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Martingale Hardy-amalgam Spaces

Presented By

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Declaration

I declare that except where due acknowledgement is made, this thesis is my own work produced from research under supervision and has never been presented wholly or partially for the award of a degree at the University of Ghana or any other University.

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Abstract

In this work, we introduce the new spaces, $H^s_{p,q}$, $H^S_{p,q}$, $H^*_{p,q}$, $\mathcal{Q}_{p,q}$, $\mathcal{P}_{p,q}$, called the martingale Hardy-amalgam spaces. We study some of the properties of these newly introduced spaces; two definitions of atoms are given and hence two atomic decompositions are given, dualities of these spaces are characterized and the martingale inequalities and embeddings of these spaces are also discussed. It is proved that the dual of $H^s_{p,q}$, $(0 < p \leq q \leq 1)$, is a Campanato-type space and the dual of $H^s_{p,q}$, $(1 < p \leq q < \infty)$, is $H^s_{p',q'}$ where $(p, p'), (q, q')$ are conjugate pairs. The variation integrable space $\mathcal{G}_{p,q}$ is also introduced and it is established that the jump bounded space $\mathcal{BD}_{p,q}$ is the dual of $\mathcal{G}_{p,q}$. To be able to characterize this duality, a larger space, which we denote by $\mathcal{K}(L_{p,q}, \ell_r)$, is introduced, such that $\mathcal{G}_{p,q}$ can be embedded into. The classical Doob's martingale inequality is also extended from the classical martingale Hardy spaces to the newly introduced martingale Hardy-amalgam spaces. The Burkholder-Davis-Gundy inequality is also extended from the classical martingale Hardy spaces to the martingale Hardy-amalgam spaces as well as the convexity inequality and the concavity inequalities involving measurable functions. The classical martingale Hardy space embeddings are also extended to the martingale Hardy-amalgam spaces. The Davis decompositions of martingales in the classical martingale Hardy spaces are also extended to the martingale Hardy-amalgam spaces. As an application of the Davis decomposition and the Garsia space, a duality theorem for $H^*_{p,q}$ $(1 \leq p, q \leq 2)$ is provided. Finally, the boundedness of martingale transforms between the martingale Hardy-amalgam spaces are also discussed. No data was collected for this study as the methodology used is purely theoretical in nature.

Dedication

I am dedicating this work to my family. Thank you for your love and support.

Acknowledgement

First, I would like to thank God for His grace that has opened doors and strengthened me on my academic journey. I would also like to thank the BANGA-Africa Project for their generosity, which made it possible for me to pursue my Doctorate in Mathematics. Finally, I would like to thank my supervisor, Dr. Benoît F. Sehba for his sound guidance on my research. I am truly grateful for all the support that I received.

Contents

Chapter 1

Introduction and Problem Statement

An overview of the three main concepts; martingales, martingale Hardy spaces and Wiener amalgam spaces, that are employed in this study are introduced in this chapter. The problem statement and motivation for this study are discussed in the last section of this chapter. This chapter is then concluded with an outline of the study.

1.1 Background of Study

1.1.1 Martingales

The concept of martingale is attributed to J. L. Doob during his seminal works in the 1950's [16]. One major contribution by Doob to the theory of martingales was that he was able to show the connection between martingales and analytic functions [17]. Afterwards, various authors including Davis B. J., Cairoli R. , Garsia A. M., and Burkholder D. L., contributed to the growth of the theory of martingale. [7, 8, 10, 14, 26, 54, 65].

A stochastic process or a random process is a mathematical object defined as a family of random variables [42, 45, 67]. A martingale is a discrete-time or continuous-time stochastic process with the property that, at every instant, given the current value and all the past values of the process, the conditional expectation of every future value is equal to the current value [19, 40, 46, 65, 67]. It is well known that the Wiener process (Brownian motion) is a typical example of a martingale [46]. Martingales can also be created from stochastic processes by suitable transformations. An example is the compensated Poisson process [43, 57]. One of the numerous reasons why martingales are important in application is its Convergence Theorem which states that martingale will converge given some conditions on their moments [19, 40, 67]. That is if $M : [0, \infty) \times \Omega \longrightarrow \mathbb{R}$ is a continuous martingale such that $\sup_{t>0} \mathbb{E}(|M_t|^p) < +\infty$ for some $p > 1$, then there exists a random variable $M \in L^p(\Omega, \mathbb{P}, \mathbb{R})$ such that $M_t \to M$ as $t \to \infty$ both $\mathbb{P}$-almost surely and in $L^p(\Omega, \mathbb{P}, \mathbb{R})$ [57] where $\mathbb{E}$ is the expectation operator. Hence problems in probability theory have been solved by first finding a martingale in the problem and studying it. The notions and notations introduced in this section will be made clearer in subsequent pages of this study. Due to their connection and application in Fourier analysis, Hardy spaces and complex analysis, martingales are in particularly interesting in their own sense. Some of these connections are discussed in [4, 7, 16, 18, 47, 65]. For instance in [7], we see that the methods developed for Banach valued martingales can be used to obtain sharp constants in some inequalities. The Riesz Theorem can also be proved in a probabilistic way as it is done in [4]. The martingale proof of $T(b)$ Theorem and some other martingale techniques in Harmonic analysis are discussed in [47, 48] and of course the main book by Weisz [65] contains applications where martingale techniques are used in Fourier analysis.

1.1.2 Hardy Spaces and Martingale Hardy Spaces

Classical Hardy space $\mathcal{H}^q := \mathcal{H}^q(\mathbb{R}^d)$ is defined as the space of tempered distribution f such that

$$M_\phi(f) = \sup_{t>0} |f * \phi_t|$$

is in $L^q(\mathbb{R}^d)$, (the space of measurable functions such that $\int_{\mathbb{R}^d} |f|^q \mathrm{d}\mu < \infty$ with norm $\|f\|_{L^q} := (\int_{\mathbb{R}^d} |f|^q \mathrm{d}\mu)^{1/q}$) that is $\|M_\phi f\|_{L^q(\mathbb{R}^d)} < \infty$, for $q > 0$ where $\phi \in C^\infty(\mathbb{R}^d)$ with support $B(0,1)$ such that $\int_{\mathbb{R}^d} \phi \mathrm{d}t = 1$ and ϕ_t is the dilated functions $\phi_t(x) = t^{-d}\phi(\frac{x}{t})$, $x \in \mathbb{R}^d$ [23, 60, 61, 62]. It is known that for $1 < q < \infty$, the Hardy space is equivalent to the Lebesgue space [23]. In the Theory of Hardy spaces, C. Fefferman and E. Stein in the paper [21], contributed immensely to the growth and development of this subject area. In recent times, Hardy spaces has become the object of studies and thus various authors have established enormous generalizations of this space (see for instance [13, 24, 37, 44, 52, 53, 59, 63]).

The Paley's inquality is known to be valid on the interval $(1, \infty)$ [55]. Motivated by Doob, when he pointed out the connection between martingale and analytic functions, Burkholder and Gundy, in the 1970's, were able to extend Paley's inequality to cover the whole $(0, \infty)$ partially. Partially in the sense that, the extension was done by considering only martingales. This is because structures defined in the martingale settings are simple and makes it easier when one wants to study some properties of function spaces such as equivalences of spaces and dual characterizations. Their ability to do this was the introduction of the measurable functions $s(f)$ and $S(f)$, (s and S are actually quasi-linear operators), where f is a martingale. These measurable functions are referred to as conditional quadratic variation and quadratic variation respectively and were first introduced by Burkholder and Gundy [8]. This insight from Burkholder and Gundy gave birth to the classical martingale Hardy spaces of which many authors have contributed immensely to the growth of this area in the past few years (see for example [8, 14, 26, 48, 65]).

These martingale techniques introduced by Burkholder and his colleagues, led to the establishments of many important results in literature. One of these results is on the equivalence result named after Burkholder, Davis and Gundy where they showed that the function space generated by the maximal function and function space generated by the quadratic variation are equivalent for $1 \leq p < \infty$ [8, 65]. Another important result obtained by applying these martingale techniques is the characterization of the dual space of the spaces generated by the maximal functions when $p = 1$. This dual space was shown to be the space of bounded mean oscillations [26]. It is also worth mention that with these martingale techniques and the use of the quasi-linear operators s, S, Burkholder and Gundy have been able to discuss the integrability of Brownian motions and stopping times (see [8]).

The spaces of all martingales whose maximal function, quadratic variation, or conditional quadratic variation belongs to the usual Lebesgue spaces L_p with a probability measure are defined as the classical martingale Hardy spaces. The atomic decompositions, martingale embeddings and dual spaces of these classical martingale Hardy spaces and related spaces are discussed by F. Weisz in [65]. Several authors have considered this type of studies for some generalizations of the classical Lebesgue spaces as Lorentz spaces, Orlicz spaces, Orlicz-Musielak spaces (see for instance [30, 38, 50, 58, 68, 69, 70]). Even though this study mainly focuses on martingale Hardy-amalgam spaces, it is also worth mentioning that atomic decompositions, martingale embeddings and dual spaces are also considered for Morrey-type spaces and its various generalizations (see for instance [15, 31, 39, 32]). Weak-type martingale Hardy spaces are also discussed in [66].

1.1.3 Amalgam Spaces and Hardy-amalgam Spaces

The amalgam space is a space of functions or distributions defined by a norm which mixes or amalgamates a local criterion for a membership with a global criterion. Many times in the literature, various versions of the amalgam space have risen independently and often provide compelling context for formulating results. Wiener amalgam spaces, as they are sometimes referred to, are good substitutions for Lebesgue spaces [12] because of the failure on the part of $L_p(\mathbb{R})$-spaces to distinguish between the local and global properties of functions [29]. Hence it is not possible to recognise from its norm whether a function is the characteristic function of an interval or the sum of many characteristic functions of small intervals spread widely over $\mathbb{R}$ [29]. Wiener amalgams were first introduced by Norbert Wiener in 1980 when Wiener aimed at the possibility of describing local and global properties of a function or distribution, [22], in connection with his development of the theory of generalized harmonic analysis [28]. In literature, the Wiener amalgam spaces are denoted $W(L^p, L^q)$ where L^p is the local and L^q is the global. That is the amalgam space, $L_{p,q}$, is the space of measurable functions f such that $\|f\mathbf{1}_{\Omega_j}\|_p < \infty$ and the sequence $\{\|f\mathbf{1}_{\Omega_j}\|_p\}_{j\in\mathbb{Z}}$ is in the sequence space where Ω_j's are disjoint such that $\cup_j\Omega_j = \Omega$, and Ω an arbitrary non-empty set. However, in the settings of this study, we shall denote $W(L^p, L^q)$ simply as $L_{p,q}$. In particular Wiener defined the discrete norm for these spaces [25, 29] as

$$\|f\|_{L_{p,q}} = \left(\sum_{n\in\mathbb{Z}}\left(\int_n^{n+1} |f(t)|^p \mathrm{d}t\right)^{\frac{q}{p}}\right)^{\frac{1}{q}}.$$

After its introduction in 1980, there has been major developments through independent studies including papers by Hans Feichtinger, who developed a comprehensive notion of amalgams which allow an extremely wide range of Banach spaces of functions or distributions on locally compact groups to be used as local or global components [22]. Among other results, Busby and Smith [9] derived convolution theorem for amalgam. J. J. Fournier and Stewarts [25] also dealt with the case of locally abelian compact groups.

The introduction of the Wiener amalgam spaces have become very useful to researchers in the past few years. For example Essen, [20], applied Wiener amalgam techniques in renewal theory. He used what we will call the space $W(M, L_w^\infty)$ where the local component is the space of locally bounded measures and the global component is the weighted $L^\infty-$spaces. Wiener amalgams are also indispensable in the theory of time-frequency analysis. In time frequency analysis, it is the space $S_0 = W(\mathcal{F}L^1, L^1)$ consisting of functions that are locally the Fourier transform of an L^1 function and globally on L^1 behaviour. S_0 is the smallest Segal algebra on which time-shift and frequency-shift act isometrically [29]. It is also known as the Modulation space, M^1, which is the proper space of window functions for time-frequency analysis [29]. Wiener amalgam spaces have been employed to study the boundedness properties of pseudo-differential operators, Fourier multipliers, Fourier integral operators and well-posedness of solutions to partial differential equations [12].

Hardy-amalgam spaces introduced by Ablé et al, [24], are generalizations to the classical Hardy spaces. The Hardy-amalgam spaces are formed by replacing the Lebesgue norm of the Hardy spaces by the Wiener amalgam norm. Various properties of this space such as atomic decompositions, duality and inequalities are discussed in [23, 24]. The authors also applied their results to investigate the boundedness of pseudo-differential operators in these spaces. A generalization of the work of Ablé et al was recently obtained by Xie et al [69].

1.2 Problem Statement and Motivation

Over the years, one major area of research that is of great importance to mathematicians is in the field of stochastic integration. The introduction of martingale Hardy spaces by Burkholder and Gundy, [8], has contributed immensely to answer some of the many questions mathematicians seek to address as far as stochastic integration is concerned. We recall that we have defined the classical martingale Hardy spaces to be the spaces of martingales whose maximal functions, quadratic variation and conditional quadratic variations all belong to the usual Lebesgue space with a probability measure. However, as pointed out in earlier sections, the Lebesgue space is not able to distinguish between the local and global properties of functions. This is so because, say we consider the set of real numbers, then it is impossible to know from the norm of a measurable function whether the function is defined with respect to an interval or is the sum of indicator functions of smaller intervals that cover $\mathbb{R}$ [29]. Hence in this regard the Wiener amalgam space is a proper substitute for the Lebesgue spaces. Therefore just as the Hardy-amalgam space [23, 24] is a generalization to the classical Hardy spaces, the martingale Hardy-amalgam spaces will be a form of generalization of the classical martingale Hardy spaces introduced by Burkholder, Gundy, Davis, F. Weisz, Garsia, [6, 7, 8, 14, 26], by replacing the Lebesgue space norm of the classical martingale Hardy spaces with the Wiener amalgam space norm. Hence the title ***martingale Hardy-amalgam spaces***. This idea was inspired by the recent works of the authors [23, 24, 69]. The martingale Hardy-amalgam spaces can be seen as a generalization of the classical martingale Hardy spaces.

Objectives of Study

After these new martingale Hardy-amalgam spaces have been introduced in the appropriate section below, some of the important properties of these newly introduced spaces will be studied. More specifically

- We seek to find the atomic decompositions of these newly introduced spaces.
- We seek to find the martingale embeddings and martingale inequalities of the various spaces. In addition, we seek to find the Davis decompositions of these spaces and discuss some of its consequences.
- As an application of the atomic decompositions of these newly introduced spaces, we seek to characterize the dual spaces of these newly introduced spaces. In addition a

Garcia-type space in the martingale Hardy-amalgam space will also be introduced and we seek to characterize its duality.

- Finally, we seek to find the martingale transforms between the martingale Hardy-amalgam space.

The motivation for this study is essentially based on two observations. As the first observation, we note that in the case of classical Hardy-amalgam spaces of [24], atomic decomposition is obtained only for the range $0 < p \leq q \leq 1$. The question that arose was to know if any answer could be obtained beyond this range. The second observation is that dyadic analogues of Hardy spaces and their dual spaces are quite practical when it comes to the study of some operators as paraproducts, Calderon-Zygmund operators and their commutators on some function spaces. It was then natural to consider dyadic analogues and more generally, martingale analogues of the Hardy-amalgam spaces of [24]. Thus all the martingales considered in this study will be defined on a dyadic filtration.

These properties of these new spaces that we want to study are very important not just in this work but for general purposes as they will play crucial roles in aiding researchers who are interested in this area of research. The atomic decompositions for instance, enables one to study other properties of the spaces such as dual characterizations. On the other hand, dualities in general, are also very crucial in applications where in recent times problems in optimal control can be reformulated as problems in dual spaces called the Dual Variational Problems in Optimal Control. Dual spaces also help to investigate other properties of the spaces such as separability. The embeddings of the various spaces have also proven to be applicable in stochastic integration, especially in the Lebesgue normed space [8], and we hope that such applications could equally be extended to the martingale Hardy-amalgam spaces. Most of the techniques employed in this study are techniques used by Weisz, Burkholder and Gundy in the classical cases.

Methodology and Outline of the Study

The methods and techniques employed in this study are theoretical in nature. Hence no data was collected for this study. In Chapter 2, we get familiar with the various concepts and notations that will be needed in subsequent chapters. After the introduction of these concepts and notations, each chapter, respectively, will address each item listed in the objectives. That is, the outline of the results obtained in this study is presented as follows. In Chapter 3, we present the results on the atomic decompositions of the newly introduced spaces. In Chapter 4, we present the results on the martingale embeddings of the martingale Hardy-amalgam spaces and the extended results of some classical martingale inequalities. In Chapter 5, we present the results on the dual characterizations of these spaces. The results presented in Chapter 3 and part of Chapter 5 are published by the author in [2]. The remainder of Chapter 5 and Chapter 4 is also published by the author in [3]. Chapter 6 presents the results on the martingale transforms between the martingale Hardy-amalgam spaces. Results presented in Chapter 6 are published by the author in [1].

Chapter 2

Preliminaries and Notations

We begin by setting down the fundamental notions and notations that will be relevant in this study. Let us start by getting familiar with the notion of dyadic intervals and dyadic filtration.

2.1 Dyadic Intervals and Dyadic Filtration

We begin with the definitions of the basic objects σ-algebra and filtration of an arbitrary non-empty set.

2.1.1 σ-algebra, Filtration and Probability Space

Let Ω be a non-empty set and let $\mathcal{F}$ be a class of subsets of Ω. Then $\mathcal{F}$ is a σ-algebra if

i. Ω and $\emptyset$ are in $\mathcal{F}$

ii. $A,\ B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$, $A \cup B \in \mathcal{F}$ and $A^c \in \mathcal{F}$

iii. for any sequence of sets $\mathcal{A}_n$ in $\mathcal{F}$ then $\bigcup_{n=1}^{\infty} \mathcal{A}_n \in \mathcal{F}$.

The sequence $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ of sub-σ-algebra of $\mathcal{F}$ is called a filtration if $\mathcal{F}_n \subset \mathcal{F}_m$ for all $m, n \in \mathbb{N}$ and $n \leq m$ [5].

A pair $(\Omega, \mathcal{F})$ consisting of an arbitrary non-empty set Ω and a σ-algebra $\mathcal{F}$ of its subsets is called a measurable space. A measure on the measurable space $(\Omega, \mathcal{F})$ is a mapping $\mu : \mathcal{F} \longrightarrow [0, \infty]$ with the property that $\mu(\emptyset) = 0$ and

$$\mu\left(\bigcup_{n\geq 1} \mathcal{A}_n\right) \leq \sum_{n\geq 1} \mu(\mathcal{A}_n).$$

If $\mu(\Omega) = 1$, then μ is called ***probability measure***. The triple $(\Omega, \mathcal{F}, \mu)$ is called a measure space. When $(\Omega_1, \mathcal{W})$ and $(\Omega_2, \mathcal{V})$ are measurable spaces, a mapping $f : \Omega_1 \longrightarrow \Omega_2$ is measurable if $f^{-1}(B) \in \mathcal{W}$ for all $B \in \mathcal{V}$. If $\Omega_2 = \mathbb{R}$, (the set of real numbers), then the measurable function f is called a ***real random variable***.

Definition 2.2 (Probability Space)**.** In particular if $\mu = \mathbb{P}$, the probability measure, then $(\Omega, \mathcal{F}, \mathbb{P})$ is called the probability space and the quadruple $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n\in\mathbb{N}}, \mathbb{P})$ is called the filtered space or more specifically a stochastic basis, where $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ is a filtration.

Let X be some random variable that is defined on the probability space. Then, the expectation operator, $\mathbb{E}$, is defined as

$$\mathbb{E}X = \int_\Omega X \mathrm{d}\mathbb{P}$$

Let X be a random variable and suppose that $\mathbb{E}|X| < \infty$ with respect to the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{Y}$ be a sub-σ-algebra of $\mathcal{F}$. The conditional expectation of X given $\mathcal{Y}$, denoted by $\mathbb{E}(X|\mathcal{Y})$, is the random variable satisfying the following conditions;

a. $\mathbb{E}(X|\mathcal{Y})$ is $\mathcal{Y}$-measurable

b. $\mathbb{E}(\mathbb{E}[X|\mathcal{Y}]) < \infty$

c. For all $Y \in \mathcal{Y}$, we have that

$$\int_Y \mathbb{E}(X|\mathcal{Y}) \mathrm{d}\mathbb{P} = \int_Y X \mathrm{d}\mathbb{P}.$$

We mostly write $\mathbb{E}_n$ to denote the conditional expectation operator.

2.2.1 Dyadic Intervals and Dyadic Filtration

Let $\mathbb{R}$ be the set of all real numbers. The dyadic interval is defined as the interval

$$\mathcal{D}^\xi = 2^j \left[[0,1) + m + (-1)^j \xi\right] \quad \text{where} \quad j, m \in \mathbb{Z} \text{ and } \xi \in \mathbb{R}$$

If $\xi = 0$, then $\mathcal{D}^\xi = \mathcal{D} = 2^j[m, m+1)$. Now fix j so that we have

$$\mathcal{D}_j = \left\{[m2^j, (m+1)2^j),\ m \in \mathbb{Z}\right\}.$$

Let $I_{m,j}$ be a member of $\mathcal{D}_j$ and divide $I_{m,j}$ into two equal halves. That is

$$I^1_{m,j} = [(2m)2^{j-1}, (2m+1)2^{j-1}), \quad I^2_{m,j} = [(2m+1)2^{j-1}, 2(m+1)2^{j-1}).$$

Then $I^1_{m,j}$ and $I^2_{m,j}$ are both dyadic intervals such that $I^1_{m,j}, I^2_{m,j} \in \mathcal{D}_{j-1}$. That is to say that for any dyadic interval $I_{m,j} \in \mathcal{D}_j$ there exist two dyadic intervals $I^1_{m,j}$ and $I^2_{m,j}$ in $\mathcal{D}_{j-1}$ such that

$$I_{m,j} = I^1_{m,j} \bigcup I^2_{m,j}.$$

Now consider the following dyadic intervals of $\mathbb{R}$;

$$I_{m,n} = \left[\frac{m}{2^n}, \frac{m+1}{2^n}\right) \quad n \in \mathbb{N},\ m \in \mathbb{Z}.$$

Let $\mathcal{D}_n = \{I_{m,n},\ m \in \mathbb{Z}\}$, $n \in \mathbb{N}$. Let $\mathcal{F}_n = \sigma(\mathcal{D}_n)$ be the σ-algebra generated by $\mathcal{D}_n$. Then $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ is a filtration. Indeed we observe that $\mathcal{D}_n \subseteq \mathcal{F}_{n+1} = \sigma(\mathcal{D}_{n+1})$ as interval in $\mathcal{D}_{n+1}$ is a union of two intervals in $\mathcal{D}_n$ and the union is in $\mathcal{F}_{n+1}$ since $\mathcal{F}_{n+1}$ is a σ-algebra. This implies that

$$\mathcal{F}_n = \sigma(\mathcal{D}_n) \subseteq \sigma(\mathcal{F}_{n+1}) = \mathcal{F}_{n+1}.$$

With this filtration, called the ***dyadic filtration***, we can define the probability space as

$$\mathbf{Ps} := (\mathbb{R}, \mathcal{F}, \{\mathcal{F}_n\}_{n\in\mathbb{N}}, \mathbb{P})$$

where $\mathbb{P}$ is the probability measure and $\mathcal{F}_n \subseteq \mathcal{F}$ for all $n \in \mathbb{N}$. Let $J_{k,n,j} \in \mathcal{D}_n$ be the dyadic interval defined as

$$J_{k,n,j} = \left[\frac{k + j2^n}{2^n}, \frac{k+1+j2^n}{2^n}\right).$$

Then

$$A_j = [j, j+1) = \bigcup_{k=0}^{2^n-1} J_{k,n,j}. \tag{2.1}$$

Therefore, $A_j \in \mathcal{F}_n$ for all n and for all j. We observe that the A_j's are dyadic intervals. Also $A_i \cap A_j = \emptyset$ for $i \neq j$ and $\bigcup_j A_j = \mathbb{R}$. This notion of dyadic intervals and dyadic filtration will become indispensable with our treatment of the martingale embeddings and inequalities in the appropriate chapter below.

2.3 Martingale and Related Concepts

In this section, we recall the definition of martingales and discuss some related concepts that will be needed in this study. We start with the definition of a martingale.

Definition 2.4 (Martingale). Let $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n\in\mathbb{N}}, \mathbb{P})$ be a probability space. An integrable sequence $f = (f_n,\ n \in \mathbb{N})$ is said to be a martingale with respect to the filtration $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ if

1. f is adapted; that is f_n is $\mathcal{F}_n$-measurable for all $n \in \mathbb{N}$ and
2. $\mathbb{E}_n f_m = f_n$ for all $n \leq m$.

If $\mathbb{E}_n f_m \geq f_n$ ($\mathbb{E}_n f_m \leq f_n$) then f is a submartingale (supermartingale). As an example, let f be integrable. Then, the sequence $g = (\mathbb{E}_n f\ n \in \mathbb{N})$ is martingale [65, 67]. We shall encounter some few examples of martingales in subsequent sections. We say that the stochastic basis is said to be *regular*, if there exists some constant $R > 0$ such that $f_n \leq R f_{n-1}$ for every martingale $f = \{f_n\}_{n\in\mathbb{N}}$. The dyadic filtration is an example of a regular stochastic basis.

Let $L_p(\Omega, \mathrm{d}\mathbb{P})$ denote the usual Lebesgue space over the non-empty set Ω (for simplicity of notation, we sometimes write $L_p(\Omega, \mathrm{d}\mathbb{P})$ simply as $L_p(\Omega)$ or L_p). That is

$$L_p(\Omega) = \left\{ f : (\mathbb{E}|f|^p)^{\frac{1}{p}} = \left(\int_\Omega |f|^p \mathrm{d}\mathbb{P} \right)^{\frac{1}{p}} < \infty \right\}$$

equipped with the norm

$$\|f\|_{L_p(\Omega)} := \|f\|_p = \left(\int_\Omega |f|^p \mathrm{d}\mathbb{P} \right)^{\frac{1}{p}}$$

where f is measurable with respect to $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n\in\mathbb{N}}, \mathbb{P})$. A martingale $f = \{f_n\}_{n\in\mathbb{N}}$ relative to the filtration $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ is said to be L_p bounded ($0 < p \leq \infty$) if $f_n \in L_p,\ n \in \mathbb{N}$ and

$$\|f\|_p := \sup_{n\in\mathbb{N}} \|f_n\|_p < \infty.$$

Definition 2.5 (Stopping Time)**.** The mapping $\nu : \Omega \longrightarrow \mathbb{N} \cup \{\infty\}$ is called a stopping time relative to the filtration $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ if

$$\{\omega \in \Omega : \nu(\omega) = n\} := \{\nu = n\} \in \mathcal{F}_n.$$

An important example of a stopping time is *hitting time* of a Borel set B. This is defined by

$$\nu_B(\omega) := \inf \{n \in \mathbb{N} : f_n(\omega) \in B\} \quad \text{for} \quad \omega \in \Omega$$

where f_n is a measurable function defined on Ω(see for example [54]). Further examples of stopping time can be found in [42, 67].

2.5.1 Difference Sequence and Stopped Processes

Definition 2.6. (Difference Sequence) Let $f = \{f_n\}_{n\in\mathbb{N}}$ be a sequence of measurable functions. Then the difference sequence of f is the sequence $\{d_n f\}_{n\in\mathbb{N}}$ where

$$d_n f = f_n - f_{n-1},\ f_0 = 0,\ \ n \in \mathbb{N}.$$

Let us discuss some useful properties of the difference sequence which will be useful in this study.

1. First we observe that if $f = \{f_n\}_{n\in\mathbb{N}}$ is a martingale with respect to some probability space, then $\{d_n f\}_{n\in\mathbb{N}}$ is adapted to the underlying filtration of f since f is adapted. Also
$$\mathbb{E}_{n-1} d_n f = \mathbb{E}_{n-1}(f_n - f_{n-1}) = f_{n-1} - f_{n-1} = 0$$
for all $n \in \mathbb{N}$. In this case $\{d_n f\}_{n\in\mathbb{N}}$ is called the martingale difference sequence.

2. We also observe that martingale differences are orthogonal. Indeed, let $m = n-1 < n$. Then since $d_m f$ is $\mathcal{F}_m$-measurable, we have that
$$\mathbb{E}_m[d_n f d_m f] = d_m f \mathbb{E}_m d_n f = 0 \tag{2.2}$$
and hence
$$\mathbb{E}[d_n f d_m f] = \mathbb{E}[\mathbb{E}_m(d_n f d_m f)] = \mathbb{E}[d_m f \mathbb{E}_m d_n f] = 0.$$

3. Let $f = \{f_n\}_{n\in\mathbb{N}}$ and $g = \{g_n\}_{n\in\mathbb{N}}$ be martingales with respect to an underlying stochastic basis. Let $h = f + g$. Then h is also adapted to the underlying filtration and for $n < m$ we have that $\mathbb{E}_n[f_m + g_m] = \mathbb{E}_n f_m + \mathbb{E}_n g_m = f_n + g_n = h_n$. Thus h is a martingale. Now
$$d_n h = h_n - h_{n-1} = (f_n + g_n) - (f_{n-1} + g_{n-1}) = d_n f + d_n g$$
and
$$d_n \alpha f = \alpha f_n - \alpha f_{n-1} = \alpha d_n f$$
for every constant α. Hence the martingale difference operator, d_n, is linear.

Definition 2.7. (Stopped Process) Let ν be a stopping time and let $f = \{f_n\}_{n\in\mathbb{N}}$ be a martingale adapted to an underlying filtration with difference sequence $d = \{d_n f\}_{n\in\mathbb{N}}$. When f_∞ exist almost everywhere, then the stopped sequence $f^\nu = \{f_n^\nu\}_{n\in\mathbb{N}}$ is defined by

$$f_n^\nu = f_{n\wedge\nu}, \quad n \in \mathbb{N}$$

where $n \wedge \nu := \min\{n, \nu\}$.

The difference sequence of the stopped process $d_n f^\nu = f_n^\nu - f_{n-1}^\nu$ is given by

$$d_n f^\nu = \begin{cases} f_n - f_{n-1} & \text{for} \quad \{n \leq \nu\} \\ f_\nu - f_\nu & \text{for} \quad \{\nu < n\} \end{cases} = \begin{cases} d_n f & \text{for} \quad \{n \leq \nu\} \\ 0 & \text{for} \quad \{n > \nu\} \end{cases} = \mathbf{1}_{\{n \leq \nu\}} d_n f.$$

Now by telescopic sum, we have that $f_n = \sum_{k=1}^n d_k f$. Hence

$$f_n^\nu = f_{n \wedge \nu} = \sum_{k=1}^n d_k f^\nu = \sum_{k=1}^n \mathbf{1}_{\{k \leq \nu\}} d_k f.$$

Since $f = \{f_n\}_{n \in \mathbb{N}}$ is adapted, the above equation implies that the stopped process is also adapted. Also $\{n \leq \nu\} = \{\nu < n-1\}^c \in \mathcal{F}_{n-1}$, since ν is a stopping time. It then implies that $\mathbf{1}_{\{n \leq \nu\}}$ is $\mathcal{F}_{n-1}$-measurable. Consequently,

$$\begin{aligned} \mathbb{E}_{n-1}(f_n^\nu - f_{n-1}^\nu) &= \mathbb{E}_{n-1} d_n f^\nu \\ &= \mathbb{E}_{n-1}(\mathbf{1}_{\{n \leq \nu\}} d_n f) \\ &= \mathbf{1}_{\{n \leq \nu\}} \mathbb{E}_{n-1}(d_n f) = 0 \end{aligned}$$

since $d_n f$ is a martingale difference sequence. Therefore

$$\mathbb{E}_{n-1}(f_n^\nu - f_{n-1}^\nu) = \mathbb{E}_{n-1}(f_n^\nu) - f_{n-1}^\nu = 0.$$

Thus the stopped process is also a martingale. Hence the following definition.

Definition 2.8. (Stopped Martingale) If ν is a stopping time and $f = \{f_n\}_{n \in \mathbb{N}}$ is a martingale adapted to an underlying filtration, then the stopped martingale $f^\nu = \{f_n^\nu\}_{n \in \mathbb{N}}$ is defined by

$$f_n^\nu = f_{n \wedge \nu} := \sum_{m=1}^n \mathbf{1}_{(\nu \geq m)} d_m f.$$

Remark 2.9. We have seen that $d_n f^\nu = \mathbf{1}_{\{n \leq \nu\}} d_n f$. Hence for $d_n f$, we can write

$$d_n f = f_n - f_{n-1} = f_{n \wedge \nu} - f_{n-1 \wedge \nu} = f_n^\nu - f_{n-1}^\nu = d_n f^\nu$$

since $n - 1 < n \leq \nu$. Thus

$$d_n f d_n f^\nu = |d_n f^\nu|^2. \tag{2.3}$$

2.9.1 Quasi-Linear Operators on Martingales

In this part of the study, we recall the quasi-linear operators which were introduced by Burkholder and Gundy [8]. These are the operators that aided Burkholder and Gundy to extend the Paley's inequality. They are denoted simply as s, S and M. Let $f = \{f_n\}_{n\in\mathbb{N}}$ be a martingale with respect to an underlying filtration. The quadratic variation, $S(f)$, and the conditional quadratic variation, $s(f)$, of $f = \{f_n\}_{n\in\mathbb{N}}$ are defined as

$$S(f) := \left(\sum_{n\in\mathbb{N}} |d_n f|^2\right)^{\frac{1}{2}} \quad \text{and} \quad s(f) := \left(\sum_{n\in\mathbb{N}} \mathbb{E}_{n-1}|d_n f|^2\right)^{\frac{1}{2}}$$

respectively, where $d_n f = f_n - f_{n-1}$ are the martingale differences. We define the stopping $\nu(\omega) \equiv n$ and $f^\nu := f^n$ and observe that

$$s^2(f^n) \;=\; \sum_{k\in\mathbb{N}} \mathbb{E}_{k-1}|d_k f^n|^2 = \sum_{k\in\mathbb{N}} \mathbb{E}_{k-1}|f_{k\wedge n} - f_{k-1\wedge n}|^2$$

Hence

$$\begin{aligned} s^2(f^n) &= \sum_{k=1}^{n} \mathbb{E}_{k-1}|f_{k\wedge n} - f_{k-1\wedge n}|^2 + \sum_{k=n+1}^{\infty} \mathbb{E}_{k-1}|f_{k\wedge n} - f_{k-1\wedge n}|^2 \\ &= \sum_{k\le n} \mathbb{E}_{k-1}|f_{k\wedge n} - f_{k-1\wedge n}|^2 + \sum_{k-1\ge n} \mathbb{E}_{k-1}|f_{k\wedge n} - f_{k-1\wedge n}|^2 \\ &= \sum_{k\le n} \mathbb{E}_{k-1}|f_k - f_{k-1}|^2 + \sum_{k-1\ge n} \mathbb{E}_{k-1}|f_n - f_n|^2 \\ &= \sum_{k=1}^{n} \mathbb{E}_{k-1}|f_k - f_{k-1}|^2 + 0 = s_n^2(f). \end{aligned}$$

Thus

$$s(f^n) = s_n(f). \tag{2.4}$$

Similarly,

$$S^2(f^n) = \sum_{k\le n} |f_k - f_{k-1}|^2 + \sum_{k-1\ge n} |f_n - f_n|^2 = S_n^2(f). \tag{2.5}$$

Consequently, we shall agree on the notation

$$S_n(f) := \left(\sum_{k=1}^{n} |d_k f|^2\right)^{\frac{1}{2}} \quad \text{and} \quad s_n(f) := \left(\sum_{k=1}^{n} \mathbb{E}_{k-1}|d_k f|^2\right)^{\frac{1}{2}}.$$

The maximal function f^* or $M(f)$ of the martingale $f = \{f_n\}_{n\in\mathbb{N}}$ is defined by

$$M(f) = f^* := \sup_{n\in\mathbb{N}} |f_n|.$$

Let us discuss some identities of these operators that will be helpful in this study.

1. Let α be some stopping time. Then we observe that

$$s^2(f - f^\alpha) = s^2(f) - s^2(f^\alpha). \tag{2.6}$$

Indeed by definition and linearity of martingale difference operator,

$$\begin{aligned} s^2(f - f^\alpha) &= \sum_{k\in\mathbb{N}} \mathbb{E}_{k-1}|d_k(f - f^\alpha)|^2 = \sum_{k\in\mathbb{N}} \mathbb{E}_{k-1}|d_k f - d_k f^\alpha|^2 \\ &= \sum_{k\in\mathbb{N}} \mathbb{E}_{k-1}\left[|d_k f|^2 + |d_k f^\alpha|^2 - 2d_k f d_k f^\alpha\right] \\ &= \sum_{k\in\mathbb{N}} \mathbb{E}_{k-1}\left[|d_k f|^2 + |d_k f^\alpha|^2 - 2|d_k f^\alpha|^2\right] \\ &= \sum_{k\in\mathbb{N}} \mathbb{E}_{k-1}|d_k f|^2 - \sum_{k\in\mathbb{N}} \mathbb{E}_{k-1}|d_k f^\alpha|^2 \end{aligned}$$

Hence by definition

$$s^2(f - f^\alpha) = s^2(f) - s^2(f^\alpha)$$

where have made use of identity (2.3). Similarly,

$$S^2(f - f^\alpha) = S^2(f) - S^2(f^\alpha). \tag{2.7}$$

2. It is also true that $s^2(f-g) + s^2(f+g) = 2(s^2(f) + s^2(g))$. Indeed by definition,

$$\begin{aligned} s^2(f - g) &= \sum_{k\in\mathbb{N}} \mathbb{E}_{k-1}|d_k f - d_k g|^2 \\ &= \sum_{k\in\mathbb{N}} \mathbb{E}_{k-1}\left[|d_k f|^2 + |d_k g|^2 - 2d_k f d_k g\right] \end{aligned}$$

$$\begin{aligned} s^2(f + g) &= \sum_{k\in\mathbb{N}} \mathbb{E}_{k-1}|d_k f + d_k g|^2 \\ &= \sum_{k\in\mathbb{N}} \mathbb{E}_{k-1}\left[|d_k f|^2 + |d_k g|^2 + 2d_k f d_k g\right] \end{aligned}$$

Hence

$$s^2(f+g)+s^2(f-g)=2\sum_{k\in\mathbb{N}}\mathbb{E}_{k-1}|d_kf|^2+2\sum_{k\in\mathbb{N}}\mathbb{E}_{k-1}|d_kg|^2$$

and thus $s^2(f-g)+s^2(f+g)=2(s^2(f)+s^2(g))$. Similarly, $S^2(f-g)+S^2(f+g)=2(S^2(f)+S^2(g))$. Also, we observe that since $s^2(f+g)$ and $s^2(f-g)$ are positive,

$$\begin{aligned} s^2(f-g) &= 2(s^2(f)+s^2(g))-s^2(f+g) \\ &\leq 2(s^2(f)+s^2(g)). \end{aligned}$$

$$\begin{aligned} s^2(f+g) &= 2(s^2(f)+s^2(g))-s^2(f-g) \\ &\leq 2(s^2(f)+s^2(g)). \end{aligned}$$

Hence

$$s(f-g)\lesssim s(f)+s(g) \quad \text{and} \quad s(f+g)\lesssim s(f)+s(g). \tag{2.8}$$

Similarly

$$S(f-g)\lesssim S(f)+S(g) \quad \text{and} \quad S(f+g)\lesssim S(f)+S(g) \tag{2.9}$$

where $A\lesssim B$ means $A\leq cB$ where c is a constant.

3. We also point out that since d_kf is adapted to the underlying filtration on which the martingale $f=\{f_k\}_{k\in\mathbb{N}}$ is defined with respect to, the quantities $s(f), S(f)$ and $M(f)$ are measurable and thus $\mathbb{E}s(f), \mathbb{E}S(f)$ and $\mathbb{E}M(f)$ are all well defined.

4. $s_n(f)$ is increasing in n. Indeed since $\mathbb{E}_{k-1}|d_kf|^2\geq 0$ for all k,

$$\begin{aligned} s_n^2(f) &= \sum_{k=1}^{n}\mathbb{E}_{k-1}|d_kf|^2 \leq \sum_{k=1}^{n}\mathbb{E}_{k-1}|d_kf|^2+\mathbb{E}_n|d_{n+1}f|^2 \\ &= \sum_{k=1}^{n+1}\mathbb{E}_{k-1}|d_kf|^2 = s_{n+1}^2(f) \end{aligned}$$

Thus $s_n(f)\leq s_{n+1}(f)$. Similarly, $S_n(f)\leq S_{n+1}(f)$.

5. Let $r\geq 2$. Then by the classical inequality $x^\beta-1\leq\beta(x-1)x^{\beta-1}$, $(1\leq x, 1\leq\beta)$, it is true that

$$s_n^{r-2}(f)[s_n^2(f)-s_{n-1}^2(f)]\geq\frac{2}{r}[s_n^r(f)-s_{n-1}^r(f)]. \tag{2.10}$$

Indeed, since $s_n(f)$ is increasing and $r \geq 2$, one can make the substitution

$$x = \frac{s_n^2(f)}{s_{n-1}^2(f)} \quad \text{and} \quad \beta = \frac{r}{2}$$

into the classical inequality and inequality (2.10) is obtained.

Remark 2.10. A martingale $f = \{f_n\}_{n \in \mathbb{N}}$ is said to be *previsible* if there exists a real number $R > 0$ such that

$$|d_n f|^2 \leq R\mathbb{E}_{n-1}|d_n f|^2. \tag{2.11}$$

It is established in [65, Proposition 2.19] that the stochastic basis is regular if and only if the martingale is previsible. Hence by summing both sides of (2.11) over $n \in \mathbb{N}$, it is true for a regular stochastic basis,

$$S(f) \leq R^{\frac{1}{2}} s(f).$$

A martingale $f = \{f_n\}_{n \in \mathbb{N}}$ is said to be predictable in L_p if there exists a sequence $\{\lambda_n\}_{n \in \mathbb{N}}$ non-decreasing, non-negative and adapted functions such that $|f_n| < \lambda_{n-1}$ and $\lambda_\infty := \sup_{n \in \mathbb{N}} \lambda_n \in L_p$.

