereas for spatial CSMA a suitable carrier sense threshold was found.

Xu et al. (2018) considered a maximization of the so called  $\alpha$ -fair utility in a spatial Aloha network. This network consisted of multiple transmitter-receiver (TR) pairs each forming a bipolar Poisson process. Communication is possible between pairs if signal to interference ratio criteria holds. They then try to optimize the medium access probability of each pair. They claim the model results in an optimization problem which is not straight forward to solve.

### 1.5.5.2 Slotted Aloha

Ma et al. (2008) used the slotted Aloha MAC protocol in implementing a system where transmission in a slot is dependent on the success of a nodes prior transmission. Using Markov chains, the throughput of the model was then studied. Jakovetić et al. (2014) considered the so called framed slotted Aloha. In this model, base stations collaborate to receive messages from some users. Within a communication radius  $r$ , transmitted messages are decoded by all base stations at each time frame.

Also, Gau (2006) considered slotted Aloha in interference induced network. He argued that an interference dominating network is a network that can simultaneously receive a number of messages from several transmitters, as long as the Signal to Noise and Interference Ratio (SINR) condition is satisfied. He employed discrete-time Markov chain for the analysis.

Li and Dai (2015) studied the performance of slotted Aloha networks. They considered a slotted Aloha model in which  $n$  nodes transmit to a single receiver with a perfect feedback after each transmission.

Some papers also focused on situations where the medium access probability adapts to the environment of the transmitting node. Hsu et al. (2012) worked on channel state information (CSI) in slotted Aloha where users are aware of their CSI. This model consist of  $N$  homogenous users who transmit messages to a common base station. Messages at one time slot are sent and the CSI is the  $SINR$  that is received. Users know their own CSI and as such adjust their transmission probability at the beginning of each time slot.

Lee et al. (2017) also studied spatial slotted-Aloha protocol in wireless networks. They proposed a slotted-Aloha medium access protocol with optimal transmission probability where each member of the network had its own transmission probability.

Coupechoux et al. (2004) analysed throughput of a multihop wireless network, where nodes transmit according to a slotted ALOHA protocol. They derived formulas for throughput

given that  $n$  messages are transmitted in a given neighbourhood with a receiver only able to decode  $k$  messages at a time. This is similar to the objectives of this study with the major difference being the mathematical tool and the quantities of interest used.

Using slotted Aloha as the medium access control and choosing an optimal transmission probability, the downward and upward link transmissions were studied using stochastic geometry. The proposed scheme ensured that the transmission probability depended on each transmitter in a group. Poisson point process was used to obtain network connectivity metrics for downward and upward links (Lee et al., 2017).

Several designs of slotted Aloha has indeed been proposed in literature. Bajović et al. (2014), made use of the slotted Aloha protocol. At each slot, network devices attempt to transmit their messages with some given probability. They analysed slotted and framed aloha in multiple base station and multiple-access systems. They analysed how at each slot, devices independently transmit messages with a certain probability. At a distance  $r$ , the messages can be intercepted by base stations. They designed it such that base stations and users are placed uniformly at random. This thesis also adopts a similar concept where users and time slots are uniformly distributed. Result of the performance metric of this study is based on simulations.

Yavascan and Uysal (2021) studied a modified version of slotted Aloha called threshold-Aloha where each terminal suspends its transmissions until the number of senders reach a certain threshold. Once number of senders reaches the threshold, transmission attempts are made with a constant probability in each slot, as in standard slotted Aloha. They then studied the performance metrics of this modified threshold slotted Aloha using Markov chains.

Munari et al. (2013) introduced and analysed a simple variation of the Slotted Aloha. The modification they did was to add multiple receivers in a way that several messages can be transmitted by a user in one slot similar to the concept in this thesis. However, for each node, the transmitted messages in each slot are assumed to be subject to only one-off independent

fading. They now compute closed form expressions for throughput and arbitrary number of users. They also demonstrated through simulations that this method produces better output than the standard slotted Aloha under similar conditions.

Given the massive interest in improving the efficiency of protocols with multiple users, Munari et al. (2020) analysed the potential of a simple slotted Aloha by assuming that there are multiple users. They characterized the performance matrix of this approach using the binomial theorem and simulations. In Yang et al. (2015), similar to the concept in this thesis, messages were allowed to be simultaneously transmitted in each time slot. They analysed the performance of such an approach by generalizing the tree based approach. However, the conclusions of the study relied on simulations.

Fereydounian et al. (2019), studied the Coded Slotted Aloha (CSA) by analysing its performance in the non-asymptotic regime. Here they assumed that number of users and frame length are finite. They established scaling laws and non-asymptotic relationship between the channel load, probability of error and the number of users. Umehara et al. (2018) also proposed a variant of slotted Aloha they called non-persistent slotted Aloha (SAloha). They showed that some tweaks in the standard Aloha could produce better performance and enhance connectivity in wireless networks.

Umehara et al. (2018), studied the advantages of the so called enhanced Slotted aloha which is known to reduce the probability of collision by 0.90 over other old versions of slotted Aloha. This clearly shows how there exist different version of the slotted Aloha and as such the renewed interest in research on mathematical formulations for such protocols.

Eroğlu and Onur (2021) proposed a version of the Slotted ALOHA Protocol called the density aware Slotted Aloha. This protocol ensures that the network density dictates a certain dynamic random access probability for each of the nodes in accessing network channels. Using Monte-Carlo simulations, they compared the performance of this approach to other existing variants of slotted Aloha. These other versions are the stabilized Slotted Aloha and

the standard Slotted Aloha.

### 1.5.6 Applications of large deviations in wireless networks

Several research papers have used large deviation theory in studying rare events. In wireless networks where this thesis is applied, there are different sub-areas such as malware propagation and other host of applications. Malware propagation is one area in wireless networks that is also gaining momentum.

The main motivation to discuss applications of large deviations in relation to wireless networks is its importance in analysing of rare events (example bad system quality which rarely happens). That is, events of the form  $Y_n \in B$  on which, for example, there is too much interference or bad connectivity. Such an event is known as a frustration event and it is of great importance in telecommunication theory. Similarly, situations where there is often good events can also be analyzed. Hence, in most fields and wireless networks which is the focus of this thesis, one is faced with typical questions like:

- what is the typical behaviour of  $Y_n$  on this event?
- what is the size of the probability of this event?

Question two investigates the behavior of  $Y_n$  under the conditional measure  $P(\cdot | Y_n \in B)$ . The most obvious reaction is to first guess that on the set  $B$ ,  $X_n$  should converge to the minimizer of  $I$ (rate function). Interestingly, this assertion is true in several instances. However, the correct formulation will require further assumptions on  $B$ . Usually, proofs for cases like this differ from situation to situation under different assumptions. The most obvious and typical technique would be to first show that  $I$  has just one minimizer  $y_n$  on  $B$  (Jahnel and König (2020)).

### 1.5.7 Conclusion from literature review

The question of cost comes into play as we may want to reduce cost as much as possible. Hence designing protocols that do not rather lead to increase in number of base stations while maintaining high throughput is of utmost importance.

Connectivity properties of wireless networks have been widely studied by engineers who rely heavily on simulations for drawing conclusions. Quite recently, mathematicians and statisticians have taken keen interest in deriving more tractable formulas for understanding the connectivity properties of medium access protocols using large deviations. For example, König and Tóbiás (2017) and Hirsch et al. (2016) used large deviations to study frustration probabilities in wireless networks. In a similar fashion, this thesis which is a motivation from the work of König and Tóbiás (2017) and Munari et al. (2013), this research uses large deviation to extend the notion in the paper to situations where interference/capacity constraint is analysed in a more realistic fashion.

## 1.6 Methodology

The mathematical concepts used in this research are described in this section. In particular, large deviation theorems and Poisson point processes are presented. The proof of Theorem 2.2.1 relies both on the Cramér and the Sanov theorem hence they are discussed in detail in this section. The contraction principle that is utilised in one of the proofs in section 2 and 3 is also discussed. The chapter begins with a formal definition of large deviation principle and then the most relevant theorems used in the study are presented.



### 1.6.1 General overview of large deviations

**Definition 1.1.** A rate function  $I$  is a lower semicontinuous mapping  $I : \mathcal{X} \rightarrow [0, \infty]$  such that for all  $\alpha \in [0, \infty)$ , the level sets  $\Psi_I(\alpha) = \{x : I(x) \leq \alpha\}$  is a closed subset of  $\mathcal{X}$ .

**Definition 1.2.** A sequence of  $\mathcal{X}$ -valued random variables  $Y_N$  with law  $\mathbb{P}_N$  satisfies a large deviation principle with rate function  $I$ , if for any open subset  $G$  and closed subset  $F$  of  $\mathcal{X}$

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{P}_N \left( Y_N \in F \right) \leq -\inf_F I, \quad (1.6.1)$$

$$\liminf_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{P}_N \left( Y_N \in G \right) \geq -\inf_G I. \quad (1.6.2)$$

(Zeitouni and Dembo, 1998). This statement is the definition of large deviation principle in a more general setting and can be adapted in a variety of ways similar (For example the definition presented in Chapter 2 ). Below are the relevant theorems that were used in the analysis of the network protocols presented in Chapter 2 and 3.

### 1.6.2 Cramér's theorem

The Cramer theorem which forms the basis of the proofs presented in Theorem 2.2.1 is discussed below.

Let the *i.i.d* random variables  $Y_1, Y_2, \dots, Y_n$  with  $Y_1$  distributed according to the probability law  $\mu$  be given. One can then define the mean of  $n$  of these random variables, where  $n \in \mathbb{N}$ , as  $S_n = \frac{1}{n} \sum_{k=1}^n Y_k$ . Let  $\mu_n$  be defined as  $\mu_n(A) = \mathbb{P}[S_n \in A]$  where  $A \in \mathcal{B}(\mathbb{R})$  and  $(P)$  is a probability measure on  $\mathcal{B}$ .

To introduce the Cramér theorem, the Fenchel-Legendre transform and the cumulant generating function (CGF) is briefly introduced. Now let, the CGF associated with the probability

measure  $\mu$  be define as:

$$\Lambda(\lambda) := \log M(\lambda) := \log \mathbb{E}[e^{\lambda Y_1}] \quad (1.6.3)$$

**Definition 3.2.** The Fenchel-Legendre transform of  $\Lambda$  (defined in (1.6.3)) is given as

$$\Lambda^* : \mathbb{R} \longrightarrow \mathbb{R} \cup \{\infty\}, \quad \Lambda^*(y) = \sup_{\lambda \in \mathbb{R}} \{\lambda y - \Lambda(\lambda)\} \quad (1.6.4)$$

for all  $y \in \mathbb{R}$

**Theorem 1.6.1.** (Cramer's theorem). Let  $\mathcal{X}$  be a topological space and  $(Y_n)_{n \in \mathbb{N}}$  be a sequence of  $\mathcal{X}$ -valued i.i.d random variables.  $(\mu_n)_{n \in \mathbb{N}}$  which is a sequence of measures satisfies an LDP with convex rate function  $\Lambda^*(\cdot)$  such that;

1. For any open set  $G \subset \mathcal{X}$

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log \mu_n(G) \geq - \inf_{y \in G} \Lambda^*(x) \quad (1.6.5)$$

2. For any closed set  $F \subset \mathcal{X}$ ,

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \mu_n(F) \leq - \inf_{y \in F} \Lambda^*(y) \quad (1.6.6)$$

### 1.6.3 Contraction principle

The contraction principle is mentioned in the proof presented in Section 2.3.3 of the alternative proof presentation of  $l = 1$  for Theorem 2.2.1.

**Theorem 1.6.2.** (Contraction principle). Let  $\psi : A \longrightarrow B$  be continuous where  $A$  and  $B$  are Polish spaces. Let the finite measures  $\mu_n$  on  $A$  satisfy an LDP with good rate function  $J$  and speed  $sn$ . Then  $\mu_n \circ \psi^{-1}$  (the image measures) also satisfies an LDP with speed  $sn$  and good

rate function  $I$  given by

$$I(y) = \inf_{x \in \psi^{-1}(\{y\})} J(x) \quad (y \in B), \quad (1.6.7)$$

where  $\inf_{x \in \emptyset} J(x) := \infty$

### 1.6.4 Sanov theorem

The proof of the alternative formulation for Theorem 2.2.1 found in section 2.3.3 is based on the Sanov theorem below.