2.10.1 Martingale Transforms

Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_n\}_{n \geq 0})$ be a probability space with the underlying filtration $\{\mathcal{F}_n\}_{n \in \mathbb{N}}$. Let $\nu = \{\nu_n\}_{n \geq 0}$ be an adapted sequence in the sense that for all n, ν_n is $\mathcal{F}_{n-1}$-measurable. This sequence is normally referred to as multiplier sequence. If $f = \{f_n\}_{n \in \mathbb{N}}$ is a martingale with respect to the filtration $\{\mathcal{F}_n\}_{n \in \mathbb{N}}$, then the following

$$g_n = \sum_{k=1}^{n} \nu_k d_k f, \quad g_0 = 0$$

is called a martingale transform of f under the sequence ν where $d_k f$ is the usual martingale difference sequence. The martingale transform need not be a martingale, however, it is a martingale if and only if $g_n \in L_1$ (that is $\mathbb{E}g_n < \infty$) [6, 48]. As an example, let τ be a stopping time. Then the following process f_n^τ, referred to as stopped process, is an example of a martingale transform since $\mathbf{1}_{\{k \leq \tau\}}$ is $\mathcal{F}_{k-1}-$measurable; $f_n^\tau = f_{n \wedge \tau} = \sum_{k=1}^{n} \mathbf{1}_{\{k \leq \tau\}} d_k f$. The usefulness of martingale transforms has helped various researchers to characterize dual spaces of some classical martingale Hardy spaces. These martingale transform techniques will be an important tool in Chapter 4 when discussing the dual characterizations of the spaces involved. It has also been employed to study the relations of the predictive spaces in their various generalizations such as the martingale transforms between Hardy-Orlicz spaces among others [34, 49, 72]. Martingale transforms also share some properties with fractional integrals [11, 51, 64]. Burkholder has established the almost everywhere convergence of the martingale transform based on the condition that the maximal function of the multiplier sequence is finite [6, 48].

2.11 Wiener Amalgam Spaces

As pointed out at the beginning of this study, the amalgam space is attributed to Norbert Wiener in 1980 when Wiener aimed at the possibility of describing local and global properties of a function or distribution, [22], in connection with his development of the theory of generalized harmonic analysis [28]. In this section, we give a formal definition of this space and discuss some of its consequences.

Let Ω be an arbitrary non-empty set and let $\{\Omega_j\}_{j\in\mathbb{Z}}$ be sequence of sets that are disjoint, i.e. $\Omega_j \cap \Omega_i = \emptyset$ for $j \neq i$, (usually we shall consider dyadic intervals), where $\Omega_j \subset \Omega$ for all j such that

$$\bigcup_{j\in\mathbb{Z}} \Omega_j = \Omega.$$

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the corresponding probability space of Ω where $\mathcal{F}$ is a σ-algebra. Let f be a measurable function defined on Ω and as usual L_p represent the usual Lebesgue space. Let $\alpha = \{\alpha_k\}_{k\in\mathbb{Z}}$ and let $q \in (0,\infty)$. Then the sequence space is the space of $\alpha = \{\alpha_k\}_{k\in\mathbb{Z}}$ such that $\sum_k |\alpha_k|^q < \infty$. The sequence space is normally denoted ℓ_q.

The classical amalgam of L_p and ℓ_q, where we agree to the notation $L_{p,q}$, on Ω consist of functions which are locally in $L_p(\Omega)$ and have ℓ_q behaviour globally, [25], in the sense that the L_p-norm over the disjoint subsets $\Omega_j \subset \Omega$ form an ℓ_q-sequence. That is, for $p, q \in (0,\infty)$,

$$L_{p,q} := \left\{ f : \left[\sum_{j\in\mathbb{Z}} \left(\int_\Omega |f|^p \mathbf{1}_{\Omega_j} d\mathbb{P} \right)^{\frac{q}{p}} \right]^{\frac{1}{q}} < \infty \right\}$$

equipped with the (quasi)-norm is given by

$$\|f\|_{p,q} := \|f\|_{L_{p,q}(\Omega)} := \left[\sum_{j\in\mathbb{Z}} \left(\int_\Omega |f|^p \mathbf{1}_{\Omega_j} d\mathbb{P} \right)^{\frac{q}{p}} \right]^{\frac{1}{q}}. \tag{2.12}$$

For $p \in (0,\infty)$ and $q = \infty$, the amalgam space is define as

$$L_{p,\infty} = \left\{ f : \sup_{j\in\mathbb{Z}} \left(\int_\Omega |f|^p \mathbf{1}_{\Omega_j} d\mathbb{P} \right)^{\frac{1}{p}} < \infty \right\}$$

equipped with the (quasi)-norm

$$\|f\|_{p,\infty} := \|f\|_{L_{p,\infty}(\Omega)} := \sup_{j\in\mathbb{Z}} \left(\int_{\Omega} |f|^p \mathbf{1}_{\Omega_j} \mathrm{d}\mathbb{P}\right)^{\frac{1}{p}}. \tag{2.13}$$

As usual, $\mathbf{1}_A$ is the indicator function of the set A. We observe that $\|f\|_{p,p} = \|f\|_p$ for $f \in L_p$. Indeed since the Ω_j's are disjoint,

$$\|f\|_{p,p} = \left[\sum_{j\in\mathbb{Z}} \left(\int_{\Omega} |f|^p \mathbf{1}_{\Omega_j} \mathrm{d}\mathbb{P}\right)^{\frac{p}{p}}\right]^{\frac{1}{p}} = \left[\sum_{j\in\mathbb{Z}} \int_{\Omega_j} |f|^p \mathrm{d}\mathbb{P}\right]^{\frac{1}{p}} = \left(\int_{\Omega} |f|^p \mathrm{d}\mathbb{P}\right)^{\frac{1}{p}}.$$

Also in [23, 24, 29], we have the following properties;

1. Endowed with the (quasi)-norm $\|\cdot\|_{p,q}$, the amalgam space $L_{p,q}$ is a complete space, and a Banach space for $1 \leq p, q \leq \infty$.
2. $\|f\|_{p,q} \leq \|f\|_p$ if $p \leq q$ and $f \in L_p$.
3. $\|f\|_p \leq \|f\|_{p,q}$ if $q \leq p$ and $f \in L_{p,q}$.

Amalgam function spaces have been essentially considered in the case $\Omega = \mathbb{R}^d$, $d \in \mathbb{N}$, and in the case $d = 1$, the subsets Ω_j are normally the intervals $[j, j+1)$, $j \in \mathbb{Z}$ [24, 25, 29].

2.12 Martingale Hardy-amalgam Spaces

As discussed above, the classical martingale Hardy spaces were introduced by Burkholder and Gundy [8]. The martingale Hardy-amalgam spaces are defined by replacing the L_p-norm in the classical martingale Hardy spaces with the (quasi)-norm of the amalgam space $L_{p,q}$. Let us now introduce the martingale Hardy-amalgam spaces more precisely.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n\in\mathbb{N}}, \mathbb{P})$ be a stochastic basis and let $f = \{f_n\}_{n\in\mathbb{N}}$ be a martingale defined with respect to the filtration $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$. Let $\mathcal{M}$ be the set of all martingales defined with respect to the stochastic basis and recall the definitions of $s(f), S(f), f^*$(a maximal function), from previous sections. Let Γ be the space of all sequences $\beta = (\beta_n)_{n\geq 0}$ of adapted (that is for all $n \in \mathbb{Z}$, β_n is $\mathcal{F}_n$-measurable), non-decreasing, non-negative functions and define

$$\beta_\infty := \lim_{n\to\infty} \beta_n.$$

We are now in the position to introduce the martingale Hardy-amalgam spaces. Let $f \in \mathcal{M}$. Then

i. $H^S_{p,q}(\Omega)$ is the space of all $f \in \mathcal{M}$ such that $S(f) \in L_{p,q}(\Omega)$ with (quasi)-norm

$$\|f\|_{H^S_{p,q}(\Omega)} := \|S(f)\|_{L_{p,q}(\Omega)}.$$

ii. $H^s_{p,q}(\Omega)$ is the space of all $f \in \mathcal{M}$ such that $s(f) \in L_{p,q}(\Omega)$ with (quasi)-norm

$$\|f\|_{H^s_{p,q}(\Omega)} := \|s(f)\|_{L_{p,q}(\Omega)}.$$

iii. $H^*_{p,q}(\Omega)$ is the space of all $f \in \mathcal{M}$ such that $f^* \in L_{p,q}(\Omega)$ with (quasi)-norm

$$\|f\|_{H^*_{p,q}(\Omega)} := \|f^*\|_{L_{p,q}(\Omega)}.$$

iv. Let $\beta = \{\beta_n\}_{n\in\mathbb{N}} \in \Gamma$. $\mathcal{Q}_{p,q}(\Omega)$ is the space of all $f \in \mathcal{M}$ such that $S_n(f) \leq \beta_{n-1}$ and $\|\beta_\infty\|_{L_{p,q}(\Omega)} < \infty$ with (quasi)-norm

$$\|f\|_{\mathcal{Q}_{p,q}(\Omega)} := \inf_{\beta\in\Gamma, S_n(f)\leq\beta_{n-1}} \|\beta_\infty\|_{L_{p,q}(\Omega)}.$$

v. Let $\beta = \{\beta_n\}_{n\in\mathbb{N}} \in \Gamma$. $\mathcal{P}_{p,q}(\Omega)$ is the space of all $f \in \mathcal{M}$ such that $|f_n| \leq \beta_{n-1}$ and $\|\beta_\infty\|_{L_{p,q}(\Omega)} < \infty$ with (quasi)-norm

$$\|f\|_{\mathcal{P}_{p,q}(\Omega)} := \inf_{\beta\in\Gamma, |f_n|\leq\beta_{n-1}} \|\beta_\infty\|_{L_{p,q}(\Omega)}.$$

These are the five martingale Hardy-amalgam spaces whose properties are studied in this work. To be more precise we study properties such as the atomic decompositions, martingale embeddings and dual characterizations of these newly introduced spaces. The spaces $\mathcal{Q}_{p,q}$ and $\mathcal{P}_{p,q}$ are called the predictive quadratic variation and predictive spaces respectively. It is worth to note that since $\|f\|_{p,p} = \|f\|_p$ for $f \in L_p(\Omega)$, then we can say that the classical martingale Hardy spaces studied by Burkholder and Gundy, Long, Weisz [8, 48, 65], are special cases of these martingale Hardy-amalgam spaces. Hence these martingale Hardy-amalgam spaces are generalizations of the classical martingale Hardy spaces.

Two more spaces considered in the classical martingale Hardy spaces are the Garsia space, $\mathcal{G}_p$, and the jump bounded space $\mathcal{BD}_p$ [65]. These spaces are very useful especially in the establishment of some classical martingale inequalities. Hence we also introduce here the spaces $\mathcal{G}_{p,q}(\Omega)$ and $\mathcal{BD}_{p,q}(\Omega)$ which we shall refer to as variation integrable space (or Garsia-type space) and jump bounded space respectively and they are generalizations to the classical cases in [48, 65]. In the appropriate chapter, we shall prove that the dual of

$\mathcal{G}_{p,q}(\Omega)$ is $\mathcal{BD}_{p',q'}(\Omega)$.

$$\mathcal{G}_{p,q}(\Omega) := \left\{ f \in \mathcal{M} : \sum_{n=0}^{\infty} |d_n f| \in L_{p,q}(\Omega) \right\}$$

endowed with the (quasi)-norm

$$\|f\|_{\mathcal{G}_{p,q}(\Omega)} = \left\| \sum_{n=0}^{\infty} |d_n f| \right\|_{L_{p,q}(\Omega)}$$

for $1 \leq p \leq q < \infty$ and

$$\mathcal{BD}_{p,q}(\Omega) := \left\{ f \in \mathcal{M} : \sup_{n \in \mathbb{N}} |d_n f| \in L_{p,q}(\Omega) \right\}$$

endowed with the (quasi)-norm

$$\|f\|_{\mathcal{BD}_{p,q}(\Omega)} = \left\| \sup_{n \in \mathbb{N}} |d_n f| \right\|_{L_{p,q}(\Omega)}$$

for $1 \leq p \leq q \leq \infty$.

2.13 A Brief Overview of the Remaining Chapters

This chapter ends with a brief overview of the remaining chapters and an important remark. In the previous chapters, we have defined and introduced the basic concepts and notations that will be appropriate in this work. In the remaining chapters, we present the results obtained in this work. We start with the atomic decompositions of $H^s_{p,q}$ and the atomic decompositions of the two predictive spaces as defined above. Following we discuss martingale inequalities and establish the relations that exists between these martingale Hardy-amalgam spaces. Next the dual spaces are characterized and we finish with the presentation on the martingale transforms. The last chapter concludes and states some recommendations for further studies.

Remark 2.14. The following remarks are in place;

- We note that the "infimum" taken in $\mathcal{P}_{p,q}$ and $\mathcal{Q}_{p,q}$ norms are attained. Indeed let $X \in \{\mathcal{P}_{p,q}, \mathcal{Q}_{p,q}\}$ and let $\beta^k = (\beta_n^{(k)})_{n \geq 0}$ be a predictable sequence of $f \in \mathcal{M}$ for $k \in \mathbb{N}$ such that
$$\|\beta_\infty^{(k)}\|_{p,q} \to \|f\|_X.$$

Set $\beta_n = \inf_k \beta_n^{(k)}, n \geq 0$, then $\beta = (\beta_n)_{n\geq 0}$ is a predictable sequence of f and

$$\|f\|_X = \|\beta_\infty\|_{p,q}.$$

Such an "infimum" sequence will be referred to as ***optimal***. This will mostly appear in Chapter 6.

- We remark here that for the sake of presentation, we sometimes write $H^S_{p,q}(\Omega)$, $H^s_{p,q}(\Omega)$, $H^*_{p,q}(\Omega)$, $\mathcal{Q}_{p,q}(\Omega)$, $\mathcal{P}_{p,q}(\Omega)$, $\mathcal{G}_{p,q}(\Omega)$, $\mathcal{BD}_{p,q}(\Omega)$ as $H^S_{p,q}$, $H^s_{p,q}$, $H^*_{p,q}$, $\mathcal{Q}_{p,q}$, $\mathcal{P}_{p,q}$, $\mathcal{G}_{p,q}$, $\mathcal{BD}_{p,q}$ respectively and $\|\cdot\|_{L_{p,q}(\Omega)}$ as $\|\cdot\|_{p,q}$.

Chapter 3

Atomic Decompositions

The atomic decompositions of Hardy spaces are very useful. For instance, with the help of atomic decompositions, other properties of the martingale Hardy spaces such as dual characterizations, martingale embeddings and martingale inequalities were studied [65]. Thus this chapter of the study is devoted to the discussion of the atomic decompositions of the martingale Hardy-amalgam spaces $H_{p,q}^s$, $\mathcal{Q}_{p,q}$ and $\mathcal{P}_{p,q}$ which were introduced in the previous chapter. With the help of stopping times, we give two definitions of atoms and hence we shall discuss two atomic decompositions of each of the spaces $H_{p,q}^s$, $\mathcal{Q}_{p,q}$ and $\mathcal{P}_{p,q}$.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n\in\mathbb{N}}, \mathbb{P})$ be a probability space and denote by $\mathcal{T}$ the set of all stopping times defined on Ω. All the martingales in this chapter are defined with respect to this probability space relative to the underlying filtration $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$. We also recall that the space of all martingales is denoted $\mathcal{M}$. The first definition of atom is the following.

Definition 3.1 (Atom)**.** Let $0 < p < \infty$, and $\max(p, 1) < r \le \infty$. A measurable function a is a $(p, r)^s$-atom (resp. $(p, r)^S$-atom, $(p, r)^*$-atom) if there exists a stopping time $\nu \in \mathcal{T}$ such that

(a1) $a_n := \mathbb{E}_n a = 0$ if $\nu \ge n$;

(a2) $\|s(a)\|_{r,r} := \|s(a)\|_r$ (resp.$\|S(f)\|_r, \|a^*\|_r) \le \mathbb{P}(B_\nu)^{\frac{1}{r}-\frac{1}{p}}$

where $B_\nu = \{\nu \ne \infty\}$.

We denote by $\mathcal{A}(p, q, r)^s$ (resp. $\mathcal{A}(p, q, r)^S$, $\mathcal{A}(p, q, r)^*$) the set of all sequences of triplets (λ_k, a^k, ν^k), where λ_k are nonnegative numbers, a^k are $(p, r)^s$-atoms (resp. $(p, r)^S$-atoms, $(p, r)^*$-atoms) and $\nu^k \in \mathcal{T}$ satisfying conditions $(a1)$ and $(a2)$ in Definition 3.1 and such that for any $0 < \eta \le 1$,

$$\sum_k \left(\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}}\right)^\eta \mathbf{1}_{B_{\nu^k}} \in L_{\frac{p}{\eta},\frac{q}{\eta}}.$$

We also have the following second definition of an atom.

Definition 3.2 (Atom)**.** Let $0 < p < \infty$, $0 < q \leq \infty$ and $\max(p, 1) < r \leq \infty$. A measurable function a is a $(p, q, r)^s$-atom (resp. $(p, q, r)^S$-atom, $(p, q, r)^*$-atom) if there exists a stopping time $\nu \in \mathcal{T}$ such that condition (a1) in Definition 3.1 is satisfied and

(a3) $\|s(a)\|_r$ (resp.$\|S(a)\|_r, \|a^*\|_r) \leq \mathbb{P}(B_\nu)^{\frac{1}{r}}\|1_{B_\nu}\|_{p,q}^{-1}$.

We denote by $\mathcal{B}(p, q, r)^s$ (resp. $\mathcal{B}(p, q, r)^S$, $\mathcal{B}(p, q, r)^*$) the set of all sequences of triplets (λ_k, a^k, ν^k), where λ_k are nonnegative numbers, a^k are $(p, q, r)^s$-atoms (resp. $(p, q, r)^S$-atoms, $(p, q, r)^*$-atoms) and $\nu^k \in \mathcal{T}$ satisfying conditions $(a1)$ and $(a3)$ in Definition 3.2 and such that for any $0 < \eta \leq 1$,

$$\sum_k \left(\frac{\lambda_k}{\|1_{B_{\nu^k}}\|_{p,q}}\right)^\eta \mathbf{1}_{B_{\nu^k}} \in L_{\frac{p}{\eta},\frac{q}{\eta}}.$$

The presentation of the atomic decompositions of the martingale Hardy-amalgam space $H^s_{p,q}$ now follows.

3.3 Atomic Decompositions of $H^s_{p,q}$

The following Theorem 3.4 describes the atomic decomposition of $H^s_{p,q}(\Omega)$ based on Definition 3.1 of an atom. We recall that $H^s_{p,q}(\Omega)$ is the space of all $f \in \mathcal{M}$ such that $s(f) \in L_{p,q}(\Omega)$ with (quasi)-norm

$$\|f\|_{H^s_{p,q}(\Omega)} := \|s(f)\|_{L_{p,q}(\Omega)}.$$

Theorem 3.4. *Let $0 < p < \infty$, $0 < q \leq \infty$ and let $\max(p, 1) < r \leq \infty$. If the martingale $f \in \mathcal{M}$ is in $H^s_{p,q}$, then there exists a sequence of triplets $(\lambda_k, a^k, \nu^k) \in \mathcal{A}(p, q, r)^s$ such that for all $n \in \mathbb{N}$,*

$$\sum_{k\in\mathbb{Z}} \lambda_k \mathbb{E}_n a^k = f_n \ \ a.e. \tag{3.1}$$

and for any $0 < \eta \leq 1$,

$$\left\|\sum_{k\in\mathbb{Z}} \left(\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}}\right)^\eta \mathbf{1}_{B_{\nu^k}}\right\|^{\frac{1}{\eta}}_{\frac{p}{\eta},\frac{q}{\eta}} \leq C\|f\|_{H^s_{p,q}}. \tag{3.2}$$

Moreover,

$$\sum_{k=l}^{m} \lambda_k a^k \longrightarrow f$$

in $H^s_{p,q}$ *as* $m \to \infty,\ l \to -\infty$.

Conversely if $f \in \mathcal{M}$ *has a decomposition as in (3.1), then for any* $0 < \eta \leq 1$,

$$\|f\|_{H^s_{p,q}} \leq C \left\| \sum_{k\in\mathbb{Z}} \left(\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} \right)^{\eta} \mathbf{1}_{B_{\nu^k}} \right\|^{\frac{1}{\eta}}_{\frac{p}{\eta},\frac{q}{\eta}}.$$

Proof. Let $0 < p < \infty$, $0 < q < \infty$, $\max(p,1) < r \leq \infty$ and let f be in $H^s_{p,q}$ and define the nonnegative and nondecreasing sequence of stopping time, $(\nu^k)_{k\in\mathbb{Z}}$, as

$$\nu^k := \inf\{n \in \mathbb{N} : s_{n+1}(f) > 2^k\}. \tag{3.3}$$

Take $\lambda_k = C2^k\mathbb{P}(\nu^k \neq \infty)^{\frac{1}{p}}$, for some constant C, to be determined later, and $(n \in \mathbb{N})$

$$a^k_n = \begin{cases} \frac{f^{\nu^{k+1}}_n - f^{\nu^k}_n}{\lambda_k}, & \text{if} \quad \lambda_k \neq 0 \\ 0, & \text{if} \quad \lambda_k = 0 \end{cases}. \tag{3.4}$$

Then, it is true that $s(f^{\nu^k}_n) \leq 2^k$, $s(a^k) \leq \mathbb{P}(\nu^k \neq \infty)^{-\frac{1}{p}}$ and $s(f) \leq \sum_{k\geq 0} \lambda_k s(a^k)$. Indeed, we recall from (2.4) that $s(f^{\nu^k}) = s_{\nu^k}(f)$. By the definition of the stopping time, $\nu^k \leq n+1$, and since $s_n(f)$ is increasing in n, and $2^k < s_{n+1}(f)$, we obtain that

$$s_{\nu^k}(f) \leq 2^k. \tag{3.5}$$

Moreover,

$$\sum_{k\in\mathbb{Z}} (f^{\nu^{k+1}}_n - f^{\nu^k}_n) = f_n \quad \text{a.e.}$$

Let us check that a^k is an $(p,r)^s$-atom. We start by showing that $s(a^k) \leq \mathbb{P}(\nu^k \neq \infty)^{-\frac{1}{p}}$. Before we do that, let us observe from the definition of stopped martingale that on the set $\{n \leq \nu^k\}$,

$$a^k_n = \frac{f^{\nu^{k+1}}_n - f^{\nu^k}_n}{\lambda_k} = \frac{f_n - f_n}{\lambda_k} = 0 \tag{3.6}$$

and thus $\mathbb{E}_n a^k = 0$ when $\nu^k \geq n$. Hence a^k satisfies condition (a1) in Definition 3.1. Also we note that equation (3.6) implies that

$$\mathbf{1}_{\{\nu^k=\infty\}}[s(a^k)]^2 \leq \sum_{n\in\mathbb{N}} \mathbf{1}_{\{\nu^k\geq n\}}\mathbb{E}_{n-1}|d_n a^k|^2 = \sum_{n\in\mathbb{N}} \mathbf{1}_{\{\nu^k\geq n\}}\mathbb{E}_{n-1}|\mathbf{1}_{\{\nu^k\geq n\}} d_n a^k|^2 = 0.$$

That is the support of $s(a^k)$ is contained in $B_{\nu^k} = \{\nu^k \neq \infty\}$. Using inequalities (2.8) and (3.5), we obtain that

$$\begin{aligned} s(a^k) &= s\left(\frac{f^{\nu^{k+1}} - f^{\nu^k}}{\lambda_k}\right) \\ &= \frac{1}{\lambda_k} s(f^{\nu^{k+1}} - f^{\nu^k}) \\ &\leq \frac{\sqrt{2}}{\lambda_k}[s(f^{\nu^{k+1}}) + s(f^{\nu^k})] \leq \frac{\sqrt{2}(2^{k+1} + 2^k)}{\lambda_k} \leq \frac{C(2^k)}{\lambda_k} \\ &= \frac{C(2^k)}{C2^k \mathbb{P}(\nu^k \neq \infty)^{\frac{1}{p}}} = \frac{1}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} \end{aligned}$$

where we take $C \geq 3\sqrt{2}$. Thus $s(a^k) \leq \mathbb{P}(B_{\nu^k})^{-\frac{1}{p}}$ and zero outside of B_{ν^k}. Therefore

$$\begin{aligned} \|s(a^k)\|_r^r &:= \int_\Omega s^r(a^k)\mathrm{d}\mathbb{P} = \int_{B_{\nu^k}} s^r(a^k)\mathrm{d}\mathbb{P} + \int_{B^c_{\nu^k}} s^r(a^k)\mathrm{d}\mathbb{P} \\ &= \int_{B_{\nu^k}} s^r(a^k)\mathrm{d}\mathbb{P} \leq \int_{B_{\nu^k}} \mathbb{P}(B_{\nu^k})^{-\frac{r}{p}}\mathrm{d}\mathbb{P} \\ &= \mathbb{P}(B_{\nu^k})^{-\frac{r}{p}} \int_{B_{\nu^k}} \mathrm{d}\mathbb{P} = \mathbb{P}(B_{\nu^k})^{-\frac{r}{p}}\mathbb{P}(B_{\nu^k}). \end{aligned}$$

Hence we obtain that

$$\|s(a^k)\|_r \leq \mathbb{P}(\nu^k \neq \infty)^{\frac{1}{r} - \frac{1}{p}}. \tag{3.7}$$

Thus condition (a2) in the definition of an $(p, r)^s$-atom is also satisfied.

We observe that, by measurability and martingale property of a^k,

$$\begin{aligned} s^2(a^k) &= \sum_n \mathbb{E}_{n-1}|d_n a^k|^2 = \sum_n \mathbb{E}_{n-1}[(a^k_n)^2 + (a^k_{n-1})^2 - 2a^k_n a^k_{n-1}] \\ &= \sum_n \mathbb{E}_{n-1}[(a^k_n)^2 + (a^k_{n-1})^2] - 2\mathbb{E}_{n-1}[a^k_n a^k_{n-1}] \\ &= \sum_n \mathbb{E}_{n-1}[(a^k_n)^2 + (a^k_{n-1})^2] - 2a^k_{n-1}\mathbb{E}_{n-1}[a^k_n] \\ &= \sum_n \mathbb{E}_{n-1}[(a^k_n)^2 + (a^k_{n-1})^2] - 2(a^k_{n-1})^2 \end{aligned}$$

and then

$$\begin{aligned}\mathbb{E}[s^2(a^k)] &= \sum_n \mathbb{E}[(\mathbb{E}_{n-1}[(a_n^k)^2 + (a_{n-1}^k)^2]) - 2\mathbb{E}[(a_{n-1}^k)^2]] \\ &= \sum_n \mathbb{E}[(a_n^k)^2 + (a_{n-1}^k)^2 - 2(a_{n-1}^k)^2] \\ &= \mathbb{E}\sum_n [(a_n^k)^2 - (a_{n-1}^k)^2] = \mathbb{E}[(a^k)^2].\end{aligned}$$

In other words, $\|s(a^k)\|_2 = \|a^k\|_2$. Thus from (3.7), we have that $\|s(a^k)\|_2 \leq \mathbb{P}(\nu^k \neq \infty)^{-\frac{1}{p}}$ and hence $\|a^k\|_2 = \|s(a^k)\|_2 \leq \mathbb{P}(\nu^k \neq \infty)^{-\frac{1}{p}}$. Which implies that $\sup_n \|a_n^k\|_2 < \mathbb{P}(\nu^k \neq \infty)^{-\frac{1}{p}} < \infty$. Hence for a fixed k, $a^k = \{a_n^k\}_{n\in\mathbb{N}}$ is an L_2-bounded martingale since $\|a_n^k\| := \|\mathbb{E}_n a^k\| \leq \|a^k\|_2$. Consequently, the limit

$$\lim_{n\to\infty} a_n^k$$

exists a.e. and in L_2[67]. Hence there exists $a^k \in L_2$ such that $\mathbb{E}_n a^k$ is well defined and since a^k is a martingale, $\mathbb{E}_n a^k = a_n^k$. Now since $\lambda_k a_n^k = f_n^{\nu^{k+1}} - f_n^{\nu^k}$, we have that

$$\sum_{k\in\mathbb{Z}} \lambda_k a_n^k = \sum_{k\in\mathbb{Z}} (f_n^{\nu^{k+1}} - f_n^{\nu^k}) = f_n \text{ a.e.}$$

and thus (3.1) is satisfied.

Let us now check that if the martingale $f = \{f_n\}_{n\in\mathbb{N}}$ has the representation as in (3.1), then $s(f) \leq \sum_{k\in\mathbb{Z}} \lambda_k s(a^k)$. Since a^k is a martingale, we have that

$$d_n f = f_n - f_{n-1} = \sum_{k\in\mathbb{Z}} \lambda_k \mathbb{E}_n a^k - \sum_{k\in\mathbb{Z}} \lambda_k \mathbb{E}_{n-1} a^k = \sum_{k\in\mathbb{Z}} \lambda_k (a_n^k - a_{n-1}^k) = \sum_{k\in\mathbb{Z}} \lambda_k d_n a^k.$$

Therefore

$$|d_n f|^2 = \left(\sum_{k\in\mathbb{Z}} \lambda_k d_n a^k\right)^2 = \sum_{k\in\mathbb{Z}} |\lambda_k d_n a^k|^2 + 2 \sum_{t<r\leq\infty} \lambda_t \lambda_r d_n a^t d_n a^r.$$

We have by orthogonality of martingale difference, Remark 2.2, that

$$\mathbb{E}_{n-1}[\lambda_t \lambda_r d_n a^t d_n a^r] = 0.$$

Hence

$$\mathbb{E}_{n-1}|d_n f|^2 = \sum_{k\in\mathbb{Z}} \lambda_k^2 \mathbb{E}_{n-1}|d_n a^k|^2 + 2 \sum_{t<r\leq\infty} \lambda_t \lambda_r \mathbb{E}_{n-1} d_n a^t d_n a^r = \sum_{k\in\mathbb{Z}} \lambda_k^2 \mathbb{E}_{n-1}|d_n a^k|^2.$$

Hence by definition,

$$\begin{aligned} s(f) &= \left(\sum_{n\in\mathbb{N}} \mathbb{E}_{n-1}|d_n f|^2\right)^{\frac{1}{2}} = \left(\sum_{n\in\mathbb{N}}\sum_{k\in\mathbb{Z}} \lambda_k^2 \mathbb{E}_{n-1}|d_n a^k|^2\right)^{\frac{1}{2}} \\ &= \left(\sum_{k\in\mathbb{Z}}\left[\lambda_k^2 \sum_{n\in\mathbb{N}} \mathbb{E}_{n-1}|d_n a^k|^2\right]\right)^{\frac{1}{2}} \leq \sum_{k\in\mathbb{Z}} \lambda_k \left(\sum_{n\in\mathbb{N}} \mathbb{E}_{n-1}|d_n a^k|^2\right)^{\frac{1}{2}} \end{aligned}$$

and therefore by definition

$$s(f) \leq \sum_{k\in\mathbb{Z}} \lambda_k \left(\sum_{n\in\mathbb{N}} \mathbb{E}_{n-1}|d_n a^k|^2\right)^{\frac{1}{2}} = \sum_{k\in\mathbb{Z}} \lambda_k s(a^k). \tag{3.8}$$

We now check that

$$\sum_k \lambda_k a^k \longrightarrow f \text{ in } H^s_{p,q}.$$

As equation (3.4) implies that $\lambda_k a^k = f^{\nu^{k+1}} - f^{\nu^k}$, we obtain that $\sum_{k=l}^m \lambda_k a^k = \sum_{k=l}^m f^{\nu^{k+1}} - f^{\nu^k} = f^{\nu^{m+1}} - f^{\nu^l}$ and therefore we have that

$$f - \sum_{k=l}^{m} \lambda_k a^k = (f - f^{\nu^{m+1}}) + f^{\nu^l}. \tag{3.9}$$

Now by definition,

$$\|f - f^{\nu^{m+1}}\|_{H^s_{p,q}} = \|s(f - f^{\nu^{m+1}})\|_{p,q}.$$

Thus for Ω_j as in the definition of amalgam space,

$$\begin{aligned} \|s(f - f^{\nu_{m+1}})\|_{p,q} &= \left[\sum_{j\in\mathbb{Z}}\left(\int_\Omega s^p(f - f^{\nu^{m+1}})\mathbf{1}_{\Omega_j} d\mathbb{P}\right)^{\frac{q}{p}}\right]^{\frac{1}{q}} \\ &= \left[\sum_{j\in\mathbb{Z}} \|s(f - f^{\nu^{m+1}})\mathbf{1}_{\Omega_j}\|_p^q\right]^{\frac{1}{q}}. \end{aligned}$$

Now by identity (2.6), we have

$$s^2(f - f^{\nu^{m+1}}) = s^2(f) - s^2(f^{\nu^{m+1}}) \leq s^2(f).$$

As $s^p(f - f^{\nu^{m+1}}) \leq s^p(f)$ and

$$\int_{\Omega_j} s^p(f) d\mathbb{P} < \infty,$$

it follows from the Dominated Convergence Theorem that

$$\|s(f - f^{\nu^{m+1}})\mathbf{1}_{\Omega_j}\|_p \longrightarrow 0$$

$m \to \infty$ and $s(f - f^{\nu^{m+1}}) \to 0$ as $m \to \infty$. Hence as $\sum_{j\in\mathbb{Z}} \|s(f)\mathbf{1}_{\Omega_j}\|_p^q = \|f\|_{H^s_{p,q}}^q < \infty$, applying the Dominated Convergence Theorem for the sequence space ℓ_q, we conclude that

$$\|f - f^{\nu^{m+1}}\|_{H^s_{p,q}} = \|s(f - f^{\nu^{m+1}})\|_{p,q} \longrightarrow 0$$

as $m \longrightarrow \infty$.

Also since $s(f^{\nu^k}) \lesssim 2^k$, we have that

$$\|f^{\nu^l}\|_{H^s_{p,q}} = \|s(f^{\nu^l})\|_{p,q} \lesssim 2^l.$$

Hence $\|f^{\nu^l}\|_{H^s_{p,q}} \longrightarrow 0$ as $l \longrightarrow -\infty$. Thus (3.9) implies that

$$\begin{aligned}
\left\| f - \sum_{k=l}^{m} \lambda_k a^k \right\|_{H^s_{p,q}} &= \left\| (f - f^{\nu^{m+1}}) + f^{\nu^l} \right\|_{H^s_{p,q}} \\
&= \left\| s((f - f^{\nu^{m+1}}) + f^{\nu^l}) \right\|_{p,q} \\
&\lesssim \left\| s(f - f^{\nu^{m+1}}) + s(f^{\nu^l}) \right\|_{p,q} \\
&\le \left\| s(f - f^{\nu^{m+1}}) \right\|_{p,q} + \left\| s(f^{\nu^l}) \right\|_{p,q} \to 0
\end{aligned}$$

as $m \longrightarrow \infty$ and $l \longrightarrow -\infty$. Hence $\sum_{k=l}^{m} \lambda_k a^k \longrightarrow f$ in $H^s_{p,q}$ as $m \to \infty,\ l \to -\infty$.

Let us now establish (3.2). Let $\Omega_j \subset \Omega$ be as it appeared in the definition of amalgam space. Then by definition,

$$\left\| \sum_{k\in\mathbb{Z}} \left(\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} \right)^{\eta} \mathbf{1}_{B_{\nu^k}} \right\|_{\frac{p}{\eta},\frac{q}{\eta}}^{\frac{1}{\eta}} = \left[\sum_{j\in\mathbb{Z}} \left(\int_{\Omega} \left(\sum_{k\in\mathbb{Z}} \left(\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} \right)^{\eta} \mathbf{1}_{B_{\nu^k}} \right)^{\frac{p}{\eta}} \mathbf{1}_{\Omega_j} d\mathbb{P} \right)^{\frac{q}{p}} \right]^{\frac{\eta}{q}\cdot\frac{1}{\eta}}.$$

Considering the inner sum, we see that

$$\left(\sum_{k\in\mathbb{Z}} \left(\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} \right)^{\eta} \mathbf{1}_{B_{\nu^k}} \right)^{\frac{p}{\eta}} = \left(\sum_{k\in\mathbb{Z}} \left(\frac{C2^k \mathbb{P}(\nu^k \neq \infty)^{\frac{1}{p}}}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} \right)^{\eta} \mathbf{1}_{B_{\nu^k}} \right)^{\frac{p}{\eta}} = \left(\sum_{k\in\mathbb{Z}} (C2^k)^{\eta}\, \mathbf{1}_{B_{\nu^k}} \right)^{\frac{p}{\eta}}.$$

We shall use some ideas from [69]. Let $G_k = B_{\nu^k} \setminus B_{\nu^{k+1}}$ where $B_{\nu^k} = \{\nu^k \neq \infty\}$. Then G_k are disjoint such that $B_{\nu^k} = \bigcup_{r=k}^{\infty} G_r$ and

$$\mathbf{1}_{B_{\nu^k}} = \sum_{r=k}^{\infty} \mathbf{1}_{G_r}. \tag{3.10}$$

Then

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\left(C2^{k}\right)^{\eta}\mathbf{1}_{B_{\nu^k}} &= \sum_{k\in\mathbb{Z}} C^{\eta}2^{k\eta}\cdot\sum_{r=k}^{\infty}\mathbf{1}_{G_r}\\ &= \sum_{r\in\mathbb{Z}}\sum_{k\leq r} C^{\eta}2^{k\eta}\mathbf{1}_{G_r} \leq \frac{C^{\eta}}{2^{\eta}-1}\sum_{k\in\mathbb{Z}}2^{(k+1)\eta}\mathbf{1}_{G_k}.\end{aligned}$$

Thus

$$\sum_{k\in\mathbb{Z}}\left(C2^{k}\right)^{\eta}\mathbf{1}_{B_{\nu^k}} \leq \frac{C^{\eta}}{2^{\eta}-1}\left(\sum_{k\in\mathbb{Z}}2^{(k+1)}\mathbf{1}_{G_k}\right)^{\eta} \leq \frac{(2C)^{\eta}}{2^{\eta}-1}\left(\sum_{k\in\mathbb{Z}}s(f)\mathbf{1}_{G_k}\right)^{\eta} \quad \text{[by disjointedness]}$$

given us

$$\sum_{k\in\mathbb{Z}}\left(C2^{k}\right)^{\eta}\mathbf{1}_{B_{\nu^k}} \leq \frac{(2C)^{\eta}s(f)^{\eta}}{2^{\eta}-1}\left(\sum_{k\in\mathbb{Z}}\mathbf{1}_{G_k}\right)^{\eta} = \frac{(2C)^{\eta}s(f)^{\eta}}{2^{\eta}-1}\sum_{k\in\mathbb{Z}}\mathbf{1}_{G_k} \quad \text{[by disjointedness].}$$

Hence

$$\begin{aligned}\left(\sum_{k\in\mathbb{Z}}\left(C2^{k}\right)^{\eta}\mathbf{1}_{B_{\nu^k}}\right)^{\frac{p}{\eta}} &\leq \left(\frac{(2C)^{\eta}s(f)^{\eta}}{2^{\eta}-1}\sum_{k\in\mathbb{Z}}\mathbf{1}_{G_k}\right)^{\frac{p}{\eta}}\\ &= \frac{(2C)^{p}s(f)^{p}}{(2^{\eta}-1)^{\frac{p}{\eta}}}\left(\sum_{k\in\mathbb{Z}}\mathbf{1}_{G_k}\right)^{\frac{p}{\eta}} = \frac{(2C)^{p}s(f)^{p}}{(2^{\eta}-1)^{\frac{p}{\eta}}}\sum_{k\in\mathbb{Z}}\mathbf{1}_{G_k}.\end{aligned}$$

We then have that

$$\begin{aligned}\left\|\sum_{k\in\mathbb{Z}}\left(\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}}\right)^{\eta}\mathbf{1}_{B_{\nu^k}}\right\|_{\frac{p}{\eta},\frac{q}{\eta}}^{\frac{1}{\eta}} &= \left[\sum_{j\in\mathbb{Z}}\left(\int_{\Omega}\left(\sum_{k\in\mathbb{Z}}\left(\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}}\right)^{\eta}\mathbf{1}_{B_{\nu^k}}\right)^{\frac{p}{\eta}}\mathbf{1}_{\Omega_j}\mathrm{d}\mathbb{P}\right)^{\frac{q}{p}}\right]^{\frac{\eta}{q}\cdot\frac{1}{\eta}}\\ &\leq \left[\sum_{j\in\mathbb{Z}}\left(\int_{\Omega}\frac{(2C)^{p}s(f)^{p}}{(2^{\eta}-1)^{\frac{p}{\eta}}}\sum_{k\in\mathbb{Z}}\mathbf{1}_{G_k}\mathbf{1}_{\Omega_j}\mathrm{d}\mathbb{P}\right)^{\frac{q}{p}}\right]^{\frac{\eta}{q}\cdot\frac{1}{\eta}}\\ &= \left[\sum_{j\in\mathbb{Z}}\left(\frac{(2C)^{p}}{(2^{\eta}-1)^{\frac{p}{\eta}}}\int_{\Omega}s^{p}(f)\sum_{k\in\mathbb{Z}}\mathbf{1}_{G_k}\mathbf{1}_{\Omega_j}\mathrm{d}\mathbb{P}\right)^{\frac{q}{p}}\right]^{\frac{\eta}{q}\cdot\frac{1}{\eta}}\\ &= \left[\sum_{j\in\mathbb{Z}}\left(\frac{(2C)^{p}}{(2^{\eta}-1)^{\frac{p}{\eta}}}\sum_{k\in\mathbb{Z}}\int_{G_k}s^{p}(f)\mathbf{1}_{\Omega_j}\mathrm{d}\mathbb{P}\right)^{\frac{q}{p}}\right]^{\frac{\eta}{q}\cdot\frac{1}{\eta}}\\ &= \left[\sum_{j\in\mathbb{Z}}\left(\frac{(2C)^{p}}{(2^{\eta}-1)^{\frac{p}{\eta}}}\int_{\Omega}s^{p}(f)\mathbf{1}_{\Omega_j}\mathrm{d}\mathbb{P}\right)^{\frac{q}{p}}\right]^{\frac{\eta}{q}\cdot\frac{1}{\eta}}.\end{aligned}$$

Hence we have that

$$\begin{aligned}\left\|\sum_{k\in\mathbb{Z}}\left(\frac{\lambda_j}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}}\right)^{\eta}\mathbf{1}_{B_{\nu^k}}\right\|_{\frac{p}{\eta},\frac{q}{\eta}}^{\frac{1}{\eta}} &\le \left[\sum_{j\in\mathbb{Z}}\left(\int_{\Omega}\frac{(2C)^p}{(2^\eta-1)^{\frac{p}{\eta}}}s^p(f)\mathbf{1}_{\Omega_j}\mathrm{d}\mathbb{P}\right)^{\frac{q}{p}}\right]^{\frac{1}{q}}\\ &= \frac{2C}{(2^\eta-1)^{\frac{1}{\eta}}}\|s(f)\|_{p,q} = \frac{2C}{(2^\eta-1)^{\frac{1}{\eta}}}\|f\|_{H^s_{p,q}}\end{aligned}$$

establishing the first part of the theorem.

Conversely, let the martingale $f=\{f_n\}_{n\in\mathbb{N}}$ have a representation as in (3.1). Then as $s(a^k)\le\mathbb{P}(\nu^k\ne\infty)^{\frac{1}{r}-\frac{1}{p}}\le\mathbb{P}(\nu^k\ne\infty)^{-\frac{1}{p}}$ with support in B_{ν^k}, we obtain from (3.8) that

$$\|f\|_{H^s_{p,q}} = \|s(f)\|_{p,q}\le\left\|\sum_{k\in\mathbb{Z}}\lambda_k s(a^k)\right\|_{p,q}\le\left\|\sum_{k\in\mathbb{Z}}\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}}\mathbf{1}_{B_{\nu^k}}\right\|_{p,q}.$$

Let us quickly check that

$$\left\|\sum_{k\in\mathbb{Z}}\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}}\mathbf{1}_{B_{\nu^k}}\right\|_{p,q}\le\left\|\sum_{k\in\mathbb{Z}}\left(\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}}\right)^{\eta}\mathbf{1}_{B_{\nu^k}}\right\|_{\frac{p}{\eta},\frac{q}{\eta}}^{\frac{1}{\eta}}$$

for $0<\eta<1$. Indeed by definition,

$$\begin{aligned}\left\|\sum_{k\in\mathbb{Z}}\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}}\mathbf{1}_{B_{\nu^k}}\right\|_{p,q} &= \left[\sum_{j\in\mathbb{Z}}\left(\int_{\Omega}\left(\sum_{k\in\mathbb{Z}}\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}}\mathbf{1}_{B_{\nu^k}}\right)^{p}\mathbf{1}_{\Omega_j}\mathrm{d}\mathbb{P}\right)^{\frac{q}{p}}\right]^{\frac{1}{q}}\\ &= \left[\sum_{j\in\mathbb{Z}}\left(\int_{\Omega}\left(\sum_{k\in\mathbb{Z}}\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}}\mathbf{1}_{B_{\nu^k}}\right)^{p\frac{\eta}{\eta}}\mathbf{1}_{\Omega_j}\mathrm{d}\mathbb{P}\right)^{\frac{q/\eta}{p/\eta}}\right]^{\frac{1}{q}\cdot\frac{\eta}{\eta}}\\ &\le \left[\sum_{j\in\mathbb{Z}}\left(\int_{\Omega}\left(\sum_{k\in\mathbb{Z}}\left(\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}}\right)^{\eta}\mathbf{1}_{B_{\nu^k}}\right)^{\frac{p}{\eta}}\mathbf{1}_{\Omega_j}\mathrm{d}\mathbb{P}\right)^{\frac{q/\eta}{p/\eta}}\right]^{\frac{\eta}{q}\cdot\frac{1}{\eta}}\\ &= \left\|\sum_{k\in\mathbb{Z}}\left(\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}}\right)^{\eta}\mathbf{1}_{B_{\nu^k}}\right\|_{\frac{p}{\eta},\frac{q}{\eta}}^{\frac{1}{\eta}}.\end{aligned}$$

Thus

$$\left\|\sum_{k\in\mathbb{Z}}\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}}\mathbf{1}_{B_{\nu^k}}\right\|_{p,q}\le\left\|\sum_{k\in\mathbb{Z}}\left(\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}}\right)^{\eta}\mathbf{1}_{B_{\nu^k}}\right\|_{\frac{p}{\eta},\frac{q}{\eta}}^{\frac{1}{\eta}}. \tag{3.11}$$

Hence

$$\|f\|_{H^s_{p,q}} \le \left\| \sum_{k\in\mathbb{Z}} \left(\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} \right)^{\eta} \mathbf{1}_{B_{\nu^k}} \right\|^{\frac{1}{\eta}}_{\frac{p}{\eta},\frac{q}{\eta}}$$

establishing the converse.

For the case $q = \infty$, we have, by definition, that

$$\|f - f^{\nu^{m+1}}\|_{H^s_{p,\infty}} = \|s(f - f^{\nu^{m+1}})\|_{p,\infty} := \sup_{j\in\mathbb{Z}} \|s(f - f^{\nu^{m+1}})\mathbf{1}_{\Omega_j}\|_p.$$

Hence, as above, $\sup_{j\in\mathbb{Z}} \|s(f - f^{\nu^{m+1}})\mathbf{1}_{\Omega_j}\|_p \to 0$ as $m \to \infty$ since $\|s(f - f^{\nu^{m+1}})\|_p \to 0$ as $m \to \infty$. We also observe that, since $s(f^{\nu^k}) \lesssim 2^k$, it is true that $\|s(f^{\nu^l})\|_{p,\infty} \lesssim 2^l \to 0$ as $l \to -\infty$. Hence $\|f^{\nu^l}\|_{H^s_{p,\infty}} \to 0$ as $l \to -\infty$. Therefore

$$\left\| f - \sum_{k=l}^{m} \lambda_k a^k \right\|_{H^s_{p,\infty}} \le \|(f - f^{\nu^{m+1}})\|_{H^s_{p,\infty}} + \|f^{\nu^l}\|_{H^s_{p,\infty}} \to 0.$$

Hence

$$\sum_{k=l}^{m} \lambda_k a^k \longrightarrow f$$

in $H^s_{p,\infty}$ as $m \to \infty$, $l \to -\infty$. It follows by definition, as we did above, that

$$\begin{aligned}
\left\| \sum_{k\in\mathbb{Z}} \left(\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} \right)^{\eta} \mathbf{1}_{B_{\nu^k}} \right\|_{\frac{p}{\eta},\infty} &= \sup_{j\in\mathbb{Z}} \left(\int_\Omega \left(\sum_{k\in\mathbb{Z}} \left(\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} \right)^{\eta} \mathbf{1}_{B_{\nu^k}} \right)^{\frac{p}{\eta}} \mathbf{1}_{\Omega_j} \mathrm{d}\mathbb{P} \right)^{\frac{\eta}{p}} \\
&\le \sup_{j\in\mathbb{Z}} \left(\int_\Omega \frac{(2C)^p s(f)^p}{(2^\eta - 1)^{\frac{p}{\eta}}} \sum_{k\in\mathbb{Z}} \mathbf{1}_{G_k} \mathbf{1}_{\Omega_j} \mathrm{d}\mathbb{P} \right)^{\frac{\eta}{p}} \\
&= \sup_{j\in\mathbb{Z}} \left(\frac{(2C)^p}{(2^\eta - 1)^{\frac{p}{\eta}}} \sum_{k\in\mathbb{Z}} \int_{G_k} s^p(f) \mathbf{1}_{\Omega_j} \mathrm{d}\mathbb{P} \right)^{\frac{\eta}{p}} \\
&= \left(\frac{(2C)^\eta}{2^\eta - 1} \right) \sup_{j\in\mathbb{Z}} \left(\int_\Omega s^p(f) \mathbf{1}_{\Omega_j} \mathrm{d}\mathbb{P} \right)^{\frac{\eta}{p}} \\
&= \left(\frac{(2C)^\eta}{2^\eta - 1} \right) \|s(f)\|^{\eta}_{p,\infty}.
\end{aligned}$$

Hence

$$\left\| \sum_{k\in\mathbb{Z}} \left(\frac{\lambda_j}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} \right)^{\eta} \mathbf{1}_{B_{\nu^k}} \right\|^{\frac{1}{\eta}}_{\frac{p}{\eta},\infty} \le \left(\frac{(2C)^\eta}{2^\eta - 1} \right)^{\frac{1}{\eta}} \|f\|_{H^s_{p,\infty}}.$$

Conversely, as in the above, let the martingale f have a representation as in (3.1). Then as $s(a^k) \leq \mathbb{P}(\nu^k \neq \infty)^{-\frac{1}{p}}$ with support in $B_{\nu^k} = \{\nu \neq \infty\}$, we obtain that

$$\begin{aligned}
\|f\|_{H^s_{p,\infty}} &= \|s(f)\|_{p,\infty} \leq \left\| \sum_{k\in\mathbb{Z}} \lambda_k s(a^k) \right\|_{p,\infty} \\
&\leq \left\| \sum_{k\in\mathbb{Z}} \lambda_k \mathbb{P}(B_{\nu^k})^{-\frac{1}{p}} \mathbf{1}_{B_{\nu^k}} \right\|_{p,\infty} \\
&= \left\| \sum_{k\in\mathbb{Z}} \frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} \mathbf{1}_{B_{\nu^k}} \right\|_{p,\infty}.
\end{aligned}$$

We check that

$$\left\| \sum_{k\in\mathbb{Z}} \frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} \mathbf{1}_{B_{\nu^k}} \right\|_{p,\infty} \leq \left\| \sum_{k\in\mathbb{Z}} \left(\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} \right)^{\eta} \mathbf{1}_{B_{\nu^k}} \right\|^{\frac{1}{\eta}}_{\frac{p}{\eta},\infty}.$$

For $0 < \eta < 1$, we get

$$\begin{aligned}
\left\| \sum_{k\in\mathbb{Z}} \frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} \mathbf{1}_{B_{\nu^k}} \right\|_{p,\infty} &= \sup_{j\in\mathbb{Z}} \left(\int_\Omega \left(\sum_{k\in\mathbb{Z}} \frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} \mathbf{1}_{B_{\nu^k}} \right)^{p} \mathbf{1}_{\Omega_j} \mathrm{d}\mathbb{P} \right)^{\frac{1}{p}} \\
&= \sup_{j\in\mathbb{Z}} \left(\int_\Omega \left(\sum_{k\in\mathbb{Z}} \frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} \mathbf{1}_{B_{\nu^k}} \right)^{p\frac{\eta}{\eta}} \mathbf{1}_{\Omega_j} \mathrm{d}\mathbb{P} \right)^{\frac{1}{p}\cdot\frac{\eta}{\eta}} \\
&\leq \sup_{j\in\mathbb{Z}} \left(\int_\Omega \left(\sum_{k\in\mathbb{Z}} \left(\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} \right)^{\eta} \mathbf{1}_{B_{\nu^k}} \right)^{\frac{p}{\eta}} \mathbf{1}_{\Omega_j} \mathrm{d}\mathbb{P} \right)^{\frac{1}{p}\cdot\frac{\eta}{\eta}} \\
&= \left\| \sum_{k\in\mathbb{Z}} \left(\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} \right)^{\eta} \mathbf{1}_{B_{\nu^k}} \right\|^{\frac{1}{\eta}}_{\frac{p}{\eta},\infty}.
\end{aligned}$$

Hence

$$\|f\|_{H^s_{p,\infty}} \leq \left\| \sum_{k\in\mathbb{Z}} \left(\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} \right)^{\eta} \mathbf{1}_{B_{\nu^k}} \right\|^{\frac{1}{\eta}}_{\frac{p}{\eta},\infty}$$

establishing the converse. The proof is complete. ☐

A second atomic decomposition of $H^s_{p,q}(\Omega)$ is described by Theorem 3.5 below. This atomic decomposition is based on Definition 3.2 of an atom.

Theorem 3.5. *Let* $0 < p < \infty$, $0 < q \leq \infty$ *and let* $\max(p,1) < r \leq \infty$. *If the martingale* $f \in \mathcal{M}$ *is in* $H^s_{p,q}$ *then there exists a sequence of triplets* $(\lambda_k, a^k, \nu^k) \in \mathcal{B}(p,q,r)^s$ *such* *that for all* $n \in \mathbb{N}$,

$$\sum_{k\in\mathbb{Z}} \lambda_k \mathbb{E}_n a^k = f_n \tag{3.12}$$

and for any $0 < \eta \leq 1$,

$$\left\| \sum_{k\geq 0} \left(\frac{\lambda_k}{\|1_{B_{\nu^k}}\|_{p,q}} \right)^{\eta} \mathbf{1}_{B_{\nu^k}} \right\|_{\frac{p}{\eta},\frac{q}{\eta}}^{\frac{1}{\eta}} \leq C\|f\|_{H^s_{p,q}}. \tag{3.13}$$

Moreover,

$$\sum_{k=l}^{m} \lambda_k a^k \longrightarrow f$$

in $H^s_{p,q}$ *as* $m \to \infty,\ l \to -\infty$.

Conversely if $f \in \mathcal{M}$ *has a decomposition as in (3.12), then for any* $0 < \eta \leq 1$,

$$\|f\|_{H^s_{p,q}} \leq C \left\| \sum_{k\in\mathbb{Z}} \left(\frac{\lambda_k}{\|1_{B_{\nu^k}}\|_{p,q}} \right)^{\eta} \mathbf{1}_{B_{\nu^k}} \right\|_{\frac{p}{\eta},\frac{q}{\eta}}^{\frac{1}{\eta}}.$$

Proof. Let $0 < p < \infty$, $0 < q < \infty$, $\max(p,1) < r \leq \infty$, and let $f \in H^s_{p,q}$ and define the stopping time as

$$\nu^k := \inf\{n \in \mathbb{N} : s_{n+1}(f) > 2^k\}. \tag{3.14}$$

We take the sequence, $\lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$, as $\lambda_k = C2^k\|\mathbf{1}_{B_{\nu^k}}\|_{p,q}$, and $(n \in \mathbb{N})$, $a = \{a^k_n\}_{k\in\mathbb{Z}}$ as

$$a^k_n = \begin{cases} \frac{f_n^{\nu^{k+1}} - f_n^{\nu^k}}{\lambda_k}, & \text{if} \quad \lambda_k \neq 0 \\ 0, & \text{if} \quad \lambda_k = 0. \end{cases} \tag{3.15}$$

Then as in the proof of Theorem 3.4 above, one obtains that $s(f_n^{\nu^k}) \lesssim 2^k$ and

$$s(a^k) = \frac{1}{\lambda_k} s(f^{\nu^{k+1}} - f^{\nu^k}) \leq \frac{C2^k}{\lambda_k} \leq \|\mathbf{1}_{B_{\nu^k}}\|_{p,q}^{-1}$$

and $s(a^k) = 0$ on $\{\nu^k = \infty\}$. Also since $\operatorname{supp}(s(a^k)) \subseteq B_{\nu^k}$, It follows that

$$\|s(a^k)\|_r \leq \left(\int_{B_{\nu^k}} \|\mathbf{1}_{B_{\nu^k}}\|_{p,q}^{-r} \mathrm{d}\mathbb{P} \right)^{\frac{1}{r}} \leq \|\mathbf{1}_{B_{\nu^k}}\|_{p,q}^{-1} \mathbb{P}(B_{\nu^k})^{\frac{1}{r}}.$$

We also observe that $\mathbb{E}_n a^k = 0$ when $\nu^k \geq n$ as in the proof of Theorem 3.4. Thus conditions (a1) and (a3) are satisfied and hence a^k is $(p,q,r)^s$-atom. It then follows from the proof of Theorem 3.4 that

$$\sum_{k=l}^{m} \lambda_k a^k \to f$$

in $H^s_{p,q}$ as $m \to \infty$ and $l \to -\infty$ since

$$\begin{aligned}
\left\| f - \sum_{k=l}^{m} \lambda_k a^k \right\|_{H^s_{p,q}} &= \left\| (f - f^{\nu^{m+1}}) + f^{\nu^l} \right\|_{H^s_{p,q}} \\
&\lesssim \left\| s(f - f^{\nu^{m+1}}) + s(f^{\nu^l}) \right\|_{p,q} \\
&\leq \left\| s(f - f^{\nu^{m+1}}) \right\|_{p,q} + \left\| s(f^{\nu^l}) \right\|_{p,q} \to 0.
\end{aligned}$$

We also observe the following as in the proof of Theorem 3.4;

$$\begin{aligned}
\left\| \sum_{k\in\mathbb{Z}} \left(\frac{\lambda_k}{\|\mathbf{1}_{B_{\nu^k}}\|_{p,q}} \right)^{\eta} \mathbf{1}_{B_{\nu^k}} \right\|_{\frac{p}{\eta},\frac{q}{\eta}}^{\frac{1}{\eta}} &= \left[\sum_{j\in\mathbb{Z}} \left(\int_{\Omega} \left(\sum_{k\in\mathbb{Z}} \left(\frac{\lambda_k}{\|\mathbf{1}_{B_{\nu^k}}\|_{p,q}} \right)^{\eta} \mathbf{1}_{B_{\nu^k}} \right)^{\frac{p}{\eta}} \mathbf{1}_{\Omega_j} \mathrm{d}\mathbb{P} \right)^{\frac{q}{p}} \right]^{\frac{\eta}{q}\cdot\frac{1}{\eta}} \\
&\leq \left[\sum_{j\in\mathbb{Z}} \left(\int_{\Omega} \frac{(2C)^p s(f)^p}{(2^\eta - 1)^{\frac{p}{\eta}}} \sum_{k\in\mathbb{Z}} \mathbf{1}_{G_k} \mathbf{1}_{\Omega_j} \mathrm{d}\mathbb{P} \right)^{\frac{q}{p}} \right]^{\frac{\eta}{q}\cdot\frac{1}{\eta}} \\
&= \left[\sum_{j\in\mathbb{Z}} \left(\frac{(2C)^p}{(2^\eta - 1)^{\frac{p}{\eta}}} \int_{\Omega} s^p(f) \mathbf{1}_{\Omega_j} \mathrm{d}\mathbb{P} \right)^{\frac{q}{p}} \right]^{\frac{\eta}{q}\cdot\frac{1}{\eta}}.
\end{aligned}$$

Hence we have that

$$\begin{aligned}
\left\| \sum_{k\in\mathbb{Z}} \left(\frac{\lambda_j}{\|\mathbf{1}_{B_{\nu^k}}\|_{p,q}} \right)^{\eta} \mathbf{1}_{B_{\nu^k}} \right\|_{\frac{p}{\eta},\frac{q}{\eta}}^{\frac{1}{\eta}} &\leq \left[\sum_{j\in\mathbb{Z}} \left(\int_{\Omega} \frac{(2C)^p}{(2^\eta - 1)^{\frac{p}{\eta}}} s^p(f) \mathbf{1}_{\Omega_j} \mathrm{d}\mathbb{P} \right)^{\frac{q}{p}} \right]^{\frac{1}{q}} \\
&= \frac{2C}{(2^\eta - 1)^{\frac{1}{\eta}}} \|s(f)\|_{p,q} = \frac{2C}{(2^\eta - 1)^{\frac{1}{\eta}}} \|f\|_{H^s_{p,q}}.
\end{aligned}$$

The first part of the theorem is then established.

Conversely, let the martingale $f = \{f_n\}_{n\in\mathbb{N}}$ have a representation as in (3.12). Then as $s(a^k) \leq \|\mathbf{1}_{B_{\nu^k}}\|_{p,q}^{-1}$ with support in B_{ν^k}, we obtain that

$$\begin{aligned}
\|f\|_{H^s_{p,q}} &= \|s(f)\|_{p,q} \leq \left\| \sum_{k\in\mathbb{Z}} \lambda_k s(a^k) \right\|_{p,q} \\
&\leq \left\| \sum_{k\in\mathbb{Z}} \frac{\lambda_k}{\|\mathbf{1}_{B_{\nu^k}}\|_{p,q}} \mathbf{1}_{B_{\nu^k}} \right\|_{p,q}.
\end{aligned}$$

For $0 < \eta < 1$, we proceed similarly as in the proof of Theorem 3.4 to obtain

$$\begin{aligned}
\left\|\sum_{k\in\mathbb{Z}} \frac{\lambda_k}{\|\mathbf{1}_{B_{\nu^k}}\|_{p,q}} \mathbf{1}_{B_{\nu^k}}\right\|_{p,q} &= \left[\sum_{j\in\mathbb{Z}} \left(\int_\Omega \left(\sum_{k\in\mathbb{Z}} \frac{\lambda_k}{\|\mathbf{1}_{B_{\nu^k}}\|_{p,q}} \mathbf{1}_{B_{\nu^k}}\right)^p \mathbf{1}_{\Omega_j} \mathrm{d}\mathbb{P}\right)^{\frac{q}{p}}\right]^{\frac{1}{q}} \\
&= \left[\sum_{j\in\mathbb{Z}} \left(\int_\Omega \left(\sum_{k\in\mathbb{Z}} \frac{\lambda_k}{\|\mathbf{1}_{B_{\nu^k}}\|_{p,q}} \mathbf{1}_{B_{\nu^k}}\right)^{p\frac{\eta}{\eta}} \mathbf{1}_{\Omega_j} \mathrm{d}\mathbb{P}\right)^{\frac{q/\eta}{p/\eta}}\right]^{\frac{1}{q}\cdot\frac{\eta}{\eta}} \\
&\leq \left[\sum_{j\in\mathbb{Z}} \left(\int_\Omega \left(\sum_{k\in\mathbb{Z}} \left(\frac{\lambda_k}{\|\mathbf{1}_{B_{\nu^k}}\|_{p,q}}\right)^\eta \mathbf{1}_{B_{\nu^k}}\right)^{\frac{p}{\eta}} \mathbf{1}_{\Omega_j} \mathrm{d}\mathbb{P}\right)^{\frac{q/\eta}{p/\eta}}\right]^{\frac{\eta}{q}\cdot\frac{1}{\eta}}.
\end{aligned}$$

Then by definition we have

$$\left\|\sum_{k\in\mathbb{Z}} \frac{\lambda_k}{\|\mathbf{1}_{B_{\nu^k}}\|_{p,q}} \mathbf{1}_{B_{\nu^k}}\right\|_{p,q} \leq \left\|\sum_{k\in\mathbb{Z}} \left(\frac{\lambda_k}{\|\mathbf{1}_{B_{\nu^k}}\|_{p,q}}\right)^\eta \mathbf{1}_{B_{\nu^k}}\right\|_{\frac{p}{\eta},\frac{q}{\eta}}^{\frac{1}{\eta}}.$$

Thus

$$\left\|\sum_{k\in\mathbb{Z}} \frac{\lambda_k}{\|\mathbf{1}_{B_{\nu^k}}\|_{p,q}} \mathbf{1}_{B_{\nu^k}}\right\|_{p,q} \leq \left\|\sum_{k\in\mathbb{Z}} \left(\frac{\lambda_k}{\|\mathbf{1}_{B_{\nu^k}}\|_{p,q}}\right)^\eta \mathbf{1}_{B_{\nu^k}}\right\|_{\frac{p}{\eta},\frac{q}{\eta}}^{\frac{1}{\eta}}. \tag{3.16}$$

Hence

$$\|f\|_{H^s_{p,q}} \leq \left\|\sum_{k\in\mathbb{Z}} \left(\frac{\lambda_k}{\|\mathbf{1}_{B_{\nu^k}}\|_{p,q}}\right)^\eta \mathbf{1}_{B_{\nu^k}}\right\|_{\frac{p}{\eta},\frac{q}{\eta}}^{\frac{1}{\eta}}$$

establishing the converse.

For the case $q = \infty$, we have, by definition, that

$$\|f - f^{\nu^{m+1}}\|_{H^s_{p,\infty}} = \|s(f - f^{\nu^{m+1}})\|_{p,\infty} := \sup_{j\in\mathbb{Z}} \|s(f - f^{\nu^{m+1}})\mathbf{1}_{\Omega_j}\|_p.$$

Hence, as above, $\sup_{j\in\mathbb{Z}} \|s(f - f^{\nu^{m+1}})\mathbf{1}_{\Omega_j}\|_p \to 0$ as $m \to \infty$ since $\|s(f - f^{\nu^{m+1}})\|_p \to 0$ as $m \to \infty$. We also observe that, since $s(f^{\nu^k}) \lesssim 2^k$, it is true that $\|s(f^{\nu^l})\|_{p,\infty} \lesssim 2^l \to 0$ as $l \to -\infty$. Hence $\|f^{\nu^l}\|_{H^s_{p,\infty}} \to 0$ as $l \to -\infty$. Therefore

$$\left\|f - \sum_{k=l}^{m} \lambda_k a^k\right\|_{H^s_{p,\infty}} \leq \|(f - f^{\nu^{m+1}})\|_{H^s_{p,\infty}} + \|f^{\nu^l}\|_{H^s_{p,\infty}} \to 0$$

Hence

$$\sum_{k=l}^{m} \lambda_k a^k \longrightarrow f$$

in $H^s_{p,\infty}$ as $m \to \infty,\ l \to -\infty$. It follows by definition, as we did above, that

$$\begin{aligned}
\left\|\sum_{k\in\mathbb{Z}}\left(\frac{\lambda_k}{\|\mathbf{1}_{B_{\nu^k}}\|_{p,q}}\right)^{\eta}\mathbf{1}_{B_{\nu^k}}\right\|_{\frac{p}{\eta},\infty} &= \sup_{j\in\mathbb{Z}}\left(\int_{\Omega}\left(\sum_{k\in\mathbb{Z}}\left(\frac{\lambda_k}{\|\mathbf{1}_{B_{\nu^k}}\|_{p,q}}\right)^{\eta}\mathbf{1}_{B_{\nu^k}}\right)^{\frac{p}{\eta}}\mathbf{1}_{\Omega_j}\mathrm{d}\mathbb{P}\right)^{\frac{\eta}{p}}\\
&\leq \sup_{j\in\mathbb{Z}}\left(\int_{\Omega}\frac{(2C)^p s(f)^p}{(2^{\eta}-1)^{\frac{p}{\eta}}}\sum_{k\in\mathbb{Z}}\mathbf{1}_{G_k}\mathbf{1}_{\Omega_j}\mathrm{d}\mathbb{P}\right)^{\frac{\eta}{p}}\\
&= \sup_{j\in\mathbb{Z}}\left(\frac{(2C)^p}{(2^{\eta}-1)^{\frac{p}{\eta}}}\sum_{k\in\mathbb{Z}}\int_{G_k}s^p(f)\mathbf{1}_{\Omega_j}\mathrm{d}\mathbb{P}\right)^{\frac{\eta}{p}}
\end{aligned}$$

and therefore

$$\begin{aligned}
\left\|\sum_{k\in\mathbb{Z}}\left(\frac{\lambda_k}{\|\mathbf{1}_{B_{\nu^k}}\|_{p,q}}\right)^{\eta}\mathbf{1}_{B_{\nu^k}}\right\|_{\frac{p}{\eta},\infty} &\leq \left(\frac{(2C)^{\eta}}{2^{\eta}-1}\right)\sup_{j\in\mathbb{Z}}\left(\int_{\Omega}s^p(f)\mathbf{1}_{\Omega_j}\mathrm{d}\mathbb{P}\right)^{\frac{\eta}{p}}\\
&= \left(\frac{(2C)^{\eta}}{2^{\eta}-1}\right)\|s(f)\|^{\eta}_{p,\infty}.
\end{aligned}$$

Hence

$$\left\|\sum_{k\in\mathbb{Z}}\left(\frac{\lambda_j}{\|\mathbf{1}_{B_{\nu^k}}\|_{p,q}}\right)^{\eta}\mathbf{1}_{B_{\nu^k}}\right\|^{\frac{1}{\eta}}_{\frac{p}{\eta},\infty} \leq \left(\frac{(2C)^{\eta}}{2^{\eta}-1}\right)^{\frac{1}{\eta}}\|f\|_{H^s_{p,\infty}}.$$

Conversely, as in the above, let the martingale f have a representation as in (3.12). Then as $s(a^k) \leq \|\mathbf{1}_{B_{\nu^k}}\|^{-1}_{p,q}$ with support in $B_{\nu^k} = \{\nu \neq \infty\}$, we obtain that

$$\begin{aligned}
\|f\|_{H^s_{p,\infty}} &= \|s(f)\|_{p,\infty} \leq \left\|\sum_{k\in\mathbb{Z}}\lambda_k s(a^k)\right\|_{p,\infty}\\
&\leq \left\|\sum_{k\in\mathbb{Z}}\frac{\lambda_k}{\|\mathbf{1}_{B_{\nu^k}}\|_{p,q}}\mathbf{1}_{B_{\nu^k}}\right\|_{p,\infty}.
\end{aligned}$$

We now check that

$$\left\|\sum_{k\in\mathbb{Z}}\frac{\lambda_k}{\|\mathbf{1}_{B_{\nu^k}}\|_{p,q}}\mathbf{1}_{B_{\nu^k}}\right\|_{p,\infty} \leq \left\|\sum_{k\in\mathbb{Z}}\left(\frac{\lambda_k}{\|\mathbf{1}_{B_{\nu^k}}\|_{p,q}}\right)^{\eta}\mathbf{1}_{B_{\nu^k}}\right\|^{\frac{1}{\eta}}_{\frac{p}{\eta},\infty}.$$

For $0 < \eta < 1$, we get

$$\begin{aligned}
\left\|\sum_{k\in\mathbb{Z}} \frac{\lambda_k}{\|\mathbf{1}_{B_{\nu^k}}\|_{p,q}}\mathbf{1}_{B_{\nu^k}}\right\|_{p,\infty} &= \sup_{j\in\mathbb{Z}}\left(\int_\Omega \left(\sum_{k\in\mathbb{Z}} \frac{\lambda_k}{\|\mathbf{1}_{B_{\nu^k}}\|_{p,q}}\mathbf{1}_{B_{\nu^k}}\right)^p \mathbf{1}_{\Omega_j}\mathrm{d}\mathbb{P}\right)^{\frac{1}{p}} \\
&= \sup_{j\in\mathbb{Z}}\left(\int_\Omega \left(\sum_{k\in\mathbb{Z}} \frac{\lambda_k}{\|\mathbf{1}_{B_{\nu^k}}\|_{p,q}}\mathbf{1}_{B_{\nu^k}}\right)^{p\frac{\eta}{\eta}} \mathbf{1}_{\Omega_j}\mathrm{d}\mathbb{P}\right)^{\frac{1}{p}\cdot\frac{\eta}{\eta}} \\
&\leq \sup_{j\in\mathbb{Z}}\left(\int_\Omega \left(\sum_{k\in\mathbb{Z}} \left(\frac{\lambda_k}{\|\mathbf{1}_{B_{\nu^k}}\|_{p,q}}\right)^{\eta}\mathbf{1}_{B_{\nu^k}}\right)^{\frac{p}{\eta}} \mathbf{1}_{\Omega_j}\mathrm{d}\mathbb{P}\right)^{\frac{1}{p}\cdot\frac{\eta}{\eta}} \\
&= \left\|\sum_{k\in\mathbb{Z}} \left(\frac{\lambda_k}{\|\mathbf{1}_{B_{\nu^k}}\|_{p,q}}\right)^{\eta}\mathbf{1}_{B_{\nu^k}}\right\|_{\frac{p}{\eta},\infty}^{\frac{1}{\eta}}.
\end{aligned}$$

Hence

$$\|f\|_{H^s_{p,\infty}} \leq \left\|\sum_{k\in\mathbb{Z}} \left(\frac{\lambda_k}{\|\mathbf{1}_{B_{\nu^k}}\|_{p,q}}\right)^{\eta}\mathbf{1}_{B_{\nu^k}}\right\|_{\frac{p}{\eta},\infty}^{\frac{1}{\eta}}$$

establishing the converse. The proof is complete. □

3.6 Atomic Decompositions of $\mathcal{Q}_{p,q}$

Recall that $\mathcal{Q}_{p,q}(\Omega)$ is the space of all $f \in \mathcal{M}$ such that $S_n(f) \leq \beta_{n-1}$ and $\|\beta_\infty\|_{L_{p,q}(\Omega)} < \infty$ with (quasi)-norm

$$\|f\|_{\mathcal{Q}_{p,q}(\Omega)} := \inf_{\beta\in\Gamma} \|\beta_\infty\|_{L_{p,q}(\Omega)}$$

where Γ be the space of all sequences $\beta = (\beta_n)_{n\geq 0}$ of adapted (that is for all $n \in \mathbb{Z}$, β_n is $\mathcal{F}_n$-measurable), non-decreasing, non-negative functions and

$$\beta_\infty := \lim_{n\to\infty} \beta_n.$$

Let us discuss the atomic decomposition of this martingale Hardy-amalgam space as stated in the Theorem below. Theorem 3.7 describes the atomic decomposition of $\mathcal{Q}_{p,q}$ based on Definition 3.1.

Theorem 3.7. *Let $0 < p < \infty$ and $0 < q \leq \infty$. If the martingale $f \in \mathcal{M}$ is in $\mathcal{Q}_{p,q}$, then there exists a sequence of triplets $(\lambda_k, a^k, \nu^k) \in \mathcal{A}(p, q, \infty)^S$ such that for any $n \in \mathbb{N}$,*

$$\sum_{k\in\mathbb{Z}} \lambda_k \mathbb{E}_n a^k = f_n \tag{3.17}$$

and for any $0 < \eta \leq 1$,

$$\left\| \sum_{k \in \mathbb{Z}} \left(\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} \right)^{\eta} \mathbf{1}_{B_{\nu^k}} \right\|_{\frac{p}{\eta},\frac{q}{\eta}}^{\frac{1}{\eta}} \leq C\|f\|_{\mathcal{Q}_{p,q}}. \tag{3.18}$$

Moreover,

$$\sum_{k=l}^{m} \lambda_k a^k \longrightarrow f$$

in $\mathcal{Q}_{p,q}$ *as* $m \to \infty,\ l \to -\infty$.

Conversely, if $f \in \mathcal{M}$ *has a decomposition as in (3.17), then for any* $0 < \eta \leq 1$,

$$\|f\|_{\mathcal{Q}_{p,q}} \leq C \left\| \sum_{k \in \mathbb{Z}} \left(\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} \right)^{\eta} \mathbf{1}_{B_{\nu^k}} \right\|_{\frac{p}{\eta},\frac{q}{\eta}}^{\frac{1}{\eta}}.$$

Proof. Let $0 < p < \infty$, $0 < q < \infty$ and let $f \in \mathcal{Q}_{p,q}$. Then there exists an optimal, non-decreasing, non-negative adapted sequence $(\beta_n)_{n\in\mathbb{N}}$ such that $S_n(f) \leq \beta_{n-1}$ and $\|f\|_{\mathcal{Q}_{p,q}} = \|\beta_\infty\|_{p,q}$. We take

$$\nu^k := \inf\{n \in \mathbb{N} : \beta_n > 2^k\} \tag{3.19}$$

to be a stopping times and define $\lambda_k = C2^k \mathbb{P}(\nu^k \neq \infty)^{\frac{1}{p}}$, and $(n \in \mathbb{N})$

$$a_n^k = \begin{cases} \frac{f_n^{\nu^{k+1}} - f_n^{\nu^k}}{\lambda_k}, & \text{if} \qquad \lambda_k \neq 0 \\ 0, & \qquad \text{otherwise.} \end{cases} \tag{3.20}$$

Now on the set $\{n \leq \nu^k\}$ we see that

$$a_n^k = \frac{f_n^{\nu^{k+1}} - f_n^{\nu^k}}{\lambda_k} = \frac{f_n - f_n}{\lambda_k} = 0$$

by definition of stopped martingales and thus $\mathbb{E}_n a^k = 0$ when $\nu^k \geq n$. Hence a^k satisfies condition (a1) in the definition of $(p,\infty)^S$-atom. Also, $a_n^k = 0$ on $\{\nu^k = \infty\}$ for all $n \geq 0$, and thus the support of $S(a^k)$ is contained in $B_{\nu^k} = \{\nu \neq \infty\}$. Also by (2.5)

$$S(f^{\nu^k}) = S_{\nu^k}(f).$$

Now by the definition of stopping time, $\nu^k = n \leq n+1$ and since $S_n(f)$ is increasing in n, we have that $S_{\nu^k}(f) \leq S_{n+1}(f) \leq \beta_n$ and $2^k < \beta_n$. Hence,

$$S_{\nu^k}(f) = S(f^{\nu^k}) \leq \beta_{\nu^k - 1} \lesssim 2^k.$$

Then by identity (2.9) we get the following

$$\begin{aligned} S(a^k) &= \frac{1}{\lambda_k} S(f^{\nu^{k+1}} - f^{\nu^k}) \\ &\lesssim \frac{1}{\lambda_k}[S(f^{\nu^{k+1}}) + S(f^{\nu^k})] \lesssim \frac{2^{k+1}+2^k}{\lambda_k} \leq \frac{C(2^k)}{\lambda_k} \\ &= \frac{C(2^k)}{C2^k \mathbb{P}(\nu^k \neq \infty)^{\frac{1}{p}}} = \frac{1}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} \end{aligned}$$

Therefore $S(a^k) \leq \mathbb{P}(B_{\nu^k})^{-\frac{1}{p}}$ and zero outside of B_{ν^k}. Therefore, we obtain that

$$\|S(a^k)\|_r \leq \mathbb{P}(\nu^k \neq \infty)^{\frac{1}{r}-\frac{1}{p}} \tag{3.21}$$

and in particular $\|S(a^k)\|_\infty \leq \mathbb{P}(\nu^k \neq \infty)^{-\frac{1}{p}}$. Thus condition (a2) in the definition of an $(p,\infty)^S$-atom also holds. Thus a^k is $(p,\infty)^S$-atom.

Moreover,

$$\sum_{k\in\mathbb{Z}} (f_n^{\nu^{k+1}} - f_n^{\nu^k}) = f_n \text{ a.e.}$$

We observe that since

$$S^2(a^k) = \sum_n |d_n a^k|^2 = \sum_n [(a_n^k)^2 + (a_{n-1}^k)^2 - 2a_n^k a_{n-1}^k],$$

we have by martingale property of a^k that

$$\begin{aligned} \mathbb{E}[S^2(a^k)] &= \sum_n \mathbb{E}([(a_n^k)^2 + (a_{n-1}^k)^2]) - 2\mathbb{E}[a_n^k a_{n-1}^k] \\ &= \sum_n \mathbb{E}([(a_n^k)^2 + (a_{n-1}^k)^2]) - 2\mathbb{E}(\mathbb{E}_{n-1}[a_n^k a_{n-1}^k]) \\ &= \sum_n \mathbb{E}[(a_n^k)^2 + (a_{n-1}^k)^2 - 2(a_{n-1}^k)^2] \\ &= \mathbb{E}\sum_n [(a_n^k)^2 - (a_{n-1}^k)^2] = \mathbb{E}[(a^k)^2]. \end{aligned}$$

In other words $\|S(a^k)\|_2 = \|a^k\|_2$. Thus from (3.21), we have that $\|S(a^k)\|_2 \leq \mathbb{P}(\nu^k \neq \infty)^{-\frac{1}{p}}$ and hence $\|a^k\|_2 = \|S(a^k)\|_2 \leq \mathbb{P}(\nu^k \neq \infty)^{-\frac{1}{p}}$. Which implies that $\sup_n \|a_n^k\|_2 < \mathbb{P}(\nu^k \neq \infty)^{-\frac{1}{p}} < \infty$. Hence for a fixed k, $a^k = \{a_n^k\}_{n\in N}$ is an L_2-bounded martingale. Consequently, the limit

$$\lim_{n\to\infty} a_n^k$$

exists a.e. in L_2. Hence there exists $a^k \in L_2$ such that $\mathbb{E}_n a^k$ is well defined and since a^k

is a martingale, $\mathbb{E}_n a^k = a^k_n$. Now since $\lambda_k a^k_n = f^{\nu^{k+1}}_n - f^{\nu^k}_n$, we have that

$$\sum_{k\in\mathbb{Z}} \lambda_k a^k_n = \sum_{k\in\mathbb{Z}} (f^{\nu^{k+1}}_n - f^{\nu^k}_n) = f_n \text{ a.e.}$$

and thus (3.17) is satisfied.

We shall now establish the inequalities in the Theorem. Before we do that let us prove that $\sum_{k=l}^m \lambda_k a^k$ converges to f in $\mathcal{Q}_{p,q}$ as $l \to -\infty$ and $m \to \infty$. We shall establish some useful results. First we show that if

$$\zeta^k_{n-1} = \mathbf{1}_{\{\nu^k \le n-1\}} \|S(a^k)\|_\infty \quad \text{and} \quad (\zeta_{n-1})^2 = \sum_{k=m+1}^{\infty} \lambda_k^2 (\zeta^k_{n-1})^2,$$

then

$$S^2_n(f - f^{\nu^{m+1}}) \le C \sum_{k=m+1}^{\infty} \lambda_k^2 (\zeta^k_{n-1})^2 = (\zeta_{n-1})^2$$

and therefore

$$S_n(f - f^{\nu^{m+1}}) \le \zeta_{n-1}. \tag{3.22}$$

Indeed from identity (2.7), we obtain

$$\begin{aligned} S^2_n(f - f^{\nu^{m+1}}) &= S^2_n(f) - S^2_n(f^{\nu^{m+1}}) \\ &= \sum_{k=m+1}^{\infty} S^2_n(f^{\nu^{k+1}}) - S^2_n(f^{\nu^k}) \\ &= \sum_{k=m+1}^{\infty} S^2_n(f^{\nu^{k+1}} - f^{\nu^k}) = \sum_{k=m+1}^{\infty} \lambda_k^2 S^2_n(a^k). \end{aligned}$$

Once again since $a^k = 0$ on the set $\{\nu^k \ge n\}$, it implies that $S(a^k) = 0$ on this set as well. Therefore $S_n(a^k) = \mathbf{1}_{B_{\nu^k}} S_n(a^k) = \mathbf{1}_{B_{\nu^k}} S(a^k_n)$, where $B_{\nu^k} = \{\nu^k \ne \infty\}$.
Hence it follows that

$$\begin{aligned} S^2_n(f - f^{\nu^{m+1}}) &\le \sum_{k=m+1}^{\infty} \lambda_k^2 \mathbf{1}_{B_{\nu^k}} \sup_{n\in\mathbb{N}} S^2(a^k_n) \\ &\le \sum_{k=m+1}^{\infty} \lambda_k^2 \mathbf{1}_{B_{\nu^k}} \|S(a^k)\|^2_\infty = \sum_{k=m+1}^{\infty} \lambda_k^2 (\zeta^k_{n-1})^2 = \zeta^2_{n-1}. \end{aligned}$$

Thus $S_n(f - f^{\nu^{m+1}}) \le \zeta_{n-1}$. Therefore

$$\zeta_{n-1} \le \sum_{k=m+1}^{\infty} \frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} \mathbf{1}_{B_{\nu^k}}. \tag{3.23}$$

It follows that

$$S(f - f^{\nu^{m+1}}) \leq \lim_{n\to\infty} \zeta_n \leq \sum_{k=m+1}^{\infty} \frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} \mathbf{1}_{B_{\nu^k}} = \sum_{k=m+1}^{\infty} C2^k \mathbf{1}_{B_{\nu^k}}$$

Hence

$$\|f - f^{\nu^{m+1}}\|_{\mathcal{Q}_{p,q}}^q \leq \|\zeta_\infty\|_{p,q}^q \leq \sum_{j\in\mathbb{Z}} \left\| \left(\sum_{k=m+1}^{\infty} C2^k \mathbf{1}_{B_{\nu^k}} \right) \mathbf{1}_{\Omega_j} \right\|_p^q .$$

Now as in the proof of Theorem 3.4, we obtain that

$$\left(\sum_{k=m+1}^{\infty} C2^k \mathbf{1}_{B_{\nu^k}} \right) \mathbf{1}_{\Omega_j} \leq C\beta_\infty \mathbf{1}_{\Omega_j}.$$

Hence as $\|\beta_\infty \mathbf{1}_{\Omega_j}\|_p < \infty$, it follows from the the Dominated Convergence Theorem that

$$\left\| \left(\sum_{k=m+1}^{\infty} C2^k \mathbf{1}_{B_{\nu^k}} \right) \mathbf{1}_{\Omega_j} \right\|_p \to 0 \text{ as } m \to \infty.$$

As

$$\left\| \left(\sum_{k=m+1}^{\infty} C2^k \mathbf{1}_{B_{\nu^k}} \right) \mathbf{1}_{\Omega_j} \right\|_p \leq C\|\beta_\infty \mathbf{1}_{\Omega_j}\|_p$$

and as

$$\sum_{j\in\mathbb{Z}} \|\beta_\infty \mathbf{1}_{\Omega_j}\|_p^q = \|\beta_\infty\|_{p,q}^q < \infty,$$

an application of the Dominated Convergence Theorem for sequence spaces leads to

$$\sum_{j\in\mathbb{Z}} \left\| \left(\sum_{k=m+1}^{\infty} C2^k \mathbf{1}_{B_{\nu^k}} \right) \mathbf{1}_{\Omega_j} \right\|_p^q \to 0 \text{ as } m \to \infty.$$

Thus $\|f - f^{\nu^{m+1}}\|_{\mathcal{Q}_{p,q}} \to 0$ as $m \to \infty$.