**Theorem 1.6.3** (Sanov's theorem). *Let the random variables  $(X_k)_{k \geq 0}$  be independent and identically distributed taking values in a Polish space  $A$ , with common law  $\mu$ . Further let*

$$M_n := \frac{1}{n} \sum_{k=1}^n \delta_{X_k}$$

*with  $n \geq 1$  be the empirical laws of the random variables  $(X_k)_{k \geq 0}$ . Hence, the laws  $\mathbb{P}(M_n \in \cdot)$  satisfy a large deviation principle with rate function  $H(\cdot | \mu)$  and speed  $n$ . Where,  $\mathbb{P}(M_n \in \cdot)$  is a probability law on the Polish space  $\mathcal{M}_1(A)$ .  $\mathcal{M}_1(A)$  are probability measures on  $A$ , equipped with the weak convergence topology.*

Proof of the Cramer theorem, Sanov theorem and the contraction principle can be found in several standard large deviation books such as (Zeitouni and Dembo, 1998). Hence the proof are not presented in this thesis.

### 1.6.5 Poisson Point Process

Poisson point process (PPP) is a random point process with intensity measure  $\mu$  (considering  $\mu$  to be a measure on  $B$ ) such that, for any  $m \in \mathbb{N}$  and any measurable pairwise disjoint

subsets  $A_1, \dots, A_k$  of  $B$  having finite  $\mu$  measure, the counts  $\Phi(A_1), \dots, \Phi(A_m)$  of the random points in the respective sets  $A_1, \dots, A_m$  are independent Poisson random variables with  $\mu(A_1), \dots, \mu(A_k)$  as parameters respectively, i.e., if for all  $n_1, \dots, n_m \in \mathbb{N}_0$ .

$$\mathbb{P}(\Phi(A_1) = n_1, \dots, \Phi(A_m) = n_k) = \prod_{i=1}^m \left[ e^{-\mu(A_i)} \frac{\mu(A_i)^{n_i}}{n_i!} \right]. \quad (1.6.8)$$

The standard Poisson point process briefly discussed below is referred to in the proof of Lemma 2.3.1.

### 1.6.5.1 Standard Poisson Point Process

The standard Poisson point process corresponds to  $B = \mathbb{R}^d$  with intensity measure  $\lambda dx$  and intensity  $\lambda \in (0, \infty)$ . The measure  $dx$  is stationary, with an underlying random point process which is also stationary and often referred to as a homogeneous PPP. Every stationary Poisson point process has a shift invariant measure. Additionally the distribution of this process is isotropic. This means that it is rotationally invariant. This is a very important characteristics of Poisson point process which makes it easier and friendly to work with.

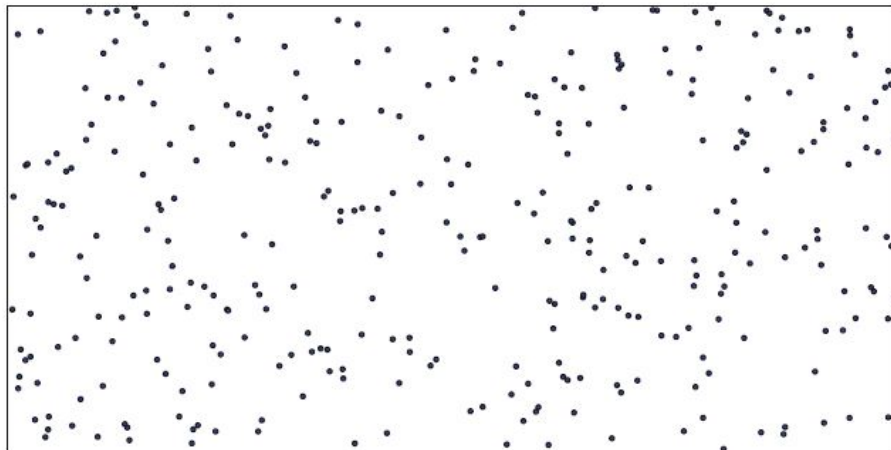


Figure 1.1: Realisation of a Poisson Point Process. Source (Jahnel and König, 2020)

### 1.6.5.2 Random thinning of Poisson Point process

Let  $\mu$  be the intensity measure of the Poisson point process  $\Phi = (X_i)_{i \in I}$ . Given  $\Phi$  and with probability  $0 \leq p \leq 1$ , a subset of the points  $X_i$  can be independently selected. Once the selection is done independently, the selected points of  $X_i$  also form a PPP with  $p\mu$  as intensity measure. A prove of this assertion is quite straight forward and can be found in for instance (Jahnel and König, 2020). The proof will not be presented here. Now, in relation to telecommunications and particularly, the ALOHA protocol (discussed in 2.3.1), devices independently chooses to transmit at a given time instant hence the property of point processes is not lost. Network devices are assumed to be identically distributed hence device positions can be seen to form a Poisson Point Process as the thinning property is preserved.

### 1.6.6 Legendre transformation

Legendre transforms are important in large deviations theory. They describe the rate functions using moment generating functions. Definition of Legendre function can be found in Definition 3.2. Below is a description of the legendre transform of the Poisson distribution.

The rate function of the Poisson distribution was directly used in the proofs presented in Section 2.3.

- **Poisson distribution.** Given that  $Y$  is a Poisson distribution with parameter  $\lambda$ , then we have that the moment generating function of  $Y$  is given by  $M(\theta) = e^{e\theta - \lambda}$ . Then the associated Legendre transform is given as

$$I(b) = \sup_{\theta} (b\theta - (e^{e\theta} - \lambda))$$

We obtain the optimal  $\theta^* = \log(b/\lambda)$  and  $I(b) = b \log(b/\lambda) - (b - \lambda)$  after setting

the derivative of  $I(b)$  with respect to  $\theta$  to zero for  $b > \lambda$ . Hence large deviations would predict that on the exponential scale,

$$P\left(\frac{\sum_{1 \leq i \leq n} Y_i}{n} > b\right) \approx e^{-(b \log(b/\lambda) - b + \lambda)n}$$

For the Poisson distribution also, one can explicitly compute the large deviations probability. Even for the sum  $Y_1 + \dots + Y_n$  of Poisson random variables, one can still compute the large deviation probability easily since the sum is also Poisson with parameter  $\lambda n$ . That is,

$$P\left(\sum_{1 \leq i \leq n} y_i > an\right) = \sum_{m > bn} \frac{(\lambda n)^m}{m!} e^{-\lambda n}$$

It is still a bit tedious to infer explicit rate of decay of certain distributions but they can be used as and when needed. In this research, the rate function of the Poisson distribution and the binomial distribution are quoted and used in proves.

### 1.6.7 Law of Large Numbers

Law of large numbers was derived using the minimisers of the large deviation rate functions (See corollary 2.2.2).

**Theorem 1.6.4.** *Let  $Y_1, Y_2, \dots, Y_n$  be independent trials, with finite expectation  $\mu = E(Y_j)$  and finite variance  $\sigma^2 = V(Y_j)$ . Let  $S_n = Y_1 + Y_2 + \dots + Y_n$ . Then for any  $\varepsilon > 0$ .*

$$\mathbb{P}\left(\left|\frac{S_n}{n} - \mu\right| \geq \varepsilon\right) \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

*Equivalently,*

$$\mathbb{P}\left(\left|\frac{S_n}{n} - \mu\right| < \varepsilon\right) \rightarrow 1 \quad \text{as } n \rightarrow \infty.$$

Proof for the above theorem can be found in standard statistical textbooks hence the proof

is left out.

### 1.6.8 Aloha/Slotted Aloha with interference constraint

In the classical ALOHA protocol each sender flips a coin and with probability  $p$  decides to send independently of all other users. The message success depends on the interference constraint. A message from transmitter  $X_i$  is successfully sent to receiver  $y$  in a reference time slot if;

$$SINR(X_i, y, \mathbb{X}) = \frac{\ell(\|X_i - y\|)}{N_0 + \sum_{j \in I \setminus \{i\}} \ell(\|X_j - y\|)} \geq \tau \quad (1.6.9)$$

and unsuccessful if  $SINR(X_i, y, \mathbb{X}) < \tau$ . Where  $\mathbb{X}$  is the set of attempting participant at a time instant with Bernoulli probability  $p$ . The function  $\ell(\|\cdot\|)$  is the path loss function that describes how the signal strength of transmission decreases with distance. The term in (1.6.9) is the quotient of the strength of the signal transmitted from  $X_i$  received at  $y$  and the sum of the strengths of all the signals sent out by any user received at  $y$ . An example of  $\ell$  is  $\ell(r) = r^{-\alpha}$  for some  $\alpha > d$ . Thorough treatment of interference through the  $SINR$  constraint can be found in (Jahnel and König, 2020).

This thesis looks at a transformation of the  $SINR$  constraint into a simple capacity constraint that a message is successfully sent if the number of attempting transmitters at a time instant is  $\leq 1/\tau = \kappa$ .

The figure below demonstrates how interference can be caused by neighbouring devices and as such sabotage the transmission of a transmitted message.

The precise choice of the interference term for any hop is not fixed, as there are some options, depending on the situation that one wants to model. For example, it is discussable to use the interference caused by all the users in the system (as in ((1.6.9))), rather than all the messages sent out (this does not have to be identical) or even all the single hops that occur at the same time. These are variants of the of network capacity/interference constraint

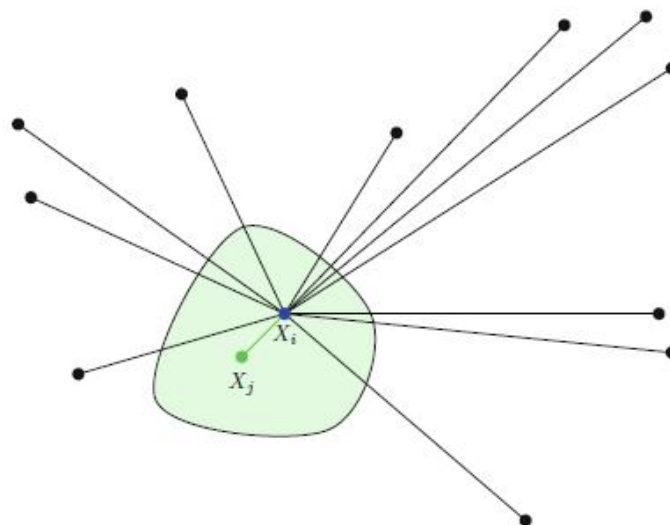


Figure 1.2: Device  $X_j$  transmits to  $X_i$  with interference from other devices transmitting to  $X_i$ . Source (Jahnel and König, 2020)

models can be addressed under any of the medium access protocols.

Figure 1.2 shows how neighbouring devices can cause interference when a device decides independently to transmit a message.

## 1.7 Motivation and goals

In communication networks, where many messages are to be transmitted via a bounded number of channels, the choice of the medium access protocol is ubiquitous and decisive. It is clear that, due to interference constraints or due to restricted availability of communication channels, the task can be successfully managed only if a suitable rule is in force that regulates which of the many messages is gaining access to the medium at a given time. Clearly, this rule should be as simple as possible, and it should not require much infrastructure nor previous investigations.

Ideal for this is a *probabilistic ansatz* in which all decisions about making sending attempts are made randomly and independently of each of the participants and with a minimal

knowledge about the current state of the system. In the models discussed in this research, only the number of participants and the number of available channels should play a role in the decision, and each decision is made with a certain fixed probability. In such a system, there may and will be also unsuccessful decisions made. The operator's task is to determine the decisive parameters optimally, i.e., in such a way that the number of successes (to mention an important quality measure) will be, on a long term, as large as possible. A probabilistic way to determine this is in terms of a law of large numbers, one of the tools used in this thesis. Another one is the analysis of events of large deviations from the usual behaviour.

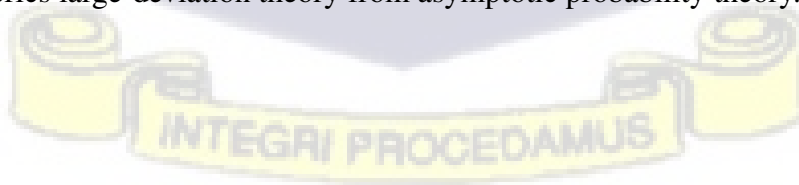
There are a number of options for medium access protocols that satisfy the above criteria. In this research, the two most fundamental ones, the (discrete-time) Aloha protocol, where new messages get access at a given time instant to a channel without knowing whether or not it is busy or idle, and a variant called the slotted Aloha protocol, where first every participant chooses whether to make an attempt and then the starting times of the message transmissions are allocated to certain slots. Slotted Aloha ensures that all network nodes are perfectly synchronized to some time slots and transmit messages according to a certain rule. Furthermore, in this thesis we restrict to discrete-time models, since all message at a given time instant (or, equivalently, in a given fixed time slot) are under a certain upper-bound constraint: if too many make a sending attempt in a slot, all fail.