Also $f \in \mathcal{Q}_{p,q}$ implies that $S(f^{\nu^l}) < \zeta_{n-1} < 2^l$. Therefore

$$\|f^{\nu^l}\|_{\mathcal{Q}_{p,q}} \leq \|2^l\|_{p,q} \longrightarrow 0$$

as $l \to -\infty$. Therefore

$$\left\| f - \sum_{k=l}^{m} \lambda_k a^k \right\|_{\mathcal{Q}_{p,q}} = \left\| (f - f^{\nu^{m+1}}) + f^{\nu^l} \right\|_{\mathcal{Q}_{p,q}} \leq \|(f - f^{\nu^{m+1}})\|_{\mathcal{Q}_{p,q}} + \|f^{\nu^l}\|_{\mathcal{Q}_{p,q}} \to 0$$

as $m \to \infty$ and $l \to -\infty$. Hence

$$\sum_{k=l}^{m} \lambda_k a^k \longrightarrow f$$

in $\mathcal{Q}_{p,q}$ as $m \to \infty,\ l \to -\infty$ and thus for all $n \in \mathbb{N}$,

$$\sum_{k\in\mathbb{Z}} \lambda_k \mathbb{E}_n a^k = f_n.$$

Now, just as we did in the prove of Theorem 3.4, we can proceed similarly to obtain that

$$\begin{aligned}
\left\| \sum_{k\in\mathbb{Z}} \left(\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} \right)^{\eta} \mathbf{1}_{B_{\nu^k}} \right\|_{\frac{p}{\eta},\frac{q}{\eta}}^{\frac{1}{\eta}} &= \left\| \sum_{k\in\mathbb{Z}} \left(\frac{C2^k \mathbb{P}(\nu^k \neq \infty)^{\frac{1}{p}}}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} \right)^{\eta} \mathbf{1}_{B_{\nu^k}} \right\|_{\frac{p}{\eta},\frac{q}{\eta}}^{\frac{1}{\eta}} \\
&= \left\| \sum_{k\in\mathbb{Z}} (C2^k)^{\eta} \mathbf{1}_{B_{\nu^k}} \right\|_{\frac{p}{\eta},\frac{q}{\eta}}^{\frac{1}{\eta}} \\
&\leq \left\| \frac{C^p}{(2^{\eta}-1)^p} \beta_\infty \right\|_{\frac{p}{\eta},\frac{q}{\eta}}^{\frac{1}{\eta}} \leq C \left\| \beta_\infty \right\|_{p,q} = C\|f\|_{\mathcal{Q}_{p,q}}.
\end{aligned}$$

Conversely let f, has representation as (3.17). Then we want to show that $f \in \mathcal{Q}_{p,q}$. By quasi-linearity of $S(\cdot)$, and $S(f_n) = S(f^n) = S_n(f)$ we obtain that

$$\begin{aligned}
S_n(f) &= S\left(\sum_{k\in\mathbb{Z}} \lambda_k \mathbb{E}_n a^k \right) \\
&\lesssim \sum_{k\in\mathbb{Z}} \lambda_k S(\mathbb{E}_n a^k) = \sum_{k\in\mathbb{Z}} \lambda_k S(a_n^k)
\end{aligned}$$

which implies that

$$S_n(f) \leq \sum_{k\in\mathbb{Z}} |\lambda_k| \sup_{n\in\mathbb{N}} S(a_n^k) \leq \sum_{k\in\mathbb{Z}} |\lambda_k| \|S(a^k)\|_\infty \mathbf{1}_{\{\nu^k < n-1\}}.$$

Hence we can choose

$$\beta_{n-1} = \sum_{k\in\mathbb{Z}} \lambda_k \|S(a^k)\|_\infty \mathbf{1}_{\{\nu^k < n-1\}}.$$

Since $\|S(a^k)\|_\infty \leq \mathbb{P}(\nu^k \neq \infty)^{-\frac{1}{p}}$ we have that

$$\beta_\infty \leq \sum_{k\in\mathbb{Z}} \frac{\lambda_k}{\mathbb{P}(\nu^k \neq \infty)^{\frac{1}{p}}} \mathbf{1}_{B_{\nu^k}}$$

where $B_{\nu^k} = \{\nu^k \neq \infty\}$. This implies that

$$\|\beta_\infty\|_{p,q} \lesssim \left\| \sum_{k\in\mathbb{Z}} \frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} \mathbf{1}_{B_{\nu^k}} \right\|_{p,q} \leq C \left\| \sum_{k\in\mathbb{Z}} \left(\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} \right)^\eta \mathbf{1}_{B_{\nu^k}} \right\|_{\frac{p}{\eta},\frac{q}{\eta}}^{\frac{1}{\eta}}$$

where the last inequality follows directly as we did in the proof of Theorem 3.4. Hence we obtain from definition that

$$\|f\|_{\mathcal{Q}_{p,q}} \leq \|\beta_\infty\|_{p,q} \leq C \left\| \sum_{k\in\mathbb{Z}} \left(\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} \right)^\eta \mathbf{1}_{B_{\nu^k}} \right\|_{\frac{p}{\eta},\frac{q}{\eta}}^{\frac{1}{\eta}}.$$

Let us now consider the case $q = \infty$. Proceeding as in the proof of Theorem 3.4,

$$\|f - f^{\nu^{m+1}}\|_{\mathcal{Q}_{p,\infty}} \leq \|\zeta_\infty\|_{p,\infty} \leq \sup_{j\in\mathbb{Z}} \left\| \left(\sum_{k=m+1}^{\infty} C2^k \mathbf{1}_{B_{\nu^k}} \right) \mathbf{1}_{\Omega_j} \right\|_p$$

and we also obtain, as in the proof of Theorem 3.4, that

$$\left(\sum_{k=m+1}^{\infty} C2^k \mathbf{1}_{B_{\nu^k}} \right) \mathbf{1}_{\Omega_j} \leq C\beta_\infty \mathbf{1}_{\Omega_j}.$$

Hence as $\|\beta_\infty \mathbf{1}_{\Omega_j}\|_p < \infty$, it follows from the the Dominated Convergence Theorem that

$$\left\| \left(\sum_{k=m+1}^{\infty} C2^k \mathbf{1}_{B_{\nu^k}} \right) \mathbf{1}_{\Omega_j} \right\|_p \to 0 \text{ as } m \to \infty.$$

As

$$\left\| \left(\sum_{k=m+1}^{\infty} C2^k \mathbf{1}_{B_{\nu^k}} \right) \mathbf{1}_{\Omega_j} \right\|_p \leq C\|\beta_\infty \mathbf{1}_{\Omega_j}\|_p$$

and as

$$\sup_{j\in\mathbb{Z}} \|\beta_\infty \mathbf{1}_{\Omega_j}\|_p = \|\beta_\infty\|_{p,\infty} < \infty,$$

an application of the Dominated Convergence Theorem leads to

$$\sup_{j\in\mathbb{Z}} \left\| \left(\sum_{k=m+1}^{\infty} C2^k \mathbf{1}_{B_{\nu^k}} \right) \mathbf{1}_{\Omega_j} \right\|_p \to 0 \text{ as } m \to \infty.$$

Thus $\|f - f^{\nu^{m+1}}\|_{\mathcal{Q}_{p,\infty}} \to 0$ as $m \to \infty$. Similarly, we obtain that $\|f^{\nu^l}\|_{\mathcal{Q}_{p,\infty}} \to 0$ as

$l \longrightarrow -\infty$. Therefore

$$\left\| f - \sum_{k=l}^{m} \lambda_k a^k \right\|_{\mathcal{Q}_{p,\infty}} \to 0$$

as $m \to \infty$ and $l \to -\infty$. Hence

$$\sum_{k=l}^{m} \lambda_k a^k \longrightarrow f$$

in $\mathcal{Q}_{p,\infty}$ as $m \to \infty$, $l \to -\infty$ and thus for all $n \in \mathbb{N}$,

$$\sum_{k\in\mathbb{Z}} \lambda_k \mathbb{E}_n a^k = f_n.$$

We also obtain, as in Theorem 3.4, that

$$\begin{aligned} \left\| \sum_{k\in\mathbb{Z}} \left(\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} \right)^{\eta} \mathbf{1}_{B_{\nu^k}} \right\|_{\frac{p}{\eta},\infty}^{\frac{1}{\eta}} &\leq \left\| \sum_{k\in\mathbb{Z}} (C2^k)^{\eta} \mathbf{1}_{B_{\nu^k}} \right\|_{\frac{p}{\eta},\infty}^{\frac{1}{\eta}} \\ &\leq C \left\| \beta_\infty \right\|_{p,\infty} \\ &\leq 2C \| f \|_{\mathcal{Q}_{p,\infty}}. \end{aligned}$$

Conversely, assume that $f \in \mathcal{M}$ has the decomposition (3.17). Define β_n by

$$\beta_n := \sum_{k\in\mathbb{Z}} \lambda_k \| S(a^k) \|_\infty \mathbf{1}_{\{\nu^k \leq n\}}.$$

Then $(\beta_n)_{n\geq}$ is a nondecreasing nonnegative adapted sequence also, for $n \geq 0$,

$$S_n(f) \lesssim \beta_{n-1}$$

as we saw above. As $\|S(a^k)\|_\infty \leq \mathbb{P}(\nu^k \neq \infty)^{-\frac{1}{p}}$, it follows that

$$\|\beta_\infty\|_{p,\infty} \leq \left\| \sum_{k\in\mathbb{Z}} \frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} \mathbf{1}_{B_{\nu^k}} \right\|_{p,\infty} \leq \left\| \sum_{k\in\mathbb{Z}} \left(\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} \right)^{\eta} \mathbf{1}_{B_{\nu^k}} \right\|_{\frac{p}{\eta},\infty}^{\frac{1}{\eta}}.$$

Then

$$\|f\|_{\mathcal{Q}_{p,\infty}} \leq \|\beta_\infty\|_{p,\infty} \leq \left\| \sum_{k\in\mathbb{Z}} \left(\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} \right)^{\eta} \mathbf{1}_{B_{\nu^k}} \right\|_{\frac{p}{\eta},\infty}^{\frac{1}{\eta}}$$

and the proof is complete. □

The second atomic decomposition of $\mathcal{Q}_{p,q}$ based on Definition 3.2 of atom is as follows.

Theorem 3.8. *Let* $0 < p < \infty$ *and* $0 < q \leq \infty$. *If the martingale* $f \in \mathcal{M}$ *is in* $\mathcal{Q}_{p,q}$, *then there exists a sequence of triplets* $(\lambda_k, a^k, \nu^k) \in \mathcal{B}(p,q,\infty)^S$ *such that for any* $n \in \mathbb{N}$,

$$\sum_{k\in\mathbb{Z}} \lambda_k \mathbb{E}_n a^k = f_n \tag{3.24}$$

and for any $0 < \eta \le 1$,

$$\left\| \sum_{k\in\mathbb{Z}} \left(\frac{\lambda_k}{\|\mathbf{1}_{B_{\nu^k}}\|_{p,q}} \right)^{\eta} \mathbf{1}_{B_{\nu^k}} \right\|_{\frac{p}{\eta},\frac{q}{\eta}}^{\frac{1}{\eta}} \le C\|f\|_{\mathcal{Q}_{p,q}}. \tag{3.25}$$

Moreover,

$$\sum_{k=l}^{m} \lambda_k a^k \longrightarrow f$$

in $\mathcal{Q}_{p,q}$ *as* $m \to \infty,\ l \to -\infty$.

Conversely, if $f \in \mathcal{M}$ *has a decomposition as in (3.24), then for any* $0 < \eta \le 1$,

$$\|f\|_{\mathcal{Q}_{p,q}} \le C \left\| \sum_{k\in\mathbb{Z}} \left(\frac{\lambda_k}{\|\mathbf{1}_{B_{\nu^k}}\|_{p,q}} \right)^{\eta} \mathbf{1}_{B_{\nu^k}} \right\|_{\frac{p}{\eta},\frac{q}{\eta}}^{\frac{1}{\eta}}.$$

Proof. Let $0 < p < \infty$, $0 < q < \infty$ and let $f \in \mathcal{Q}_{p,q}$. Then there exists an optimal, adapted non-decreasing, non-negative adapted sequence $(\beta_n)_{n\in\mathbb{N}}$ such that $S_n(f) \le \beta_{n-1}$ and $\|f\|_{\mathcal{Q}_{p,q}} = \|\beta_\infty\|_{p,q}$. We take

$$\nu^k := \inf\{n \in \mathbb{N} : \beta_n > 2^k\} \tag{3.26}$$

to be a stopping time and define $\lambda_k = C2^k\|\mathbf{1}_{B_{\nu^k}}\|_{p,q}$, and $(n \in \mathbb{N})$

$$a_n^k = \frac{f_n^{\nu^{k+1}} - f_n^{\nu^k}}{\lambda_k} \text{ if } \lambda_k \ne 0, \text{ and } a^k = 0 \text{ otherwise.} \tag{3.27}$$

Now on the set $\{n \le \nu^k\}$ we see that

$$a_n^k = \frac{f_n^{\nu^{k+1}} - f_n^{\nu^k}}{\lambda_k} = \frac{f_n - f_n}{\lambda_k} = 0$$

by definition of stopped martingales and thus $\mathbb{E}_n a^k = 0$ when $\nu^k > n$. Hence a^k satisfies condition (a1). Also, $a_n^k = 0$ on $\{\nu^k = \infty\}$ for all $n \ge 0$, and the support of $S(a^k)$ is contained in $B_{\nu^k} = \{\nu \ne \infty\}$. Hence as in the proof of Theorem 3.7,

$$S_{\nu^k}(f) = S(f^{\nu^k}) \le \beta_{\nu^k - 1} \lesssim 2^k$$

and by identity (2.9) we obtain

$$S(a^k) \;=\; \frac{1}{\lambda_k} S(f^{\nu^{k+1}} - f^{\nu^k}) \leq \frac{C(2^k)}{\lambda_k} = \|\mathbf{1}_{B_{\nu^k}}\|_{p,q}^{-1}. \tag{3.28}$$

Therefore $S(a^k) \leq \|\mathbf{1}_{B_{\nu^k}}\|_{p,q}^{-1}$ and zero outside of B_{ν^k} and thus, we obtain that

$$\|S(a^k)\|_\infty \leq \|\mathbf{1}_{B_{\nu^k}}\|_{p,q}^{-1}.$$

Thus condition (a3) also holds for $r = \infty$ and thus a^k is $(p,q,\infty)^S$-atom. Moreover,

$$\sum_{k\in\mathbb{Z}} (f_n^{\nu^{k+1}} - f_n^{\nu^k}) = f_n \;\text{ a.e.}$$

From (3.28), we have that $\|S(a^k)\|_2 \leq \mathbb{P}(B_{\nu^k})^{\frac{1}{2}} \|\mathbf{1}_{B_{\nu^k}}\|_{p,q}^{-1}$ and thus $\|a^k\|_2 = \|S(a^k)\|_2 \leq \|\mathbf{1}_{B_{\nu^k}}\|_{p,q}^{-1}$. Hence for a fixed k, $a^k = \{a_n^k\}_{n\in\mathbb{N}}$ is an L_2-bounded martingale. Consequently, the limit

$$\lim_{n\to\infty} a_n^k$$

exists a.e. in L_2. Hence there exists $a^k \in L_2$ such that $\mathbb{E}_n a^k$ is well defined and since a^k is a martingale, $\mathbb{E}_n a^k = a_n^k$. Now since $\lambda_k a_n^k = f_n^{\nu^{k+1}} - f_n^{\nu^k}$, we have that

$$\sum_{k\in\mathbb{Z}} \lambda_k a_n^k = \sum_{k\in\mathbb{Z}} (f_n^{\nu^{k+1}} - f_n^{\nu^k}) = f_n \;\text{ a.e.}$$

and thus (3.24) is satisfied.

We recall from the proof of Theorem 3.7 that if

$$\zeta_{n-1}^k = \mathbf{1}_{\{\nu^k \leq n-1\}} \|S(a^k)\|_\infty \quad \text{and} \quad (\zeta_{n-1})^2 = \sum_{k=m+1}^{\infty} \lambda_k^2 (\zeta_{n-1}^k)^2,$$

then

$$S_n^2(f - f^{\nu^{m+1}}) \leq C \sum_{k=m+1}^{\infty} \lambda_k^2 (\zeta_{n-1}^k)^2 = (\zeta_{n-1})^2$$

and therefore

$$S_n(f - f^{\nu^{m+1}}) \leq \zeta_{n-1}. \tag{3.29}$$

Hence

$$\|f - f^{\nu^{m+1}}\|_{\mathcal{Q}_{p,q}}^q \leq \|\zeta_\infty\|_{p,q}^q \leq \sum_{j\in\mathbb{Z}} \left\| \left(\sum_{k=m+1}^{\infty} C2^k \mathbf{1}_{B_{\nu^k}} \right) \mathbf{1}_{\Omega_j} \right\|_p^q.$$

Now as in the proof of Theorem 3.4, we obtain that

$$\left(\sum_{k=m+1}^{\infty} C2^k \mathbf{1}_{B_{\nu k}}\right)\mathbf{1}_{\Omega_j} \leq C\beta_\infty \mathbf{1}_{\Omega_j}.$$

Hence as $\|\beta_\infty \mathbf{1}_{\Omega_j}\|_p < \infty$, it follows from the the Dominated Convergence Theorem that

$$\left\|\left(\sum_{k=m+1}^{\infty} C2^k \mathbf{1}_{B_{\nu k}}\right)\mathbf{1}_{\Omega_j}\right\|_p \to 0 \text{ as } m \to \infty.$$

As

$$\left\|\left(\sum_{k=m+1}^{\infty} C2^k \mathbf{1}_{B_{\nu k}}\right)\mathbf{1}_{\Omega_j}\right\|_p \leq C\|\beta_\infty \mathbf{1}_{\Omega_j}\|_p$$

and as

$$\sum_{j\in\mathbb{Z}} \|\beta_\infty \mathbf{1}_{\Omega_j}\|_p^q = \|\beta_\infty\|_{p,q}^q < \infty,$$

an application of the Dominated Convergence Theorem for sequence spaces leads to

$$\sum_{j\in\mathbb{Z}}\left\|\left(\sum_{k=m+1}^{\infty} C2^k \mathbf{1}_{B_{\nu k}}\right)\mathbf{1}_{\Omega_j}\right\|_p^q \to 0 \text{ as } m \to \infty.$$

Thus $\|f - f^{\nu^{m+1}}\|_{\mathcal{Q}_{p,q}} \to 0$ as $m \to \infty$.

Also $f \in \mathcal{Q}_{p,q}$ implies that $S(f^{\nu^l}) < \zeta_{n-1} < 2^l$. Therefore

$$\|f^{\nu^l}\|_{\mathcal{Q}_{p,q}} \leq \|2^l\|_{p,q} \longrightarrow 0$$

as $l \to -\infty$. Therefore

$$\left\|f - \sum_{k=l}^{m} \lambda_k a^k\right\|_{\mathcal{Q}_{p,q}} = \left\|(f - f^{\nu^{m+1}}) + f^{\nu^l}\right\|_{\mathcal{Q}_{p,q}} \leq \|(f - f^{\nu^{m+1}})\|_{\mathcal{Q}_{p,q}} + \|f^{\nu^l}\|_{\mathcal{Q}_{p,q}} \to 0$$

as $m \to \infty$ and $l \to -\infty$. Hence

$$\sum_{k=l}^{m} \lambda_k a^k \longrightarrow f$$

in $\mathcal{Q}_{p,q}$ as $m \to \infty,\ l \to -\infty$ and thus for all $n \in \mathbb{N}$,

$$\sum_{k\in\mathbb{Z}} \lambda_k \mathbb{E}_n a^k = f_n.$$

Now, just as we did in the prove of Theorem 3.7, we proceed similarly to obtain that

$$\begin{aligned}\left\|\sum_{k\in\mathbb{Z}}\left(\frac{\lambda_k}{\|\mathbf{1}_{B_{\nu^k}}\|_{p,q}}\right)^{\eta}\mathbf{1}_{B_{\nu^k}}\right\|^{\frac{1}{\eta}}_{\frac{p}{\eta},\frac{q}{\eta}} &= \left\|\sum_{k\in\mathbb{Z}}(C2^k)^{\eta}\mathbf{1}_{B_{\nu^k}}\right\|^{\frac{1}{\eta}}_{\frac{p}{\eta},\frac{q}{\eta}}\\ &\leq \left\|\frac{C^p}{(2^{\eta}-1)^p}\beta_\infty\right\|^{\frac{1}{\eta}}_{\frac{p}{\eta},\frac{q}{\eta}} \leq C\,\|\beta_\infty\|_{p,q} = C\|f\|_{\mathcal{Q}_{p,q}}\end{aligned}$$

Conversely let f, has representation as (3.24). Proceeding as in the proof of Theorem 3.7, we observe that

$$S(f) \leq \sum_{k\in\mathbb{Z}}|\lambda_k|S(a_n^k)$$

which implies that

$$S_n(f) \leq \sum_{k\in\mathbb{Z}}|\lambda_k|\|S(a^k)\|_\infty\mathbf{1}_{\{\nu^k<n-1\}}.$$

Hence we can choose

$$\beta_{n-1} = \sum_{k\in\mathbb{Z}}\lambda_k\|S(a^k)\|_\infty\mathbf{1}_{\{\nu^k<n-1\}}.$$

Since $\|S(a^k)\|_\infty \leq \|\mathbf{1}_{B_{\nu^k}}\|^{-1}_{p,q}$ we have that

$$\beta_\infty \leq \sum_{k\in\mathbb{Z}}\frac{\lambda_k}{\|\mathbf{1}_{B_{\nu^k}}\|_{p,q}}\mathbf{1}_{B_{\nu^k}}$$

where $B_{\nu^k} = \{\nu^k \neq \infty\}$. This implies that

$$\|\beta_\infty\|_{p,q} \lesssim \left\|\sum_{k\in\mathbb{Z}}\frac{\lambda_k}{\|\mathbf{1}_{B_{\nu^k}}\|_{p,q}}\mathbf{1}_{B_{\nu^k}}\right\|_{p,q} \leq C\left\|\sum_{k\in\mathbb{Z}}\left(\frac{\lambda_k}{\|\mathbf{1}_{B_{\nu^k}}\|_{p,q}}\right)^{\eta}\mathbf{1}_{B_{\nu^k}}\right\|^{\frac{1}{\eta}}_{\frac{p}{\eta},\frac{q}{\eta}}$$

where the last inequality follows directly as we did in the proof of Theorem 3.7. And thus we see that

$$\|\beta_\infty\|_{p,q} \leq C\left\|\sum_{k\in\mathbb{Z}}\left(\frac{\lambda_k}{\|\mathbf{1}_{B_{\nu^k}}\|_{p,q}}\right)^{\eta}\mathbf{1}_{B_{\nu^k}}\right\|^{\frac{1}{\eta}}_{\frac{p}{\eta},\frac{q}{\eta}}.$$

Hence

$$\|f\|_{\mathcal{Q}_{p,q}} \leq C\left\|\sum_{k\in\mathbb{Z}}\left(\frac{\lambda_k}{\|\mathbf{1}_{B_{\nu^k}}\|_{p,q}}\right)^{\eta}\mathbf{1}_{B_{\nu^k}}\right\|^{\frac{1}{\eta}}_{\frac{p}{\eta},\frac{q}{\eta}}.$$

Let us now consider the case $q = \infty$. Proceeding as in the proof of Theorem 3.7,

$$\|f - f^{\nu^{m+1}}\|_{\mathcal{Q}_{p,\infty}} \leq \|\zeta_\infty\|_{p,\infty} \leq \sup_{j\in\mathbb{Z}} \left\| \left(\sum_{k=m+1}^{\infty} C2^k \mathbf{1}_{B_{\nu^k}} \right) \mathbf{1}_{\Omega_j} \right\|_p .$$

As in the proof of Theorem 3.7, we obtain that

$$\left(\sum_{k=m+1}^{\infty} C2^k \mathbf{1}_{B_{\nu^k}} \right) \mathbf{1}_{\Omega_j} \leq C\beta_\infty \mathbf{1}_{\Omega_j}.$$

Hence as $\|\beta_\infty \mathbf{1}_{\Omega_j}\|_p < \infty$, it follows from the the Dominated Convergence Theorem that

$$\left\| \left(\sum_{k=m+1}^{\infty} C2^k \mathbf{1}_{B_{\nu^k}} \right) \mathbf{1}_{\Omega_j} \right\|_p \to 0 \text{ as } m \to \infty.$$

As

$$\left\| \left(\sum_{k=m+1}^{\infty} C2^k \mathbf{1}_{B_{\nu^k}} \right) \mathbf{1}_{\Omega_j} \right\|_p \leq C\|\beta_\infty \mathbf{1}_{\Omega_j}\|_p$$

and as

$$\sup_{j\in\mathbb{Z}} \|\beta_\infty \mathbf{1}_{\Omega_j}\|_p = \|\beta_\infty\|_{p,\infty} < \infty,$$

an application of the Dominated Convergence Theorem leads to

$$\sup_{j\in\mathbb{Z}} \left\| \left(\sum_{k=m+1}^{\infty} C2^k \mathbf{1}_{B_{\nu^k}} \right) \mathbf{1}_{\Omega_j} \right\|_p \to 0 \text{ as } m \to \infty.$$

Thus $\|f - f^{\nu^{m+1}}\|_{\mathcal{Q}_{p,\infty}} \to 0$ as $m \to \infty$. Similarly, we obtain that $\|f^{\nu^l}\|_{\mathcal{Q}_{p,\infty}} \to 0$ as $l \longrightarrow -\infty$. Therefore

$$\|f - \sum_{k=l}^{m} \lambda_k a^k\|_{\mathcal{Q}_{p,\infty}} \to 0$$

as $m \to \infty$ and $l \to -\infty$. Hence

$$\sum_{k=l}^{m} \lambda_k a^k \longrightarrow f$$

in $\mathcal{Q}_{p,\infty}$ as $m \to \infty$, $l \to -\infty$ and thus for all $n \in \mathbb{N}$,

$$\sum_{k\in\mathbb{Z}} \lambda_k \mathbb{E}_n a^k = f_n.$$

We also obtain, as in Theorem 3.7, that

$$\begin{aligned}\left\|\sum_{k\in\mathbb{Z}}\left(\frac{\lambda_k}{\|\mathbf{1}_{B_{\nu^k}}\|_{p,q}}\right)^{\eta}\mathbf{1}_{B_{\nu^k}}\right\|_{\frac{p}{\eta},\infty}^{\frac{1}{\eta}} &\leq \left\|\sum_{k\in\mathbb{Z}}(C2^k)^{\eta}\mathbf{1}_{B_{\nu^k}}\right\|_{\frac{p}{\eta},\infty}^{\frac{1}{\eta}}\\ &\leq C\,\|\beta_\infty\|_{p,\infty}\\ &\leq 2C\|f\|_{\mathcal{Q}_{p,\infty}}.\end{aligned}$$

Conversely, assume that $f\in\mathcal{M}$ has the decomposition (3.24). Define β_n by

$$\beta_n := \sum_{k\in\mathbb{Z}}\lambda_k\|S(a^k)\|_\infty\mathbf{1}_{\{\nu^k\leq n\}}.$$

Then $(\beta_n)_{n\geq}$ is a nondecreasing nonnegative adapted sequence also, for $n\geq 0$,

$$S_n(f)\leq\beta_{n-1}.$$

As $\|S(a^k)\|_\infty\leq\|\mathbf{1}_{B_{\nu^k}}\|_{p,q}^{-1}$, it follows that

$$\|\beta_\infty\|_{p,\infty}\leq\left\|\sum_{k\in\mathbb{Z}}\frac{\lambda_k}{\|\mathbf{1}_{B_{\nu^k}}\|_{p,q}}\mathbf{1}_{B_{\nu^k}}\right\|_{p,\infty}\leq\left\|\sum_{k\in\mathbb{Z}}\left(\frac{\lambda_k}{\|\mathbf{1}_{B_{\nu^k}}\|_{p,q}}\right)^{\eta}\mathbf{1}_{B_{\nu^k}}\right\|_{\frac{p}{\eta},\infty}^{\frac{1}{\eta}}.$$

Then

$$\|f\|_{\mathcal{Q}_{p,\infty}}\leq\|\beta_\infty\|_{p,\infty}\leq\left\|\sum_{k\in\mathbb{Z}}\left(\frac{\lambda_k}{\|\mathbf{1}_{B_{\nu^k}}\|_{p,q}}\right)^{\eta}\mathbf{1}_{B_{\nu^k}}\right\|_{\frac{p}{\eta},\infty}^{\frac{1}{\eta}}$$

and the proof is complete. □

3.9 Atomic Decomposition of $\mathcal{P}_{p,q}$

Finally we discuss the atomic decomposition of $\mathcal{P}_{p,q}$. The first atomic decomposition of $\mathcal{P}_{p,q}$, as stated in the Theorem 3.10 below, is in relation to Definition 3.1. We recall that $\mathcal{P}_{p,q}(\Omega)$ is the space of all $f\in\mathcal{M}$ such that $|f_n|\leq\beta_{n-1}$ and $\|\beta_\infty\|_{L_{p,q}(\Omega)}<\infty$ with (quasi)-norm

$$\|f\|_{\mathcal{P}_{p,q}(\Omega)} := \inf_{\beta\in\Gamma}\|\beta_\infty\|_{L_{p,q}(\Omega)}$$

Γ be the space of all sequences $\beta=(\beta_n)_{n\geq 0}$ of adapted (that is for all $n\in\mathbb{Z}$, β_n is $\mathcal{F}_n$-measurable), non-decreasing, non-negative functions and

$$\beta_\infty := \lim_{n\to\infty}\beta_n.$$

Theorem 3.10. *Let $0 < p < \infty$ and $0 < q \leq \infty$. If the martingale $f \in \mathcal{M}$ is in $\mathcal{P}_{p,q}$, then there exists a sequence of triplets $(\lambda_k, a^k, \nu^k) \in \mathcal{A}(p, q, \infty)^*$ such that for any $n \in \mathbb{N}$,*

$$\sum_{k\in\mathbb{Z}} \lambda_k \mathbb{E}_n a^k = f_n \tag{3.30}$$

and for any $0 < \eta \leq 1$,

$$\left\| \sum_{k\in\mathbb{Z}} \left(\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} \right)^{\eta} \mathbf{1}_{B_{\nu^k}} \right\|_{\frac{p}{\eta},\frac{q}{\eta}}^{\frac{1}{\eta}} \leq C\|f\|_{\mathcal{P}_{p,q}}. \tag{3.31}$$

Moreover,

$$\sum_{k=l}^{m} \lambda_k a^k \longrightarrow f$$

in $\mathcal{P}_{p,q}$ as $m \to \infty$, $l \to -\infty$.

Conversely, if $f \in \mathcal{M}$ has a decomposition as in (3.30), then for any $0 < \eta \leq 1$,

$$\|f\|_{\mathcal{P}_{p,q}} \leq C \left\| \sum_{k\in\mathbb{Z}} \left(\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} \right)^{\eta} \mathbf{1}_{B_{\nu^k}} \right\|_{\frac{p}{\eta},\frac{q}{\eta}}^{\frac{1}{\eta}}.$$

Proof. Let $0 < p < \infty$, $0 < q < \infty$ and let $f \in \mathcal{P}_{p,q}$. Then there exists an optimal, adapted non-decreasing, non-negative adapted sequence $(\beta_n)_{n\in\mathbb{N}}$ such that $|f_n| \leq \beta_{n-1}$ and $\|f\|_{\mathcal{P}_{p,q}} = \|\beta_\infty\|_{p,q}$. We take

$$\nu^k := \inf\{n \in \mathbb{N} : \beta_n > 2^k\} \tag{3.32}$$

to be a stopping times and define $\lambda_k = C2^k\mathbb{P}(\nu^k \neq \infty)^{\frac{1}{p}}$, and $(n \in \mathbb{N})$

$$a_n^k = \frac{f_n^{\nu^{k+1}} - f_n^{\nu^k}}{\lambda_k} \text{ if } \lambda_k \neq 0, \text{ and } a^k = 0 \text{ otherwise.} \tag{3.33}$$

Now on the set $\{n \leq \nu^k\}$ we see that

$$a_n^k = \frac{f_n^{\nu^{k+1}} - f_n^{\nu^k}}{\lambda_k} = \frac{f_n - f_n}{\lambda_k} = 0$$

by definition of stopped martingales and thus $\mathbb{E}_n a^k = 0$ when $\nu^k > n$. Hence a^k satisfies condition (a1) in the definition of $(p, \infty)^*$-atom. Also, $a_n^k = 0$ on $\{\nu^k = \infty\}$ for all $n \geq 0$, and the support of $(a^k)^*$ is contained in $B_{\nu^k} = \{\nu \neq \infty\}$. Recall that $(a^k)^* = \sup_{n\in\mathbb{N}} |a_n^k|$. Now from definition of stopping time, $\nu^k = n$ and also $2^k < \beta_n$. Also $|f_n| = |f_{\nu^k}| \leq \beta_{n-1} = \beta_{\nu^k-1} \leq \beta_n$ and thus $|f^{\nu^k}| \lesssim 2^k$. Therefore

$$|a^k| = \left|\frac{f^{\nu^{k+1}} - f^{\nu^k}}{\lambda_k}\right| \leq \left|\frac{|f^{\nu^{k+1}}| + |f^{\nu^k}|}{\lambda_k}\right| \lesssim \left|\frac{2^{k+1} + 2^k}{\lambda_k}\right| \leq \frac{C(2^k)}{C2^k\mathbb{P}(\nu^k \neq \infty)^{\frac{1}{p}}} = \mathbb{P}(B_{\nu^k})^{-\frac{1}{p}}.$$

Thus $(a^k)^* = \sup_{n\in\mathbb{N}} |a_n^k| \leq \mathbb{P}(B_{\nu^k})^{-\frac{1}{p}}$. This implies that

$$\|(a^k)^*\|_r \leq \mathbb{P}(\nu^k \neq \infty)^{\frac{1}{r}-\frac{1}{p}} \tag{3.34}$$

and in particular $\|(a^k)^*\|_\infty \leq \mathbb{P}(\nu^k \neq \infty)^{-\frac{1}{p}}$. Thus condition (a2) in the definition of an $(p,\infty)^*$-atom also holds. Thus a^k is $(p,\infty)^*$-atom.

Moreover,

$$\sum_{k\in\mathbb{Z}} (f_n^{\nu^{k+1}} - f_n^{\nu^k}) = f_n \text{ a.e.}$$

From (3.34), we have that $\|a^k\|_2 \leq \mathbb{P}(\nu^k \neq \infty)^{-\frac{1}{p}}$. Hence for a fixed k, $a^k = \{a_n^k\}_{n\in\mathbb{N}}$ is an L_2-bounded martingale. Consequently, the limit

$$\lim_{n\to\infty} a_n^k$$

exists a.e. in L_2. Hence there exists $a^k \in L_2$ such that $\mathbb{E}_n a^k$ is well defined and since a^k is a martingale, $\mathbb{E}_n a^k = a_n^k$. Now since $\lambda_k a_n^k = f_n^{\nu^{k+1}} - f_n^{\nu^k}$, we have that

$$\sum_{k\in\mathbb{Z}} \lambda_k a_n^k = \sum_{k\in\mathbb{Z}} (f_n^{\nu^{k+1}} - f_n^{\nu^k}) = f_n \text{ a.e.}$$

and thus (3.30) is satisfied.