In all such protocols, the use of stochasticity and a high degree of independence of the decisions turns out to be extremely helpful. Each of the messages is equipped with its own decision making algorithm, based on randomness, and involving only a small set of parameters that may depend on the current situation. In the situation(s) that this thesis focuses on, it is assumed that the current number of messages that are to be transmitted is known to every user equipment involved, and that there is an upper bound  $\kappa \in \mathbb{N}$  for the number of messages that can be successfully transmitted at any given time. The reason

for this upper bound may be either a restriction of the number of available channels or an interference constraint.

The main system parameter is the probability parameter  $p$ , the *medium access probability (MAP)*, with which each of the messages tries randomly to gain access to the system. If  $p$  is too large, then it is likely that the system exceeds the upper bound  $\kappa$ , which results into failures of many message transmissions. On the other hand, if  $p$  is too small, then a part of the possible capacity is not exhausted, and the system underachieves. Similar arguments can for example be found in Baccelli and Błaszczyszyn (2009). One of the goals is to quantify an optimal choice of  $p$ . The main quantity for this criterion is the *throughput*, the number of successfully transmitted messages per time unit. This thesis also analyses other quantities like the number of message attempts.

For any of these protocols, we are interested in the large-scale performance of the system, i.e., in a large number of messages during a large number of time instances. We decided to consider just one time interval  $[0, 1]$  and to assume that a successful message transmission requires a very short time of length  $1/N$ , where  $N \in \mathbb{N}$  is a scale parameter that is proportional to the number of messages. We will calculate and analyse the distribution of the main quantities in the limit  $N \rightarrow \infty$ . An important aspect of this research is that, focus is not merely on calculating expectations, but on the asymptotics of probabilities of *large deviations*, i.e., events that deviate from the expected behaviour, like the event that the number of successfully delivered messages has a lower linear rate than the expectation. It is also an interesting and important question to get some useful information about ‘bad’ events of that sort, and to characterise this event. The main methods used in this thesis consist of combinatorics large-deviation theory from asymptotic probability theory.



## 1.8 Contribution

This thesis proposes simple and easy to implement protocols in wireless networks. The theorems in chapter four and their proofs are a result of the extension of the work of (König and Tóbiás, 2019). The results also serve as a natural extension to most works that mainly analyse single message transmissions in multi-hop network channels. This is because the results generalises the single hop message transmissions in many channels to  $\kappa$  number of messages. This thesis also provides a much stronger newer approach to deriving large deviation results which is a much stronger statement than the usual LDP. The result also shows that, one can, for example, use a cutting argument to bypass some technicalities in contraction principle.

This thesis also leaves room for several open problems that can be further researched. For instance, it will be interesting to analyse the case where  $\kappa \rightarrow \infty$  in a multi-channel network. Some of these open problems are further discussed in Chapter four.

## 1.9 Organisation of thesis

There are four chapters in this thesis. Chapter one is the introduction and this is mainly made up of the background of the study which briefly introduces the subject matter. It also contains the problem statement, objectives and relevance of the study. It also consist review of literature. It presents some articles related to the subject matter. It also briefly discusses the applications of large deviation theory which is the mathematical tool used in analysing the models. Lastly, it contains the methodology. Here, the main tools that were adopted and used for the proofs are discussed. Sanov and Cramer's Theorems were used hence they are presented in detail. Law of large numbers and signal to interference and noise ratio (SINR) are also discussed.

Chapters two and three are the contribution of the thesis. These chapters introduce and analyse the models in the thesis. Large deviation results are presented in Chapter two and optimisation results are presented in Chapter three.

Chapter four contains the conclusion, summary and recommendations of the thesis. One detailed model that can be implemented and analysed in the future is also briefly presented.



## Chapter 2

# Large deviation for the joint probability of throughput in Slotted Aloha

### 2.1 Description of the models

We now introduce the models that were analysed in this research. Each of the models has a large parameter  $N \in \mathbb{N}$ . Each successful transmission of a message requires precisely  $1/N$  time units. We consider a reference time interval, which we pick as  $[0, 1]$ ; hence the system can manage the delivery of  $\asymp N$  messages during this time interval. To ease the notation, the integer-part brackets is waived. Assume that a large number  $bN$  (where  $b \in (0, \infty)$  is the message number parameter) of messages of unit size are to be transmitted. The number of messages that can be transmitted at a given time is bounded by  $\kappa \in \mathbb{N}$  (This is e.g. due to interference constraints or to the fact that only  $\kappa$  channels are available.). Each of the messages makes attempts at random times to be admitted to the communication system. According to certain rules (to be specified below), the attempt is admitted or not, and if the upper bound rule is not violated, then it is successfully delivered  $1/N$  time units later. In this case, we say that this messages has gained access to the medium.

We write  $[k] = \{1, \dots, k\}$  for  $k \in \mathbb{N}$ . The medium access protocols that we consider are the following:

- (1) *Slotted Aloha*: At each time in  $\{0, \frac{1}{N}, \frac{2}{N}, \dots, \frac{N-1}{N}\}$ , any message decides with probability  $p/N$ , independently over all the messages and over all the times, whether to make an immediate sending attempt or not. For any  $j \in [N]$ , if not more than  $\kappa$  of all the messages decided at time  $(j-1)/N$  to do so, then all these messages are successfully delivered during the time slot  $[\frac{1}{N}(j-1), \frac{j}{N})$ , otherwise all are cancelled.
- (2) *Organized Aloha*: At the beginning of the time stretch  $[0, 1]$ , any of the messages decides with probability  $p$  whether to make an attempt during  $[0, 1]$ , independently over all messages. In a second step, all these attempts are made during the time stretch  $[0, 1]$  at random times, independently and uniformly distributed over  $[0, 1]$ . In each time slot  $[\frac{1}{N}(j-1), \frac{j}{N})$  with  $j \in [N]$ , all messages that have been assigned to this interval are successfully delivered if they are not more than  $\kappa$ , otherwise they will all be cancelled.

Both protocols are effectively in discrete time as we are only interested in the number of sending attempts and of successful emissions in the time slots  $[\frac{1}{N}(j-1), \frac{j}{N})$  with  $j \in [N]$ .

In protocol (2), any participant has only at most one chance during  $[0, 1]$ , while in protocol (1), every message has an unbounded number of trials and can be successful several times. We assume that each message has an unbounded number of packages to be sent, i.e., it makes successively an unbounded number of sending attempts. Protocol (2) has a two-step random strategy, at first each message randomly decides whether to attempt a transmission, and then picks randomly a microscopic time slot. Here the number of random variables that need to be sampled is much smaller than in protocol (1), and the probability parameter is of finite order in  $N$ , in contrast to protocol (1). We therefore see substantial practical advantages in protocol (2) over protocol (1).

There are three parameters in these simple models:

- $p > 0$  the message emission attempt probability parameter,
- $b \in (0, \infty)$  the rate of messages that would like to be transmitted during  $[0, 1]$ ,
- $\kappa \in \mathbb{N}$  the threshold for the success criterion.

Note that  $p$  is indeed a probability (i.e., lies in  $[0, 1]$ ) in protocol (2), while it can be any positive number in protocol (1).

### 2.1.1 Quantities and questions of interest

The quantities that we are interested in are the following.

- $A_N$  = the number of message sending attempts,
- $S_N$  = number of successfully sent messages,
- $R_N$  = number of successful micro slots, that is, slots  $(\frac{j-1}{N}, \frac{j}{N}]$  in which all messages are successfully transmitted, each during the entire interval  $[0, 1]$ .

The most important quantity is the *throughput*, the number of successfully sent messages per time unit, which is  $S_N/N$  in our model. But we find it also important to consider the number of unsuccessful sending attempts, in order to be able to say something about the frustration of the participants of the system.

## 2.2 Large deviation results for throughput

The first main result is on the asymptotics as  $N \rightarrow \infty$  of the joint distribution of  $(S_N, A_N, R_N)$ , both in the sense of a law of large numbers and a large-deviation principle. We denote by  $\mathbb{P}_N^{(l)}$  the probability measures for the two models according to protocol  $(l)$  for any  $l \in \{1, 2\}$ .

The following are introduction of objects that are necessary for formulating the results of the study. Introduce, for  $x \in [0, \infty)$ ,

$$I_{\leq \kappa}(x) = \sup_{z \in \mathbb{R}} \left( xz - \log \sum_{i \in \mathbb{N}_0: i \leq \kappa} \frac{e^{zi}}{i!} \right) \quad \text{and} \quad I_{> \kappa}(x) = \sup_{z \in \mathbb{R}} \left( xz - \log \sum_{i \in \mathbb{N}_0: i > \kappa} \frac{e^{zi}}{i!} \right) \quad (2.2.1)$$

and

$$F_{\kappa}(a, s, r) = r \log r + (1 - r) \log(1 - r) + r I_{\leq \kappa} \left( \frac{s}{r} \right) + (1 - r) I_{> \kappa} \left( \frac{a-s}{1-r} \right). \quad (2.2.2)$$

Then our main result is as follows.

**Theorem 2.2.1** (LDP for  $\frac{1}{N}(A_N, S_N, R_N)$ ). *Fix the model parameters  $b, p > 0$  and  $\kappa \in \mathbb{N}$ , where we assume  $p \leq 1$  for Model (2). Fix  $a, s \in [0, b]$  satisfying  $s \leq a$  and fix  $r \in [0, 1]$ . Pick sequences  $a_N, s_N, r_N \in \frac{1}{N}\mathbb{N}_0$  such that  $a_N \rightarrow a$ ,  $s_N \rightarrow s$  and  $r_N \rightarrow r$  as  $N \rightarrow \infty$ . Then for  $l \in \{1, 2\}$ ,*

$$I^{(l)}(a, s, r) = - \lim_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{P}_N^{(l)} \left( A_N = Na_N, S_N = Ns_N, R_N = Nr_N \right) \quad (2.2.3)$$

exists, and the rate function  $I^{(l)} = I_{p,b,\kappa}^{(l)}$  is given by

$$I^{(l)}(a, s, r) = J_{b,p}^{(l)}(a) + a - a \log a + F_{\kappa}(a, s, r), \quad (2.2.4)$$

where

$$J_{b,p}^{(1)}(a) = pb - a + a \log \frac{a}{pb}, \quad (2.2.5)$$

$$J_{b,p}^{(2)}(a) = a \log \frac{a}{p} + (b - a) \log \frac{b - a}{1 - p} - b \log b. \quad (2.2.6)$$

There is the following alternative representation for  $I^{(1)}$ :

$$I^{(1)}(a, s, r) = \inf \left\{ H(\mu | \text{Poi}_{bp}) : \mu \in \mathcal{M}_1(\mathbb{N}_0), \sum_{k \in \mathbb{N}_0} f(k) \mu_k = (a, s, r) \right\} \quad (2.2.7)$$

where  $H(\mu | \text{Poi}_{bp}) = \sum_{k \in \mathbb{N}_0} \mu_k \log \frac{\mu_k}{q_k}$  denotes the entropy of  $\mu$  with respect to the Poisson distribution with parameter  $bp$ , where  $q_k = e^{-pb} (pb)^k / k!$ , and  $f(k) = (k, k \mathbb{1}\{k \leq \kappa\}, \mathbb{1}\{k \leq \kappa\})$ .

The proof of Theorem 2.2.1 for  $l = 2$  is in Section 2.3.1 and for  $l = 1$  in Section 2.3.3. In Section 2.3.3 a proof of the representation in (2.2.7) is presented.

It turns out in Theorem 2.2.1 that the difference between protocol (1) and (2) is only in the rate function for the number of sending attempts. Indeed, this number is binomial with parameters  $bN$  and  $p$  in case (2) (and  $J_{b,p}^{(2)}(a)$  is the negative exponential rate of  $\binom{bN}{aN}$ ), but a sum of  $N$  independent binomials with parameters  $Nb$  and  $p/N$  (approaching the Poisson distribution  $\text{Poi}_{bp}$ ) in case (1) (with  $J_{b,p}^{(1)}$  being the rate function for the average of  $N$   $\text{Poi}_{bp}$ -distributed random variables). Both rate functions have an infinite slope at  $a \downarrow 0$ , but only  $J_{b,p}^{(2)}$  has for  $a \uparrow b$ .

It is now a technical task to derive from Theorem 2.2.1 that  $\frac{1}{N}(A_N, S_N, R_N)$  satisfies large-deviation principles (LDPs) with rate functions  $I^{(l)}$  under  $\mathbb{P}_N^{(l)}$ , which says that for any open set  $G \subset [0, b] \times [0, b] \times [0, 1]$  and closed set  $F \subset [0, b] \times [0, b] \times [0, 1]$ ,

$$\begin{aligned} \limsup_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{P}_N^{(l)} \left( \frac{1}{N} (A_N, S_N, R_N) \in F \right) &\leq -\inf_F I^{(l)}, \\ \liminf_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{P}_N^{(l)} \left( \frac{1}{N} (A_N, S_N, R_N) \in G \right) &\geq -\inf_G I^{(l)}. \end{aligned}$$

It is a standard conclusion from the LDP that, if the rate function has a unique minimizer at  $(a_p, s_p, r_p)$ , a law of large numbers (LLN) follows, i.e.,  $\frac{1}{N}(A_N, S_N, R_N) \rightarrow (a_p, s_p, r_p)$  in  $\mathbb{P}_N^{(l)}$ -probability with exponential decay of the probability of being outside a neighbourhood

of  $(a_p, s_p, r_p)$ . Hence, the following statement implies two LLNs.

**Corollary 2.2.2** (LLN for the throughput). *For  $l \in \{1, 2\}$ , the rate function  $I^{(l)}$  is strictly convex and possesses the same unique minimizer  $(a_p, s_p, r_p)$  given by  $a_p = pb$  and*

$$s_p = e^{-bp} \sum_{i \in \mathbb{N}_0: i \leq \kappa} i \frac{(bp)^i}{i!} = \mathbb{E}_{\text{Poi}_{bp}}[X \mathbb{1}\{X \leq \kappa\}] = bp e^{-bp} \sum_{i=0}^{\kappa-1} \frac{(bp)^i}{i!}, \quad (2.2.8)$$

$$r_p = e^{-bp} \sum_{i \in \mathbb{N}_0: i \leq \kappa} \frac{(bp)^i}{i!} = \text{Poi}_{bp}([0, \kappa]). \quad (2.2.9)$$

Another standard corollary of Theorem 2.2.1 is an LDP for the number  $S_N$  of successes, which follows directly from the contraction principle (see Theorem 1.6.2); indeed  $\frac{1}{N}S_N$  satisfies an LDP under  $\mathbb{P}_N^{(l)}$  with rate function  $s \mapsto \inf_{a,r} I^{(l)}(a, s, r)$ . There is no closed form for this rate function; only for  $l = 1$ , we can write  $\inf_{a,r} I^{(l)}(a, s, r) = \inf\{H(\mu | \text{Poi}_{bp}) : \mu \in \mathcal{M}_1(\mathbb{N}_0), \sum_{k=0}^{\kappa} k\mu(k) = s\}$ . This variational formula is analysed as a by-product of the proof of Lemma 2.3.1 below.

## 2.3 Proofs of the LDPs

### 2.3.1 Proof of Theorem 2.2.1 for $l = 2$

In this section, we work under protocol (2), i.e., at the beginning of the time stretch  $[0, 1]$ , each of the  $bN$  message holders chooses with probability  $p$  whether to make a sending attempt during any of the  $N$  time slots in  $[0, 1]$ , and then all those who do this are uniformly independently distributed over all the  $N$  micro time slots. In each micro slot, all the messages are successfully transmitted if their number is  $\leq \kappa$ , otherwise all are unsuccessful.

For any  $k \in \mathbb{N}$ , we write  $[k]$  for the set  $\{1, \dots, k\}$ . Let  $X_i \in \{0, 1\}$  for  $i \in [bN]$  be the indicator on the event that the  $i$ -th message chooses to attempt to send, then  $(X_i)_{i \in [bN]}$  is a family of

independent Bernoulli variables with parameter  $p$ . Let

$$\mathcal{A}_N = \{i \in [bN] : X_i = 1\}$$

be the set of those that attempt, i.e.,  $A_N = |\mathcal{A}_N|$ . For  $i \in \mathcal{A}_N$  let  $Y_i \in [N]$  be the index of the microscopic interval that is assigned to the  $i$ -th message for the sending instant, then, given  $\mathcal{A}_N$ ,  $(Y_i)_{i \in \mathcal{A}_N}$  is a family of independent  $\mathcal{U}_N$ -distributed random variables, where  $\mathcal{U}_N$  is the uniform distribution on  $[N] = \{1, \dots, N\}$ . Let  $K_j = \sum_{i \in \mathcal{A}_N} \mathbb{1}\{Y_i = j\}$  denote the number of attempts in the  $j$ -th time slot for  $j \in [N]$ . By

$$\mathcal{R}_N = \{j \in [N] : K_j \leq \kappa\}$$

we denote the set of indices of all the ‘successful micro slots’, i.e., of the intervals in which not more than  $\kappa$  of the messages make a transmission attempt and will therefore be successfully transmitted. Clearly  $R_N = |\mathcal{R}_N|$ . Then the number of successful messages is equal to

$$S_N = \sum_{j \in \mathcal{R}_N} K_j = \#\{i \in \mathcal{A}_N : K_{Y_i} \leq \kappa\}.$$

Fix parameters  $a, s \in [0, b]$  satisfying  $s \leq a$ , and  $r \in [0, 1]$  and sequences  $a = a_N, s = s_N, r = r_N \in \frac{1}{N}\mathbb{N}_0$  such that  $a_N \rightarrow a$ , and  $s_N \rightarrow s$  and  $r_N \rightarrow r$  as  $N \rightarrow \infty$ . The goal is to find the logarithmic asymptotics of

$$q_N = \mathbb{P}_N(A_N = Na_N, S_N = Ns_N, R_N = Nr_N).$$

Note that the number  $A_N$  of messages that attempt to send is binomially distributed with parameters  $bN$  and  $p$ . Conditional on the event  $\{A_N = Na_N\}$ , the joint distribution of  $(Y_i)_{i \in \mathcal{A}_N}$  is equal to  $\mathcal{U}_N^{\otimes Na_N}$  (the exchangeability property of  $X_i$  ensures this), and we may

take  $\mathcal{A}_N = [Na_N]$  as the index set. Hence for  $N$  large enough,

$$\begin{aligned}
 q_N &= \mathbb{P}_N(A_N = Na_N) \mathbb{P}_N(S_N = Ns_N, R_N = Nr_N | A_N = Na_N) \\
 &= \binom{bN}{Na_N} p^{Na_N} (1-p)^{bN-Na_N} \mathcal{U}_N^{\otimes Na_N} \left( R_N = Nr_N, \sum_{j \in \mathcal{R}_N} K_j = Ns_N \right) \\
 &= e^{N[J_{b,p}^{(2)}(a)+o(1)]} \mathcal{U}_N^{\otimes Na_N} \left( R_N = Nr_N, \sum_{j \in \mathcal{R}_N} K_j = Ns_N \right), \quad N \rightarrow \infty,
 \end{aligned} \tag{2.3.10}$$

where  $J_{b,p}^{(2)}$  is defined in (2.2.6). In the third equality we have used the Stirling's approximation to the factorial of a large enough positive integer.

We turn to the last term on the right of (2.3.10). We need to decompose  $\mathcal{R}_N$  into the part of those  $j$  such that  $K_j \leq \kappa$  and the others. Hence,

$$\begin{aligned}
 &\mathcal{U}_N^{\otimes Na_N} \left( R_N = Nr_N, \sum_{j \in \mathcal{R}_N} K_j = Ns_N \right) \\
 &= \sum_{B \subset [N]: |B|=Nr_N} \mathcal{U}_N^{\otimes Na_N} \left( B = \{j: K_j \leq \kappa\}, \sum_{j \in B} K_j = Ns_N \right) \\
 &= \binom{N}{Nr_N} \mathcal{U}_N^{\otimes Na_N} \left( [Nr_N] = \{j: K_j \leq \kappa\}, \sum_{j \in [Nr_N]} K_j = Ns_N \right),
 \end{aligned} \tag{2.3.11}$$

where we took without loss of generality  $\mathcal{R}_N$  as the set  $[Nr_N] = \{1, \dots, Nr_N\}$ . In the next step, we make explicit the set of indices  $i$  with cardinality  $Ns_N$  such that  $Y_i \in [Nr_N]$ :

$$\begin{aligned}
 &\mathcal{U}_N^{\otimes Na_N} \left( [Nr_N] = \{j: K_j \leq \kappa\}, \sum_{j \in [Nr_N]} K_j = Ns_N \right) \\
 &= \sum_{D \subset [Na_N]: |D|=Ns_N} \mathcal{U}_N^{\otimes Na_N} \left( D = \{i \in [Na_N]: Y_i \in [Nr_N]\}, [Nr_N] = \{j: K_j \leq \kappa\} \right) \\
 &= \binom{Na_N}{Ns_N} \mathcal{U}_N^{\otimes Na_N} \left( [Ns_N] = \{i \in [Na_N]: Y_i \in [Nr_N]\}, [Nr_N] = \{j: K_j \leq \kappa\} \right) \\
 &= \binom{Na_N}{Ns_N} \mathcal{U}_N^{\otimes Ns_N} \left( Y_1, \dots, Y_{Ns_N} \in [Nr_N], K_j \leq \kappa \forall j \in [Nr_N] \right) \\
 &\quad \times \mathcal{U}_N^{\otimes (Na_N - Ns_N)} \left( Y_{Ns_N+1}, \dots, Y_{Na_N} \in [Nr_N]^c, K_j > \kappa \forall j \in [Nr_N]^c \right),
 \end{aligned} \tag{2.3.12}$$

where we took  $[Ns_N]$  as the set of indices  $i$  such that  $Y_i \in [Nr_N]$ , and we recall that  $K_j = \sum_{i=1}^{Ns_N} \mathbb{1}\{Y_i = j\}$  in the first probability term, and analogously  $K_j = \sum_{i=1}^{Na_N - Ns_N} \mathbb{1}\{Y_i = j\}$  in the second. The complement is taken within the set  $[N] = \{1, \dots, N\}$ .

Now we make a change of measure from the uniform distribution on  $[N]$  to the uniform distribution on  $[Nr_N]$  respectively on  $[Nr_N]^c$ , and we use that all these probabilities depend only on the cardinality of  $[Nr_N]$ . Hence, substituting in (2.3.10), we see that we have

$$q_N = \binom{bN}{Na_N} p^{Na_N} (1-p)^{Nb-Na_N} \binom{N}{Nr_N} \binom{Na_N}{Ns_N} \left(\frac{Nr_N}{N}\right)^{Ns_N} \left(\frac{N-Nr_N}{N}\right)^{Na_N-Ns_N} \mathcal{U}_{Nr_N}^{\otimes Ns_N}(K_j \leq \kappa, \forall j \in [Nr_N]) \mathcal{U}_{N-Nr_N}^{\otimes (Na_N-Ns_N)}(K_j > \kappa, \forall j \in [N-Nr_N]). \quad (2.3.13)$$

Now we evaluate the large deviations of each of the two last terms. Recall the two functions  $I_{\leq \kappa}$  and  $I_{> \kappa}$  from (2.2.1).

**Lemma 2.3.1.** *Suppose  $s_N \rightarrow s > 0$ , and  $a_N \rightarrow a > 0$  and  $r_N \rightarrow r \in [0, 1]$ , then for  $N$  large enough,*

$$\mathcal{U}_{Nr_N}^{\otimes Ns_N}(K_j \leq \kappa \forall j \in [Nr_N]) = \frac{(Ns_N)!}{(Nr_N)^{Ns_N}} e^{-Nr_N I_{\leq \kappa}(s/r)} e^{o(N)}, \quad (2.3.14)$$

and

$$\mathcal{U}_{N-Nr_N}^{\otimes (Na_N-Ns_N)}(K_j > \kappa \forall j \in [N-Nr_N]) = \frac{(N(a_N - s_N))!}{(N(1-r_N))^{N(a_n-s_N)}} e^{-N(1-r)I_{> \kappa}((a-s)/(1-r))} e^{o(N)}. \quad (2.3.15)$$

*Proof.* We prove only (2.3.14); the proof of (2.3.15) is analogous.