We have from (3.33) that

$$f_n - f_n^{\nu^{m+1}} = \sum_{k=m+1}^{\infty} f_n^{\nu^{k+1}} - f_n^{\nu^k} = \sum_{k=m+1}^{\infty} \lambda_k a_n^k$$

and since $a^k = 0$ on the set $\{\nu^k > n\}$, we can write $a^k = \mathbf{1}_{\{\nu\leq n-1\}} a^k$ and hence

$$|f_n - f_n^{\nu^{m+1}}| \leq \sum_{k=m+1}^{\infty} \lambda_k \mathbf{1}_{\{\nu^k\leq n-1\}} |a_n^k| \leq \sum_{k=m+1}^{\infty} \lambda_k \mathbf{1}_{\{\nu_k\leq n-1\}} \sup_{n\in\mathbb{N}} |a_n^k|.$$

Thus

$$|f_n - f_n^{\nu^{m+1}}| \leq \sum_{k=m+1}^{\infty} \lambda_k \mathbf{1}_{\{\nu_k\leq n-1\}} \|(a^k)^*\|_\infty.$$

Let $\zeta_{n-1} = \sum_{k=m+1}^{\infty} \lambda_k \mathbf{1}_{\{\nu_k\leq n-1\}} \|(a^k)^*\|_\infty$. Then since $\|(a^k)^*\|_\infty \leq \mathbb{P}(\nu^k \neq \infty)^{-\frac{1}{p}}$ we obtain

$$\zeta_{n-1} \le \sum_{k=m+1}^{\infty} \frac{\lambda_k}{\mathbb{P}(\nu^k \neq \infty)^{\frac{1}{p}}} \mathbf{1}_{\{\nu_k \le n-1\}} \le \sum_{k=m+1}^{\infty} C2^k \mathbf{1}_{B_{\nu^k}}.$$

It follows that

$$(f - f^{\nu^{m+1}})^* \le \lim_{n\to\infty} \zeta_n \le \sum_{k=m+1}^{\infty} C2^k \mathbf{1}_{B_{\nu^k}}.$$

Hence

$$\|f - f^{\nu^{m+1}}\|_{\mathcal{P}_{p,q}}^q \le \|\zeta_\infty\|_{p,q}^q \le \sum_{j\in\mathbb{Z}} \left\| \left(\sum_{k=m+1}^{\infty} C2^k \mathbf{1}_{B_{\nu^k}} \right) \mathbf{1}_{\Omega_j} \right\|_p^q .$$

Proceeding as in the proof of Theorem 3.4, we obtain that

$$\left(\sum_{k=m+1}^{\infty} C2^k \mathbf{1}_{B_{\nu^k}} \right) \mathbf{1}_{\Omega_j} \le C\beta_\infty \mathbf{1}_{\Omega_j}.$$

Hence as $\|\beta_\infty \mathbf{1}_{\Omega_j}\|_p < \infty$, it follows from the the Dominated Convergence Theorem that

$$\left\| \left(\sum_{k=m+1}^{\infty} C2^k \mathbf{1}_{B_{\nu^k}} \right) \mathbf{1}_{\Omega_j} \right\|_p \to 0 \text{ as } m \to \infty.$$

As

$$\left\| \left(\sum_{k=m+1}^{\infty} C2^k \mathbf{1}_{B_{\nu^k}} \right) \mathbf{1}_{\Omega_j} \right\|_p \le C\|\beta_\infty \mathbf{1}_{\Omega_j}\|_p$$

and as

$$\sum_{j\in\mathbb{Z}} \|\beta_\infty \mathbf{1}_{\Omega_j}\|_p^q = \|\beta_\infty\|_{p,q}^q < \infty,$$

an application of the Dominated Convergence Theorem for sequence spaces leads to

$$\sum_{j\in\mathbb{Z}} \left\| \left(\sum_{k=m+1}^{\infty} C2^k \mathbf{1}_{B_{\nu^k}} \right) \mathbf{1}_{\Omega_j} \right\|_p^q \to 0 \text{ as } m \to \infty.$$

Thus $\|f - f^{\nu^{k m+1}}\|_{\mathcal{P}_{p,q}} \to 0$ as $m \to \infty$. Also since $f^{\nu^l} \le 2^l$, it is true that

$$\|f^{\nu^l}\|_{\mathcal{P}_{p,q}} \le \|2^l\|_{\mathcal{P}_{p,q}} \longrightarrow 0 \quad \text{as} \quad l \longrightarrow -\infty.$$

It then implies that

$$\begin{aligned} \left\| f - \sum_{k=l}^{m} \lambda_k a^k \right\|_{\mathcal{P}_{p,q}} &= \left\| (f - f^{\nu^{m+1}}) + f^{\nu^l} \right\|_{\mathcal{P}_{p,q}} \\ &\le \|(f - f^{\nu^{m+1}})\|_{\mathcal{P}_{p,q}} + \|f^{\nu^l}\|_{\mathcal{P}_{p,q}} \\ &\longrightarrow 0 \end{aligned}$$

as $m \longrightarrow \infty$ and $l \longrightarrow -\infty$. Hence

$$\sum_{k=l}^{m} \lambda_k a^k \longrightarrow f$$

in $\mathcal{P}_{p,q}$ as $m \to \infty,\ l \to -\infty$ and thus for all $n \in \mathbb{N}$,

$$\sum_{k\in\mathbb{Z}} \lambda_k \mathbb{E}_n a^k = f_n.$$

Now, as in Theorem 3.4, we obtain

$$\begin{aligned}
\left\| \sum_{k\in\mathbb{Z}} \left(\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} \right)^{\eta} \mathbf{1}_{B_{\nu^k}} \right\|_{\frac{p}{\eta},\frac{q}{\eta}}^{\frac{1}{\eta}} &\leq \left\| \sum_{k\in\mathbb{Z}} (C2^k)^{\eta} \mathbf{1}_{B_{\nu^k}} \right\|_{\frac{p}{\eta},\frac{q}{\eta}}^{\frac{1}{\eta}} \\
&\leq C \left\| \beta_\infty \right\|_{p,q} \\
&\leq C \| f \|_{\mathcal{P}_{p,q}}.
\end{aligned}$$

Conversely, let f be represented as equation (3.30). Then it is true that

$$|f_n| = \left| \sum_{k\in\mathbb{Z}} \lambda_k a_n^k \right| \leq \sum_{k\in\mathbb{Z}} \lambda_k |a_n^k| \leq \sum_{k\in\mathbb{Z}} \lambda_k \sup_{n\in\mathbb{N}} |a_n^k| \leq \sum_{k\in\mathbb{Z}} \lambda_k \|(a^k)^*\|_\infty \mathbf{1}_{\{\nu^k < n-1\}}.$$

Hence by choosing

$$\beta_{n-1} = \sum_{k\in\mathbb{Z}} \lambda_k \|(a^k)^*\|_\infty \mathbf{1}_{\{\nu^k < n-1\}}$$

the inequality

$$\|f\|_{\mathcal{P}_{p,q}} \leq C \left\| \sum_{k\in\mathbb{Z}} \left(\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} \right)^{\eta} \mathbf{1}_{B_{\nu^k}} \right\|_{\frac{p}{\eta},\frac{q}{\eta}}^{\frac{1}{\eta}}$$

follows established as in the proof of Theorem 3.7. Indeed since $\|(a^k)^*\|_\infty \leq \mathbb{P}(\nu^k \neq \infty)^{-\frac{1}{p}}$ we have that

$$\beta_\infty \leq \sum_{k\in\mathbb{Z}} \lambda_k \mathbb{P}(\nu^k \neq \infty)^{-\frac{1}{p}} \mathbf{1}_{B_{\nu^k}} = \sum_{k\in\mathbb{Z}} \frac{\lambda_k}{\mathbb{P}(\nu^k \neq \infty)^{\frac{1}{p}}} \mathbf{1}_{B_{\nu^k}}$$

where $B_{\nu^k} = \{\nu^k < \infty\} = \{\nu^k \neq \infty\}$. This implies that

$$\|\beta_\infty\|_{p,q} \leq \left\| \sum_{k\in\mathbb{Z}} \frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} \mathbf{1}_{B_{\nu^k}} \right\|_{p,q} \leq C \left\| \sum_{k\in\mathbb{Z}} \left(\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} \right)^{\eta} \mathbf{1}_{B_{\nu^k}} \right\|_{\frac{p}{\eta},\frac{q}{\eta}}^{\frac{1}{\eta}}$$

where the last inequality is as we did in the proof of Theorem 3.7. And thus we have that

$$\|\beta_\infty\|_{p,q} \leq C \left\| \sum_{k\in\mathbb{Z}} \left(\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} \right)^\eta \mathbf{1}_{B_{\nu^k}} \right\|_{\frac{p}{\eta},\frac{q}{\eta}}^{\frac{1}{\eta}}.$$

Hence

$$\|f\|_{\mathcal{P}_{p,q}} \leq C \left\| \sum_{k\in\mathbb{Z}} \left(\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} \right)^\eta \mathbf{1}_{B_{\nu^k}} \right\|_{\frac{p}{\eta},\frac{q}{\eta}}^{\frac{1}{\eta}}.$$

Let us now consider the case $q = \infty$. Proceeding as in the proof of Theorem 3.4,

$$\|f - f^{\nu^{m+1}}\|_{\mathcal{P}_{p,\infty}} \leq \sup_{j\in\mathbb{Z}} \left\| \left(\sum_{k=m+1}^{\infty} C2^k \mathbf{1}_{B_{\nu^k}} \right) \mathbf{1}_{\Omega_j} \right\|_p.$$

Proceeding as in the proof of Theorem 3.4, we obtain that

$$\left(\sum_{k=m+1}^{\infty} C2^k \mathbf{1}_{B_{\nu^k}} \right) \mathbf{1}_{\Omega_j} \leq C\beta_\infty \mathbf{1}_{\Omega_j}.$$

Hence as $\|\beta_\infty \mathbf{1}_{\Omega_j}\|_p < \infty$, it follows from the the Dominated Convergence Theorem that

$$\left\| \left(\sum_{k=m+1}^{\infty} C2^k \mathbf{1}_{B_{\nu^k}} \right) \mathbf{1}_{\Omega_j} \right\|_p \to 0 \text{ as } m \to \infty.$$

As

$$\left\| \left(\sum_{k=m+1}^{\infty} C2^k \mathbf{1}_{B_{\nu^k}} \right) \mathbf{1}_{\Omega_j} \right\|_p \leq C\|\beta_\infty \mathbf{1}_{\Omega_j}\|_p$$

and as

$$\sup_{j\in\mathbb{Z}} \|\beta_\infty \mathbf{1}_{\Omega_j}\|_p = \|\beta_\infty\|_{p,\infty} < \infty,$$

an application of the Dominated Convergence Theorem leads to

$$\sup_{j\in\mathbb{Z}} \left\| \left(\sum_{k=m+1}^{\infty} C2^k \mathbf{1}_{B_{\nu^k}} \right) \mathbf{1}_{\Omega_j} \right\|_p \to 0 \text{ as } m \to \infty.$$

Thus $\|f - f^{\nu^{m+1}}\|_{\mathcal{P}_{p,\infty}} \to 0$ as $m \to \infty$. Similarly, we obtain that

$$\|f - \sum_{k=l}^{m} \lambda_k a^k\|_{\mathcal{P}_{p,\infty}} \to 0$$

as $m \to \infty$ and $l \to -\infty$. Hence

$$\sum_{k=l}^{m} \lambda_k a^k \longrightarrow f$$

in $\mathcal{P}_{p,\infty}$ as $m \to \infty,\ l \to -\infty$ and thus for all $n \in \mathbb{N}$,

$$\sum_{k\in\mathbb{Z}} \lambda_k \mathbb{E}_n a^k = f_n.$$

We also obtain, as in Theorem 3.4, that

$$\begin{aligned}\left\| \sum_{k\in\mathbb{Z}} \left(\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} \right)^{\eta} \mathbf{1}_{B_{\nu^k}} \right\|_{\frac{p}{\eta},\infty}^{\frac{1}{\eta}} &\leq \left\| \sum_{k\in\mathbb{Z}} (C2^k)^{\eta} \mathbf{1}_{B_{\nu^k}} \right\|_{\frac{p}{\eta},\infty}^{\frac{1}{\eta}} \\ &\leq C \left\| \beta_\infty \right\|_{p,\infty} \\ &\leq C \|f\|_{\mathcal{P}_{p,\infty}}.\end{aligned}$$

Conversely, assume that $f \in \mathcal{M}$ has the decomposition (3.30). Define β_n by

$$\beta_n := \sum_{k\in\mathbb{Z}} \lambda_k \|(a^k)^*\|_\infty \mathbf{1}_{\{\nu^k \leq n\}}.$$

Then $(\beta_n)_{n\geq}$ is a nondecreasing nonnegative adapted sequence also, for $n \geq 0$,

$$|f_n| \leq \beta_{n-1}.$$

As $\|(a^k)^*\|_\infty \leq \mathbb{P}(\nu^k \neq \infty)^{-\frac{1}{p}}$, it follows that

$$\|\beta_\infty\|_{p,\infty} \leq \left\| \sum_{k\in\mathbb{Z}} \frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} \mathbf{1}_{B_{\nu^k}} \right\|_{p,\infty} \leq \left\| \sum_{k\in\mathbb{Z}} \left(\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} \right)^{\eta} \mathbf{1}_{B_{\nu^k}} \right\|_{\frac{p}{\eta},\infty}^{\frac{1}{\eta}}.$$

Then

$$\|f\|_{\mathcal{P}_{p,\infty}} \leq \|\beta_\infty\|_{p,\infty} \leq \left\| \sum_{k\in\mathbb{Z}} \left(\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} \right)^{\eta} \mathbf{1}_{B_{\nu^k}} \right\|_{\frac{p}{\eta},\infty}^{\frac{1}{\eta}}.$$

The proof is complete. □

Let us now discuss the second atomic decomposition of $\mathcal{P}_{p,q}$ based on Definition 3.2.

Theorem 3.11. *Let $0 < p < \infty$ and $0 < q \leq \infty$. If the martingale $f \in \mathcal{M}$ is in $\mathcal{P}_{p,q}$, then there exists a sequence of triplets $(\lambda_k, a^k, \nu^k) \in \mathcal{B}(p, q, \infty)^*$ such that for any $n \in \mathbb{N}$,*

$$\sum_{k\in\mathbb{Z}} \lambda_k \mathbb{E}_n a^k = f_n \tag{3.35}$$

and for any $0 < \eta \le 1$,

$$\left\| \sum_{k\in\mathbb{Z}} \left(\frac{\lambda_k}{\|\mathbf{1}_{B_{\nu^k}}\|_{p,q}} \right)^{\eta} \mathbf{1}_{B_{\nu^k}} \right\|_{\frac{p}{\eta},\frac{q}{\eta}}^{\frac{1}{\eta}} \le C\|f\|_{\mathcal{P}_{p,q}}. \tag{3.36}$$

Moreover,

$$\sum_{k=l}^{m} \lambda_k a^k \longrightarrow f$$

in $\mathcal{P}_{p,q}$ *as* $m \to \infty$, $l \to -\infty$.

Conversely, if $f \in \mathcal{M}$ *has a decomposition as in (3.35), then for any* $0 < \eta \le 1$,

$$\|f\|_{\mathcal{P}_{p,q}} \le C \left\| \sum_{k\in\mathbb{Z}} \left(\frac{\lambda_k}{\|\mathbf{1}_{B_{\nu^k}}\|_{p,q}} \right)^{\eta} \mathbf{1}_{B_{\nu^k}} \right\|_{\frac{p}{\eta},\frac{q}{\eta}}^{\frac{1}{\eta}}.$$

Proof. Let $0 < p < \infty$, $0 < q < \infty$ and let $f \in \mathcal{P}_{p,q}$. Then there exists an optimal, adapted non-decreasing, non-negative adapted sequence $(\beta_n)_{n\in\mathbb{N}}$ such that $|f_n| \le \beta_{n-1}$ and $\|f\|_{\mathcal{P}_{p,q}} = \|\beta_\infty\|_{p,q}$. We take

$$\nu^k := \inf\{n \in \mathbb{N} : \beta_n > 2^k\} \tag{3.37}$$

to be a stopping times and define $\lambda_k = C2^k\|\mathbf{1}_{B_{\nu^k}}\|_{p,q}$, and $(n \in \mathbb{N})$

$$a_n^k = \frac{f_n^{\nu^{k+1}} - f_n^{\nu^k}}{\lambda_k} \text{ if } \lambda_k \ne 0, \text{ and } a^k = 0 \text{ otherwise.} \tag{3.38}$$

Now on the set $\{n \le \nu^k\}$ we see that

$$a_n^k = \frac{f_n^{\nu^{k+1}} - f_n^{\nu^k}}{\lambda_k} = \frac{f_n - f_n}{\lambda_k} = 0$$

by definition of stopped martingales and thus $\mathbb{E}_n a^k = 0$ when $\nu^k > n$. Hence a^k satisfies condition (a1). Also, $a_n^k = 0$ on $\{\nu^k = \infty\}$ for all $n \ge 0$, and the support of $(a^k)^*$ is contained in $B_{\nu^k} = \{\nu \ne \infty\}$. Now from definition of stopping time, $\nu^k = n$ and also $2^k < \beta_n$. Also $|f_n| = |f_{\nu^k}| \le \beta_{n-1} = \beta_{\nu^k-1} \le \beta_n$ and thus $|f^{\nu^k}| \lesssim 2^k$. Therefore

$$|a^k| = \left| \frac{f^{\nu^{k+1}} - f^{\nu^k}}{\lambda_k} \right| \lesssim \left| \frac{2^{k+1} + 2^k}{\lambda_k} \right| \le \frac{C(2^k)}{C2^k\|\mathbf{1}_{B_{\nu^k}}\|_{p,q}} = \|\mathbf{1}_{B_{\nu^k}}\|_{p,q}^{-1}. \tag{3.39}$$

Thus $(a^k)^* = \sup_{n\in\mathbb{N}} |a_n^k| \leq \|\mathbf{1}_{B_{\nu^k}}\|_{p,q}^{-1}$. This implies that

$$\|(a^k)\|_\infty \leq \|\mathbf{1}_{B_{\nu^k}}\|_{p,q}^{-1}.$$

Hence condition (a3) also holds and thus a^k is $(p, q, \infty)^*$-atom. Moreover,

$$\sum_{k\in\mathbb{Z}} (f_n^{\nu^{k+1}} - f_n^{\nu^k}) = f_n \text{ a.e.}$$

From (3.39), we have that $\|(a^k)\|_2 \leq \|\mathbf{1}_{B_{\nu^k}}\|_{p,q}^{-1}$. Hence for a fixed k, $a^k = \{a_n^k\}_{n\in\mathbb{N}}$ is an L_2-bounded martingale. Consequently, the limit

$$\lim_{n\to\infty} a_n^k$$

exists a.e. in L_2. Hence there exists $a^k \in L_2$ such that $\mathbb{E}_n a^k$ is well defined and since a^k is a martingale, $\mathbb{E}_n a^k = a_n^k$. Now since $\lambda_k a_n^k = f_n^{\nu^{k+1}} - f_n^{\nu^k}$, we have that

$$\sum_{k\in\mathbb{Z}} \lambda_k a_n^k = \sum_{k\in\mathbb{Z}} (f_n^{\nu^{k+1}} - f_n^{\nu^k}) = f_n \text{ a.e.}$$

and thus (3.35) is satisfied.

We have from (3.38) that

$$f_n - f_n^{\nu^{m+1}} = \sum_{k=m+1}^{\infty} f_n^{\nu^{k+1}} - f_n^{\nu^k} = \sum_{k=m+1}^{\infty} \lambda_k a_n^k$$

and since $a^k = 0$ on the set $\{\nu^k > n\}$, we can write $a^k = \mathbf{1}_{\{\nu \leq n-1\}} a^k$ and hence

$$|f_n - f_n^{\nu^{m+1}}| \leq \sum_{k=m+1}^{\infty} \lambda_k \mathbf{1}_{\{\nu^k \leq n-1\}} |a_n^k| \leq \sum_{k=m+1}^{\infty} \lambda_k \mathbf{1}_{\{\nu_k \leq n-1\}} \sup_{n\in\mathbb{N}} |a_n^k|.$$

Thus

$$|f_n - f_n^{\nu^{m+1}}| \leq \sum_{k=m+1}^{\infty} \lambda_k \mathbf{1}_{\{\nu_k \leq n-1\}} \|(a^k)^*\|_\infty.$$

Let $\zeta_{n-1} = \sum_{k=m+1}^{\infty} \lambda_k \mathbf{1}_{\{\nu_k \leq n-1\}} \|(a^k)^*\|_\infty$. Then since $\|(a^k)^*\|_\infty \leq \|\mathbf{1}_{B_{\nu^k}}\|_{p,q}^{-1}$ we obtain

$$\zeta_{n-1} \leq \sum_{k=m+1}^{\infty} \frac{\lambda_k}{\|\mathbf{1}_{B_{\nu^k}}\|_{p,q}} \mathbf{1}_{\{\nu_k \leq n-1\}} \leq \sum_{k=m+1}^{\infty} C 2^k \mathbf{1}_{B_{\nu^k}}.$$

It follows that

$$(f - f^{\nu^{m+1}})^* \leq \lim_{n\to\infty} \zeta_n \leq \sum_{k=m+1}^{\infty} C 2^k \mathbf{1}_{B_{\nu^k}}.$$

Hence

$$\|f - f^{\nu^{m+1}}\|_{\mathcal{P}_{p,q}}^q \leq \|\zeta_\infty\|_{p,q}^q \leq \sum_{j\in\mathbb{Z}} \left\| \left(\sum_{k=m+1}^{\infty} C2^k \mathbf{1}_{B_{\nu k}} \right) \mathbf{1}_{\Omega_j} \right\|_p^q .$$

Proceeding as in the proof of Theorem 3.4, we obtain that

$$\left(\sum_{k=m+1}^{\infty} C2^k \mathbf{1}_{B_{\nu k}} \right) \mathbf{1}_{\Omega_j} \leq C\beta_\infty \mathbf{1}_{\Omega_j}.$$

Hence as $\|\beta_\infty \mathbf{1}_{\Omega_j}\|_p < \infty$, it follows from the the Dominated Convergence Theorem that

$$\left\| \left(\sum_{k=m+1}^{\infty} C2^k \mathbf{1}_{B_{\nu k}} \right) \mathbf{1}_{\Omega_j} \right\|_p \to 0 \text{ as } m \to \infty.$$

As

$$\left\| \left(\sum_{k=m+1}^{\infty} C2^k \mathbf{1}_{B_{\nu k}} \right) \mathbf{1}_{\Omega_j} \right\|_p \leq C\|\beta_\infty \mathbf{1}_{\Omega_j}\|_p$$

and as

$$\sum_{j\in\mathbb{Z}} \|\beta_\infty \mathbf{1}_{\Omega_j}\|_p^q = \|\beta_\infty\|_{p,q}^q < \infty,$$

an application of the Dominated Convergence Theorem for sequence spaces leads to

$$\sum_{j\in\mathbb{Z}} \left\| \left(\sum_{k=m+1}^{\infty} C2^k \mathbf{1}_{B_{\nu k}} \right) \mathbf{1}_{\Omega_j} \right\|_p^q \to 0 \text{ as } m \to \infty.$$

Thus $\|f - f^{\nu^{k}m+1}\|_{\mathcal{P}_{p,q}} \to 0$ as $m \to \infty$. Also since $f^{\nu^l} \leq 2^l$, it is true that

$$\|f^{\nu^l}\|_{\mathcal{P}_{p,q}} \leq \|2^l\|_{\mathcal{P}_{p,q}} \longrightarrow 0 \quad \text{as} \quad l \longrightarrow -\infty.$$

It then implies that

$$\begin{aligned} \left\| f - \sum_{k=l}^{m} \lambda_k a^k \right\|_{\mathcal{P}_{p,q}} &= \left\| (f - f^{\nu^{m+1}}) + f^{\nu^l} \right\|_{\mathcal{P}_{p,q}} \\ &\leq \|(f - f^{\nu^{m+1}})\|_{\mathcal{P}_{p,q}} + \|f^{\nu^l}\|_{\mathcal{P}_{p,q}} \\ &\longrightarrow 0 \end{aligned}$$

as $m \longrightarrow \infty$ and $l \longrightarrow -\infty$. Hence

$$\sum_{k=l}^{m} \lambda_k a^k \longrightarrow f$$

in $\mathcal{P}_{p,q}$ as $m \to \infty,\ l \to -\infty$ and thus for all $n \in \mathbb{N}$,

$$\sum_{k\in\mathbb{Z}} \lambda_k \mathbb{E}_n a^k = f_n.$$

Now, as in Theorem 3.4, we obtain

$$\begin{aligned}
\left\| \sum_{k\in\mathbb{Z}} \left(\frac{\lambda_k}{\|\mathbf{1}_{B_{\nu^k}}\|_{p,q}^{-1}} \right)^{\eta} \mathbf{1}_{B_{\nu^k}} \right\|_{\frac{p}{\eta},\frac{q}{\eta}}^{\frac{1}{\eta}} &\leq \left\| \sum_{k\in\mathbb{Z}} (C2^k)^{\eta} \mathbf{1}_{B_{\nu^k}} \right\|_{\frac{p}{\eta},\frac{q}{\eta}}^{\frac{1}{\eta}} \\
&\leq C\,\|\beta_\infty\|_{p,q} \\
&\leq C\|f\|_{\mathcal{P}_{p,q}}.
\end{aligned}$$

Conversely, let f be represented as equation (3.35). Then it is true that

$$|f_n| = \left| \sum_{k\in\mathbb{Z}} \lambda_k a_n^k \right| \leq \sum_{k\in\mathbb{Z}} \lambda_k |a_n^k| \leq \sum_{k\in\mathbb{Z}} \lambda_k \sup_{n\in\mathbb{N}} |a_n^k| \leq \sum_{k\in\mathbb{Z}} \lambda_k \|(a^k)^*\|_\infty \mathbf{1}_{\{\nu^k < n-1\}}$$

Hence by choosing

$$\beta_{n-1} = \sum_{k\in\mathbb{Z}} \lambda_k \|(a^k)^*\|_\infty \mathbf{1}_{\{\nu^k < n-1\}}$$

the inequality

$$\|f\|_{\mathcal{P}_{p,q}} \leq C \left\| \sum_{k\in\mathbb{Z}} \left(\frac{\lambda_k}{\|\mathbf{1}_{B_{\nu^k}}\|_{p,q}} \right)^{\eta} \mathbf{1}_{B_{\nu^k}} \right\|_{\frac{p}{\eta},\frac{q}{\eta}}^{\frac{1}{\eta}}$$

follows established as in the proof of Theorem 3.10. Indeed since $\|(a^k)^*\|_\infty \leq \|\mathbf{1}_{B_{\nu^k}}\|_{p,q}^{-1}$, we have that

$$\beta_\infty \leq \sum_{k\in\mathbb{Z}} \frac{\lambda_k}{\|\mathbf{1}_{B_{\nu^k}}\|_{p,q}} \mathbf{1}_{B_{\nu^k}}$$

where $B_{\nu^k} = \{\nu^k < \infty\} = \{\nu^k \neq \infty\}$. This implies that

$$\|\beta_\infty\|_{p,q} \leq \left\| \sum_{k\in\mathbb{Z}} \frac{\lambda_k}{\|\mathbf{1}_{B_{\nu^k}}\|_{p,q}} \mathbf{1}_{B_{\nu^k}} \right\|_{p,q} \leq C \left\| \sum_{k\in\mathbb{Z}} \left(\frac{\lambda_k}{\|\mathbf{1}_{B_{\nu^k}}\|_{p,q}} \right)^{\eta} \mathbf{1}_{B_{\nu^k}} \right\|_{\frac{p}{\eta},\frac{q}{\eta}}^{\frac{1}{\eta}}$$

where the last inequality is as we did in the proof of Theorem 3.10. And thus we obtain that

$$\|\beta_\infty\|_{p,q} \leq C \left\| \sum_{k\in\mathbb{Z}} \left(\frac{\lambda_k}{\|\mathbf{1}_{B_{\nu^k}}\|_{p,q}} \right)^{\eta} \mathbf{1}_{B_{\nu^k}} \right\|_{\frac{p}{\eta},\frac{q}{\eta}}^{\frac{1}{\eta}}.$$

Hence

$$\|f\|_{\mathcal{P}_{p,q}} \leq C \left\| \sum_{k\in\mathbb{Z}} \left(\frac{\lambda_k}{\|\mathbf{1}_{B_{\nu k}}\|_{p,q}} \right)^{\eta} \mathbf{1}_{B_{\nu k}} \right\|_{\frac{p}{\eta},\frac{q}{\eta}}^{\frac{1}{\eta}}.$$

Let us now consider the case $q = \infty$. Proceeding as in the proof of Theorem 3.10,

$$\|f - f^{\nu^{m+1}}\|_{\mathcal{P}_{p,\infty}} \leq \sup_{j\in\mathbb{Z}} \left\| \left(\sum_{k=m+1}^{\infty} C2^k \mathbf{1}_{B_{\nu k}} \right) \mathbf{1}_{\Omega_j} \right\|_p$$

and

$$\left(\sum_{k=m+1}^{\infty} C2^k \mathbf{1}_{B_{\nu k}} \right) \mathbf{1}_{\Omega_j} \leq C\beta_\infty \mathbf{1}_{\Omega_j}.$$

Hence as $\|\beta_\infty \mathbf{1}_{\Omega_j}\|_p < \infty$, it follows from the the Dominated Convergence Theorem that

$$\left\| \left(\sum_{k=m+1}^{\infty} C2^k \mathbf{1}_{B_{\nu k}} \right) \mathbf{1}_{\Omega_j} \right\|_p \to 0 \text{ as } m \to \infty.$$

As

$$\left\| \left(\sum_{k=m+1}^{\infty} C2^k \mathbf{1}_{B_{\nu k}} \right) \mathbf{1}_{\Omega_j} \right\|_p \leq C\|\beta_\infty \mathbf{1}_{\Omega_j}\|_p$$

and as

$$\sup_{j\in\mathbb{Z}} \|\beta_\infty \mathbf{1}_{\Omega_j}\|_p = \|\beta_\infty\|_{p,\infty} < \infty,$$

an application of the Dominated Convergence Theorem leads to

$$\sup_{j\in\mathbb{Z}} \left\| \left(\sum_{k=m+1}^{\infty} C2^k \mathbf{1}_{B_{\nu k}} \right) \mathbf{1}_{\Omega_j} \right\|_p \to 0 \text{ as } m \to \infty.$$

Thus $\|f - f^{\nu^{m+1}}\|_{\mathcal{P}_{p,\infty}} \to 0$ as $m \to \infty$. Similarly, we obtain that $\|f - \sum_{k=l}^{m} \lambda_k a^k\|_{\mathcal{P}_{p,\infty}} \to 0$ as $m \to \infty$ and $l \to -\infty$. Hence $\sum_{k=l}^{m} \lambda_k a^k \longrightarrow f$ in $\mathcal{P}_{p,\infty}$ as $m \to \infty,\ l \to -\infty$ and thus for all $n \in \mathbb{N}$,

$$\sum_{k\in\mathbb{Z}} \lambda_k \mathbb{E}_n a^k = f_n.$$

We also obtain, as in Theorem 3.4, that

$$\begin{aligned} \left\| \sum_{k\in\mathbb{Z}} \left(\frac{\lambda_k}{\|\mathbf{1}_{B_{\nu k}}\|_{p,q}} \right)^{\eta} \mathbf{1}_{B_{\nu k}} \right\|_{\frac{p}{\eta},\infty}^{\frac{1}{\eta}} &\leq \left\| \sum_{k\in\mathbb{Z}} (C2^k)^{\eta} \mathbf{1}_{B_{\nu k}} \right\|_{\frac{p}{\eta},\infty}^{\frac{1}{\eta}} \\ &\leq C\, \|\beta_\infty\|_{p,\infty} \\ &\leq C\|f\|_{\mathcal{P}_{p,\infty}}. \end{aligned}$$

Conversely, assume that $f \in \mathcal{M}$ has the decomposition (3.35). Define β_n by

$$\beta_n := \sum_{k\in\mathbb{Z}} \lambda_k \|(a^k)^*\|_\infty \mathbf{1}_{\{\nu^k \le n\}}.$$

Then $(\beta_n)_{n\geq}$ is a nondecreasing nonnegative adapted sequence also, for $n \geq 0$,

$$|f_n| \leq \beta_{n-1}.$$

As $\|(a^k)^*\|_\infty \leq \|\mathbf{1}_{B_{\nu^k}}\|_{p,q}^{-1}$, it follows that

$$\|\beta_\infty\|_{p,\infty} \leq \left\| \sum_{k\in\mathbb{Z}} \frac{\lambda_k}{\|\mathbf{1}_{B_{\nu^k}}\|_{p,q}} \mathbf{1}_{B_{\nu^k}} \right\|_{p,\infty} \leq \left\| \sum_{k\in\mathbb{Z}} \left(\frac{\lambda_k}{\|\mathbf{1}_{B_{\nu^k}}\|_{p,q}} \right)^\eta \mathbf{1}_{B_{\nu^k}} \right\|_{\frac{p}{\eta},\infty}^{\frac{1}{\eta}}.$$

Then

$$\|f\|_{\mathcal{P}_{p,\infty}} \leq \|\beta_\infty\|_{p,\infty} \leq \left\| \sum_{k\in\mathbb{Z}} \left(\frac{\lambda_k}{\|\mathbf{1}_{B_{\nu^k}}\|_{p,q}} \right)^\eta \mathbf{1}_{B_{\nu^k}} \right\|_{\frac{p}{\eta},\infty}^{\frac{1}{\eta}}.$$

The proof is complete. ☐

We conclude this chapter by noting that upon the imposition of the condition that the stochastic basis is regular, then all the five martingale Hardy-amalgam spaces are equivalent and hence they will all have the same atomic decomposition. This is the result stated and proved in Theorem 4.17.

Chapter 4

Martingale Inequalities and Embeddings

Martingale inequalities are ubiquitous, and the famous Doob's martingale inequality is one of the many examples. In recent past times, there have been various studies about the embeddings of classical martingale Hardy spaces. Some of these discussions were carried out by various authors including Ferenc, Garsia and Rulin [26, 48, 65]. Other important inequalities involving martingales are also available in [6, 8, 35, 36, 41]. For instance in Ferenc, [65], we can find the discussion on the Doob's inequality, the convexity and concavity inequalities for martingales in the classical martingale Hardy spaces. We can also find the discussion of norm inequalities for operators of matrix type on martingales and proof of Burkholder-Davis-Gundy inequality in [8, 65]. A discussion on the weighted norm inequality similar to the Burkholder-Davis-Gundy inequality can also be found in [35] and inequalities of operators of non-matrix type on martingales with a weighted probability measure is available in [41]. It is also interesting to mention that an analogue of weighted norm inequality for the Hardy maximal function result is also valid in the settings of martingale theory [36]. Some of these classical results will be very useful in this chapter and for consistency purposes, these classical results will be restated when needed in the appropriate section below. In recent times, new martingale inequalities have been obtained which has proven to be very useful in applications, for example in Fourier analysis [69].

Martingale inequalities have proven to be very useful in various applications. For instance, the justification of martingale convergence theorems both forward and backward convergence have been established by application of these classical martingale inequalities [33]. In Fourier analysis, we have the involvement of martingale inequalities in the establishment of the boundedness of the maximal Fejér operator [69]. Martingale inequalities also play some important roles in the study of properties of Brownian motions [8].

In this chapter, we are interested in studying the embeddings relations between the martingale Hardy-amalgam spaces, $H_{p,q}^S$, $H_{p,q}^s$, $H_{p,q}^*$, $\mathcal{Q}_{p,q}$ and $\mathcal{P}_{p,q}$, introduced in Chapter 2. That is, we extend the embeddings of the classical martingale Hardy spaces, [65, Theorem 2.11], to the martingale Hardy-amalgam spaces. We will also extend the Doob's inequality, the convexity inequality, the concavity inequality and the Burkholder-Davis-Gundy inequality of the classical martingale Hardy spaces to the martingale Hardy-amalgam spaces. We will also discuss the Davis decompositions of martingales in the martingale Hardy-amalgam spaces. As an application of the Davis decomposition, the dual space of $H_{p,q}^*$ is characterized in the next chapter.

In this chapter, all the martingales are defined with respect to the stochastic basis $\mathbf{Ps} := (\mathbb{R}, \mathcal{F}, \{\mathcal{F}_n\}_{n\in\mathbb{N}}, \mathbb{P})$ where $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ is the dyadic filtration. We also recall from Chapter 2 that the dyadic intervals $A_j = [j, j+1)$ are covered by $J_{k,n,j}$ and that $A_j \in \mathcal{F}_n$ for all n and for all j. We also recall from Chapter 2 that the stochastic basis is regular if there exists $R > 0$ such that $f_n \leq R f_{n-1}$. The following Lemma will be relevant in most parts of the remaining work.

Lemma 4.1. *Assume that* $A_j \in \mathcal{F}_n$ *for all* $n \geq 1$. *Then if* $f \in \mathcal{M}$, *then* $f\mathbf{1}_{A_j} = (f_n\mathbf{1}_{A_j})_{n\geq 0}$ *is also a martingale in* $\mathcal{M}$ *(*$f\mathbf{1}_{A_j}$ *is a sub-martingale if* f *is a sub-martingale). Moreover, if* T *is any of the operators* s, S *and* M *(the maximal operator), then*

$$T(f\mathbf{1}_{A_j}) = T(f)\mathbf{1}_{A_j}.$$

Proof. Let $(\mathbb{R}, \mathcal{F}, \{\mathcal{F}_n\}_{n\in\mathbb{N}}, \mathbb{P})$ be a stochastic basis with underlying dyadic filtration. Then we know that the dyadic interval $A_j = [j, j+1)$ is such that $A_j \in \mathcal{F}_n$ for all $n \in \mathbb{N}$ and for all j. In other words, the characteristic function $\mathbf{1}_{A_j}$ is $\mathcal{F}_n$-measurable for all n. Let $f = \{f_n\}_{n\in\mathbb{N}}$ be a martingale (or a sub-martingale). Then f is $\mathcal{F}_n$-measurable for all n. Now since $\mathbf{1}_{A_j}$ is also $\mathcal{F}_n$-measurable for all n, the product $f\mathbf{1}_{A_j} = \{f_n\mathbf{1}_{A_j}\}_{n\in\mathbb{N}}$ is also adapted. Also for $m \leq n$, we have, (replacing the last equality by $\geq$ if f is sub-martingale),

$$\mathbb{E}_m(f\mathbf{1}_{A_j}) = \mathbf{1}_{A_j}\mathbb{E}_m f_n = \mathbf{1}_{A_j} f_m$$

since $\mathbf{1}_{A_j}$ is $\mathcal{F}_m$-measurable and f is a martingale. Thus $f\mathbf{1}_{A_j} = \{f_n\mathbf{1}_{A_j}\}_{n\in\mathbb{N}}$ is also a martingale. Now since the martingale difference operator is linear, we have that

$$d_n(f\mathbf{1}_{A_j}) = \mathbf{1}_{A_j} f_n - \mathbf{1}_{A_j} f_{n-1} = \mathbf{1}_{A_j} d_n f.$$

Therefore

$$s^2(f\mathbf{1}_{A_j}) = \sum_{n\in\mathbb{N}} \mathbb{E}_{n-1}|d_n(f\mathbf{1}_{A_j})|^2 = \sum_{n\in\mathbb{N}} \mathbb{E}_{n-1}\mathbf{1}_{A_j}|d_n(f)|^2 = \sum_{n\in\mathbb{N}} \mathbf{1}_{A_j}\mathbb{E}_{n-1}|d_n(f)|^2$$

where in the last equality, we have used the fact that the characteristic function is $\mathcal{F}_{n-1}$-measurable. Thus

$$s(f\mathbf{1}_{A_j}) = \mathbf{1}_{A_j} s(f). \tag{4.1}$$

Similarly we shall have that

$$S(f\mathbf{1}_{A_j}) = \mathbf{1}_{A_j} S(f) \quad \text{and} \quad M(f\mathbf{1}_{A_j}) = \mathbf{1}_{A_j} M(f) \tag{4.2}$$

and the proof is complete. □

4.2 Martingale Inequalities

In this section, we aim to extend some of the important classical martingale inequalities; Doob's inequality, the convexity inequality, concavity inequalities and Burkholder-Davis-Gundy inequality, in the classical case, to the martingale Hardy-amalgam space. The following classical result, Doob's inequality, can be found in [65, Proposition 2.6].

Proposition 4.3 ([65]). *Let $f = \{f_n\}_{n\in\mathbb{N}}$ be a non-negative L_p-bounded submartingale. Then for $p > 1$, we have that*

$$\left(\mathbb{E}\left[\left(\sup_{n\in\mathbb{N}} f_n\right)^p\right]\right)^p \leq \frac{p}{p-1} \sup_{n\in\mathbb{N}} (\mathbb{E}[f_n^p])^p.$$

The Burkholder-Davis-Gundy inequality is a well known inequality which describe the equivalence of the the classical spaces $H_p^*(\mathbb{R})$ and $H_p^S(\mathbb{R})$. The discussion of the following classical result can be found in [8, Theorem 5.1] or [65, Theorem 2.12].

Theorem 4.4 ([8, 65]). *The spaces $H_p^*(\mathbb{R})$ and $H_p^S(\mathbb{R})$ are equivalent for $1 \leq p < \infty$.*

We also have the following classical convexity and concavity inequalities from [65, Theorem 2.10].

Theorem 4.5 ([65]). *Let $(\mathcal{B}_t,\ t \in \mathcal{U})$ be a sequence of σ-algebra (not necessarily monotone) such that $\sigma(\cup_{t\in\mathcal{U}}\mathcal{B}_t) = \mathcal{B}_t$ where $\mathcal{U}$ is a countable index set. Suppose that $\forall u \in L_p,\ (p > 1)$, the following inequality holds*

$$\left(\mathbb{E}\left[\left(\sup_{t\in\mathbb{U}} |\mathbb{E}_t u|\right)^p\right]\right)^p \lesssim (\mathbb{E}[|u|^p])^p$$

where $\mathbb{E}_t$ is the conditional operator relative to $\mathcal{B}_t$. Let $(h_t)_{t\in\mathcal{U}}$ be sequence of non-negative *measurable functions. Then for* $p \in [1, \infty)$, *we have*

$$\left\| \sum_{t \in \mathcal{U}} \mathbb{E}_t h_t \right\|_{L_p} \lesssim \left\| \sum_{t \in \mathcal{U}} h_t \right\|_{L_p}$$

and for $p \in (0, 1]$, *we have the reverse of the inequality*

$$\left\| \sum_{t \in \mathcal{U}} h_t \right\|_{L_p} \lesssim \left\| \sum_{t \in \mathcal{U}} \mathbb{E}_t h_t \right\|_{L_p}.$$

The following Theorem is the extension of Proposition 4.3 from the classical martingale Hardy space to the martingale Hardy-amalgam space.

Theorem 4.6. *Let* $0 < q < \infty$ *and* $p > 1$. *For every non-negative* $L_{p,q}$*-bounded submartingale* $f = \{f_n\}_{n \in \mathbb{N}}$, *we have that*

$$\left\| \sup_{n \in \mathbb{N}} f_n \right\|_{L_{p,q}(\mathbb{R})} \leq \frac{p}{p-1} \sup_{n \in \mathbb{N}} \|f_n\|_{L_{p,q}(\mathbb{R})}.$$

Proof. Let A_j be defined as equation (2.1). Let $g_n = f_n \mathbf{1}_{A_j}$. Then by Lemma 4.1, $g = (g_n)_{n \in \mathbb{N}}$ is a submartingale. We observe that

$$\sup_{n \in \mathbb{N}} \|g_n\|_p^q = \sup_{n \in \mathbb{N}} \|f_n \mathbf{1}_{A_j}\|_p^q \leq \sup_{n \in \mathbb{N}} \sum_j \|f_n \mathbf{1}_{A_j}\|_p^q = \sup_{n \in \mathbb{N}} \|f_n\|_{p,q}^q < \infty.$$

Hence g_n is a non-negative L_p-bounded submartingale. Therefore by Proposition 4.3 we have that

$$\left\| \sup_{n \in \mathbb{N}} f_n \mathbf{1}_{A_j} \right\|_{L_p(\mathbb{R})} \leq \frac{p}{p-1} \sup_{n \in \mathbb{N}} \|f_n \mathbf{1}_{A_j}\|_{L_p(\mathbb{R})}.$$

Therefore by definition of amalgam space,

$$\begin{aligned}
\left\| \sup_{n \in \mathbb{N}} f_n \right\|_{L_{p,q}(\mathbb{R})}^q &= \sum_j \left\| \sup_{n \in \mathbb{N}} f_n \mathbf{1}_{A_j} \right\|_p^q \\
&\leq \left(\frac{p}{p-1} \right)^q \sum_j \left(\sup_{n \in \mathbb{N}} \|f_n \mathbf{1}_{A_j}\|_p \right)^q \\
&= \left(\frac{p}{p-1} \right)^q \sum_j \sup_{n \in \mathbb{N}} \|f_n \mathbf{1}_{A_j}\|_p^q = \left(\frac{p}{p-1} \right)^q \sup_{n \in \mathbb{N}} \sum_j \|f_n \mathbf{1}_{A_j}\|_p^q \\
&= \left(\frac{p}{p-1} \right)^q \sup_{n \in \mathbb{N}} \sum_j \|f_n \mathbf{1}_{A_j}\|_p^q
\end{aligned}$$

Thus

$$\|\sup_{n\in\mathbb{N}} f_n\|^q_{L_{p,q}(\mathbb{R})} \leq \left(\frac{p}{p-1}\right)^q \sup_{n\in\mathbb{N}} \sum_j \|f_n \mathbf{1}_{A_j}\|^q_p = \left(\frac{p}{p-1}\right)^q \sup_{n\in\mathbb{N}} \|f\|^q_{L_{p,q}(\mathbb{R})}.$$

The proof is complete. $\square$

The next Theorem is the extension of Burkholder-Davis-Gundy inequality from the classical martingale Hardy spaces to the the martingale Hardy-amalgam space.

Theorem 4.7. *The spaces $H^S_{p,q}(\mathbb{R})$ and $H^*_{p,q}(\mathbb{R})$ are equivalent for $1 \leq p, q \leq \infty$, namely,*

$$c_p\|f\|_{H^S_{p,q}(\mathbb{R})} \leq \|f\|_{H^*_{p,q}(\mathbb{R})} \leq C_p\|f\|_{H^S_{p,q}(\mathbb{R})} \quad (1 \leq p, q < \infty)$$

and

$$c_p\|f\|_{H^S_{p,\infty}(\mathbb{R})} \leq \|f\|_{H^*_{p,\infty}(\mathbb{R})} \leq C_p\|f\|_{H^S_{p,\infty}(\mathbb{R})} \quad (1 \leq p < \infty).$$

Proof. The proof follows from Theorem 4.4 and Lemma 4.1. Let $f = \{f_n\}_{n\in\mathbb{N}}$ be a martingale and define $g = \{f_n \mathbf{1}_{A_j}\}_{n\in\mathbb{N}}$ where A_j be defined as equation (2.1). Then since g is a martingale, we have by Lemma 4.1 that $S(g) = S(f\mathbf{1}_{A_j}) = \mathbf{1}_{A_j}S(f)$. We have from Theorem 4.4 that $\|S(g)\|_p \lesssim \|g^*\|_p \lesssim \|S(g)\|_p$. Therefore

$$\|\mathbf{1}_{A_j}S(f)\|_p \lesssim \|\mathbf{1}_{A_j}f^*\|_p \lesssim \|\mathbf{1}_{A_j}S(f)\|_p. \tag{4.3}$$

To get the first equivalence, we take the ℓ_q norm on both sides of the equivalence (4.3). That is

$$\sum_{j\in\mathbb{Z}} \|\mathbf{1}_{A_j}S(f)\|^q_p \lesssim \sum_{j\in\mathbb{Z}} \|\mathbf{1}_{A_j}f^*\|^q_p \lesssim \sum_{j\in\mathbb{Z}} \|\mathbf{1}_{A_j}S(f)\|^q_p.$$

Hence by definition of amalgam space $c_p\|f\|_{H^S_{p,q}(\mathbb{R})} \leq \|f\|_{H^*_{p,q}(\mathbb{R})} \leq C_p\|f\|_{H^S_{p,q}(\mathbb{R})}$.