Let us identify the joint distribution of  $K_1, \dots, K_{Nr_N}$  under  $\mathcal{U}_{Nr_N}^{\otimes Ns_N}$  in terms of standard Poisson point process (see definition in 1.6.5.1) with parameter one on  $[0, Nr_N]$ . Conditional on having precisely  $Ns_N$  Poisson points in that interval, their distribution is i.i.d. and uniform. Then the distribution of  $Y_1, \dots, Y_{Ns_N}$  under  $\mathcal{U}_{Nr_N}^{\otimes Ns_N}$  is equal to that of the integer up rounded values of these conditional Poisson points. Hence, the distribution of  $K_j$  is the distribution

of the number of these Poisson points that fall into  $[j-1, j]$ . Summarizing, the joint distribution of  $K_1, \dots, K_{Nr_N}$  under  $\mathcal{U}_{Nr_N}^{\otimes N s_N}$  is equal to that of  $Nr_N$  i.i.d. Poisson distributed random variables with parameter one conditioned on having their sum equal to  $Ns_N$ . Hence, writing  $\text{Poi} = \text{Poi}_1$ ,

$$\begin{aligned} \mathcal{U}_{Nr_N}^{\otimes N s_N}(K_j \leq \kappa, \forall j \in [Nr_N]) &= \text{Poi}^{\otimes Nr_N} \left( K_j \leq \kappa \forall j \in [Nr_N], \sum_{j=1}^{Nr_N} K_j = Ns_N \right) \frac{1}{\text{Poi}_{Nr_N}(Ns_N)} \\ &= \text{Poi}([0, \kappa])^{Nr_N} \text{Poi}_{\leq \kappa}^{\otimes Nr_N} \left( \frac{1}{Nr_N} \sum_{j=1}^{Nr_N} K_j = \frac{s_N}{r_N} \right) e^{r_N N} \frac{(Ns_N)!}{(Nr_N)^{Ns_N}}, \end{aligned} \quad (2.3.16)$$

where  $\text{Poi}_{\leq \kappa}$  is Poi, conditioned on  $[0, \kappa]$ . Now we can use Cramér's theorem (see Theorem 1.6.1) for the second term on the right-hand side. Indeed, the average of the  $K_1, \dots, K_{Nr_N}$  satisfies an LDP with speed  $rN$  and rate function equal to the Legendre transform (see, 1.6.6 and 1.6.4) of  $z \mapsto \log \mathbb{E}_{\leq \kappa}[e^{zK_1}]$ , where  $\mathbb{E}_{\leq \kappa}$  is the expectation with respect to  $\text{Poi}_{\leq \kappa}$ , and  $K_1$  is a corresponding random variable. It is only a small technicality to see that the negative exponential rate of the second term on the right-hand side of (2.3.16) on the scale  $rN$  is equal to

$$\begin{aligned} \sup_{z \in \mathbb{R}} \left( \frac{s}{r} z - \log \mathbb{E}_{\leq \kappa}[e^{zK_1}] \right) &= \sup_{z \in \mathbb{R}} \left( \frac{s}{r} z - \log \sum_{i \in \mathbb{N}_0: i \leq \kappa} e^{-1} \frac{e^{zi}}{i!} \right) + \log \text{Poi}([0, \kappa]) \\ &= 1 + I_{\leq \kappa} \left( \frac{s}{r} \right) + \log \text{Poi}([0, \kappa]). \end{aligned}$$

Note that the first and the third term on the right-hand side of this cancel the third and the first term, respectively, on the right-hand side of (2.3.16), on the exponential scale. This directly implies (2.3.14).  $\square$

Substituting (2.3.14) and (2.3.15) into (2.3.13), we obtain

$$q_N = \text{Bin}_{bN, p}(Na_N) \frac{(Na_N)!}{N^{Na_N}} \binom{N}{Nr_N} e^{-Nr_N I_{\leq \kappa}(s/r)} e^{-N(1-r)I_{> \kappa}((a-s)/(1-r))} e^{o(N)} \quad (2.3.17)$$

Using Stirling's approximation  $n! = (n/e)^n e^{o(n)}$  and the facts  $a_N \rightarrow a$ , and  $s_N \rightarrow s$  and  $r_N \rightarrow r$

as  $N \rightarrow \infty$ , we arrive at the assertion. This finishes the proof of Theorem 2.2.1 for  $l = 2$ .

### 2.3.2 Proof of Theorem 2.2.1 for $l = 1$

We present a proof of Theorem 2.2.1 for  $l = 1$  using Crámer's theorem. We are working for protocol (1), where in each of the micro slots  $(\frac{j-1}{N}, \frac{j}{N}]$ ,  $j \in [N]$ , the sending decision is made with probability  $p/N$ , independently for all the message holders and all the micro time slots. Each time slot is successful, i.e., all its messages are successfully transmitted, if and only if the number of transmission attempts in that slot is  $\leq \kappa$ . Hence, all the three quantities under interest,  $S_N$ ,  $A_N$  and  $R_N$ , are equal to sums of  $N$  i.i.d. random quantities. Here it is assumed that any message can attempt any number of times. There are  $j \in [N]$  micro time slots. Again, we write  $[k] = \{1, \dots, k\}$  for any  $k \in \mathbb{N}$ .

For  $i \in [bN]$  and  $j \in [N]$ , we let  $X_i^{(j)} \in \{0, 1\}$  be the indicator on the event that the  $i$ -th message chooses to attempt to send a message in the  $j$ -th time slot. Let

$$A_N^{(j)} = \sum_{i \in [bN]} \mathbb{1}\{X_i^{(j)} = 1\}, \quad R_N^{(j)} = \mathbb{1}\{A_N^{(j)} \leq \kappa\}, \quad S_N^{(j)} = A_N^{(j)} \mathbb{1}\{A_N^{(j)} \leq \kappa\}.$$

Then  $A_N^{(j)}$  is the number of transmission attempts,  $R_N^{(j)}$  the indicator on the event that the  $j$ -th micro slot is successful and  $S_N^{(j)}$  is the number of successfully sent messages during that time slot. Hence,  $A_N = \sum_{j=1}^N A_N^{(j)}$ ,  $R_N = \sum_{j=1}^N R_N^{(j)}$  and  $S_N = \sum_{j=1}^N S_N^{(j)}$ . Furthermore,  $A_N^{(1)}, \dots, A_N^{(N)}$  are i.i.d., and each of them is binomially distributed with parameter  $bN$  and  $p/N$ , hence converges towards the Poisson distribution ( $\text{Poi}_{bp}$ ) with parameter  $pb$  as  $N \rightarrow \infty$ . Again, the goal is to find the logarithmic asymptotics of

$$d_N = \mathbb{P}_N \left( A_N = Na_N, S_N = Ns_N, R_N = Nr_N \right).$$

One short proof, which uses a lot of the proofs given in Section 2.3.3, is via the conditioning

$$d_N = \mathbb{P}(A_N = Na_N) \mathbb{P}(S_N = Ns_N, R_N = Nr_N | A_N = Na_N)$$

and noting that the second term is identical to the corresponding one for protocol (2), i.e., equal to the second term on the right-hand side of the first line of (2.3.10). The only difference between the two protocols is the first term, which was identified as  $\text{Bin}_{bN,p}(Na_N)$  in (2.3.10). Here, however, it is identified as the probability that the sum of  $bN$  independent  $\text{Bin}_{bN,p/N}$ -distributed variables is equal to  $Na_N$ . The logarithmic asymptotics of the latter can be identified in terms of the well-known large-deviation rate function,  $J_{b,p}^{(1)}$ , for the sum of i.i.d.  $\text{Poi}_{bp}$ -distributed variables. The only point that is left to do is to check whether the approximation of  $\text{Poi}_{bp}$  by  $\text{Bin}_{bN,p/N}$ , due to the Poisson limit theorem, is good enough for deducing that  $A_N/N$  satisfies the same LDP.

Nevertheless, we provide another, independent proof. Distinguishing the  $r_N N$  micro slots with  $\leq \kappa$  message attempts, we observe that

$$\begin{aligned} d_N &= \binom{N}{r_N N} \mathbb{P}\left(A_N^{(j)} \leq \kappa \forall j \in [r_N N], \sum_{j \in [r_N N]} A_N^{(j)} = s_N N\right) \\ &\quad \times \mathbb{P}\left(A_N^{(j)} > \kappa \forall j \in [(1-r_N)N], \sum_{j \in [(1-r_N)N]} A_N^{(j)} = (a_N - s_N)N\right) \end{aligned} \quad (2.3.18)$$

where we used the fact that, micro slots with not more than  $\kappa$  messages are independent of slots with more than  $\kappa$  messages and hence we can use independence. Hence, we can proceed with

$$\begin{aligned} d_N &= \binom{N}{r_N N} \text{Bin}_{bN,p/N}([0, \kappa])^{r_N N} \mathbb{P}_{\leq \kappa} \left( \frac{1}{r_N N} \sum_{j \in [r_N N]} A_N^{(j)} = \frac{s_N}{r_N} \right) \\ &\quad \times \text{Bin}_{bN,p/N}((\kappa, \infty))^{(1-r_N)N} \mathbb{P}_{> \kappa} \left( \frac{1}{(1-r_N)N} \sum_{j \in [(1-r_N)N]} A_N^{(j)} = \frac{a_N - s_N}{1-r_N} \right), \end{aligned} \quad (2.3.19)$$

where  $\mathbb{P}_{\leq \kappa}$  is the probability with respect to independent  $\text{Bin}_{bN,p/N}$ -distributed variables,

conditioned on being  $\leq \kappa$ , and  $\mathbb{P}_{>\kappa}$  is defined analogously.

Now we again use the Poisson limit theorem to see that for  $N$  large enough  $\text{Bin}_{bN,p/N}([0, \kappa])^{rN} = [\text{Poi}_{pb}([0, \kappa])]^{rN} e^{o(N)}$  and the analogous statement for the other probability term. The average of  $A_N^{(j)}$  under  $\mathbb{P}_{\leq\kappa}$  satisfy the same LDP as the average of i.i.d.  $\text{Poi}_{bp}$ -distributed random variables, conditioned on being  $\leq \kappa$  (analogously with  $> \kappa$  instead of  $\leq \kappa$ ). The latter do satisfy an LDP, according to Cramér's theorem (see Theorem 1.6.1), with rate function equal to the Legendre transform of  $y \mapsto \log \mathbb{E}_{bp, \leq\kappa}[e^{yX_1}]$ , where  $\mathbb{E}_{bp, \leq\kappa}$  is the expectation with respect to  $\text{Poi}_{bp}$  restricted to  $[0, \kappa]$ , and  $X_1$  is a corresponding random variable. Hence we have that

$$-\lim_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{P}_{\leq\kappa} \left( \frac{1}{rN} \sum_{j \in [rN]} A_N^{(j)} = \frac{s}{r} \right) = r \sup_{y \in \mathbb{R}} \left( \frac{s}{r} y - \log \mathbb{E}_{bp, \leq\kappa}[e^{yX_1}] \right).$$

This, together with the negative large- $N$  logarithmic asymptotics of the probability term in the first line of (2.3.19), give the term

$$\begin{aligned} r \sup_{y \in \mathbb{R}} \left( \frac{s}{r} y - \log \mathbb{E}_{bp, \leq\kappa}[e^{yX_1}] \right) - r \log \text{Poi}_{pb}([0, \kappa]) &= rbp + r \sup_{y \in \mathbb{R}} \left( \frac{s}{r} y - \log \sum_{i \in \mathbb{N}_0: i \leq \kappa} \frac{(e^y bp)^i}{i!} \right) \\ &= rbp - s \log(bp) + rI_{\leq\kappa} \left( \frac{s}{r} \right), \end{aligned}$$

as the substitution  $e^z = bpe^y$ , i.e,  $y = z - \log(bp)$ , gives. Hence, this is the negative exponential rate of the two last terms on the first line of (2.3.19). An analogous formula holds for the two terms in the second line. Summarizing and using Stirling's approximation for the binomial term implies the assertion of Theorem 2.2.1 for  $l = 1$ .

### 2.3.3 Identification of the rate function in Theorem 2.2.1 for $l = 1$

Here we again work under the protocol (1) as in Section 2.3.2. We will present another large-deviation approach using Sanov's theorem instead of Cramér's theorem. This does

not only give an alternate proof of the LDP in Theorem 2.2.1, but also an alternate formula for the rate function, which will be used later.

We use the same notation as in Section 2.3.2. We introduce the empirical measure

$$\mu_N := \frac{1}{N} \sum_{j=1}^N \delta_{A_N^{(j)}},$$

which is a random member of the set  $\mathcal{M}_1(\mathbb{N}_0)$  of probability measures on  $\mathbb{N}_0$ . We denote by  $H(\mu|\nu) = \sum_{k \in \mathbb{N}_0} \mu_k \log \frac{\mu_k}{\nu_k}$  the entropy of  $\mu \in \mathcal{M}_1(\mathbb{N}_0)$  with respect to  $\nu \in \mathcal{M}_1(\mathbb{N}_0)$ . Furthermore, we abbreviate  $q_k = \text{Poi}_{bp}(k) = e^{-pb} (pb)^k / k!$  and  $q = (q_k)_{k \in \mathbb{N}_0}$ .

If  $A_N^{(j)}$  would be exactly  $\text{Poi}_{bp}$ -distributed, then Sanov's theorem would imply that  $(\mu_N)_{N \in \mathbb{N}}$  satisfies an LDP with rate function  $\mu \mapsto H(\mu|\text{Poi}_{bp})$ . However, it is not difficult by explicit calculation to see that the empirical measure of  $A_N^{(1)}, \dots, A_N^{(N)}$  and the empirical measure of  $N$  independent  $\text{Poi}_{bp}$ -distributed random variables are exponentially equivalent as  $N \rightarrow \infty$  and therefore satisfy the same LDP.