For the second equivalence, we replace the summation with the supremum. That is

$$\sup_{j\in\mathbb{Z}} \|\mathbf{1}_{A_j}S(f)\|^q_p \lesssim \sup_{j\in\mathbb{Z}} \|\mathbf{1}_{A_j}f^*\|^q_p \lesssim \sup_{j\in\mathbb{Z}} \|\mathbf{1}_{A_j}S(f)\|^q_p$$

and thus by definition of amalgam space, $c_p\|f\|_{H^S_{p,\infty}(\mathbb{R})} \leq \|f\|_{H^*_{p,\infty}(\mathbb{R})} \leq C_p\|f\|_{H^S_{p,\infty}(\mathbb{R})}$. $\square$

Let us now find an extension of Theorem 4.5.

Theorem 4.8. *Let $(\mathcal{B}_t,\ t \in \mathcal{U})$ be a sequence of σ-algebra (not necessarily monotone) such that $\sigma(\cup_{t\in\mathcal{U}}\mathcal{B}_t) = \mathcal{B}_t$ where $\mathcal{U}$ is a countable index set. Suppose that $\forall u \in L_p$, $(p >$*

1), *the following Doob's inequality holds*

$$\left(\mathbb{E}\left[\left(\sup_{t\in\mathbb{U}}|\mathbb{E}_t u|\right)^p\right]\right)^p \leq \theta_p(\mathbb{E}[|u|^p])^p$$

where $\mathbb{E}_t$ *is the conditional operator relative to* $\mathcal{B}_t$. *Let* $(h_t)_{t\in\mathcal{U}}$ *be sequence of non-negative measurable functions. Then for* $p \in [1, \infty)$, *we have*

$$\left\|\sum_{t\in\mathcal{U}} \mathbb{E}_t h_t\right\|_{L_{p,q}} \leq \theta_{p'}^{p} \left\|\sum_{t\in\mathcal{U}} h_t\right\|_{L_{p,q}}$$

and for $p \in (0, 1]$, *we have the reverse of the inequality;*

$$\left\|\sum_{t\in\mathcal{U}} h_t\right\|_{L_{p,q}} \leq \Theta_p \left\|\sum_{t\in\mathcal{U}} \mathbb{E}_t h_t\right\|_{L_{p,q}}$$

where (p, p') *are conjugate pairs and* Θ_p, θ_p *are positive constants.*

Proof. Let A_j be the usual interval and let $g_t = f_t\mathbf{1}_{A_j}$. Then g_t is also non-negative and measurable. Hence the conclusion of Theorem 4.5 holds for g_t. Hence

$$\begin{aligned}
\left\|\sum_t \mathbb{E}_t f_t\right\|_{p,q}^q &= \sum_j \left\|\sum_t \mathbb{E}_t f_t \mathbf{1}_{A_j}\right\|_p^q \\
&\leq \sum_j \theta_{p'}^{pq} \left\|\sum_t f_t \mathbf{1}_{A_j}\right\|_p^q = \theta_{p'}^{pq} \sum_j \left\|\sum_t f_t \mathbf{1}_{A_j}\right\|_p^q \\
&= \theta_{p'}^{pq} \left\|\sum_t f_t\right\|_{p,q}^q .
\end{aligned}$$

The first inequality follows by raising to the power $1/q$. The second inequality can be proved similarly by invoking the second inequality of Theorem 4.5. That is

$$\begin{aligned}
\left\|\sum_t \mathbb{E}_t f_t\right\|_{p,q}^q &= \sum_j \left\|\sum_t \mathbb{E}_t f_t \mathbf{1}_{A_j}\right\|_p^q \\
&\geq \sum_j \Theta_{p'}^{pq} \left\|\sum_t f_t \mathbf{1}_{A_j}\right\|_p^q = \Theta_{p'}^{pq} \sum_j \left\|\sum_t f_t \mathbf{1}_{A_j}\right\|_p^q \\
&= \Theta_{p'}^{pq} \left\|\sum_t f_t\right\|_{p,q}^q .
\end{aligned}$$

Raising to the power $1/q$ the second inequality is obtained. The proof is complete. ☐

4.9 Davis Decompositions

In this section, we want to focus on the Davis decomposition of martingales in the martingale Hardy-amalgam spaces $H^S_{p,q}$ and $H^*_{p,q}$. Let $f = \{f_n\}_{n\in\mathbb{N}}$ be a martingale. Then Davis decomposition asserts that if $f \in H^S_{pq}$ (resp. $H^*_{p,q}$), then $f = \{f_n\}_{n\in\mathbb{N}}$ can be written as the sum of two martingales $f = \{f_n = h_n + g_n\}_{n\in\mathbb{N}}$ where $h = \{h_n\}_{n\in\mathbb{N}} \in \mathcal{G}_{p,q}$ and $g = \{g_n\}_{n\in\mathbb{N}} \in \mathcal{Q}_{p,q}$ (resp. $g = \{g_n\}_{n\in\mathbb{N}} \in \mathcal{P}_{p,q}$). Let us start with the Davis decompositions of the martingale Hardy-amalgam space $H^S_{p,q}$.

Davis Decomposition of $H^S_{p,q}$

Theorem 4.10. *Let* $f = \{f_n\}_{n\in\mathbb{N}} \in H^S_{p,q}$ $(1 \leq p < \infty,\ 1 \leq q < \infty)$. *Then there exists* $h = \{h_n\}_{n\in\mathbb{N}} \in \mathcal{G}_{p,q}$ *and* $g = \{g_n\}_{n\in\mathbb{N}} \in \mathcal{Q}_{p,q}$ *such that* $f = \{f_n = h_n + g_n\}_{n\in\mathbb{N}}$ *for all* $n \in \mathbb{N}$ *and*

$$\|h\|_{\mathcal{G}_{p,q}} \leq (2 + 2C)\|f\|_{H^S_{p,q}}$$

and

$$\|g\|_{\mathcal{Q}_{p,q}} \leq (7 + 2C)\|f\|_{H^S_{p,q}}$$

Proof. Let $f = \{f_n\}_{n\in\mathbb{N}} \in H^S_{p,q}$. Suppose there exists an adapted increasing nonnegative sequence $\beta = \{\beta_n\}_{n\in\mathbb{N}}$ such that $S_n(f) \leq \beta_n$ and $\beta_\infty \in L_{p,q}$. Now since β is an increasing adapted sequence, there exists β_{n-1} such that $2\beta_{n-1} < \beta_n$ or $\beta_n \leq 2\beta_{n-1}$. Thus we can write the martingale difference $d_n f$ as

$$d_n f = d_n f \mathbf{1}_{\{\beta_n > 2\beta_{n-1}\}} + d_n f \mathbf{1}_{\{\beta_n \leq 2\beta_{n-1}\}}.$$

Also since f is a martingale, $\mathbb{E}_{k-1} d_k f = 0$. Therefore

$$\begin{aligned} f_n &= \sum_{k=1}^n f_k - f_{k-1} = \sum_{k=1}^n d_k f \\ &= \sum_{k=1}^n \left(d_k f \mathbf{1}_{\{\beta_k > 2\beta_{k-1}\}} + d_k f \mathbf{1}_{\{\beta_k \leq 2\beta_{k-1}\}} - \mathbb{E}_{k-1} d_k f\right) \\ &= \sum_{k=1}^n \left(d_k f \mathbf{1}_{\{\beta_k > 2\beta_{k-1}\}} + d_k f \mathbf{1}_{\{\beta_k \leq 2\beta_{k-1}\}} - \mathbb{E}_{k-1}(d_k f \mathbf{1}_{\{\beta_k > 2\beta_{k-1}\}} + d_k f \mathbf{1}_{\{\beta_k \leq 2\beta_{k-1}\}})\right) \\ &= \sum_{k=1}^n \left(d_k f \mathbf{1}_{\{\beta_k > 2\beta_{k-1}\}} - \mathbb{E}_{k-1}(d_k f \mathbf{1}_{\{\beta_k > 2\beta_{k-1}\}})\right) + \sum_{k=1}^n \left(d_k f \mathbf{1}_{\{\beta_k \leq 2\beta_{k-1}\}} - \mathbb{E}_{k-1} d_k f \mathbf{1}_{\{\beta_k \leq 2\beta_{k-1}\}})\right) \\ &= h_n + g_n \end{aligned}$$

where

$$h_n = \sum_{k=1}^{n} \left(d_k f \mathbf{1}_{\{\beta_k > 2\beta_{k-1}\}} - \mathbb{E}_{k-1}(d_k f \mathbf{1}_{\{\beta_k > 2\beta_{k-1}\}}) \right)$$

and

$$g_n = \sum_{k=1}^{n} \left(d_k f \mathbf{1}_{\{\beta_k \le 2\beta_{k-1}\}} - \mathbb{E}_{k-1} d_k f \mathbf{1}_{\{\beta_k \le 2\beta_{k-1}\}}) \right).$$

Let us now show that $h = \{h_n\}_{n \in \mathbb{N}} \in \mathcal{G}_{p,q}$ and $g = \{g_n\}_{n \in \mathbb{N}} \in \mathcal{Q}_{p,q}$. Now $S_n(f) \le \beta_n$ implies from definition that $\sum_{k=1}^{n} |d_k f|^2 \le \beta_n^2$. Therefore $|d_n f|^2 \le \beta_n^2$ and hence $|d_k f| \le \beta_k$ for all k. Also, $|d_k f| \mathbf{1}_{\{\beta_k > 2\beta_{k-1}\}} \le \beta_k \mathbf{1}_{\{\beta_k > 2\beta_{k-1}\}} \le \beta_k$. Now $\beta_k > 2\beta_{k-1}$ implies that $\beta_k < 2(\beta_k - \beta_{k-1})$ and therefore

$$|d_k f| \mathbf{1}_{\{\beta_k > 2\beta_{k-1}\}} \le \beta_k < 2(\beta_k - \beta_{k-1}).$$

We observe from h_n above that

$$\begin{aligned} h_n &= \sum_{k=1}^{n} \left(d_k f \mathbf{1}_{\{\beta_k > 2\beta_{k-1}\}} - \mathbb{E}_{k-1}(d_k f \mathbf{1}_{\{\beta_k > 2\beta_{k-1}\}}) \right) \\ &= d_n f \mathbf{1}_{\{\beta_n > 2\beta_{n-1}\}} - \mathbb{E}_{n-1}(d_n f \mathbf{1}_{\{\beta_n > 2\beta_{n-1}\}}) + \sum_{k=1}^{n-1} \left(d_k f \mathbf{1}_{\{\beta_k > 2\beta_{k-1}\}} - \mathbb{E}_{k-1}(d_k f \mathbf{1}_{\{\beta_k > 2\beta_{k-1}\}}) \right) \\ &= d_n f \mathbf{1}_{\{\beta_n > 2\beta_{n-1}\}} - \mathbb{E}_{n-1}(d_n f \mathbf{1}_{\{\beta_n > 2\beta_{n-1}\}}) + h_{n-1}. \end{aligned}$$

We then have that $h_n - h_{n-1} = d_n f \mathbf{1}_{\{\beta_n > 2\beta_{n-1}\}} - \mathbb{E}_{n-1}(d_n f \mathbf{1}_{\{\beta_n > 2\beta_{n-1}\}})$ and thus

$$\begin{aligned} |d_n h| &= |d_n f \mathbf{1}_{\{\beta_n > 2\beta_{n-1}\}} - \mathbb{E}_{n-1}(d_n f \mathbf{1}_{\{\beta_n > 2\beta_{n-1}\}})| \\ &\le |d_n f| \mathbf{1}_{\{\beta_n > 2\beta_{n-1}\}} + \mathbb{E}_{n-1}(|d_n f| \mathbf{1}_{\{\beta_n > 2\beta_{n-1}\}}) \\ &\le 2(\beta_n - \beta_{n-1}) + 2\mathbb{E}_{n-1}(\beta_n - \beta_{n-1}) \end{aligned}$$

and then

$$\begin{aligned} \sum_{k=1}^{\infty} |d_k h| &\le 2\sum_{k=1}^{\infty} (\beta_k - \beta_{k-1}) + 2\sum_{k=1}^{\infty} \mathbb{E}_{k-1}(\beta_k - \beta_{k-1}) \qquad (4.4) \\ &= 2\beta_\infty + 2\sum_{k=1}^{\infty} \mathbb{E}_{k-1}(\beta_k - \beta_{k-1}). \end{aligned}$$

Therefore

$$\begin{aligned} \left\| \sum_{k=1}^{\infty} |d_k h| \right\|_{p,q} &\le 2 \left\| \beta_\infty + \sum_{k=1}^{\infty} \mathbb{E}_{k-1}(\beta_k - \beta_{k-1}) \right\|_{p,q} \\ &\le 2 \left\| \beta_\infty \right\|_{p,q} + 2 \left\| \sum_{k=1}^{\infty} \mathbb{E}_{k-1}(\beta_k - \beta_{k-1}) \right\|_{p,q} \end{aligned}$$

and by Theorem 4.8 we have

$$\left\| \sum_{k=1}^{\infty} |d_k h| \right\|_{p,q} \leq 2 \left\| \beta_\infty \right\|_{p,q} + 2C \left\| \sum_{k=1}^{\infty} (\beta_k - \beta_{k-1}) \right\|_{p,q} .$$

Hence

$$\left\| \sum_{k=1}^{\infty} |d_k h| \right\|_{p,q} \leq 2 \left\| \beta_\infty \right\|_{p,q} + 2C \left\| \beta_\infty \right\|_{p,q} = (2+2C) \left\| \beta_\infty \right\|_{p,q} .$$

Now $\beta = \{\beta_n\}_{n\in\mathbb{N}}$ is arbitrary and since $S_n(f)$ is increasing, nonnegative and measurable for all n, we can set $\beta_n = S_n(f)$ and thus $\|\beta_\infty\|_{p,q} = \|S(f)\|_{p,q} := \|f\|_{H^S_{p,q}}$. Hence by definition

$$|h\|_{\mathcal{G}_{p,q}} \leq (2+2C)\|f\|_{H^S_{p,q}} < \infty$$

since $f = \{f_n\}_{n\in\mathbb{N}} \in H^S_{p,q}$. Thus $h = \{h_n\}_{n\in\mathbb{N}} \in \mathcal{G}_{p,q}$.

Now $|d_k f| \leq \beta_k$ for all k also implies that $|d_k f|\mathbf{1}_{\{\beta_k \leq 2\beta_{k-1}\}} \leq \beta_k \mathbf{1}_{\{\beta_k \leq 2\beta_{k-1}\}} \leq \beta_k$ and since $\beta_k \leq 2\beta_{k-1}$, we have $|d_k f|\mathbf{1}_{\{\beta_k \leq 2\beta_{k-1}\}} \leq 2\beta_{k-1}$. We also have that

$$g_n = g_{n-1} + d_n f \mathbf{1}_{\{\beta_n \leq 2\beta_{n-1}\}} - \mathbb{E}_{n-1}(|d_n f|\mathbf{1}_{\{\beta_n \leq 2\beta_{n-1}\}})$$

and thus

$$d_n g = g_n - g_{n-1} = d_n f \mathbf{1}_{\{\beta_n \leq 2\beta_{n-1}\}} - \mathbb{E}_{n-1}(d_n f \mathbf{1}_{\{\beta_n \leq 2\beta_{n-1}\}}).$$

Hence since β_{k-1} is $\mathcal{F}_{k-1}$-measurable

$$\begin{aligned} |d_n g| &= |d_n f \mathbf{1}_{\{\beta_n \leq 2\beta_{n-1}\}} - \mathbb{E}_{n-1}(d_n f \mathbf{1}_{\{\beta_n \leq 2\beta_{n-1}\}})| \\ &\leq |d_n f|\mathbf{1}_{\{\beta_n \leq 2\beta_{n-1}\}} + \mathbb{E}_{n-1}(|d_n f|\mathbf{1}_{\{\beta_n \leq 2\beta_{n-1}\}}) \\ &\leq 2\beta_{n-1} + 2\mathbb{E}_{n-1}\beta_{n-1} \\ &= 2\beta_{n-1} + 2\beta_{n-1} = 4\beta_{n-1}. \end{aligned}$$

Therefore $|d_k g| \leq 4\beta_{k-1}$ for all k. We have, by definition, that

$$\begin{aligned} S_n(g) &= \left(\sum_{k=1}^{n} |d_k g|^2 \right)^{\frac{1}{2}} = \left(\sum_{k=1}^{n-1} |d_k g|^2 + |d_n g|^2 \right)^{\frac{1}{2}} \\ &\leq \left(\sum_{k=1}^{n-1} |d_k g|^2 \right)^{\frac{1}{2}} + |d_n g| \leq S_{n-1}(g) + 4\beta_{n-1}. \end{aligned}$$

Now $f = h + g$ implies that $g = f - h$ and by identity (2.9), we have $S_{n-1}(f-h) \lesssim S_{n-1}(f) + S_{n-1}(h)$. Hence

$$S_n(g) \lesssim S_{n-1}(f) + S_{n-1}(h) + 4\beta_{n-1}.$$

By (4.4), we also get that

$$\begin{aligned} S_{n-1}(h) &= \left(\sum_{k=1}^{n-1}|d_k h|^2\right)^{\frac{1}{2}} \leq \sum_{k=1}^{n-1}|d_k h| \\ &\leq 2\beta_{n-1} + 2\sum_{k=1}^{n-1}\mathbb{E}_{k-1}(\beta_k - \beta_{k-1}). \end{aligned}$$

Thus since $S_{n-1}(f) \leq \beta_{n-1}$,

$$\begin{aligned} S_n(g) &\leq \beta_{n-1} + 2\beta_{n-1} + 2\sum_{k=1}^{n-1}\mathbb{E}_{k-1}(\beta_k - \beta_{k-1}) + 4\beta_{n-1} \\ &= 7\beta_{n-1} + 2\sum_{k=1}^{n-1}\mathbb{E}_{k-1}(\beta_k - \beta_{k-1}). \end{aligned}$$

Define $\gamma = \{\gamma_n\}_{n\in\mathbb{N}}$ as the sequence given by

$$\gamma_n = 7\beta_n + 2\sum_{k=1}^{n}\mathbb{E}_{k-1}(\beta_k - \beta_{k-1}).$$

Then as $\beta = \{\beta_n\}_{n\in\mathbb{N}}$ is an increasing, nonnegative adapted sequence, so is $\gamma = \{\gamma_n\}_{n\in\mathbb{N}}$. Therefore $S_n(g) \leq \gamma_{n-1}$. Hence

$$\begin{aligned} \|\gamma_\infty\|_{p,q} &= \left\|7\beta_\infty + 2\sum_{k=1}^{\infty}\mathbb{E}_{k-1}(\beta_k - \beta_{k-1})\right\|_{p,q} \\ &\leq \|7\beta_\infty\|_{p,q} + 2\left\|\sum_{k=1}^{\infty}\mathbb{E}_{k-1}(\beta_k - \beta_{k-1})\right\|_{p,q} \end{aligned}$$

and by Theorem 4.8 we have

$$\|\gamma_\infty\|_{p,q} \leq \|7\beta_\infty\|_{p,q} + 2C\left\|\sum_{k=1}^{\infty}(\beta_k - \beta_{k-1})\right\|_{p,q}.$$

Hence

$$\|\gamma_\infty\|_{p,q} \leq 7\|\beta_\infty\|_{p,q} + 2C\|\beta_\infty\|_{p,q} = (7+2C)\|\beta_\infty\|_{p,q}.$$

Now $\beta = \{\beta_n\}_{n\in\mathbb{N}}$ is arbitrary and since $S_n(f)$ is increasing, nonnegative and measurable for all n, we can set $\beta_n = S_n(f)$ and thus $\|\beta_\infty\|_{p,q} = \|S(f)\|_{p,q} := \|f\|_{H^S_{p,q}}$. Hence by definition

$$\|g\|_{\mathcal{Q}_{p,q}} \leq (7+2C)\|f\|_{H^S_{p,q}} < \infty$$

since $f = \{f_n\}_{n\in\mathbb{N}} \in H^S_{p,q}$. Thus $g = \{g_n\}_{n\in\mathbb{N}} \in \mathcal{Q}_{p,q}$. The proof is complete. □

Davis Decomposition of $H^*_{p,q}$

Theorem 4.11. *Let* $f = \{f_n\}_{n\in\mathbb{N}} \in H^*_{p,q}$ $(1 \le p < \infty,\ 1 \le q < \infty)$. *Then there exists* $h = \{h_n\}_{n\in\mathbb{N}} \in \mathcal{G}_{p,q}$ *and* $g = \{g_n\}_{n\in\mathbb{N}} \in \mathcal{P}_{p,q}$ *such that* $f = \{f_n = h_n + g_n\}_{n\in\mathbb{N}}$ *for all* $n \in \mathbb{N}$ *and*

$$\|h\|_{\mathcal{G}_{p,q}} \le (4 + 4C)\|f\|_{H^*_{p,q}}$$

and

$$\|g\|_{\mathcal{P}_{p,q}} \le (13 + 4C)\|f\|_{H^*_{p,q}}.$$

Proof. Let $f = \{f_n\}_{n\in\mathbb{N}} \in H^*_{p,q}$. Suppose there exists an adapted non-decreasing non-negative sequence $\beta = \{\beta_n\}_{n\in\mathbb{N}}$ such that $|f_n| \le \beta_n$ and $\beta_\infty \in L_{p,q}$. We choose h and g as we did in the proof of Theorem 4.10. Let us now show that $h = \{h_n\}_{n\in\mathbb{N}} \in \mathcal{G}_{p,q}$ and $g = \{g_n\}_{n\in\mathbb{N}} \in \mathcal{P}_{p,q}$. Now $|d_k f| = |f_k - f_{k-1}| \le |f_k| + |f_{k-1}| \le \beta_k + \beta_{k-1} \le \beta_k + \beta_k \le 2\beta_k$ and since $\beta_k \le 2(\beta_k - \beta_{k-1})$ we have that $|d_k f|\mathbf{1}_{\{\beta_k > 2\beta_{k-1}\}} \le 2\beta_k \le 4(\beta_k - \beta_{k-1})$. Then as in the proof of Theorem 4.10, we get that

$$|d_n h| \le 4(\beta_n - \beta_{n-1}) + 4\mathbb{E}_{n-1}(\beta_n - \beta_{n-1})$$

and therefore

$$\sum_{k=1}^{\infty} |d_k f| \le 4\beta_\infty + 4\sum_{k=1}^{\infty} \mathbb{E}_{n-1}(\beta_k - \beta_{k-1}).$$

Hence

$$\left\| \sum_{k=1}^{\infty} |d_k f| \right\|_{p,q} \le (4 + 4C)\|\beta_\infty\|_{p,q}.$$

Now since $\|f\|_{H^*_{p,q}} < \infty$ and $\beta = \{\beta_n\}_{n\in\mathbb{N}}$ is arbitrary, we can set $\beta_n = \sup_n |f_n|$ and thus $\|\beta_\infty\|_{p,q} = \|f^*\|_{p,q} := \|f\|_{H^*_{p,q}}$. Hence by definition

$$\|h\|_{\mathcal{G}_{p,q}} \le (4 + 4C)\|f\|_{H^*_{p,q}} < \infty$$

since $f = \{f_n\}_{n\in\mathbb{N}} \in H^*_{p,q}$. Thus $h = \{h_n\}_{n\in\mathbb{N}} \in \mathcal{G}_{p,q}$.

Now $|d_k f| \le 2\beta_k$ for all k also implies that $|d_k f|\mathbf{1}_{\{\beta_k \le 2\beta_{k-1}\}} \le 2\beta_k \mathbf{1}_{\{\beta_k \le 2\beta_{k-1}\}} \le 2\beta_k$ and since $\beta_k \le 2\beta_{k-1}$, we have $|d_k f|\mathbf{1}_{\{\beta_k \le 2\beta_{k-1}\}} \le 4\beta_{k-1}$. We also have that

$$g_n = g_{n-1} + d_n f \mathbf{1}_{\{\beta_n \le 2\beta_{n-1}\}} - \mathbb{E}_{n-1}(|d_n f|\mathbf{1}_{\{\beta_n \le 2\beta_{n-1}\}})$$

and thus

$$d_n g = g_n - g_{n-1} = d_n f \mathbf{1}_{\{\beta_n \le 2\beta_{n-1}\}} - \mathbb{E}_{n-1}(d_n f \mathbf{1}_{\{\beta_n \le 2\beta_{n-1}\}}).$$

Hence since β_{k-1} is $\mathcal{F}_{k-1}$-measurable

$$\begin{aligned} |d_n g| &= |d_n f \mathbf{1}_{\{\beta_n \le 2\beta_{n-1}\}} - \mathbb{E}_{n-1}(d_n f \mathbf{1}_{\{\beta_n \le 2\beta_{n-1}\}})| \\ &\le |d_n f| \mathbf{1}_{\{\beta_n \le 2\beta_{n-1}\}} + \mathbb{E}_{n-1}(|d_n f| \mathbf{1}_{\{\beta_n \le 2\beta_{n-1}\}}) \\ &\le 4\beta_{n-1} + 4\mathbb{E}_{n-1}\beta_{n-1} = 4\beta_{n-1} + 4\beta_{n-1} = 8\beta_{n-1}. \end{aligned}$$

Therefore $|d_k g| \le 8\beta_{k-1}$ for all k. Now

$$|g_n| = |g_{n-1} + g_n - g_{n-1}| \le |g_{n-1}| + |d_n g| \le |g_{n-1}| + 8\beta_{n-1}.$$

Since $g = f - h$, we have

$$|g_{n-1}| = |f_{n-1} - h_{n-1}| \le |f_{n-1}| + |h_{n-1}|.$$

Now

$$h_{n-1} \le \sum_{k=1}^{n-1} \left(|d_k f| \mathbf{1}_{\{\beta_k > 2\beta_{k-1}\}} + \mathbb{E}_{k-1}(|d_k f| \mathbf{1}_{\{\beta_k > 2\beta_{k-1}\}})\right)$$

and since $|d_k f| \mathbf{1}_{\{\beta_k > 2\beta_{k-1}\}} \le 4(\beta_k - \beta_{k-1})$ and the right side of the above inequality is positive, we have

$$\begin{aligned} |h_{n-1}| &\le \sum_{k=1}^{n-1} (4(\beta_k - \beta_{k-1}) + 4\mathbb{E}_{k-1}(\beta_k - \beta_{k-1})) \\ &= 4\beta_{n-1} + 4\sum_{k=1}^{n-1} \mathbb{E}_{k-1}(\beta_k - \beta_{k-1}). \end{aligned}$$

Therefore since $|f_{n-1}| < \beta_{n-1}$, we have

$$\begin{aligned} |g_n| &\le |g_{n-1}| + 8\beta_{n-1} \le |f_{n-1}| + |h_{n-1}| + 8\beta_{n-1} \\ &\le \beta_{n-1} + 4\beta_{n-1} + 4\sum_{k=1}^{n-1} \mathbb{E}_{k-1}(\beta_k - \beta_{k-1}) + 8\beta_{n-1} \\ &= 13\beta_{n-1} + 4\sum_{k=1}^{n-1} \mathbb{E}_{k-1}(\beta_k - \beta_{k-1}). \end{aligned}$$

Define $\gamma = \{\gamma_n\}_{n\in\mathbb{N}}$ as the sequence given by

$$\gamma_n = 13\beta_n + 4\sum_{k=1}^{n} \mathbb{E}_{k-1}(\beta_k - \beta_{k-1}).$$

Then as $\beta = \{\beta_n\}_{n\in\mathbb{N}}$ is an increasing, nonnegative adapted sequence, so is $\gamma = \{\gamma_n\}_{n\in\mathbb{N}}$.

Therefore $|g_n| \leq \gamma_{n-1}$. Hence

$$\|\gamma_\infty\|_{p,q} = \left\|13\beta_\infty + 4\sum_{k=1}^{\infty}\mathbb{E}_{k-1}(\beta_k - \beta_{k-1})\right\|_{p,q} \leq \|13\beta_\infty\|_{p,q} + 4\left\|\sum_{k=1}^{\infty}\mathbb{E}_{k-1}(\beta_k - \beta_{k-1})\right\|_{p,q}$$

and by Theorem 4.8 we have

$$\|\gamma_\infty\|_{p,q} \leq \|13\beta_\infty\|_{p,q} + 4C\left\|\sum_{k=1}^{\infty}(\beta_k - \beta_{k-1})\right\|_{p,q}.$$

Hence

$$\|\gamma_\infty\|_{p,q} \leq 13\,\|\beta_\infty\|_{p,q} + 4C\,\|\beta_\infty\|_{p,q} = (13+4C)\,\|\beta_\infty\|_{p,q}.$$

ow since $\|f\|_{H^*_{p,q}} < \infty$ and $\beta = \{\beta_n\}_{n\in\mathbb{N}}$ is arbitrary, we can set $\beta_n = \sup_n |f_n|$ and thus $\|\beta_\infty\|_{p,q} = \|f^*\|_{p,q} := \|f\|_{H^*_{p,q}}$. Hence by definition

$$\|g\|_{\mathcal{P}_{p,q}} \leq (13+4C)\|f\|_{H^*_{p,q}} < \infty$$

since $f = \{f_n\}_{n\in\mathbb{N}} \in H^*_{p,q}$. Thus $g = \{g_n\}_{n\in\mathbb{N}} \in \mathcal{P}_{p,q}$. The proof is complete. □

4.12 Martingale Embeddings

In this section we will discuss the various inclusions of the martingale Hardy-amalgam spaces $H^S_{p,q}(\mathbb{R})$, $H^s_{p,q}(\mathbb{R})$, $H^*_{p,q}(\mathbb{R})$, $\mathcal{Q}_{p,q}(\mathbb{R})$ and $\mathcal{P}_{p,q}(\mathbb{R})$. We recall here that the disjoint cover $(A_j)_{j\in\mathbb{Z}}$ is such that $A_j \in \mathcal{F}_n$ for all $j \in \mathbb{Z}$ and all $n \geq 1$.

The following classical result whose proof can be found in [65, Theorem 2.11] will become indispensable in the sequel.

Proposition 4.13 ([65]). *For any $f \in \mathcal{M}$, the following hold.*

(i) $\|f\|_{H^*_p(\mathbb{R})} \leq C_p\|f\|_{H^s_p(\mathbb{R})}$, $\|f\|_{H^S_p(\mathbb{R})} \leq C_p\|f\|_{H^s_p(\mathbb{R})}$ $(0 < p \leq 2)$

(ii) $\|f\|_{H^s_p(\mathbb{R})} \leq C_p\|f\|_{H^*_p(\mathbb{R})}$, $\|f\|_{H^s_p}(\mathbb{R}) \leq C_p\|f\|_{H^S_p(\mathbb{R})}$ $(2 \leq p < \infty)$

(iii) $\|f\|_{H^*_p(\mathbb{R})} \leq C_p\|f\|_{\mathcal{P}_p(\mathbb{R})}$, $\|f\|_{H^S_p(\mathbb{R})} \leq C_p\|f\|_{\mathcal{Q}_p(\mathbb{R})}$ $(0 < p < \infty)$

(iv) $\|f\|_{H^*_p(\mathbb{R})} \leq C_p\|f\|_{\mathcal{Q}_p(\mathbb{R})}$, $\|f\|_{H^S_p(\mathbb{R})} \leq C_p\|f\|_{\mathcal{P}_p(\mathbb{R})}$ $(0 < p < \infty)$

(v) $\|f\|_{H^s_p(\mathbb{R})} \leq C_p\|f\|_{\mathcal{P}_p(\mathbb{R})}$, $\|f\|_{H^s_p(\mathbb{R})} \leq C_p\|f\|_{\mathcal{Q}_p(\mathbb{R})}$ $(0 < p < \infty)$.

Moreover, for a regular stochastic basis, the five spaces are equivalent.

Let us now state and prove the theorem that extends the above embeddings to the martingale Hardy-amalgam space. We recall that all the martingales in this section are defined with respect to an underlying dyadic filtration.

Theorem 4.14 (Martingale Embeddings)**.** *Let* $0 < q \leq \infty$. *Then*

(i) $\|f\|_{H^*_{p,q}(\mathbb{R})} \leq C\|f\|_{H^s_{p,q}(\mathbb{R})}, \quad \|f\|_{H^S_{p,q}(\mathbb{R})} \leq C\|f\|_{H^s_{p,q}(\mathbb{R})} \qquad (0 < p \leq 2),$

(ii) $\|f\|_{H^s_{p,q}(\mathbb{R})} \leq C\|f\|_{H^*_{p,q}(\mathbb{R})}, \quad \|f\|_{H^s_{p,q}(\mathbb{R})} \leq C\|f\|_{H^S_{p,q}(\mathbb{R})} \qquad (2 \leq p < \infty).$

(iii) $\|f\|_{H^*_{p,q}(\mathbb{R})} \leq C\|f\|_{\mathcal{P}_{p,q}(\mathbb{R})}, \quad \|f\|_{H^S_{p,q}(\mathbb{R})} \leq C\|f\|_{\mathcal{Q}_{p,q}(\mathbb{R})} \qquad (0 < p < \infty),$

(iv) $\|f\|_{H^*_{p,q}(\mathbb{R})} \leq C\|f\|_{\mathcal{Q}_{p,q}(\mathbb{R})}, \quad \|f\|_{H^S_{p,q}(\mathbb{R})} \leq C\|f\|_{\mathcal{P}_{p,q}(\mathbb{R})} \qquad (0 < p < \infty),$

(v) $\|f\|_{H^s_{p,q}(\mathbb{R})} \leq C\|f\|_{\mathcal{P}_{p,q}(\mathbb{R})}, \quad \|f\|_{H^s_{p,q}(\mathbb{R})} \leq C\|f\|_{\mathcal{Q}_{p,q}(\mathbb{R})} \qquad (0 < p < \infty).$

Proof. We shall make use of Proposition 4.13. Let A_j be defined as equation (2.1).

(i) By definition,

$$\|f\|_{H^*_{p,q}} = \|f^*\|_{p,q} = \left(\sum_j \|f^*\mathbf{1}_{A_j}\|_p^q\right)^{\frac{1}{q}}.$$

We have seen that $g = f\mathbf{1}_{A_j}$ is a martingale. Also $g^* = f^*\mathbf{1}_{A_j}$. By Proposition 4.13, $\|g^*\|_p \leq C\|s(g)\|_p$ for $0 < p \leq 2$. Hence using equation (4.1),

$$\begin{aligned}
\|f\|_{H^*_{p,q}} &= \left(\sum_j \|f^*\mathbf{1}_{A_j}\|_p^q\right)^{\frac{1}{q}} \\
&= \left(\sum_j \|g^*\|_p^q\right)^{\frac{1}{q}} \leq C\left(\sum_j \|s(g)\|_p^q\right)^{\frac{1}{q}} = \left(\sum_j \|s(f)\mathbf{1}_{A_j}\|_p^q\right)^{\frac{1}{q}}.
\end{aligned}$$

But by definition

$$\left(\sum_j \|s(f)\mathbf{1}_{A_j}\|_p^q\right)^{\frac{1}{q}} = \|s(f)\|_{p,q} = \|f\|_{H^s_{p,q}}.$$

The first inequality is proved. To prove the second inequality, we let $g = f\mathbf{1}_{A_j}$. Then by Proposition 4.13, we have that $\|S(g)\|_p \leq C\|s(g)\|_p$ for $0 < p \leq 2$. Hence using equations (4.1) and (4.2),

$$\begin{aligned}
\|f\|_{H^S_{p,q}} &= \left(\sum_j \|S(f)\mathbf{1}_{A_j}\|_p^q\right)^{\frac{1}{p}} = \left(\sum_j \|S(f\mathbf{1}_{A_j})\|_p^q\right)^{\frac{1}{p}} \\
&\lesssim \left(\sum_j \|s(f\mathbf{1}_{A_j})\|_p^q\right)^{\frac{1}{p}} = \left(\sum_j \|s(f)\mathbf{1}_{A_j}\|_p^q\right)^{\frac{1}{p}} = \|f\|_{H^s_{p,q}}.
\end{aligned}$$

The second inequality is proved.

(ii) Considering the martingale $g = f\mathbf{1}_{A_j}$, we have by Proposition 4.13 that $\|s(f\mathbf{1}_{A_j})\|_p \leq \|f^*\mathbf{1}_{A_j}\|$ for $2 \leq p < \infty$. Hence

$$\|f\|_{H^s_{p,q}} = \left(\sum_j \|s(f\mathbf{1}_{A_j})\|_p^q\right)^{\frac{1}{p}} \leq \left(\sum_j \|f^*\mathbf{1}_{A_j}\|_p^q\right)^{\frac{1}{p}} = \|f^*\|_{p,q} = \|f\|_{H^*_{p,q}}$$

To prove the second inequality, we let $g = f\mathbf{1}_{A_j}$. Then by Proposition 4.13, we have that $\|s(g)\|_p \leq C\|S(g)\|_p$ for $2 \leq p < \infty$. Hence using equations (4.1) and (4.2),

$$\begin{aligned}\|f\|_{H^s_{p,q}} &= \left(\sum_j \|s(f)\mathbf{1}_{A_j}\|_p^q\right)^{\frac{1}{p}} = \left(\sum_j \|s(f\mathbf{1}_{A_j})\|_p^q\right)^{\frac{1}{p}} \\ &\leq \left(\sum_j \|S(f\mathbf{1}_{A_j})\|_p^q\right)^{\frac{1}{p}} = \left(\sum_j \|S(f)\mathbf{1}_{A_j}\|_p^q\right)^{\frac{1}{p}} = \|S(f)\|_{p,q}.\end{aligned}$$

The second is proved.

Let T be any of the operators s, S and M, and $H_p^T(\mathbb{R})$, $H_{p,q}^T(\mathbb{R})$ the corresponding martingale spaces, then since $A_j \in \mathcal{F}_n$ for all $j \in \mathbb{Z}$ and all $n \geq 1$ and by Lemma 4.1, we have that for any martingale $f \in \mathcal{M}$ and any $j \in \mathbb{Z}$,

$$\int_{\mathbb{R}} T(f)^p \mathbf{1}_{A_j} d\mathbb{P} = \int_{\mathbb{R}} T(f\mathbf{1}_{A_j})^p d\mathbb{P} = \|f\mathbf{1}_{A_j}\|^p_{H_p^T(\mathbb{R})},$$

and consequently,

$$\sum_j \|T(f)\mathbf{1}_{A_j}\|^q_{L_p(\mathbb{R})} = \sum_j \|f\mathbf{1}_{A_j}\|^q_{H_p^T(\mathbb{R})}. \tag{4.5}$$

To obtain the other assertions, we first prove the following

$$\sum_j \|f\mathbf{1}_{A_j}\|^q_{\mathcal{Q}_p(\mathbb{R})} \leq C\|f\|^q_{\mathcal{Q}_{p,q}(\mathbb{R})} \tag{4.6}$$

and

$$\sum_j \|f\mathbf{1}_{A_j}\|^q_{\mathcal{P}_p(\mathbb{R})} \leq C\|f\|^q_{\mathcal{P}_{p,q}(\mathbb{R})}. \tag{4.7}$$

We start with the proof of (4.6) as (4.7) follows similarly.