We introduce

$$f: \mathbb{N}_0 \rightarrow \mathbb{N}_0 \times [\kappa] \times \{0, 1\}, \quad f(a) = \left( a, a \mathbb{1}\{a \leq \kappa\}, \mathbb{1}\{a \leq \kappa\} \right).$$

Note that the triple under interest,  $(A_N, R_N, S_N)$ , is nothing but  $N \langle f, \mu_N \rangle$ , i.e., the image of  $\mu_N$  under the map  $\mu \mapsto \langle f, \mu \rangle$ . Hence, if this map would be continuous in the weak topology on  $\mathcal{M}_1(\mathbb{N}_0)$ , then the contraction principle (see Theorem 1.6.2) immediately would give the LDP of Theorem 2.2.1 for  $l = 1$  with rate function  $I^{(1)}$  given as in (2.2.7). However, clearly  $f$  is not bounded, hence the map  $\mu \mapsto \langle f, \mu \rangle$  is not continuous in the weak topology on  $\mathcal{M}_1(\mathbb{N}_0)$ . Hence we cannot directly apply the contraction principle. However, it is clear that the second and third argument in the function are bounded. It is clear that a sufficient

truncation argument is required by proving that

$$\lim_{C \rightarrow \infty} \limsup_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{P}_N(A_N > CN) = -\infty. \quad (2.3.20)$$

It is easy to see that (2.3.20) is true, using the exponential Chebyshev inequality ( $\mathbb{P}_N(A_N > CN) \leq e^{-C\alpha N} \mathbb{E}_N[e^{\alpha A_N}]$  for  $\alpha > 0$ ) and that  $A_N$  is a sum of  $N$  independent  $\text{Bin}_{bN, p/N}$ -distributed random variables  $A_N^{(j)}$  and that  $\mathbb{E}[e^{\alpha A_N^{(1)}}] \rightarrow e^{pb(e^\alpha - 1)}$  as  $N \rightarrow \infty$ .

## 2.4 Analysis of the rate functions

### 2.4.1 Description of the minimizer(s) $a, s, r$

We now prove Corollary 2.2.2. We rewrite the function defined in (2.2.1), for  $x \in \mathbb{R}$ , as

$$I_{\leq \kappa}(x) = \sup_{y \in (0, \infty)} \left( x \log y - \log f_{\leq}(y) \right) \quad \text{with} \quad f_{\leq}(y) = \sum_{i \in \mathbb{N}_0: i \leq \kappa} \frac{y^i}{i!}. \quad (2.4.21)$$

The following lemma implies Corollary 2.2.2.

**Lemma 2.4.1.** *For  $l \in \{1, 2\}$ , the rate function  $I^{(l)}$  defined in (2.2.4) is convex and has precisely one minimiser  $(a, r, s)$ , depending on  $(b, p, \kappa)$ . It is characterized by*

$$a = pb, \quad r = \frac{f_{\leq}(a)}{e^a}, \quad s = r \frac{af'_{\leq}(a)}{f_{\leq}(a)} = \frac{af'_{\leq}(a)}{e^a}. \quad (2.4.22)$$

*Proof.* Let us first argue the convexity. There is a simple standard proof, based on (2.2.7), that shows the convexity of  $I^{(1)}$ , since the entropy  $H(\mu|v)$  is convex in  $\mu$  and the constraint is linear in  $\mu$ . Now observe that  $I^{(2)} - I^{(1)}$  is also convex; indeed it depends only on  $a$ :

$$H(a) = I^{(2)}(a, s, r) - I^{(1)}(a, s, r) = a - pb - (b - a) \log \frac{b(1-p)}{b-a}, \quad (2.4.23)$$

and one easily calculates that  $H''(a) = \frac{1}{b-a} > 0$  for any  $a \in (0, b)$ . The existence and uniqueness of a minimizer hence follows from the uniqueness of zeros of the gradient of  $I^{(1)}$ , which we will show now.

The maximizer  $Y_{\leq}(x)$  in (2.4.21) is characterized by  $f'_{\leq}(Y_{\leq}(x))Y_{\leq}(x) = x f_{\leq}(Y_{\leq}(x))$  and satisfies  $I_{\leq \kappa}(x) = x \log Y_{\leq}(x) - \log f_{\leq}(Y_{\leq}(x))$ .

We write  $I_{> \kappa}$  in a similar manner, i.e.,  $I_{> \kappa}(x) = x \log Y_{>}(x) - \log f_{>}(Y_{>}(x))$ , and  $Y_{> x}$  is characterized by  $f'_{>}(Y_{>}(x))Y_{>}(x) = x f_{>}(Y_{>}(x))$ , where  $f_{>}(y) = \sum_{i \in \mathbb{N}_0: i > \kappa} \frac{y^i}{i!}$ .

In order to characterize the minimizer of  $I = I^{(1)}$ , we equate its gradient in  $(a, s, r)$  to zero and obtain

$$0 = \frac{\partial I(a, s, r)}{\partial a} = -\log \frac{p(b-a)}{1-p} + \log Y_{>}\left(\frac{a-s}{1-r}\right), \quad (2.4.24)$$

$$0 = \frac{\partial I(a, s, r)}{\partial r} = \log \frac{r}{1-r} - \log \frac{f_{\leq}\left(Y_{\leq}\left(\frac{s}{r}\right)\right)}{f_{>}\left(Y_{>}\left(\frac{a-s}{1-r}\right)\right)}, \quad (2.4.25)$$

$$0 = \frac{\partial I(a, s, r)}{\partial s} = \log Y_{\leq}\left(\frac{s}{r}\right) - \log Y_{>}\left(\frac{a-s}{1-r}\right). \quad (2.4.26)$$

From (2.4.24) and (2.4.26), we have that

$$Y_{\leq}\left(\frac{s}{r}\right) = Y_{>}\left(\frac{a-s}{1-r}\right) = \frac{p(b-a)}{1-p}. \quad (2.4.27)$$

We abbreviate  $c = \frac{p(b-a)}{1-p}$ , i.e.,  $a = b - \frac{(a-p)c}{p}$ , then we have  $c = Y_{\leq}\left(\frac{s}{r}\right)$ . Using the defining property of  $Y_{\leq}(x)$  from above, we get that  $f'_{\leq}(c)c = \frac{s}{r} f_{\leq}(c)$ .

Substituting (2.4.27) into (2.4.25) yields

$$f_{\leq}(c) = f_{>}(c) \frac{r}{1-r} = (e^c - f_{\leq}(c)) \frac{r}{1-r}, \quad (2.4.28)$$

that is,  $f_{\leq}(c) = r e^c$  and  $f_{>}(c) = (1-r)e^c$ . Combining this with  $f'_{\leq}(c)c = \frac{s}{r} f_{\leq}(c)$  from

above, we see that  $a = c$  and therefore  $a = pb$ .

From (2.4.27) we also have that

$$f'_{\leq}(c) = f'_{>}(c)s/(a-s) \quad (2.4.29)$$

and analogously  $f'_{\leq}(c) = \frac{s}{a}e^c$  and  $f'_{>}(c) = \frac{a-s}{a}e^c$ .

Also from the characterisation of  $Y_{\leq}(\cdot)$ , we have that

$$f'_{\leq}(c)/f_{\leq}(c) = s/r \quad (2.4.30)$$

Altogether, we arrive at (2.4.22). This ends the proof of the Lemma 2.4.1.  $\square$



## Chapter 3

# Optimisation result for the throughput probability parameter ( $p$ ) in Aloha/Slotted Aloha

### 3.1 Optimal $p$

A natural and important question is about that value of  $p$  that maximizes the expected throughput per micro slot,  $s_p$ . It is clear that the optimal value should be such that  $bp$  is smaller than  $\kappa$ , since otherwise the number of attempts per time slot is larger than the success threshold. But the question is how much below  $\kappa$  should one go in order not to underachieve more than necessary.

Let us call the optimal value  $p_*^{(l)}$ . Recall that it is the same function  $s_p$  that is to be optimized for both  $l = 1$  and  $l = 2$ , but the domain for  $p_*^{(2)}$  is restricted to  $[0, 1]$ . The map  $k \mapsto \text{Poi}_a(k)$  increases on  $[0, a] \cap \mathbb{N}_0$  and decreases on  $[a, \infty) \cap \mathbb{N}_0$ . Hence we expect that the optimal value of  $a = a_* \in (0, \infty)$  for  $a \mapsto \sum_{k=0}^{\kappa} k \text{Poi}_a(k) = s_{a/b}$  will be smaller than  $\kappa$ . Indeed, it is even smaller than  $\kappa - 1$ :

**Lemma 3.1.1** (Optimal  $p$ ). *For  $l = 1$ , there is precisely one  $p_* = p_*^{(1)} \in (0, \infty)$  that maximizes the map  $(0, \infty) \ni p \mapsto s_p$ . It is characterised by*

$$\frac{(a_*)^\kappa}{(\kappa - 1)!} = \sum_{i \in \mathbb{N}_0: i \leq \kappa - 1} \frac{(a_*)^i}{i!}, \quad a_* = bp_*, \quad (3.1.1)$$

*and it satisfies  $bp_* < \kappa - 1$  and  $bp_* \sim \kappa$  as  $\kappa \rightarrow \infty$ . Furthermore,  $p \mapsto s_p$  strictly increases in  $[0, p_*]$  and strictly decreases in  $[p_*, \infty)$ . Hence,  $p_*^{(2)} = \min\{p_*, 1\}$ .*

The proof of Lemma 3.1.1 is in Section 3.3.

## 3.2 Conditioning the number of attempts on the number of successes

In this section we discuss an interesting question that can be answered with the help of large-deviation theory, combined with an analysis of the rate functions: What is the most likely reason for a deviation event of the form that the throughput is below the theoretically optimal one? Is it that too few attempts have been made or too many?

In order to formalize this question, we write  $\mathbb{P}_{N,p}^{(l)}$  for the probability measure of the protocol  $l$  with parameter  $p$ , pick some  $0 < s \leq a$ , then it is clear that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{P}_{N,p}^{(l)} \left( A_N = \lfloor aN \rfloor \mid S_N = \lfloor Ns \rfloor \right) = - \inf_r I_p^{(l)}(a, s, r) + \inf_{\tilde{a}, r} I_p^{(l)}(\tilde{a}, s, r).$$

From this, we see that

$$\lim_{N \rightarrow \infty} \mathbb{E}_{N,p}^{(l)} \left( \frac{A_N}{N} \mid S_N = \lfloor Ns \rfloor \right) = \operatorname{argmin}_a \left( \inf_r I_p^{(l)}(a, s, r) \right).$$

Given  $s$ , we now define  $a_p(s)$  as a minimizer of the map  $a \mapsto \inf_r I_p^{(l)}(a, s, r)$ , i.e., the typical

number of sending attempts, conditional on having  $\approx sN$  successes. It will turn out that  $a_p(s)$  is well-defined at least in a neighbourhood of  $a_p(s_p)$  if  $p$  is close enough to  $p^*$ , where we recall that  $s_p$  is the minimizer that is established in Corollary 2.2.2, and  $p^*$  is the maximizing  $p$  for  $p \mapsto s_p$  characterized by (3.1.1). In terms of these quantities, the question now reads: Given  $s < s_p$ , is it true that  $a_p(s) < a_p(s_p)$ ?

**Theorem 3.2.1.** *For  $l = 1$ , fix  $\kappa$ . Then, for any  $p \in (0, \infty)$  and for any  $s$  in some neighbourhood of  $s_p$ , we have*

$$p < p^* \implies \left[ s < s_p \implies a_p(s) < a_p(s_p) \right] \text{ and } \left[ s > s_p \implies a_p(s) > a_p(s_p) \right], \quad (3.2.2)$$

$$p > p^* \implies \left[ s < s_p \implies a_p(s) > a_p(s_p) \right] \text{ and } \left[ s > s_p \implies a_p(s) < a_p(s_p) \right]. \quad (3.2.3)$$

Furthermore, for  $p = p_*$ , for any  $s \in [0, p_*b] \setminus \{s_{p_*}\}$ , we have  $a_{p_*}(s) > a_{p_*}(s_{p_*})$ .

The proof is in Section 3.4. Theorem 3.2.1 says that, for non-optimal  $p$ , if  $s$  sufficiently close to the optimal  $s_p$ , then the attempt number  $a_p(s)$  deviates to the same side of  $a_p(s_p)$  as  $s$  is with respect to  $s_p$ , while in the optimal  $p_*$ , the typical attempt number for non-optimal success number is always larger than the optimal one. The latter means that, for the optimal choice  $p = p_*$ , the event of non-optimal throughput always comes with overwhelming probability from too many attempts. Apparently, here the conditional probability for having too many attempts is much larger than the one for having too few.