Let $(\varrho_n)_{n\geq 0}$ be an arbitrary nonnegative nondecreasing adapted sequence such that

$$S_n(f) \leq \varrho_{n-1}, \text{ and } \|\varrho_\infty\|_{L_{p,q}(\mathbb{R})} < \infty.$$

We have that the sequence $(\gamma_n^j)_{n\geq 0} = (\varrho_n\mathbf{1}_{A_j})_{n\geq 0}$ is also nonnegative nondecreasing and adapted, and

$$S_n(f\mathbf{1}_{A_j}) = S_n(f)\mathbf{1}_{A_j} \leq \varrho_{n-1}\mathbf{1}_{A_j} = \gamma^j_{n-1}$$

and

$$\|\gamma^j_\infty\|_{L_p(\mathbb{R})} = \|\varrho_\infty \mathbf{1}_{A_j}\|_{L_p(\mathbb{R})} \leq \|\varrho_\infty\|_{L_{p,q}(\mathbb{R})} < \infty.$$

It follows that

$$\sum_j \|f\mathbf{1}_{A_j}\|^q_{\mathcal{Q}_p(\mathbb{R})} \leq \sum_j \|\gamma^j_\infty\|^q_{L_p(\mathbb{R})} = \sum_j \|\varrho_\infty \mathbf{1}_{A_j}\|^q_{L_p(\mathbb{R})} = \|\varrho_\infty\|^q_{L_{p,q}(\mathbb{R})}.$$

As the sequence $(\varrho_n)_{n\geq 0}$ was chosen arbitrarily, we conclude that

$$\sum_j \|f\mathbf{1}_{A_j}\|^q_{\mathcal{Q}_p(\mathbb{R})} \leq \inf_{\varrho\in\rho} \|\varrho_\infty\|^q_{L_{p,q}(\mathbb{R})} = \|f\|^q_{\mathcal{Q}_{p,q}(\mathbb{R})}.$$

Also let $(\varrho_n)_{n\geq 0}$ be an arbitrary nonnegative nondecreasing adapted sequence such that

$$|f_n| \leq \varrho_{n-1}, \text{ and } \|\varrho_\infty\|_{L_{p,q}(\mathbb{R})} < \infty.$$

We have that the sequence $(\gamma^j_n)_{n\geq 0} = (\varrho_n \mathbf{1}_{A_j})_{n\geq 0}$ is also nonnegative nondecreasing and adapted, and

$$|f_n|\mathbf{1}_{A_j} \leq \varrho_{n-1}\mathbf{1}_{A_j} = \gamma^j_{n-1}$$

and

$$\|\gamma^j_\infty\|_{L_p(\mathbb{R})} = \|\varrho_\infty \mathbf{1}_{A_j}\|_{L_p(\mathbb{R})} \leq \|\varrho_\infty\|_{L_{p,q}(\mathbb{R})} < \infty.$$

It follows that

$$\sum_j \|f\mathbf{1}_{A_j}\|^q_{\mathcal{P}_p(\mathbb{R})} \leq \sum_j \|\gamma^j_\infty\|^q_{L_p(\mathbb{R})} = \sum_j \|\varrho_\infty \mathbf{1}_{A_j}\|^q_{L_p(\mathbb{R})} = \|\varrho_\infty\|^q_{L_{p,q}(\mathbb{R})}.$$

As the sequence $(\varrho_n)_{n\geq 0}$ was chosen arbitrarily, we conclude that

$$\sum_j \|f\mathbf{1}_{A_j}\|^q_{\mathcal{P}_p(\mathbb{R})} \leq \inf_{\varrho\in\rho} \|\varrho_\infty\|^q_{L_{p,q}(\mathbb{R})} = \|f\|^q_{\mathcal{P}_{p,q}(\mathbb{R})}.$$

(iii) Now from Proposition 4.13, we have that $\|f\|_{H^*_p(\mathbb{R})} \leq C_p\|f\|_{\mathcal{P}_p(\mathbb{R})}$, $\|f\|_{H^S_p(\mathbb{R})} \leq C_p\|f\|_{\mathcal{Q}_p(\mathbb{R})}$ $(0 < p < \infty)$. Following (4.5) and (4.7), we have

$$\|f\|^q_{H^*_{p,q}} = \sum_{j\in\mathbb{Z}} \|f^*\mathbf{1}_{A_j}\|^q_p \lesssim \sum_{j\in\mathbb{Z}} \|f\mathbf{1}_{A_j}\|^q_{\mathcal{P}_p} \lesssim \|f\|^q_{\mathcal{P}_{p,q}}.$$

Also from (4.5) and (4.6), we have

$$\|f\|^q_{H^S_{p,q}} = \sum_{j\in\mathbb{Z}} \|S(f)\mathbf{1}_{A_j}\|^q_p \lesssim \sum_{j\in\mathbb{Z}} \|f\mathbf{1}_{A_j}\|^q_{\mathcal{Q}_p} \lesssim \|f\|^q_{\mathcal{Q}_{p,q}}.$$

(iv) Also, from Proposition 4.13, we have that $\|f\|_{H^*_p(\mathbb{R})} \leq C_p\|f\|_{\mathcal{Q}_p(\mathbb{R})}$, $\|f\|_{H^S_p(\mathbb{R})} \leq C_p\|f\|_{\mathcal{P}_p(\mathbb{R})}$ $(0 < p < \infty)$. Following (4.5) and (4.7), we have

$$\|f\|^q_{H^*_{p,q}} = \sum_{j\in\mathbb{Z}} \|f^*\mathbf{1}_{A_j}\|^q_p \lesssim \sum_{j\in\mathbb{Z}} \|f\mathbf{1}_{A_j}\|^q_{\mathcal{Q}_p} \lesssim \|f\|^q_{\mathcal{Q}_{p,q}}.$$

Also from (4.5) and (4.6), we have

$$\|f\|^q_{H^S_{p,q}} = \sum_{j\in\mathbb{Z}} \|S(f)\mathbf{1}_{A_j}\|^q_p \lesssim \sum_{j\in\mathbb{Z}} \|f\mathbf{1}_{A_j}\|^q_{\mathcal{P}_p} \lesssim \|f\|^q_{\mathcal{P}_{p,q}}.$$

(v) Finally, from Proposition 4.13, we have that $\|f\|_{H^s_p(\mathbb{R})} \leq C_p\|f\|_{\mathcal{P}_p(\mathbb{R})}$, $\|f\|_{H^s_p(\mathbb{R})} \leq C_p\|f\|_{\mathcal{Q}_p(\mathbb{R})}$ $(0 < p < \infty)$. Following (4.5) and (4.7), we have

$$\|f\|^q_{H^s_{p,q}} = \sum_{j\in\mathbb{Z}} \|s(f)\mathbf{1}_{A_j}\|^q_p \lesssim \sum_{j\in\mathbb{Z}} \|f\mathbf{1}_{A_j}\|^q_{\mathcal{P}_p} \lesssim \|f\|^q_{\mathcal{P}_{p,q}}.$$

Also from (4.5) and (4.6), we have

$$\|f\|^q_{H^s_{p,q}} = \sum_{j\in\mathbb{Z}} \|s(f)\mathbf{1}_{A_j}\|^q_p \lesssim \sum_{j\in\mathbb{Z}} \|f\mathbf{1}_{A_j}\|^q_{\mathcal{Q}_p} \lesssim \|f\|^q_{\mathcal{Q}_{p,q}}.$$

The proof is complete. □

As an application of Davis decomposition, we can improve upon Theorem 4.14 (ii). This is stated as the corollary below.

Corollary 4.15. *Let* $f = \{f_n\}_{n\in\mathbb{N}} \in \Psi$ *where* $\Psi \in \{H^*_{p,q}, H^S_{p,q}\}$ $(1 \leq p < \infty,\ 0 < q < \infty)$. *Then there exists* $g \in H^s_{p,q}$ *and* $h \in \mathcal{G}_{p,q}$ *such that* $f_n = h_n + g_n$ *for all* $n \in \mathbb{N}$ *and*

$$\|g\|_{H^s_{p,q}} \lesssim \|f\|_\Psi$$

Proof. Let $\Psi = H^*_{p,q}$. Then by Theorem 4.11, $f_n = h_n + g_n$ where $h_n \in \mathcal{G}_{p,q}$ and $g_n \in \mathcal{P}_{p,q}$ and also

$$\|g\|_{\mathcal{P}_{p,q}} \leq (13 + 4C)\|f\|_{H^*_{p,q}}.$$

Now by (v) of Theorem 4.14

$$\|g\|_{H^s_{p,q}} \lesssim \|g\|_{\mathcal{P}_{p,q}}.$$

Thus

$$\|g\|_{H^s_{p,q}} \lesssim \|g\|_{\mathcal{P}_{p,q}} \lesssim \|f\|_{H^*_{p,q}}.$$

Let $\Psi = H^S_{p,q}$. Then by Theorem 4.10, $f_n = h_n + g_n$ where $h_n \in \mathcal{G}_{p,q}$ and $g_n \in \mathcal{Q}_{p,q}$ and also

$$\|g\|_{\mathcal{Q}_{p,q}} \leq (7 + 2C)\|f\|_{H^S_{p,q}}.$$

Now by (v) of Theorem 4.14

$$\|g\|_{H^s_{p,q}} \lesssim \|g\|_{\mathcal{Q}_{p,q}}.$$

Thus

$$\|g\|_{H^s_{p,q}} \lesssim \|g\|_{\mathcal{Q}_{p,q}} \lesssim \|f\|_{H^S_{p,q}}.$$

□

Let us now look at Theorem 4.17 that establishes the equivalence of the five martingale Hardy-amalgam spaces. Lemma 4.16 below is essentially [65, Lemma 2.20]. It is restated for completeness sake since it will be essential in the proof of Theorem 4.17 that follows.

Lemma 4.16 ([65]). *Let* $0 < p < \infty$. *Then for an arbitrary martingale* $f = \{f_n\}_{n\in\mathbb{N}}$,

$$\left\|\sup_{n\in\mathbb{N}} \mathbb{E}_{n-1}|f_n|^p\right\|_{L_1} \leq 2\|f^*\|^p_{L_p}$$

and

$$\left\|\sup_{n\in\mathbb{N}} \mathbb{E}_{n-1}S^p_n(f)\right\|_{L_1} \leq 2\|S(f))\|^p_{L_p}.$$

Theorem 4.17. *For* $0 < p < \infty$ *and* $0 < q < \infty$, *if the stochastic basis is regular, then the spaces* $H^S_{p,q}$, $H^s_{p,q}$, $H^*_{p,q}$, $\mathcal{Q}_{p,q}$ *and* $\mathcal{P}_{p,q}$ *are equivalent.*

Proof. Suppose that the stochastic basis be regular and let that $f \in \mathcal{M}$ where f is defined with respect to a dyadic filtration. Then by Remark 2.10, f is previsible. Hence we have that for some $R > 0$, $|d_nf|^p \leq R\mathbb{E}_{n-1}|d_nf|^p$. Consequently, since $f_n \leq f_{n-1} + |d_nf|$, we have that for $0 < p < \infty$, $|f_n|^p \leq C(|f_{n-1}|^p + |d_nf|^p)$ and thus by regularity,

$$|f_n|^p \leq C\left(|f_{n-1}|^p + R\mathbb{E}_{n-1}|d_nf|^p\right) \leq C\left(f^{*p}_{n-1} + \mathbb{E}_{n-1}|f_n|^p\right).$$

Let $\beta^p_{n-1} = f^{*p}_{n-1} + \mathbb{E}_{n-1}|f_n|^p$. Then $\beta^p_\infty \lesssim f^{*p} + \sup_{n\in\mathbb{N}} \mathbb{E}_{n-1}|f_n|^p$. Multiplying through by $\mathbf{1}_{A_j}$, we obtian $\beta^p_\infty \mathbf{1}_{A_j} \lesssim f^{*p}\mathbf{1}_{A_j} + \sup_{n\in\mathbb{N}} \mathbb{E}_{n-1}|f_n\mathbf{1}_{A_j}|^p$. Hence

$$\mathbb{E}(\beta^p_\infty \mathbf{1}_{A_j}) \lesssim \mathbb{E}(f^{*p}\mathbf{1}_{A_j}) + \mathbb{E}\left(\sup_{n\in\mathbb{N}} \mathbb{E}_{n-1}|f_n\mathbf{1}_{A_j}|^p\right).$$

Therefore by Lemma 4.16, we obtain

$$\mathbb{E}(\beta_\infty^p \mathbf{1}_{A_j}) \lesssim \mathbb{E}(f^{*p}\mathbf{1}_{A_j}) + 2\mathbb{E}(f^{*p}\mathbf{1}_{A_j}) = 3C\mathbb{E}(f^{*p}\mathbf{1}_{A_j}).$$

In other words $\|\beta_\infty \mathbf{1}_{A_j}\|_p^p \le 3C\|f^*\mathbf{1}_{A_j}\|_p^p$ and thus

$$\|\beta_\infty \mathbf{1}_{A_j}\|_p^q \le (3C)^{\frac{q}{p}}\|f^*\mathbf{1}_{A_j}\|_p^q.$$

Therefore

$$\left(\sum_{j\in\mathbb{Z}} \|\beta_\infty \mathbf{1}_{A_j}\|_p^q\right)^{\frac{1}{q}} \le (3C)^{\frac{1}{p}} \left(\sum_{j\in\mathbb{Z}} \|f^*\mathbf{1}_{A_j}\|_p^q\right)^{\frac{1}{q}}.$$

Hence

$$\|\beta_\infty\|_{p,q} \le (3C)^{\frac{1}{p}}\|f^*\|_{p,q}.$$

Since $|f_n| \le \beta_{n-1}$, we have by definition that

$$\|f\|_{\mathcal{P}_{p,q}} \le (3C)^{\frac{1}{p}}\|f\|_{H^*_{p,q}}. \tag{4.8}$$

Now by definition, $S_n(f) - S_{n-1}(f) = |d_n f|^2 \le R\mathbb{E}_{n-1}|d_n f|^2$. Thus

$$S_n(f) \le S_{n-1}(f) + R\mathbb{E}_{n-1}|d_n f|^2 = S_{n-1}(f) + R\mathbb{E}_{n-1}[S_n(f) - S_{n-1}(f)].$$

Hence we obtain that

$$S_n(f)^p \le C\left(S_{n-1}(f)^p + \mathbb{E}_{n-1}S_n(f)^p\right).$$

Let $\beta_{n-1}^p = S_{n-1}(f)^p + \mathbb{E}_{n-1}S_n(f)^p$. Then $\beta_\infty^p \lesssim S^p(f) + \sup_{n\in\mathbb{N}}\mathbb{E}_{n-1}S_n^p(f)$. Multiplying through by $\mathbf{1}_{A_j}$, we obtian $\beta_\infty^p\mathbf{1}_{A_j} \lesssim S^p(f)\mathbf{1}_{A_j} + \sup_{n\in\mathbb{N}}\mathbb{E}_{n-1}|S_n(f)\mathbf{1}_{A_j}|^p$. Hence

$$\mathbb{E}(\beta_\infty^p\mathbf{1}_{A_j}) \lesssim \mathbb{E}(S^p(f)\mathbf{1}_{A_j}) + \mathbb{E}\left(\sup_{n\in\mathbb{N}}\mathbb{E}_{n-1}|S_n(f)\mathbf{1}_{A_j}|^p\right).$$

Therefore by Lemma 4.16, we obtain

$$\mathbb{E}(\beta_\infty^p\mathbf{1}_{A_j}) \lesssim \mathbb{E}(S^p(f)\mathbf{1}_{A_j}) + 2\mathbb{E}(S^p(f)\mathbf{1}_{A_j}) = 3C\mathbb{E}(S^p(f)\mathbf{1}_{A_j}).$$

In other words $\|\beta_\infty\mathbf{1}_{A_j}\|_p^p \le 3C\|S(f)\mathbf{1}_{A_j}\|_p^p$ and thus

$$\|\beta_\infty\mathbf{1}_{A_j}\|_p^q \le (3C)^{\frac{q}{p}}\|S(f)\mathbf{1}_{A_j}\|_p^q.$$

Therefore

$$\left(\sum_{j\in\mathbb{Z}} \|\beta_\infty \mathbf{1}_{A_j}\|_p^q\right)^{\frac{1}{q}} \le (3C)^{\frac{1}{p}} \left(\sum_{j\in\mathbb{Z}} \|S(f)\mathbf{1}_{A_j}\|_p^q\right)^{\frac{1}{q}}.$$

Hence

$$\|\beta_\infty\|_{p,q} \le (3C)^{\frac{1}{p}} \|S(f)\|_{p,q}.$$

Since $S_n(f) \le \beta_{n-1}$, we have by definition that

$$\|f\|_{\mathcal{Q}_{p,q}} \le (3C)^{\frac{1}{p}} \|f\|_{H^*_{p,q}}. \tag{4.9}$$

Also, by Remark 2.10,

$$S(f) \le R^{\frac{1}{2}} s(f) \Longrightarrow \|S(f)\|_p \le C\|s(f)\|_p \Longrightarrow \|S(f)\|_{p,q} \le C\|s(f)\|_{p,q}.$$

Thus, by definition,

$$\|f\|_{H^S_{p,q}} \le C\|f\|_{H^s_{p,q}}. \tag{4.10}$$

Combining inequalities (4.8), (4.9) and (4.10), together with Theorem 4.14, we obtain the following

$$\|f\|_{H^s_{p,q}} \le C\|f\|_{H^S_{p,q}} \le C\|f\|_{H^*_{p,q}} \le C\|f\|_{\mathcal{P}_{p,q}} \le C\|f\|_{\mathcal{Q}_{p,q}} \le C\|f\|_{H^s_{p,q}}. \tag{4.11}$$

We have thus established the theorem. □

Chapter 5

Dual Space Characterizations

Let $\mathcal{X}$ be an arbitrary Banach space. Then we say that the dual of $\mathcal{X}$ is the space of all linear functionals on $\mathcal{X}$. In this chapter, we will characterize the dual spaces of the martingale Hardy-amalgams spaces $H^s_{p,q}$ and $\mathcal{G}_{p,q}$. The Garsia-type space, $\mathcal{G}_{p,q}$, is a component of the Davis decomposition of martingales and a justification of this was provided in the previous chapter. We shall, however, show in this chapter that the dual of the Garsia-type space $\mathcal{G}_{p,q}$ is the jump bounded space $\mathcal{BD}_{p',q'}$. As an application of the Garsia-type space in conjunction with the Davis decomposition of $H^*_{p,q}$, the dual space of $H^*_{p,q}$ is characterized as well in this chapter. The atomic decomposition of $H^s_{p,q}$ discussed in the previous chapter will play a crucial role in characterizing the dual space of $H^s_{p,q}$ $(0 < p \leq q \leq 1)$. Before we start characterzing the dual space of $H^s_{p,q}$, there is one more important space, the Campanato space, that we have to recall [56]. This space happens to be the dual space of $H^s_{p,q}$ when $0 < p \leq q \leq 1$.

The Campanato Space

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Denote by L^0_2 the set of all $f \in L_2$ such that $\mathbb{E}_0 f = 0$. For $f \in L^0_2$, put $f_n = \mathbb{E}_n f$. We recall that $(f_n)_{n\geq 0}$ is in $\mathcal{M}$ and L_2-bounded. Moreover, $(f_n)_{n\geq 0}$ converges to f in L_2 as pointed out in [54]. Define the function $\phi : \mathcal{F} \longrightarrow (0, \infty)$ by

$$\phi(A) = \frac{\|\mathbf{1}_A\|_{p,q}}{\mathbb{P}(A)}$$

for all $A \in \mathcal{F}$, $P(A) \neq 0$.

The *Campanato space*, $\mathcal{L}_{2,\phi}$, is then defined to be the set

$$\mathcal{L}_{2,\phi} := \left\{ f \in L^0_2 : \|f\|_{\mathcal{L}_{2,\phi}} := \sup_{\nu \in \mathcal{T}} \frac{1}{\phi(B_\nu)} \left(\frac{1}{\mathbb{P}(B_\nu)} \int_{B_\nu} |f - f^\nu|^2 d\mathbb{P} \right)^{\frac{1}{2}} < \infty \right\}.$$

5.1 Dual Space Characterization of $H^s_{p,q}$

We recall that $H^s_{p,q}(\Omega)$ is the space of all $f \in \mathcal{M}$ such that $s(f) \in L_{p,q}(\Omega)$ with (quasi)-norm

$$\|f\|_{H^s_{p,q}(\Omega)} := \|s(f)\|_{L_{p,q}(\Omega)}.$$

In this section, the dual space characterization of $H^s_{p,q}$ is discussed. The method employed does not permit the establishment of the dual space of martingale Hardy-amalgam space $H^s_{p,q}$ on the whole range $0 < p, q < \infty$. Instead, the range $0 < p, q < \infty$ is divided into subintervals and the dual spaces are obtained for each subinterval. We first characterize the dual for the interval $0 < p \le q \le 1$ in Section 5.1.1. In Section 5.3.1, we will characterize the dual space of $H^s_{p,q}$ for $1 < q \le p < 2$ or $2 \le p \le q < \infty$ (see Theorem 5.6). It is emphasized here that martingales defined in Section 5.3.1 are with respect to an underlying dyadic filtration.

5.1.1 Duality of $H^s_{p,q}$ for $0 < p \le q \le 1$

We recall from the atomic decomposition of $H^s_{p,q}$ that, for a $(p,r)^s$-atom $a = \{a^k\}_{k\in\mathbb{Z}}$, we have that $s(a^k) \le \mathbb{P}(B_{\nu^k})^{-\frac{1}{p}}$ and $\operatorname{supp}(s(a^k)) \subseteq B_{\nu^k}$ for some stopping time ν^k where $B_{\nu^k} = \{\nu^k \ne \infty\}$. Hence $s(a^k) = s(a^k)\mathbf{1}_{B_{\nu^k}}$ and thus $s(a^k) = s(a^k)\mathbf{1}_{B_{\nu^k}} \le \mathbb{P}(B_{\nu^k})^{-\frac{1}{p}}\mathbf{1}_{B_{\nu^k}}$. Therefore

$$\|s(a^k)\|_r \le \|\mathbb{P}(B_{\nu^k})^{-\frac{1}{p}}\mathbf{1}_{B_{\nu^k}}\|_r = \mathbb{P}(B_{\nu^k})^{\frac{1}{r}-\frac{1}{p}}$$

for $r > \max(p, 1)$. We will show that the dual of $H^s_{p,q}$ $(0 < p \le q \le 1)$, is the Campanato space, $\mathcal{L}_{2,\phi}$. We start by introducing the following result which is essentially [24, Proposition 2.1]. We provide a proof for the sake of completeness.

Proposition 5.2 ([24]). *Let $0 < p < 1$ and $0 < q \le 1$. For all finite sequence $\{f_n\}_{n=-m}^m$ of elements in $L_{p,q}(\Omega)$, we have*

$$\sum_{n=-m}^{m} \|f_n\|_{p,q} \le \left\| \sum_{n=-m}^{m} |f_n| \right\|_{p,q}.$$

Proof. Let $0 < p < 1$, $0 < q \le 1$ and let $\{f_n\}_{n=0}^m$ be a finite sequence of elements of $L_{p,q}(\Omega)$. For $q = 1$, using the reverse Minkowski's inequality in L_p ([27, p. 11-12]), we obtain

$$\sum_{n=-m}^{m} \|f_n\|_{p,1} = \sum_{j\in\mathbb{Z}} \left\| \sum_{n=-m}^{m} |f_n \mathbf{1}_{\Omega_j}| \right\|_p \le \left\| \sum_{n=-m}^{m} |f_n| \right\|_{p,1}.$$

Now assume that $0 < q < 1$ and set

$$x_n := \left\{ \|f_n \mathbf{1}_{\Omega_j}\|_p \right\}_{j \in \mathbb{Z}} \quad \forall n = -m, \dots, m.$$

Applying the reverse Minkowski's inequality in ℓ^q and L_p, we obtain

$$\begin{aligned} \sum_{n=-m}^{m} \|f_n\|_{p,q} &= \sum_{n=-m}^{m} \|x_n\|_{\ell^q} \leq \left\| \left\{ \sum_{n=-m}^{m} \|f_n \mathbf{1}_{\Omega_j}\|_p \right\}_{j\in\mathbb{Z}} \right\|_{\ell^q} \\ &\leq \left\| \left\{ \left\| \sum_{n=-m}^{m} f_n \mathbf{1}_{\Omega_j} \right\|_p \right\}_{j\in\mathbb{Z}} \right\|_{\ell^q} = \left\| \sum_{n=-m}^{m} |f_n| \right\|_{p,q}. \end{aligned}$$

□

The characterization of the dual space of $H^s_{p,q}$ for $0 < p \leq q \leq 1$ is as follows.

Theorem 5.3. *Let $0 < p \leq q \leq 1$. For $\kappa \in (H^s_{p,q})^*$, the dual space of $H^s_{p,q}$, there exists $g \in \mathcal{L}_{2,\phi}$ such that*

$$\kappa(f) = \mathbb{E}[fg] \quad \text{for all} \quad f \in H^s_{p,q}$$

and

$$\|g\|_{\mathcal{L}_{2,\phi}} \leq c\|\kappa\|.$$

Conversely, let $g \in \mathcal{L}_{2,\phi}$. Then the mapping

$$\kappa_g(f) = \mathbb{E}[fg] = \int_\Omega fg \, d\mathbb{P}, \quad \forall f \in L_2(\Omega)$$

can be extended to a continuously linear functional on $H^s_{p,q}$ such that

$$\|\kappa\| \leq c\|g\|_{\mathcal{L}_{2,\phi}}.$$

Proof. Let us start by defining some spaces. For ν a stopping time, we define

$$L_2^\nu(\Omega) := \{f \in L_2(\Omega) : \ \mathbb{E}_n(f) = 0, \ \text{for } \nu \geq n, \ n \in \mathbb{N}\}$$

and

$$L_2^\nu(B_\nu) := \{f \in L_2^\nu(\Omega) : \text{supp}(f) \subseteq B_\nu\}.$$

We endow $L_2^\nu(B_\nu)$ with

$$\|f\|_{L_2^\nu(B_\nu)} := \left(\int_{B_\nu} |f|^2 d\mathbb{P} \right)^{\frac{1}{2}} < \infty.$$

We will first prove that any continuous linear functional on $H^s_{p,q}(\Omega)$ is also continuous on $L^\nu_2(B_\nu)$.

Let $f \in L^\nu_2(B_\nu) \setminus \{0\}$ and consider

$$a(\omega) := \frac{C\mathbb{P}(B_\nu)^{\frac{1}{2}-\frac{1}{p}}}{\|f\|_{L^\nu_2(B_\nu)}} f(\omega), \quad \omega \in \Omega.$$

Then a is zero outside of B_ν and $s(a) = \frac{C\mathbb{P}(B_\nu)^{\frac{1}{2}-\frac{1}{p}}}{\|f\|_{L^\nu_2(B_\nu)}} s(f)$. Hence

$$\|s(a)\|_2 = \frac{C\mathbb{P}(B_\nu)^{\frac{1}{2}-\frac{1}{p}}}{\|f\|_{L^\nu_2(B_\nu)}} \|s(f)\|_2.$$

Then for an appropriate choice of the constant, we can choose $C \leq \frac{\|f\|_{L^\nu_2(B_\nu)}}{\|s(f)\|_2}$, we obtain $\|s(a)\|_2 \leq \mathbb{P}(B_\nu)^{\frac{1}{2}-\frac{1}{p}}$. Thus a is an $(p,2)^s$-atom associated to $\nu \in \mathcal{T}$. Since a is $(p,2)^s$-atom, we observe from Theorem 3.4 that $s(a) \lesssim \mathbb{P}(B_\nu)^{-\frac{1}{p}}$ and thus $\|s(a)\|_{p,q} \lesssim \|\mathbf{1}_{B_\nu}\|_{p,q}\mathbb{P}(B_\nu)^{-\frac{1}{p}}$. Hence by definition,

$$\|a\|_{H^s_{p,q}(\Omega)} \lesssim \|\mathbf{1}_{B_\nu}\|_{p,q}\mathbb{P}(B_\nu)^{-\frac{1}{p}},$$

and recalling from the hypothesis that $p \leq q$, we obtain

$$\begin{aligned}\|f\|_{H^s_{p,q}(\Omega)} &= C^{-1}\|f\|_{L^\nu_2(B_\nu)}\mathbb{P}(B_\nu)^{\frac{1}{p}-\frac{1}{2}}\|a\|_{H^s_{p,q}(\Omega)} \\ &\lesssim \|f\|_{L^\nu_2(B_\nu)}\mathbb{P}(B_\nu)^{\frac{1}{p}-\frac{1}{2}}\|\mathbf{1}_{B_\nu}\|_{p,q}\mathbb{P}(B_\nu)^{-\frac{1}{p}} \\ &\lesssim \|f\|_{L^\nu_2(B_\nu)}\mathbb{P}(B_\nu)^{\frac{1}{p}-\frac{1}{2}}.\end{aligned}$$

It follows that for any continuous linear functional κ on $H^s_{p,q}(\Omega)$ with operator norm $\|\kappa\|$,

$$|\kappa(f)| \leq \|\kappa\|\|f\|_{H^s_{p,q}(\Omega)} \lesssim \|\kappa\|\mathbb{P}(B_\nu)^{\frac{1}{p}-\frac{1}{2}}\|f\|_{L^\nu_2(B_\nu)}.$$

Hence κ is continuous on $L^\nu_2(B_\nu)$ with operator norm

$$\|\kappa\|_{(L^\nu_2(B_\nu))^*} := \sup_{\substack{f\in L^\nu_2(B_\nu)\\ \|f\|_{L^\nu_2(B_\nu)}\leq 1}} \|\kappa(f)\| \lesssim \mathbb{P}(B_\nu)^{\frac{1}{p}-\frac{1}{2}}\|\kappa\|.$$

As $L^\nu_2(B_\nu)$ is a subspace of $L_2(B_\nu) = L_2(B_\nu, d\mathbb{P})$, it follows from the above observation and the Hahn-Banach Theorem that any continuous linear functional κ on $H^s_{p,q}(\Omega)$ can be extended to a continuous linear functional κ_ν on $L_2(B_\nu)$. As $L_2(B_\nu)$ is auto-dual, it follows that there exists $g \in L_2(B_\nu)$ such that

$$\kappa_\nu(f) = \int_{B_\nu} fg\, d\mathbb{P}, \quad \forall f \in L_2(B_\nu).$$

Consequently,

$$\kappa(f) = \kappa_\nu(f) = \int_{B_\nu} fg\,d\mathbb{P}, \quad \forall f \in L_2^\nu(B_\nu).$$

Next, as $L_2(\Omega)$ is a dense in $H^s_{p,q}(\Omega)$ (this follows from the fact that $p \leq q < 2$ and Theorem 3.4), we have that any element κ of the dual space of $H^s_{p,q}(\Omega)$ can be represented by

$$\kappa(f) = \int_{\Omega} fg\,d\mathbb{P}, \quad \forall f \in L_2(\Omega). \tag{5.1}$$

We are going to prove that the function g in (5.1) is in $\mathcal{L}_{2,\phi}(\Omega)$.

Let $\nu \in \mathcal{T}$ and let $f \in L_2^\nu(B_\nu)$ with $\|f\|_{L_2^\nu(B_\nu)} \leq 1$. Define

$$\rho(\omega) = C\mathbb{P}(B_\nu)^{\frac{1}{2}-\frac{1}{p}} \frac{(f-f^\nu)\mathbf{1}_{B_\nu}(\omega)}{\|(f-f^\nu)\mathbf{1}_{B_\nu}\|_{L_2(\Omega)}}, \quad \omega \in \Omega.$$

Then for an appropriate choice of the constant, we can choose $C \leq \frac{\|(f-f^\nu)\mathbf{1}_{B_\nu}\|_{L_2(\Omega)}}{\|s((f-f^\nu)\mathbf{1}_{B_\nu})\|_{L_2(\Omega)}}$, ρ is $(p,2)^s$-atom associated to the stopping time ν and $\rho \in L_2(\Omega)$. Hence

$$\kappa(\rho) = \int_{\Omega} \rho g\,d\mathbb{P} = \int_{B_\nu} \rho g\,d\mathbb{P}.$$

Thus

$$\begin{aligned}
\left|\int_{B_\nu} \rho(g-g^\nu)\,d\mathbb{P}\right| &= \left|\int_{B_\nu} \rho g\,d\mathbb{P}\right| \\
&= |\kappa(\rho)| \\
&\leq \|\kappa\|\|\rho\|_{H^s_{p,q}(\Omega)} \\
&\lesssim \|\kappa\|\|\mathbf{1}_{B_\nu}\|_{p,q}\mathbb{P}(B_\nu)^{-\frac{1}{p}}.
\end{aligned}$$

Hence

$$\begin{aligned}
\left|\int_{B_\nu} f(g-g^\nu)\,d\mathbb{P}\right| &= \left|\int_{B_\nu} (f-f^\nu)(g-g^\nu)\,d\mathbb{P}\right| \\
&\lesssim C^{-1}\|(f-f^\nu)\mathbf{1}_{B_\nu}\|_{L_2(\Omega)}\mathbb{P}(B_\nu)^{\frac{1}{p}-\frac{1}{2}}\|\kappa\|\|\mathbf{1}_{B_\nu}\|_{p,q}\mathbb{P}(B_\nu)^{-\frac{1}{p}} \\
&\lesssim \mathbb{P}(B_\nu)^{-\frac{1}{2}}\|\mathbf{1}_{B_\nu}\|_{p,q}\|\kappa\|.
\end{aligned}$$

Thus

$$\begin{aligned}
\left(\int_{B_\nu} |g-g^\nu|^2\,d\mathbb{P}\right)^{\frac{1}{2}} &:= \sup_{\substack{f \in L_2^\nu(B_\nu) \\ \|f\|_{L_2^\nu(B_\nu)} \leq 1}} \left|\int_{B_\nu} f(g-g^\nu)\,d\mathbb{P}\right| \\
&\lesssim \mathbb{P}(B_\nu)^{-\frac{1}{2}}\|\mathbf{1}_{B_\nu}\|_{p,q}\|\kappa\|.
\end{aligned}$$

This gives us

$$\frac{1}{\phi(B_\nu)}\left(\frac{1}{\mathbb{P}(B_\nu)}\int_{B_\nu}|g-g^\nu|^2\,d\mathbb{P}\right)^{\frac{1}{2}}\lesssim\|\kappa\|,\quad\forall\nu\in\mathcal{T}.$$

Hence $g\in\mathcal{L}_{2,\phi}(\Omega)$, and the proof of the first part of the theorem is complete.

Conversely, let $g\in\mathcal{L}_{2,\phi}(\Omega)$. Let $f\in H^s_{p,q}(\Omega)$. We know that for the stopping times

$$\nu^k:=\inf\{n\in\mathbb{N}:s_{n+1}(f)>2^k\},\ k\in\mathbb{Z},$$

$$\left\|\sum_{k\in\mathbb{Z}}\left(\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}}\right)^\eta\mathbf{1}_{B_{\nu^k}}\right\|^{\frac{1}{\eta}}_{\frac{p}{\eta},\frac{q}{\eta}}\leq C\|f\|_{H^s_{p,q}}.\tag{5.2}$$

and moreover,

$$\sum_{k=l}^{m}\lambda_ka^k\longrightarrow f$$

in $H^s_{p,q}$ as $m\to\infty,\ l\to-\infty$, where $(\lambda_k,a^k,\nu_k)\in\mathcal{A}(p,q,2)^s$. Also since a^k is L_2-bounded, for $f\in H^s_{p,q}(\Omega)$,

$$\kappa_g(f)=\mathbb{E}[fg]=\sum_{k\geq0}\mathbb{E}[\lambda_ka^kg]$$

is well defined and linear. Using this, Schwartz's inequality, and the fact that $\|s(a^k)\|_2\leq\mathbb{P}(B_{\nu^k})^{\frac{1}{2}-\frac{1}{p}}$, we obtain

$$\begin{aligned}|\kappa_g(f)| &\leq \sum_{k\in\mathbb{Z}}\lambda_k\left|\int_\Omega a^k(g-g^{\nu^k})\mathrm{d}\mathbb{P}\right|\leq\sum_{k\in\mathbb{Z}}\lambda_k\|a^k\|_2\left(\int_{B_{\nu^k}}|g-g^{\nu^k}|^2\mathrm{d}\mathbb{P}\right)^{\frac{1}{2}}\\ &\lesssim \sum_{k\in\mathbb{Z}}\lambda_k\|s(a^k)\|_2\left(\int_{B_{\nu^k}}|g-g^{\nu^k}|^2\mathrm{d}\mathbb{P}\right)^{\frac{1}{2}}\\ &= \sum_{k\in\mathbb{Z}}\lambda_k\frac{\|\mathbf{1}_{B_{\nu^k}}\|_{p,q}}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}}\frac{1}{\phi(B_{\nu^k})}\left(\frac{1}{\mathbb{P}(B_{\nu^k})}\int|g-g^{\nu^k}|^2\mathrm{d}\mathbb{P}\right)^{\frac{1}{2}}.\end{aligned}$$

Hence using Proposition 5.2, inequalities (3.16) and (5.2), we deduce that

$$\begin{aligned}|\kappa_g(f)| &\lesssim \|g\|_{\mathcal{L}_{2,\phi}}\sum_{k\in\mathbb{Z}}\left\|\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}}\mathbf{1}_{B_{\nu^k}}\right\|_{p,q}\leq\|g\|_{\mathcal{L}_{2,\phi}}\left\|\sum_{k\in\mathbb{Z}}\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}}\mathbf{1}_{B_{\nu^k}}\right\|_{p,q}\\ &\leq \|g\|_{2,\phi}\left\|\sum_{k\geq0}\left(\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}}\right)^\eta\mathbf{1}_{B_{\nu^k}}\right\|^{\frac{1}{\eta}}_{\frac{p}{\eta},\frac{q}{\eta}}\lesssim\|f\|_{H^s_{p,q}}\|g\|_{2,\phi}.\end{aligned}$$

Thus $\kappa_g(f)=\mathbb{E}[fg]$ extends continuously on $H^s_{p,q}(\Omega)$ and the proof is complete. □

5.3.1 Duality of $H^s_{p,q}$ for $1 < p, q < \infty$

With the help of martingale transform, we are able to characterize the dual space of $H^s_{p,q}$ where $1 < p \leq q < \infty$. In this part of the study, all the martingales are defined relatively to the stochastic basis $\mathbf{Ps} := (\mathbb{R}, \mathcal{F}, \{\mathcal{F}_n\}_{n \in \mathbb{N}}, \mathbb{P})$ where $\mathcal{F}_n$ is the σ-algebra generated by the dyadic intervals $\mathcal{D}_n$ as defined in Chapter 2. We also recall from Chapter 2 that the dyadic intervals $A_j = [j, j+1)$ is covered by $J_{k,n,j}$ and therefore $A_j \in \mathcal{F}_n$ for all n and for all j.

The following lemma and its corollary shows that the martingale Hardy-amalgam space $H^s_{p,q}$ is uniformly convex and reflexive. These properties will be employed to characterize the dual space of $H^s_{p,q}$ when $1 < p, q < \infty$.

Lemma 5.4. *Let $2 \leq p \leq q < \infty$. Then the space $H^s_{p,q}(\mathbb{R})$ is uniformly convex.*

Proof. We recall that a Banach space $\mathcal{H}$ is uniformly convex if for any $\epsilon > 0$, there exists $\delta > 0$ such that if $x, y \in \mathcal{H}$ with $\|x\|_{\mathcal{H}} \leq 1$, $\|y\|_{\mathcal{H}} \leq 1$ and $\|x - y\|_{\mathcal{H}} \geq \epsilon$, then $\|x + y\|_{\mathcal{H}} \leq 2(1 - \delta)$.

We recall that for $1 \leq r < \infty$ and for $a, b > 0$,

$$(a+b)^r \leq 2^{r-1}(a^r + b^r) \quad \text{and} \quad a^r + b^r \leq (a+b)^r.$$

Let $\epsilon > 0$, and assume that $f, g \in H^s_{p,q}$ with $\|f\|_{H^s_{p,q}} \leq 1$, $\|g\|_{H^s_{p,q}} \leq 1$ and $\|f - g\|_{H^s_{p,q}} \geq \epsilon$. We start by observing that

$$s^2(f+g) + s^2(f-g) = 2(s^2(f) + s^2(g)).$$

We then obtain

$$\left(s^2(f+g)\right)^{\frac{p}{2}} + \left(s^2(f-g)\right)^{\frac{p}{2}} \leq \left(s^2(f+g) + s^2(f-g)\right)^{\frac{p}{2}} \leq 2^{p-1}\left[s^p(f) + s^p(g)\right].$$

Hence for any $j \in \mathbb{Z}$,

$$\|s(f+g)\mathbf{1}_{A_j}\|_p^p + \|s(f+g)\mathbf{1}_{A_j}\|_p^p \leq 2^{p-1}\left(\|s(f)\mathbf{1}_{A_j}\|_p^p + \|s(g)\mathbf{1}_{A_j}\|_p^p\right).$$

Raising both members of the last inequality to the power $\frac{q}{p} \geq 1$, we obtain

$$\begin{aligned}
\|s(f+g)\mathbf{1}_{A_j}\|_p^q + \|s(f+g)\mathbf{1}_{A_j}\|_p^q &\leq \left(\|s(f+g)\mathbf{1}_{A_j}\|_p^p + \|s(f+g)\mathbf{1}_{A_j}\|_p^p\right)^{\frac{q}{p}} \\
&\leq 2^{\frac{q}{p}(p-1)}\left(\|s(f)\mathbf{1}_{A_j}\|_p^p + \|s(g)\mathbf{1}_{A_j}\|_p^p\right)^{\frac{q}{p}} \\
&\leq 2^{\frac{q}{p}(p-1)}2^{\frac{q}{p}-1}\left(\|s(f)\mathbf{1}_{A_j}\|_p^q + \|s(g)\mathbf{1}_{A_j}\|_p^q\right).
\end{aligned}$$

Hence taking the sum over $j \in \mathbb{Z}$, we obtain

$$\|s(f+g)\|_{p,q}^q + \|s(f-g)\|_{p,q}^q \leq 2^{q-1}\left(\|s(f)\|_{p,q}^q + \|s(g)\|_{p,q}^q\right)$$

and so

$$\|s(f+g)\|_{p,q}^q \quad \leq \quad 2^{q-1}\left(\|s(f)\|_{p,q}^q + \|s(g)\|_{p,q}^q\right) - \|s(f-g)\|_{p,q}^q \leq 2^q - \epsilon^q.$$

Thus

$$\|s(f+g)\|_{p,q} \leq 2(1-\delta)$$

where

$$\delta = 1 - \left(1 - \frac{\epsilon^q}{2^q}\right)^{\frac{1}{q}}$$

and the proof is complete. □

From the above Lemma and Milman's Theorem (see [71]), we deduce the following.

Corollary 5.5. *Let $2 \leq p \leq q < \infty$. Then the space $H_{p,q}^s(\mathbb{R})$ is reflexive.*

The following Theorem characterizes the dual of $H_{p,q}^s$ when $1 < p \leq q < \infty$.

Theorem 5.6. *If either $1 < q \leq p \leq 2$ or $2 \leq p \leq q < \infty$, then the dual space of $H_{p,q}^s(\mathbb{R})$ identifies with $H_{p',q'}^s(\mathbb{R})$ where $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$.*

Proof. It follows from Corallary 5.5 above that we only need to prove for the case $1 < q \leq p \leq 2$. Let $g \in H_{p',q'}^s(\mathbb{R})$ and

$$\kappa_g(f) := \mathbb{E}\left(\sum_{n=0}^{\infty} d_n f d_n g\right) \quad \left(f \in H_{p,q}^s(\mathbb{R})\right).$$

Hence by Hölder's inequality, we have that

$$\begin{aligned}
|\kappa_g(f)| &\leq \int_{\mathbb{R}} \sum_{n=0}^{\infty} \mathbb{E}_{n-1}|d_n f||d_n g| d\mathbb{P} \\
&= \sum_{j\in\mathbb{Z}} \int_{A_j} \sum_{n=0}^{\infty} \mathbb{E}_{n-1}|d_n f||d_n g| d\mathbb{P} \\
&\leq \sum_{j\in\mathbb{Z}} \int_{A_j} \sum_{n=0}^{\infty} \left(\mathbb{E}_{n-1}|d_n f|^2\right)^{\frac{1}{2}} \left(\mathbb{E}_{n-1}|d_n g|^2\right)^{\frac{1}{2}} d\mathbb{P} \\
&\leq \sum_{j\in\mathbb{Z}} \int_{A_j} \left(\sum_{n=0}^{\infty} \mathbb{E}_{n-1}|d_n f|^2\right)^{\frac{1}{2}} \left(\sum_{n=0}^{\infty} \mathbb{E}_{n-1}|d_n g|^2\right)^{\frac{1}{2}} d\mathbb{P}
\end{aligned}$$

$$
\begin{aligned}
|\kappa_g(f)| &= \sum_{j\in\mathbb{Z}} \int_{A_j} s(f)s(g)\mathrm{d}\mathbb{P} \\
&\leq \sum_{j\in\mathbb{Z}} \|s(f)\mathbf{1}_{A_j}\|_p \|s(g)\mathbf{1}_{A_j}\|_{p'} \\
&\leq \left(\sum_{j\in\mathbb{Z}} \|s(f)\mathbf{1}_{A_j}\|_p^q\right)^{\frac{1}{q}} \left(\sum_{j\in\mathbb{Z}} \|s(g)\mathbf{1}_{A_j}\|_{p'}^{q'}\right)^{\frac{1}{q'}} = \|f\|_{H^s_{p,q}(\mathbb{R})}\|g\|_{H^s_{p',q'}(\mathbb{R})}.
\end{aligned}
$$

Thus

$$|\kappa_g(f)| \leq C\|g\|_{H^s_{p',q'}(\mathbb{R})}.$$

Conversely, let κ be a continuous linear functional on $H^s_{p,q}(\mathbb{R})$. Then as $H^s_{p,q}(\mathbb{R})$ embeds continuously into $H^s_p(\mathbb{R})$ (since $q < p$), we have by the Hahn-Banach theorem that κ can be extended to a continuous linear functional $\tilde{\kappa}$ on $H^s_p(\mathbb{R})$ having the same operator norm as κ. It follows from [65, Theorem 2.26] that there exists some $g \in H^s_{p'}(\mathbb{R})$ such that

$$\tilde{\kappa}(f) = \mathbb{E}(fg) \quad \left(\forall f \in H^s_p(\mathbb{R})\right).$$

In particular

$$\kappa(f) = \tilde{\kappa}(f) = \mathbb{E}(fg) \quad \left(\forall f \in H^s_{p,q}(\mathbb{R})\right). \tag{5.3}$$

Let us prove that

$$\|g\|_{H^s_{p',q'}(\mathbb{R})} \lesssim \sup_{f\in H^s_{p,q}(\mathbb{R}),\, \|f\|_{H^s_{p,q}(\mathbb{R})}\leq 1} |\kappa(f)| < \infty. \tag{5.4}$$

Obviously, this holds if $\|g\|_{H^s_{p',q'}(\mathbb{R})} = 0$. Hence we assume that $\|g\|_{H^s_{p',q'}(\mathbb{R})} \neq 0$.