### 3.3 Optimizing $p \mapsto s_p$

In this section, we prove Lemma 3.1.1, that is, we analyse the maximizer of the map  $p \mapsto s_p$ , the optimal throughput, for both protocols (1) and (2), the only difference being that the domain for protocol (1) is  $p \in [0, \infty)$  and for (2) it is  $p \in [0, 1]$ .

Let us first consider optimization over  $(0, \infty)$ . The analytic function  $g(a) = s_{a/b} = ae^{-a} \sum_{i=0}^{\kappa-1} \frac{a^i}{i!}$

is positive in  $(0, \infty)$  with limits 0 at  $a \downarrow 0$  and  $a \rightarrow \infty$ , hence it has at least one maximizer  $a_*$ , which is characterised by  $g'(a_*) = 0$ . We see that (with  $f_{\leq}(a) = \sum_{i=0}^{\kappa} \frac{a^i}{i!}$ )

$$\frac{d}{dp} s_p = b e^{-ap} \left( f'_{\leq}(a_p) + a_p f''_{\leq}(a_p) - a_p f'_{\leq}(a_p) \right) = b e^{-bp} \left[ \sum_{i \leq \kappa-1} \frac{(bp)^i}{i!} - \frac{(bp)^{\kappa}}{(\kappa-1)!} \right], \quad p > 0. \quad (3.3.4)$$

Hence, (3.1.1) characterizes  $p_*^{(1)}$ .

Using elementary calculus, we see that a solution  $a_*$  to (3.1.1) exists since the polynomial  $f(a) = -(\kappa-1)! b e^{-a} \frac{d}{dp} s_p = a^{\kappa} - \sum_{i \leq \kappa-1} a^i \frac{(\kappa-1)!}{i!}$  starts with  $f(0) < 0$  and satisfies  $f(a) \rightarrow \infty$  as  $a \rightarrow \infty$ . Note that, for any  $a > 0$ , we have

$$\begin{aligned} f(a) &\geq a^{\kappa} - \sum_{i \leq \kappa-1} a^i (\kappa-1)^{\kappa-1-i} = a^{\kappa} - (\kappa-1)^{\kappa-1} \sum_{i \leq \kappa-1} \left( \frac{a}{\kappa-1} \right)^i = a^{\kappa} + \frac{(\kappa-1)^{\kappa} - a^{\kappa}}{a - (\kappa-1)} \\ &= \frac{a^{\kappa}(a - \kappa) + (\kappa-1)^{\kappa}}{a - (\kappa-1)}, \end{aligned}$$

and the latter is positive for any  $a > \kappa - 1$ . Hence, we even have that  $a_* \leq \kappa - 1$ . Furthermore, there is only one solution, since  $f'(a) = \kappa a^{\kappa-1} - \sum_{i \leq \kappa-1} a^i \frac{(\kappa-1)!}{i!} + a^{\kappa-1}$  for any  $a$ , and for any solution  $a_*$  we see that  $f'(a_*) = (\kappa+1)a_*^{\kappa-1} - a_*^{\kappa} = a_*^{\kappa-1}[\kappa+1-a_*]$ , which is positive. Hence,  $f$  has precisely one zero in  $[0, \infty)$ . It is negative left of  $a_*$  and positive right of it. Accordingly,  $p \mapsto s_p$  is increasing in  $[0, p_*]$  and decreasing in  $[p_*, \infty)$ . We obtain a lower bound for  $a_*$  by

$$f(a) \leq a^{\kappa} - a^i \frac{(\kappa-1)!}{i!} < a^i \left( a^{\kappa-i} - (i+1)^{\kappa-i-1} \right), \quad a > 0, i \in \{0, \dots, \kappa-1\}.$$

This upper bound is zero for  $a = (i+1)^{1-1/(\kappa-i)}$ , hence  $a_* \geq \max_{i=0}^{\kappa-1} (i+1)^{1-1/(\kappa-i)}$ . Taking  $i = \kappa - \sqrt{\kappa}$  gives  $a_* \geq (\kappa - \sqrt{\kappa})^{1-\kappa^{-1/2}} = \kappa(1+o(1))$  as  $\kappa \rightarrow \infty$ .

### 3.4 Conditioning on successes

In this section, we prove Theorem 3.2.1. Recall that we conceive the maximal throughput per micro slot,  $s = s_p$ , as a function of  $p$ . Recall from Lemma 3.1.1 that the maximal  $p^*$  for  $p \mapsto s_p$  is characterized by

$$\frac{a_p^\kappa}{(\kappa - 1)!} = \sum_{i=0}^{\kappa-1} \frac{a_p^i}{i!}, \quad a_p = bp. \quad (3.4.5)$$

Furthermore recall that  $a_p^{(l)}(s)$  denotes the minimising  $a$  for the map  $a \mapsto \inf_r I^{(l)}(a, s, r)$  for  $l \in \{1, 2\}$ , and note that  $a_p = a_p(s_p) = a_p^{(1)}(s_p) = a_p^{(2)}(s_p) = bp$ . Here we answer the question of the reason for few number of successes. The following lemma proves Theorem 3.2.1 for  $l = 1$ .

**Lemma 3.4.1.** *Let us consider the rate function for the protocol  $l = 1$ . Then, for any  $p \in (0, \infty)$ , we have  $(a_p^{(1)})'(s_p) < 0$  for  $p < p_*$  and  $(a_p^{(1)})'(s_p) > 0$  for  $p > p_*$ . In particular, for  $s$  in a neighbourhood of  $s_p$ , (3.2.2) and (3.2.3) hold.*

*Furthermore, for  $p = p_*$ , we have  $a_{p_*}^{(1)}(s) > a_{p_*}^{(1)}(s_{p_*}) = bp_*$  for any  $s \in [0, b] \setminus \{bp_*\}$ .*

*Proof.* Let us first analyse  $\inf_r I_p^{(1)}(a, s, r)$  for fixed  $a, s$ . We benefit from the representation in (2.2.4): We have that

$$\begin{aligned} \inf_r I_p^{(1)}(a, s, r) &= \inf_r \inf \{H(\mu | \text{Poi}_{pb}) : \langle f, \mu \rangle = (a, s, r)\} \\ &= \inf \left\{ H(\mu | \text{Poi}_{pb}) : \sum_{k=0}^{\infty} k\mu_k = a, \sum_{k=0}^{\kappa} k\mu_k = s \right\} \\ &= \inf \left\{ H(\mu | \text{Poi}_{pb}) : \langle \mu, \text{id} \rangle = a, \langle \mu, \text{id}|_{\leq \kappa} \rangle = s \right\}, \end{aligned}$$

where  $\text{id}$  is the identity function on  $\mathbb{N}_0$  and  $\text{id}|_{\leq \kappa}(k) = k \mathbb{1}_{[0, \kappa]}(k)$ ; and we used the notation  $\langle \mu, f \rangle$  for the integral of a function  $f$  with respect to a measure  $\mu$ . Now we apply standard variational calculus. Consider a minimizer  $\mu$  of the last formula. A standard argument

shows that  $\mu_k > 0$  for any  $k$ . Fix some  $\gamma: \mathbb{N}_0 \rightarrow \mathbb{R}$  satisfying  $\gamma \perp \mathbb{1}, \gamma \perp \text{id}$  and  $\gamma \perp \text{id}|_{\leq \kappa}$ . Then, for any  $\varepsilon \in \mathbb{R}$  with sufficiently small  $|\varepsilon|$ , the measure  $\mu + \varepsilon\gamma$  is admissible. From minimality, we deduce that

$$0 = \partial_\varepsilon|_{\varepsilon=0} H(\mu + \varepsilon\gamma | \text{Poi}_{pb}) = \sum_k \left( \gamma_k \log \frac{\mu_k}{q_k} + \mu_k \frac{\gamma_k}{\mu_k} \right) = \left\langle \gamma, \log \frac{\mu}{q} \right\rangle,$$

where we put  $q_k = \text{Poi}_{pb}(k)$ . Hence,  $\log \frac{\mu}{q}$  is a linear combination of  $\mathbb{1}, \text{id}$  and  $\text{id}|_{\leq \kappa}$ . That is, there are  $A, B, C \in \mathbb{R}$  such that

$$\mu_k = q_k e^{Ae^{Bk}} \times \begin{cases} e^{Ck} & \text{for } k \leq \kappa, \\ 1 & \text{for } k > \kappa, \end{cases} \quad k \in \mathbb{N}_0. \quad (3.4.6)$$

We note that  $A, B$  and  $C$  are well-defined functions of  $a$  and  $s$ , since  $\mathbb{1}, \text{id}$  and  $\text{id}|_{\leq \kappa}$  are linearly independent.

Now using that  $\langle \mu, \mathbb{1} \rangle = 1$  and  $\langle \mu, \text{id} \rangle = a$  and  $\langle \mu, \text{id}|_{\leq \kappa} \rangle = s$ , and introducing the notation

$$\varphi(B, C) := \log \left( \sum_{k=0}^{\kappa} q_k e^{(B+C)k} + \sum_{k>\kappa} q_k e^{Bk} \right), \quad B, C \in \mathbb{R}, \quad (3.4.7)$$

we see that  $B = B(a, s)$  and  $C = C(a, s)$  are characterised by

$$a = \frac{\sum_{k \leq \kappa} k q_k e^{(B+C)k} + \sum_{k > \kappa} k q_k e^{Bk}}{\sum_{k \leq \kappa} q_k e^{(B+C)k} + \sum_{k > \kappa} q_k e^{Bk}} = \partial_B \varphi(B, C), \quad (3.4.8)$$

$$s = \frac{\sum_{k \leq \kappa} k q_k e^{(B+C)k}}{\sum_{k \leq \kappa} q_k e^{(B+C)k} + \sum_{k > \kappa} q_k e^{Bk}} = \partial_C \varphi(B, C), \quad (3.4.9)$$

while  $A(a, s) = -\varphi(B(a, s), C(a, s))$ . Furthermore,

$$\inf_r I_p^{(1)}(a, s, r) = \sum_k \mu_k \log \frac{\mu_k}{q_k} = Ba + Cs - \varphi(B, C). \quad (3.4.10)$$

This finishes the characterisation of  $\inf_r I_p^{(1)}(a, s, r)$  for any fixed  $a, s$ .

Now we optimise over  $a$  with  $s$  fixed. We recall that  $a_p^{(1)}(s)$  denotes the minimizing  $a$  of  $\inf_r I_p^{(1)}(a, s, r)$ . Recalling that  $B$  and  $C$  are functions of  $a$  and  $s$ , we differentiate (3.4.10) with respect to  $a$  and use it for  $a = a_p^{(1)}(s)$  to obtain

$$\begin{aligned} 0 &= \left( a_p^{(1)}(s) - \partial_B \varphi(B(a_p^{(1)}(s), s), C(a_p^{(1)}(s), s)) \right) \frac{d}{ds} B(a_p^{(1)}(s), s) \\ &\quad + \left( s - \partial_C \varphi(B(a_p^{(1)}(s), s), C(a_p^{(1)}(s), s)) \right) \frac{d}{ds} C(a_p^{(1)}(s), s) + B(a_p^{(1)}(s), s) \quad (3.4.11) \\ &= B(a_p^{(1)}(s), s), \end{aligned}$$

also using (3.4.8) and (3.4.9). Differentiating this with respect to  $s$  produces

$$(a_p^{(1)})'(s) = - \frac{\partial_s B(a_p^{(1)}(s), s)}{\partial_a B(a_p^{(1)}(s), s)}. \quad (3.4.12)$$

A tedious calculation, starting from differentiating both (3.4.8) and (3.4.9) both with respect to  $a$  and to  $s$ , gives, for  $B = B(a, s)$  and any  $a$  and  $s$ ,

$$\partial_a B = \frac{\partial_C^2 \varphi}{\partial_B^2 \varphi \partial_C^2 \varphi - (\partial_C \partial_B \varphi)^2} \quad \text{and} \quad \partial_s B = - \frac{\partial_B \partial_C \varphi}{\partial_B^2 \varphi \partial_C^2 \varphi - (\partial_C \partial_B \varphi)^2}$$

and hence

$$(a_p^{(1)})'(s) = \frac{\partial_B \partial_C \varphi(B, C)}{\partial_C^2 \varphi(B, C)} \quad \text{with } B = B(a_p^{(1)}(s), s) = 0 \text{ and } C = C(a_p^{(1)}(s), s). \quad (3.4.13)$$



First we show that the denominator is positive:

$$\begin{aligned} \partial_C^2 \varphi(B, C) &= \frac{\sum_{k \leq \kappa} k^2 q_k e^{(B+C)k} \left( \sum_{k \leq \kappa} q_k e^{(B+C)k} + \sum_{k > \kappa} q_k e^{Bk} \right) - \left( \sum_{k \leq \kappa} k q_k e^{(B+C)k} \right)^2}{\left( \sum_{k \leq \kappa} q_k e^{(B+C)k} + \sum_{k > \kappa} q_k e^{Bk} \right)^2} \\ &\geq \frac{\left( \sum_{k \leq \kappa} k^2 q_k e^{(B+C)k} \right) \left( \sum_{k \leq \kappa} q_k e^{(B+C)k} \right) - \left( \sum_{k \leq \kappa} k q_k e^{(B+C)k} \right)^2}{\left( \sum_{k \leq \kappa} q_k e^{(B+C)k} + \sum_{k > \kappa} q_k e^{Bk} \right)^2} > 0, \quad B, C \in \mathbb{R}, \end{aligned}$$

as a standard symmetrisation shows. Next we consider the numerator in (3.4.13):

$$\begin{aligned} \partial_B \partial_C \varphi(0, C) &= \left( \sum_{k \leq \kappa} q_k e^{Ck} + \sum_{k > \kappa} q_k \right)^{-2} \\ &\quad \left[ \sum_{k \leq \kappa} k^2 q_k e^{Ck} \left( \sum_{k \leq \kappa} q_k e^{Ck} + \sum_{k > \kappa} q_k \right) - \left( \sum_{k \leq \kappa} k q_k e^{Ck} + \sum_{k > \kappa} k q_k \right) \sum_{k \leq \kappa} k q_k e^{Ck} \right]. \end{aligned} \quad (3.4.14)$$

No we use the facts that  $\sum_{k \leq \kappa} q_k + \sum_{k > \kappa} q_k = 1$  (since  $(q_k)_{k \in \mathbb{N}_0}$  is a probability distribution) and  $\sum_{k \in \mathbb{N}_0} k q_k = bp = a_p^{(1)} = a_p^{(1)}(s_p)$  (see Corollary 2.2.2;  $(q_k)_{k \in \mathbb{N}_0} = \text{Poi}_{pb}$  has expectation  $pb$ ). Furthermore, note that  $C(a_p^{(1)}(s_p), s_p) = 0$  by optimality (which can be seen in the same way as the fact that  $B(a_p^{(1)}(s), s) = 0$  above). Then we get

$$\begin{aligned} (a_p^{(1)})'(s_p) &= \partial_B \partial_C \varphi(0, 0) = \sum_{k \leq \kappa} k^2 q_k - bp \sum_{k \leq \kappa} k q_k \\ &= bpe^{-bp} \left[ \sum_{k \leq \kappa-1} (k+1) \frac{(bp)^k}{k!} - bp \sum_{k \leq \kappa-1} \frac{(bp)^k}{k!} \right] = p \frac{d}{dp} s_p, \end{aligned}$$

as we see from (3.3.4). Recall that  $p_*$  is the unique maximizer for  $p \mapsto s_p$ . According to Lemma 3.1.1, this (and therefore  $(a_p^{(1)})'(s_p)$ ) is positive if  $p < p_*$  and negative if  $p > p_*$ . This implies all assertions of Lemma 3.4.1 for  $p \neq p_*$ .

Now we consider the case  $p = p_*$  characterised in (3.4.5). Here it will not be successful to rely on the characterisation of  $a_p^{(1)}(s)$  by  $0 = B(a_p^{(1)}(s), s)$  and to consider the derivative with respect to  $s$  in  $s = s_{p_*}$  only, since  $\partial_B \partial_C \varphi(0, 0) = 0$  for  $p = p_*$ . Instead, we use (3.4.8) and

explicitly look at the difference

$$\begin{aligned} a_{p^*}(s) - a_{p^*}(s_{p^*}) &= \partial_B \varphi(0, C) - bp^* = \frac{\sum_{k \leq \kappa} q_k e^{Ck} [k - a_{p^*}] + \sum_{k > \kappa} q_k [k - a_{p^*}]}{\sum_{k \leq \kappa} q_k e^{Ck} + \sum_{k > \kappa} q_k} \\ &= \frac{\sum_{k \leq \kappa} q_k [e^{Ck} - 1] [k - a_{p^*}]}{\sum_{k \leq \kappa} q_k e^{Ck} + \sum_{k > \kappa} q_k}, \end{aligned} \quad (3.4.15)$$

with  $C = C(a_{p^*}^{(1)}(s), s)$ . We used in the last step that  $\sum_{k > \kappa} kq_k = a_{p^*} - \sum_{k \leq \kappa} kq_k$  and  $\sum_{k \in \mathbb{N}_0} q_k = 1$ . Note that  $C < 0$  for  $s < s_{p^*}$  and  $C > 0$  for  $s > s_{p^*}$ . Indeed, a similar calculation as in (3.4.11) shows that

$$\frac{d}{ds} \inf_{r,a} I_{p^*}^{(1)}(a, s, r) = \frac{d}{ds} \left[ sC(a_{p^*}^{(1)}(s), s) - \varphi\left(0, C(a_{p^*}^{(1)}(s), s)\right) \right] = C(a_{p^*}^{(1)}(s), s), \quad s \in (0, \infty).$$

Now note that  $s_{p^*}$  is defined as the minimizer of the function  $s \mapsto \inf_{r,a} I_{p^*}^{(1)}(a, s, r)$ ; hence it is decreasing left of the minimal point and increasing right of it.

Write  $g(C) = \sum_{k=1}^{\kappa} q_k [e^{Ck} - 1] [k - a_{p^*}]$  for the numerator of the right-hand side of (3.4.15). Clearly  $g(0) = 0$ . Recall from above that  $g'(0) = 0$  and observe that, for any  $C < 0$ ,

$$g'(C) = \sum_{k=0}^{\kappa} kq_k e^{Ck} (k - a_{p^*}) < e^{Ca_{p^*}} \sum_{k \leq a_{p^*}} kq_k (k - a_{p^*}) + e^{Ca_{p^*}} \sum_{k: a_{p^*} < k \leq \kappa} kq_k (k - a_{p^*}) = 0.$$

Hence,  $g$  is strictly decreasing in  $(-\infty, 0]$  and hence positive in  $(-\infty, 0)$ . An analogous argument shows that  $g'(C) > 0$  for  $C > 0$ :

$$g'(C) = \sum_{k=0}^{\kappa} kq_k e^{Ck} (k - a_{p^*}) > \sum_{k \leq a_{p^*}} kq_k (k - a_{p^*}) + \sum_{k: a_{p^*} < k \leq \kappa} kq_k (k - a_{p^*}) = 0.$$

Hence  $g$  is strictly increasing and positive in  $(0, \infty)$ . This implies that  $a_{p^*}(s) > a_{p^*}(s_{p^*})$  for any  $s \neq s_{p^*}$  and finishes the proof of the lemma.  $\square$

## Chapter 4

# Summary, conclusion and recommendations

### 4.1 Summary

This research extended the general concept of considering interference/capacity constraint only at the first instance a message attempts to makes a hop. An ideal and more realistic approach is to consider interference/capacity constraint anytime a message makes a hop until it gets to the intended receiver.

Each of the objectives outlined in Chapter one has been duly answered in this thesis. More specifically,

- Objective one which seeks to propose a version of Aloha and slotted Aloha has been duly described in Section 2.1. Both models have been described in relation to the number of sending attempts, medium access probability and how messages are able to access a communication channel. In both models, it is assumed each channel can only transmit if there are not more than  $\kappa$  number of message.

- The goal of objective two was to derive large deviation results for the proposed models described in Section 2.1. Theorem 2.2.1 shows the large deviation results for the two models. The proof of Theorem 2.2.1 for  $l = 2$  is in Section 2.3.1 and for  $l = 1$  is in Section 2.3.3. In Section 2.3.3 a proof of an alternative representation (2.2.7) for the rate function of  $l = 1$  can also be found.
- The goal of objective three was to derive minimisers for the large deviation rate functions. Corollary 2.2.2 which is a consequence of the proof in Section 2.3.1 answers this objective. This is the statement of Lemma 2.4.1 which has also been rigorously proved.
- Objective four was duly answered in Section 3.2. A direct consequence of this objective is Lemma 3.1.1 and Theorem 3.2.1. This objective answers a very important question. What is the most likely reason that the throughput is below the theoretically optimal one? Is the reason due to too few or too many attempts? Theorem 3.2.1, clearly answers this question using the large deviation results.

## 4.2 Conclusion

In the Aloha model it is assumed that messages decide independently with probability  $p$  whether to transmit to an intended receiver or not. In the event that too many messages are sent out with a high probability, this results in the failure of all the messages. A good way to reduce this total cancellation of messages is the introduction of the slotted Aloha. An important question of optimisation then is, what appropriate transmission probability  $p$  should be chosen such that very large number of messages can be transmitted at a time. It turns out that, the optimal value of  $p$  must be such that, the sending attempts  $bp$  is smaller than  $\kappa$ . It is clear that Aloha and slotted Aloha can be described using large deviation techniques. The result shows that there exist unique minimisers for the large deviation rate

functions. These are stated and proved in Chapter three. It was also possible to derive an alternative representation of the results from the Cramer theorem using the Sanov theorem by using a cutting argument.

The existence of a unique optimal message transmission  $p$  was also proved. This means that, it is possible to choose a transmission probability that can ensure that maximum throughput is realised. This optimal probability was also shown to be unique. From the result, one can deduce tractable large deviation result for the number of successes  $S_N$  for the case  $l = 1$  using the alternative formulation of Theorem 2.2.1 for  $l = 1$ , but not for  $l = 2$ . Theorem 3.2.1 clearly shows that, for the optimal choice  $p = p_*$ , the event of non-optimal throughput always comes with overwhelming probability from too many message attempts.

This research is different in many ways to other similar models which dealt with only one channel. This is because most of these papers considered purely simulated results and computation of expectations for even the simple case. No large deviation result has so far been derived for the multichannel models using large deviations in the past.

### 4.3 Recommendations for future research

There is quite a number of discussable extensions to the models presented in this thesis. A few options are itemized below;

- One can look at the case where the upper limit of the number of messages through each channel is allowed to go to infinity.
- One can also look at the case where the  $\kappa$  number of messages is allowed to depend on the spatial distribution of the network nodes. Meaning each  $\kappa$  can depend on how many messages are at a particular place at a given time.
- One can also look at the case where instead of considering Poisson points, we extend

to the Cox process. This will add another layer of randomness to the model to make it even more realistic.

- One could also extend the same ideas in this thesis to another protocol. Particularly, the CSMA protocol, where each sending attempt requires some minimal pre-information like the existence of an idle channel.
- A continuous time version which I am currently working on could be applied to the Aloha and the CSMA protocols.
- Another version could be, to directly look at interference constraints in the Aloha and slotted Aloha models using large deviations.
- Theorem 3.2.1 for  $l = 2$  could not be solved hence can be taken on as further research topic.

Below I briefly describe another model that was brainstormed and could be adopted and researched on using other large deviation theorems or techniques. This is a model of message trajectories which like this research depends on message hops and as such has some similarities with this current work.

#### 4.3.0.1 Trajectory of messages

Define a trajectory of messages from  $X_i$  to  $Y_i$  as  $T_i : X_i \rightarrow Y_i$ . Now for any  $i \in N$ , express the vector of message trajectories from user  $X_i$  to  $Y_i$  as

$$T_i = (X_i, T_i^{(1)}, \dots, T_i^{k_i-1}, Y_i) = (T_i^{(s)})_{s \in 0, \dots, k_i}, \quad k_i \in N$$

such that  $T_i^{(s)} = T_i^{(s+1)}$  is possible where  $s$  is the time for a hop and  $k_i$  is the length of the  $i^{th}$  trajectory. Each message has a function of relays and that only one message is sent out at a time. Suppose we define the empirical measure of message hops from  $X$  to  $Y$  and also

the number of possible trajectories per unit time. A question such as the appropriate hop length can also be analysed. We can now study the following empirical measure of messages trajectories;

$$L_N = \frac{1}{N} \sum_{x,y} \sum_{i=1}^N \delta_{T_{x,y}}^{(i)} \quad (4.3.1)$$

A natural question is, what is the large deviation behaviour of of this empirical measure under different assumptions of the number of messages from  $X$  to  $Y$ . The number of messages can be such that it is either deterministic or random depending on how complicated one wants it to be. Several questions can be formulated and an attempt can be made at answering them using large deviations. Questions such as; what is the minimum hop for a typical trajectory, the length of a typical trajectory and average number of trajectories conditioned on the number of participants in the network can be investigated.



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