We recall that, $A_j \in \mathcal{F}_n$ for all $j \in \mathbb{Z}$ and $n \geq 1$. Set

$$\mu_n = \sum_{j\in\mathbb{Z}} \frac{s_n^{p'-2}(g)\mathbf{1}_{A_j}}{\|s(g)\|_{p',q'}^{q'-1}\|s(g)\mathbf{1}_{A_j}\|_{p'}^{p'-q'}}. \tag{5.5}$$

Since the A_j's are pairwise disjoint, we have that

$$\mu_n^2 = \sum_{j\in\mathbb{Z}} \frac{s_n^{2p'-4}(g)\mathbf{1}_{A_j}}{\|s(g)\|_{p',q'}^{2q'-2}\|s(g)\mathbf{1}_{A_j}\|_{p'}^{2(p'-q')}}.$$

From the definition of $s(\cdot)$, we have that μ_n is $\mathcal{F}_{n-1}$-measurable. We define h as the martingale transform of g by μ_n. That is

$$h_n = \sum_{k=1}^{n} \mu_k d_k g \text{ (in other words } d_n h = \mu_n d_n g). \tag{5.6}$$

We then obtain

$$\sum_{n=0}^{\infty} \mathbb{E}_{n-1}|d_n h|^2 = \sum_{n=0}^{\infty} \mu_n^2 \mathbb{E}_{n-1}|d_n g|^2$$

or equivalently

$$s^2(h) = \sum_{n=0}^{\infty} \sum_{j\in\mathbb{Z}} \frac{s_n^{2p'-4}(g)\mathbf{1}_{A_j}}{\|s(g)\|_{p',q'}^{2q'-2}\|s(g)\mathbf{1}_{A_j}\|_{p'}^{2(p'-q')}} \mathbb{E}_{n-1}|d_n g|^2.$$

Therefore

$$\begin{aligned} s^2(h) &= \sum_{j\in\mathbb{Z}} \frac{\mathbf{1}_{A_j}}{\|s(g)\|_{p',q'}^{2q'-2}\|s(g)\mathbf{1}_{A_j}\|_{p'}^{2(p'-q')}} \sum_{n=0}^{\infty} s_n^{2p'-4}(g)\mathbb{E}_{n-1}|d_n g|^2 \\ &= \sum_{j\in\mathbb{Z}} \frac{\mathbf{1}_{A_j}}{\|s(g)\|_{p',q'}^{2q'-2}\|s(g)\mathbf{1}_{A_j}\|_{p'}^{2(p'-q')}} \sum_{n=0}^{\infty} s_n^{2p'-4}(g)(s_n^2(g) - s_{n-1}^2(g)) \\ &= \frac{1}{\|s(g)\|_{p',q'}^{2q'-2}} \sum_{j\in\mathbb{Z}} \frac{\mathbf{1}_{A_j}}{\|s(g)\mathbf{1}_{A_j}\|_{p'}^{2(p'-q')}} \sum_{n=0}^{\infty} [s_n^{2p'-2}(g) - s_n^{2p'-4}(g)s_{n-1}^2(g)]. \end{aligned}$$

It follows that

$$\begin{aligned} s^2(h) &\le \frac{1}{\|s(g)\|_{p',q'}^{2q'-2}} \sum_{j\in\mathbb{Z}} \frac{\mathbf{1}_{A_j}}{\|s(g)\mathbf{1}_{A_j}\|_{p'}^{2(p'-q')}} \sum_{n=0}^{\infty} [s_n^{2p'-2}(g) - s_{n-1}^{2p'-2}(g)] \\ &= \frac{1}{\|s(g)\|_{p',q'}^{2q'-2}} \sum_{j\in\mathbb{Z}} \frac{s^{2p'-2}(g)\mathbf{1}_{A_j}}{\|s(g)\mathbf{1}_{A_j}\|_{p'}^{2(p'-q')}}. \end{aligned}$$

Thus, by disjointedness of the A_j's,

$$s(h) \le \frac{s^{p'-1}(g)}{\|s(g)\|_{p',q'}^{q'-1}} \sum_{j\in\mathbb{Z}} \frac{\mathbf{1}_{A_j}}{\|s(g)\mathbf{1}_{A_j}\|_{p'}^{p'-q'}}. \tag{5.7}$$

We also have that for any $k \in \mathbb{Z}$,

$$s(h)\mathbf{1}_{A_k} \le \sum_{j\in\mathbb{Z}} \frac{s^{p'-1}(g)}{\|s(g)\|_{p',q'}^{q'-1}} \frac{\mathbf{1}_{A_j}}{\|s(g)\mathbf{1}_{A_j}\|_{p'}^{p'-q'}} \mathbf{1}_{A_k} = \frac{s^{p'-1}(g)}{\|s(g)\|_{p',q'}^{q'-1}} \frac{\mathbf{1}_{A_k}}{\|s(g)\mathbf{1}_{A_k}\|_{p'}^{p'-q'}}.$$

Therefore

$$\|s(h)\mathbf{1}_{A_k}\|_p \le \frac{\|s^{p'-1}(g)\mathbf{1}_{A_k}\|_p}{\|s(g)\|_{p',q'}^{q'-1}\|s(g)\mathbf{1}_{A_k}\|_{p'}^{p'-q'}} = \frac{\|s(g)\mathbf{1}_{A_k}\|_{p'}^{p'-1}}{\|s(g)\|_{p',q'}^{q'-1}\|s(g)\mathbf{1}_{A_k}\|_{p'}^{p'-q'}} = \frac{\|s(g)\mathbf{1}_{A_k}\|_{p'}^{q'-1}}{\|s(g)\|_{p',q'}^{q'-1}}.$$

Hence

$$\sum_{k\in\mathbb{Z}} \|s(h)\mathbf{1}_{A_k}\|_p^q \leq \sum_{k\in\mathbb{Z}} \frac{\|s(g)\mathbf{1}_{A_k}\|_{p'}^{q(q'-1)}}{\|s(g)\|_{p',q'}^{q(q'-1)}} = \sum_{k\in\mathbb{Z}} \frac{\|s(g)\mathbf{1}_{A_k}\|_{p'}^{q'}}{\|s(g)\|_{p',q'}^{q'}} = \frac{\|s(g)\|_{p',q'}^{q'}}{\|s(g)\|_{p',q'}^{q'}} = 1.$$

That is

$$\|h\|_{H^s_{p,q}(\mathbb{R})} \leq 1.$$

We now test (5.4) with the martingale h above. First proceeding as in [65, p.37] and with the application of inequality (2.10) (this is why we need p to be smaller than 2), we obtain

$$\begin{aligned}
|\kappa(h)| &= \mathbb{E}\left(\sum_{n=0}^{\infty} d_n h d_n g\right) = \mathbb{E}\left(\sum_{n=0}^{\infty} \mu_n |d_n g|^2\right) \\
&= \frac{1}{\|s(g)\|_{p',q'}^{q'-1}} \mathbb{E}\left(\sum_{n=0}^{\infty} \sum_{j\in\mathbb{Z}} \frac{s_n^{p'-2}(g)\mathbf{1}_{A_j}}{\|s(g)\mathbf{1}_{A_j}\|_{p'}^{p'-q'}} \mathbb{E}_{n-1}|d_n g|^2\right) \\
&= \frac{1}{\|s(g)\|_{p',q'}^{q'-1}} \mathbb{E}\left(\sum_{n=0}^{\infty} \sum_{j\in\mathbb{Z}} \frac{s_n^{p'-2}(g)\mathbf{1}_{A_j}}{\|s(g)\mathbf{1}_{A_j}\|_{p'}^{p'-q'}} (s_n^2(g) - s_{n-1}^2(g))\right)
\end{aligned}$$

It follows that

$$\begin{aligned}
|\kappa(h)| &\geq \frac{2}{p'} \frac{1}{\|s(g)\|_{p',q'}^{q'-1}} \sum_{j\in\mathbb{Z}} \frac{1}{\|s(g)\mathbf{1}_{A_j}\|_{p'}^{p'-q'}} \mathbb{E}\left(\mathbf{1}_{A_j} \sum_{n=0}^{\infty} s_n^{p'}(g) - s_{n-1}^{p'}(g)\right) \\
&= \frac{2}{p'} \frac{1}{\|s(g)\|_{p',q'}^{q'-1}} \sum_{j\in\mathbb{Z}} \frac{1}{\|s(g)\mathbf{1}_{A_j}\|_{p'}^{p'-q'}} \mathbb{E}\left(\mathbf{1}_{A_j} s^{p'}(g)\right) \\
&= \frac{2}{p'} \frac{1}{\|s(g)\|_{p',q'}^{q'-1}} \sum_{j\in\mathbb{Z}} \frac{1}{\|s(g)\mathbf{1}_{A_j}\|_{p'}^{p'-q'}} \int_{\mathbb{R}} \mathbf{1}_{A_j} s^{p'}(g) d\mathbb{P} \\
&= \frac{2}{p'} \frac{1}{\|s(g)\|_{p',q'}^{q'-1}} \sum_{j\in\mathbb{Z}} \frac{1}{\|s(g)\mathbf{1}_{A_j}\|_{p'}^{p'-q'}} \|s(g)\mathbf{1}_{A_j}\|_{p'}^{p'} \\
&= \frac{2}{p'} \frac{1}{\|s(g)\|_{p',q'}^{q'-1}} \sum_{j\in\mathbb{Z}} \|s(g)\mathbf{1}_{A_j}\|_{p'}^{q'} \\
&= \frac{2}{p'} \frac{\|s(g)\|_{p',q'}^{q'}}{\|s(g)\|_{p',q'}^{q'-1}} = \frac{2}{p'} \|s(g)\|_{L_{p',q'}(\mathbb{R})}.
\end{aligned}$$

The proof is complete. □

Remark 5.7. We note that because of the equivalence in (4.11), the dual of the of the spaces $H^S_{p,q}$, $H^s_{p,q}$, $H^*_{p,q}$, $\mathcal{Q}_{p,q}$ and $\mathcal{P}_{p,q}$ will be $\mathcal{L}_{2,\phi}$ for $0 < p \leq q < 1$ and $H^s_{p',q'}$ for $1 \leq q \leq p < \infty$ or $2 \leq p \leq q < \infty$ when the stochastic basis is regular.

5.8 Dual Space Characterization of the Garsia-type Space

In this section, all the martingales are defined with respect to the stochastic basis $\mathbf{Ps} := (\mathbb{R}, \mathcal{F}, \{\mathcal{F}_n\}_{n\in\mathbb{N}}, \mathbb{P})$ where $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ is the dyadic filtration. We recall from Chapter 2 that the variation integrable space (or Garsia-type space) is the space

$$\mathcal{G}_{p,q} := \left\{ f \in \mathcal{M} : \sum_{n=0}^{\infty} |d_n f| \in L_{p,q} \right\}$$

equipped with the norm

$$\|f\|_{\mathcal{G}_{p,q}} = \left\| \sum_{n=0}^{\infty} |d_n f| \right\|_{p,q}$$

for $1 \leq p, q < \infty$. This space has proven to be very useful in application. For instance, as pointed out earlier in this study, $\mathcal{G}_{p,q}$ is a component of the Davis decomposition of martingales. This decomposition was discussed in detail in the previous chapter.

However, in this section, we will characterize the dual space of $\mathcal{G}_{p,q}$. We will establish in this section that the dual of $\mathcal{G}_{p,q}$ is the jump bounded space. We recall from Chapter 2 that the jump bounded space is the space

$$\mathcal{BD}_{p,q} := \left\{ f \in \mathcal{M} : \left\| \sup_{n\in\mathbb{N}} |d_n f| \right\|_{p,q} < \infty \right\}$$

equipped with the norm

$$\|f\|_{\mathcal{BD}_{p,q}} = \left\| \sup_{n\in\mathbb{N}} |d_n f| \right\|_{p,q}$$

for $1 \leq p, q \leq \infty$.

Before we characterize this duality, let us introduce a larger space $\mathcal{K}(L_{p,q}, \ell_r)$ for which $\mathcal{G}_{p,q}$ can be embedded into.

Definition 5.9. Let $n \in \mathbb{N}_0$ and let $1 \leq p, q, r < \infty$. We define the space $\mathcal{K}(L_{p,q}, \ell_r)$ by

$$\mathcal{K}(L_{p,q}, \ell_r) = \left\{ \text{measurable process } \epsilon = (\epsilon_n)_{n\geq 0} : \|\epsilon\|_{\mathcal{K}(L_{p,q},\ell_r)} < \infty \right\}$$

where

$$\|\epsilon\|_{\mathcal{K}(L_{p,q},\ell_r)} = \left\| \left(\sum_{n\geq 0} |\epsilon_n|^r \right)^{\frac{1}{r}} \right\|_{L_{p,q}(\mathbb{R})}.$$

We observe that $\mathcal{G}_{p,q} \subseteq \mathcal{K}(L_{p,q}, \ell_1)$. Indeed let f be a martingale. Then it is measurable with respect to the underlining filtration hence it increment, $d_n f$, is also measurable. Thus we can take $\epsilon_n = d_n f$ and it follows by setting $r = 1$. Also since $\ell_\infty = \{\{t_n\}_{n\geq 0},\ t_n \in \mathbb{R}$ for any $n \in \mathbb{N}_0,\ \sup_n |t_n| < \infty\}$, if we set $t_n = \epsilon_n = d_n f$ then we see that $\mathcal{BD}_{p,q} \subseteq \mathcal{K}(L_{p,q}, \ell_\infty)$.

We also observe that since $L_{p,p}(\mathbb{R}) = L_p(\mathbb{R})$, then $\mathcal{K}(L_{p,p}, \ell_r)$ is the space defined by Ferenc ([65]). The following lemma is part of the proof of Propositions 5.11 and 5.12 that follows, but for the sake of the presentation, we isolate it.

Lemma 5.10. *Let* $1 < p, q < \infty$, $1 \leq r < \infty$ *and let* (p, p'), (q, q'), (r, r') *be their respective conjugate pairs. Let* $\eta \in \mathcal{K}(L_{p',q'}, \ell_{r'})$. *For* $1 < r < \infty$, *define the sequence* $h = \{h_k\}_{k\geq 0}$ *by*

$$h_k = \begin{cases} \sum_{i\geq 0} \frac{|\eta_k|^{r'}}{\eta_k} \frac{\|\eta\|_{\ell_{r'}}^{p'-r'}}{\|\eta\|_{\mathcal{K}(L_{p',q'},\ell_{r'})}^{q'-1}} \frac{\mathbf{1}_{A_i}}{\|\|\eta\|_{\ell_{r'}}\mathbf{1}_{A_i}\|_{L_{p'}}^{p'-q'}} & , \quad \eta_k \neq 0 \\ 0 & , \quad \textit{otherwise} \end{cases} \tag{5.8}$$

and for $r = 1$, *define the sequence* $h = \{h_k\}_{k\geq 0}$ *by*

$$h_k = \begin{cases} \sum_{i\geq 0} \frac{\operatorname{sign}(\eta_k)}{2^{k+1}} \frac{\|\eta\|_{\ell_\infty}^{p'-1}}{\|\eta\|_{\mathcal{K}(L_{p',q'},\ell_\infty)}^{q'-1}} \frac{\mathbf{1}_{A_i}}{\|\|\eta\|_{\ell_\infty}\mathbf{1}_{A_i}\|_{L_{p'}}^{p'-q'}} & , \quad \eta_k \neq 0 \\ 0 & , \quad \textit{otherwise} \end{cases} \tag{5.9}$$

Then h *has a unit norm in* $\mathcal{K}(L_{p,q}, \ell_r)$. *Consequently* $h \in \mathcal{K}(L_{p,q}, \ell_r)$.

Proof. We start the proof with the case $1 < r < \infty$. By definition,

$$\begin{aligned} \|h\|_{\mathcal{K}(L_{p,q},\ell_r)} &= \left(\sum_{j\geq 0} \left(\int_{\mathbb{R}} \left(\sum_k |h_k|^r \right)^{\frac{p}{r}} \mathbf{1}_{A_j} d\mathbb{P} \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \\ &= \left(\sum_{j\geq 0} \|\|h\|_{\ell_r}\mathbf{1}_{A_j}\|_{L_p}^q \right)^{\frac{1}{q}}. \end{aligned}$$

Now

$$|h_k|^r = |\eta_k|^{r'} \frac{\|\eta\|_{\ell_{r'}}^{(p'-r')r}}{\|\eta\|_{\mathcal{K}(L_{p',q'},\ell_{r'})}^{(q'-1)r}} \left(\sum_{i\geq 0} \frac{\mathbf{1}_{A_i}}{\|\|\eta\|_{\ell_{r'}}\mathbf{1}_{A_i}\|_{L_{p'}}^{p'-q'}} \right)^r$$

so that

$$\sum_k |h_k|^r = \sum_k |\eta_k|^{r'} \frac{\|\eta\|_{\ell_{r'}}^{(p'-r')r}}{\|\eta\|_{\mathcal{K}(L_{p',q'},\ell_{r'})}^{(q'-1)r}} \left(\sum_{i\geq 0} \frac{\mathbf{1}_{A_i}}{\|\|\eta\|_{\ell_{r'}}\mathbf{1}_{A_i}\|_{L_{p'}}^{p'-q'}} \right)^r$$

and hence

$$\begin{aligned}
\|h\|_{\ell_r}^{r} &= \|\eta\|_{\ell_{r'}}^{r'} \frac{\|\eta\|_{\ell_{r'}}^{(p'-r')r}}{\|\eta\|_{\mathcal{K}(L_{p',q'},\ell_{r'})}^{(q'-1)r}} \left(\sum_{i\geq 0} \frac{\mathbf{1}_{A_i}}{\big\|\|\eta\|_{\ell_{r'}}\mathbf{1}_{A_i}\big\|_{L_{p'}}^{p'-q'}}\right)^{r} \\
&= \frac{\|\eta\|_{\ell_{r'}}^{r'+(p'-r')r}}{\|\eta\|_{\mathcal{K}(L_{p',q'},\ell_{r'})}^{(q'-1)r}} \left(\sum_{i\geq 0} \frac{\mathbf{1}_{A_i}}{\big\|\|\eta\|_{\ell_{r'}}\mathbf{1}_{A_i}\big\|_{L_{p'}}^{p'-q'}}\right)^{r} \\
&= \frac{\|\eta\|_{\ell_{r'}}^{(p'-1)r}}{\|\eta\|_{\mathcal{K}(L_{p',q'},\ell_{r'})}^{(q'-1)r}} \left(\sum_{i\geq 0} \frac{\mathbf{1}_{A_i}}{\big\|\|\eta\|_{\ell_{r'}}\mathbf{1}_{A_i}\big\|_{L_{p'}}^{p'-q'}}\right)^{r}
\end{aligned}$$

and then

$$\|h\|_{\ell_r} = \frac{\|\eta\|_{\ell_{r'}}^{p'-1}}{\|\eta\|_{\mathcal{K}(L_{p',q'},\ell_{r'})}^{q'-1}} \sum_{i\geq 0} \frac{\mathbf{1}_{A_i}}{\big\|\|\eta\|_{\ell_{r'}}\mathbf{1}_{A_i}\big\|_{L_{p'}}^{p'-q'}}.$$

By disjointedness,

$$\begin{aligned}
\|h\|_{\ell_r}\mathbf{1}_{A_j} &= \frac{\|\eta\|_{\ell_{r'}}^{p'-1}}{\|\eta\|_{\mathcal{K}(L_{p',q'},\ell_{r'})}^{q'-1}} \sum_{i\geq 0} \frac{\mathbf{1}_{A_i}}{\big\|\|\eta\|_{\ell_{r'}}\mathbf{1}_{A_i}\big\|_{L_{p'}}^{p'-q'}}\mathbf{1}_{A_j} \\
&= \frac{\|\eta\|_{\ell_{r'}}^{p'-1}}{\|\eta\|_{\mathcal{K}(L_{p',q'},\ell_{r'})}^{q'-1}} \frac{\mathbf{1}_{A_j}}{\big\|\|\eta\|_{\ell_{r'}}\mathbf{1}_{A_j}\big\|_{L_{p'}}^{p'-q'}}. \qquad (5.10)
\end{aligned}$$

We now take $L_p(\mathbb{R})$ norm of both sides.

$$\begin{aligned}
\int_{\mathbb{R}} \|h\|_{\ell_r}^{p}\mathbf{1}_{A_j}\mathrm{d}\mathbb{P} &= \int_{\mathbb{R}} \frac{\|\eta\|_{\ell_{r'}}^{(p'-1)p}}{\|\eta\|_{\mathcal{K}(L_{p',q'},\ell_{r'})}^{(q'-1)p}} \frac{\mathbf{1}_{A_j}}{\big\|\|\eta\|_{\ell_{r'}}\mathbf{1}_{A_j}\big\|_{L_{p'}}^{(p'-q')p}}\mathrm{d}\mathbb{P} \\
&= \frac{1}{\|\eta\|_{\mathcal{K}(L_{p',q'},\ell_{r'})}^{(q'-1)p}} \int_{\mathbb{R}} \frac{\|\eta\|_{\ell_{r'}}^{(p'-1)p}\mathbf{1}_{A_j}}{\big\|\|\eta\|_{\ell_{r'}}\mathbf{1}_{A_j}\big\|_{L_{p'}}^{(p'-q')p}}\mathrm{d}\mathbb{P} \\
&= \frac{1}{\|\eta\|_{\mathcal{K}(L_{p',q'},\ell_{r'})}^{(q'-1)p}} \frac{1}{\big\|\|\eta\|_{\ell_{r'}}\mathbf{1}_{A_j}\big\|_{L_{p'}}^{(p'-q')p}} \int_{\mathbb{R}} \|\eta\|_{\ell_{r'}}^{p'}\mathbf{1}_{A_j}\mathrm{d}\mathbb{P} \\
&= \frac{1}{\|\eta\|_{\mathcal{K}(L_{p',q'},\ell_{r'})}^{(q'-1)p}} \frac{\big\|\|\eta\|_{\ell_{r'}}\mathbf{1}_{A_j}\big\|_{L_{p'}}^{p'}}{\big\|\|\eta\|_{\ell_{r'}}\mathbf{1}_{A_j}\big\|_{L_{p'}}^{(p'-q')p}} \\
&= \frac{1}{\|\eta\|_{\mathcal{K}(L_{p',q'},\ell_{r'})}^{(q'-1)p}} \big\|\|\eta\|_{\ell_{r'}}\mathbf{1}_{A_j}\big\|_{L_{p'}}^{(q'-1)p}.
\end{aligned}$$

Therefore

$$\big\|\|h\|_{\ell_r}\mathbf{1}_{A_j}\big\|_{L_p} = \frac{1}{\|\eta\|_{\mathcal{K}(L_{p',q'},\ell_{r'})}^{q'-1}} \big\|\|\eta\|_{\ell_{r'}}\mathbf{1}_{A_j}\big\|_{L_{p'}}^{q'-1}.$$

Hence

$$\begin{aligned}\sum_{j\geq 0}\left\|\|h\|_{\ell_r}\mathbf{1}_{A_j}\right\|_{L_p}^{q} &= \sum_{j\geq 0}\frac{1}{\|\eta\|^{(q'-1)q}_{\mathcal{K}(L_{p',q'},\ell_{r'})}}\left\|\|\eta\|_{\ell_{r'}}\mathbf{1}_{A_j}\right\|_{L_{p'}}^{(q'-1)q}\\ &= \frac{1}{\|\eta\|^{q'}_{\mathcal{K}(L_{p',q'},\ell_{r'})}}\sum_{j\geq 0}\left\|\|\eta\|_{\ell_{r'}}\mathbf{1}_{A_j}\right\|_{L_{p'}}^{q'}\end{aligned}$$

and then

$$\sum_{j\geq 0}\left\|\|h\|_{\ell_r}\mathbf{1}_{A_j}\right\|_{L_p}^{q} = \frac{1}{\|\eta\|^{q'}_{\mathcal{K}(L_{p',q'},\ell_{r'})}}\|\eta\|^{q'}_{\mathcal{K}(L_{p',q'},\ell_{r'})} = 1.$$

Therefore

$$\|h\|_{\mathcal{K}(L_{p,q},\ell_r)} = \left(\sum_{j\geq 0}\left\|\|h\|_{\ell_r}\mathbf{1}_{A_j}\right\|_{L_p}^{q}\right)^{\frac{1}{q}} = 1.$$

For the case $r = 1$, we observe that

$$|h_k| = \frac{1}{2^{k+1}}\frac{\|\eta\|^{p'-1}_{\ell_\infty}}{\|\eta\|^{q'-1}_{\mathcal{K}(L_{p',q'},\ell_\infty)}}\sum_{i\geq 0}\frac{\mathbf{1}_{A_i}}{\left\|\|\eta\|_{\ell_\infty}\mathbf{1}_{A_i}\right\|_{p'}^{p'-q'}}$$

and hence

$$\sum_{k\geq 0}|h_k| = \sum_{k\geq 0}\frac{1}{2^{k+1}}\frac{\|\eta\|^{p'-1}_{\ell_\infty}}{\|\eta\|^{q'-1}_{\mathcal{K}(L_{p',q'},\ell_\infty)}}\sum_{i\geq 0}\frac{\mathbf{1}_{A_i}}{\left\|\|\eta\|_{\ell_\infty}\mathbf{1}_{A_i}\right\|_{p'}^{p'-q'}}.$$

Thus

$$\|h\|_{\ell_1} = \frac{\|\eta\|^{p'-1}_{\ell_\infty}}{\|\eta\|^{q'-1}_{\mathcal{K}(L_{p',q'},\ell_\infty)}}\sum_{i\geq 0}\frac{\mathbf{1}_{A_i}}{\left\|\|\eta\|_{\ell_\infty}\mathbf{1}_{A_i}\right\|_{p'}^{p'-q'}}$$

since $\sum_{k\geq 0}\frac{1}{2^{k+1}} = 1$. Hence by disjointedness,

$$\|h\|_{\ell_1}\mathbf{1}_{A_j} = \frac{\|\eta\|^{p'-1}_{\ell_\infty}}{\|\eta\|^{q'-1}_{\mathcal{K}(L_{p',q'},\ell_\infty)}}\frac{\mathbf{1}_{A_j}}{\left\|\|\eta\|_{\ell_\infty}\mathbf{1}_{A_j}\right\|_{p'}^{p'-q'}}.$$

We also observe that this equation is the same as equation (5.10) above with $r = 1$ and $r' = \infty$. Hence following as above, we obtain that

$$\|h\|_{\mathcal{K}(L_{p,q},\ell_1)} = 1.$$

Thus $h = (h_k)_{k\geq 0} \in \mathcal{K}(L_{p,q},\ell_r)$ since $h = (h_k)_{k\geq 0}$ is measurable and the proof is complete.

$\square$

Proposition 5.11 below characterizes the dual of $\mathcal{K}(L_{p,q},\ell_r)$ when $1 < r < \infty$ and Proposition 5.12 below also characterizes the dual of $\mathcal{K}(L_{p,q},\ell_r)$ when $r = 1$

Proposition 5.11. *For* $1 < p, q < \infty$ *and* $1 < r < \infty$, *the dual space,* $\mathcal{K}(L_{p,q}, \ell_r)^*$, *of* $\mathcal{K}(L_{p,q}, \ell_r)$ *is* $\mathcal{K}(L_{p',q'}, \ell_{r'})$ *where*

$$\frac{1}{p} + \frac{1}{p'} = 1, \quad \frac{1}{q} + \frac{1}{q'} = 1, \quad \frac{1}{r} + \frac{1}{r'} = 1.$$

Proof. Let $\eta = (\eta_k)_{k\geq 0} \in \mathcal{K}(L_{p',q'}, \ell_{r'})$ and $\epsilon = (\epsilon_k)_{k\geq 0} \in \mathcal{K}(L_{p,q}, \ell_r)$. Let $\langle \cdot, \cdot \rangle$ be the usual inner product, that is,

$$\langle \eta, \epsilon \rangle = \sum_k \eta_k \epsilon_k.$$

and define the functional, Λ, by

$$\Lambda_\eta(\epsilon) = \mathbb{E}\langle \eta, \epsilon \rangle = \int_{\mathbb{R}} \sum_{k\geq 0} \epsilon_k \eta_k \mathrm{d}\mathbb{P} = \sum_{j\geq 0} \int_{A_j} \sum_{k\geq 0} \epsilon_k \eta_k \mathrm{d}\mathbb{P}$$

for all $\eta = (\eta_k)_{k\geq 0} \in \mathcal{K}(L_{p',q'}, \ell_{r'})$ measurable and $\epsilon = (\epsilon_k)_{k\geq 0} \in \mathcal{K}(L_{p,q}, \ell_r)$. Then by Hölder inequality,

$$|\Lambda_\eta(\epsilon)| = \left| \sum_{j\geq 0} \int_{A_j} \sum_{k\geq 0} \epsilon_k \eta_k \mathrm{d}\mathbb{P} \right| \leq \|\epsilon\|_{\mathcal{K}(L_{p,q},\ell_r)} \|\eta\|_{\mathcal{K}(L_{p',q'},\ell_{r'})}. \tag{5.11}$$

and since $\Lambda_\eta(\cdot)$ is linear and bounded, it is a continuous linear functional on $\mathcal{K}(L_{p,q}, \ell_r)$. From inequality (5.11), we observe that

$$|\Lambda_\eta(\epsilon)| \leq C \|\eta\|_{\mathcal{K}(L_{p',q'},\ell_{r'})} \tag{5.12}$$

Conversely, let Λ be a continuous linear functional on $\mathcal{K}(L_{p,q}, \ell_r)$. Then as $\mathcal{K}(L_{p,q}, \ell_r)$ embeds continuously into $\mathcal{K}(L_p, \ell_r)$ (since $q < p$), we have by Hahn-Banach Theorem that Λ can be extended to a continuous linear functional $\tilde{\Lambda}$ on $\mathcal{K}(L_p, \ell_r)$ having the same operator norm as Λ. It follows from ([65, Lemma 2.9]) that there exists some $\eta \in \mathcal{K}(L_{p'}, \ell_{r'})$ such that

$$\tilde{\Lambda}_\eta(\epsilon) = \mathbb{E}\langle \eta, \epsilon \rangle$$

for all $\epsilon \in \mathcal{K}(L_p, \ell_r)$. In particular

$$\Lambda_\eta(\epsilon) = \tilde{\Lambda}_\eta(\epsilon) = \mathbb{E}\langle \eta, \epsilon \rangle$$

for all $\epsilon \in \mathcal{K}(L_{p,q}, \ell_r)$. Let us now show that

$$\|\eta\|_{\mathcal{K}(L_{p',q'},\ell_{r'})} \lesssim \sup_{\epsilon \in \mathcal{K}(L_{p,q},\ell_r),\, \|\epsilon\|_{\mathcal{K}(L_{p,q},\ell_r)} \leq 1} |\Lambda_\eta(\epsilon)|.$$

Set $h = (h_k)_{k\geq 0}$ to be the sequence (5.8) defined in Lemma 5.10. Since $h = (h_k)_{k\geq 0} \in$

$\mathcal{K}(L_{p,q}, \ell_r)$ and with a unit norm, by linearity of the expectation operator, we have that

$$\begin{aligned}
\|\Lambda\| &\geq |\Lambda_\eta(h)| = \frac{1}{\|\eta\|^{q'-1}_{\mathcal{K}(L_{p',q'},\ell_{r'})}} \mathbb{E}\left(\left(\sum_{k\geq 0} |\eta_k|^{r'}\right)^{\frac{p'}{r'}} \sum_{j\geq 0} \frac{\mathbf{1}_{A_j}}{\|\|\eta\|_{\ell_{r'}} \mathbf{1}_{A_j}\|^{p'-q'}_{L_{p'}}}\right) \\
&= \frac{1}{\|\eta\|^{q'-1}_{\mathcal{K}(L_{p',q'},\ell_{r'})}} \mathbb{E}\left(\|\eta\|^{p'}_{\ell_{r'}} \sum_{j\geq 0} \frac{\mathbf{1}_{A_j}}{\|\|\eta\|_{\ell_{r'}} \mathbf{1}_{A_j}\|^{p'-q'}_{L_{p'}}}\right) \\
&\geq \frac{1}{\|\eta\|^{q'-1}_{\mathcal{K}(L_{p',q'},\ell_{r'})}} \sum_{j\geq 0} \mathbb{E}\left(\frac{\|\eta\|^{p'}_{\ell_{r'}} \mathbf{1}_{A_j}}{\|\|\eta\|_{\ell_{r'}} \mathbf{1}_{A_j}\|^{p'-q'}_{L_{p'}}}\right) \\
&= \frac{1}{\|\eta\|^{q'-1}_{\mathcal{K}(L_{p',q'},\ell_{r'})}} \sum_{j\geq 0} \frac{\mathbb{E}(\|\eta\|^{p'}_{\ell_{r'}} \mathbf{1}_{A_j})}{\|\|\eta\|_{\ell_{r'}} \mathbf{1}_{A_j}\|^{p'-q'}_{L_{p'}}} \\
&= \frac{1}{\|\eta\|^{q'-1}_{\mathcal{K}(L_{p',q'},\ell_{r'})}} \left[\sum_{j\in\mathbb{Z}} \frac{\|\|\eta\|_{\ell_{r'}} \mathbf{1}_{A_j}\|^{p'}_{L_{p'}}}{\|\|\eta\|_{\ell_{r'}} \mathbf{1}_{A_j}\|^{p'-q'}_{L_{p'}}}\right] \\
&= \frac{1}{\|\eta\|^{q'-1}_{\mathcal{K}(L_{p',q'},\ell_{r'})}} \left[\sum_{j\in\mathbb{Z}} \|\|\eta\|_{\ell_{r'}} \mathbf{1}_{A_j}\|^{q'}_{L_{p'}}\right].
\end{aligned}$$

Therefore

$$\|\Lambda\| \geq \frac{1}{\|\eta\|^{q'-1}_{\mathcal{K}(L_{p',q'},\ell_{r'})}} \|\eta\|^{q'}_{\mathcal{K}(L_{p',q'},\ell_{r'})} = \|\eta\|_{\mathcal{K}(L_{p',q'},\ell_{r'})}.$$

Thus

$$\|\Lambda\| \geq \|\eta\|_{\mathcal{K}(L_{p',q'},\ell_{r'})} \tag{5.13}$$

completing the proof. ☐

Proposition 5.12. *For* $1 < p, q < \infty$, *the dual space,* $\mathcal{K}(L_{p,q}, \ell_1)^*$, *of* $\mathcal{K}(L_{p,q}, \ell_1)$ *is* $\mathcal{K}(L_{p',q'}, \ell_\infty)$ *where*

$$\frac{1}{p} + \frac{1}{p'} = 1, \quad \frac{1}{q} + \frac{1}{q'} = 1.$$

Proof. Let $\eta = (\eta_k)_{k\geq 0} \in \mathcal{K}(L_{p',q'}, \ell_\infty)$ and $\epsilon = (\epsilon_k)_{k\geq 0} \in \mathcal{K}(L_{p,q}, \ell_1)$. Let $\langle \cdot, \cdot \rangle$ be the usual inner product, that is,

$$\langle \eta, \epsilon \rangle = \sum_k \eta_k \epsilon_k.$$

and define the functional, Λ, by

$$\Lambda_\eta(\epsilon) = \mathbb{E}\langle \eta, \epsilon \rangle = \int_{\mathbb{R}} \sum_{k\geq 0} \epsilon_k \eta_k \mathrm{d}\mathbb{P} = \sum_{j\geq 0} \int_{A_j} \sum_{k\geq 0} \epsilon_k \eta_k \mathrm{d}\mathbb{P}$$

for all $\eta = (\eta_k)_{k\geq 0} \in \mathcal{K}(L_{p',q'}, \ell_\infty)$ measurable and $\epsilon = (\epsilon_k)_{k\geq 0} \in \mathcal{K}(L_{p,q}, \ell_1)$. Then by Hölder inequality,

$$|\Lambda_\eta(\epsilon)| = \left| \sum_{j\geq 0} \int_{A_j} \sum_{k\geq 0} \epsilon_k \eta_k \mathrm{d}\mathbb{P} \right| \leq \|\epsilon\|_{\mathcal{K}(L_{p,q},\ell_1)} \|\eta\|_{\mathcal{K}(L_{p',q'},\ell_\infty)}. \tag{5.14}$$

and since $\Lambda_\eta(\cdot)$ is linear and bounded, it is a continuous linear functional on $\mathcal{K}(L_{p,q}, \ell_r)$. From inequality (5.14), we observe that

$$|\Lambda_\eta(\epsilon)| \leq C\|\eta\|_{\mathcal{K}(L_{p',q'},\ell_\infty)} \tag{5.15}$$

Conversely, let Λ be a continuous linear functional on $\mathcal{K}(L_{p,q}, \ell_1)$. Then as $\mathcal{K}(L_{p,q}, \ell_1)$ embeds continuously into $\mathcal{K}(L_p, \ell_1)$ (since $q < p$), we have by Hahn-Banach Theorem that Λ can be extended to a continuous linear functional $\tilde{\Lambda}$ on $\mathcal{K}(L_p, \ell_1)$ having the same operator norm as Λ. It follows from ([65, Lemma 2.9]) that there exists some $\eta \in \mathcal{K}(L_{p'}, \ell_\infty)$ such that

$$\tilde{\Lambda}_\eta(\epsilon) = \mathbb{E}\langle \eta, \epsilon \rangle$$

for all $\epsilon \in \mathcal{K}(L_p, \ell_r)$. In particular

$$\Lambda_\eta(\epsilon) = \tilde{\Lambda}_\eta(\epsilon) = \mathbb{E}\langle \eta, \epsilon \rangle$$

for all $\epsilon \in \mathcal{K}(L_{p,q}, \ell_r)$. Let us now show that

$$\|\eta\|_{\mathcal{K}(L_{p',q'},\ell_\infty)} \lesssim \sup_{\epsilon\in\mathcal{K}(L_{p,q},\ell_1),\, \|\epsilon\|_{\mathcal{K}(L_{p,q},\ell_r)}\leq 1} |\Lambda_\eta(\epsilon)|.$$

Set $h = (h_k)_{k\geq 0}$ to be the sequence (5.9) defined in Lemma 5.10 and observe the following. For some fixed integer k_0, $\frac{1}{2}\|\eta\|_{\ell_\infty} \leq |\eta_{k_0}|$. It follows that $\frac{1}{2^{k_0+2}}\|\eta\|_{\ell_\infty} \leq \frac{|\eta_{k_0}|}{2^{k_0+1}}$ and also since $\frac{|\eta_{k_0}|}{2^{k_0+1}} \leq \sum_{k\geq 0} \frac{|\eta_k|}{2^{k+1}}$, we have that

$$\frac{1}{2^{k_0+2}}\|\eta\|_{\ell_\infty} \leq \sum_{k\geq 0} \frac{|\eta_k|}{2^{k+1}} \tag{5.16}$$

Since $h = (h_k)_{k\geq 0} \in \mathcal{K}(L_{p,q}, \ell_1)$ and with a unit norm, by linearity of the expectation operator, and using equation (5.16) we have that

$$\|\Lambda\| \quad \geq \quad |\Lambda_\eta(h)| = \mathbb{E}\left(\sum_{k\geq 0} \frac{|\eta_k|}{2^{k+1}} \frac{\|\eta\|_{\ell_\infty}^{p'-1}}{\|\eta\|^{q'-1}_{\mathcal{K}(L_{p',q'},\ell_\infty)}} \sum_{j\geq 0} \frac{\mathbf{1}_{A_j}}{\big\| \|\eta\|_{\ell_\infty} \mathbf{1}_{A_j} \big\|^{p'-q'}_{L_{p'}}} \right)$$

$$\begin{aligned}
\|\Lambda\| &\geq \frac{C}{\|\eta\|^{q'-1}_{\mathcal{K}(L_{p',q'},\ell_\infty)}}\mathbb{E}\left(\|\eta\|^{p'}_{\ell_\infty}\sum_{j\geq 0}\frac{\mathbf{1}_{A_j}}{\|\|\eta\|_{\ell_\infty}\mathbf{1}_{A_j}\|^{p'-q'}_{L_{p'}}}\right)\\
&= \frac{C}{\|\eta\|^{q'-1}_{\mathcal{K}(L_{p',q'},\ell_\infty)}}\sum_{j\geq 0}\mathbb{E}\left(\frac{\|\eta\|^{p'}_{\ell_\infty}\mathbf{1}_{A_j}}{\|\|\eta\|_{\ell_\infty}\mathbf{1}_{A_j}\|^{p'-q'}_{L_{p'}}}\right)\\
&= \frac{C}{\|\eta\|^{q'-1}_{\mathcal{K}(L_{p',q'},\ell_\infty)}}\sum_{j\geq 0}\frac{\mathbb{E}(\|\eta\|^{p'}_{\ell_\infty}\mathbf{1}_{A_j})}{\|\|\eta\|_{\ell_\infty}\mathbf{1}_{A_j}\|^{p'-q'}_{L_{p'}}}\\
&\geq \frac{C}{\|\eta\|^{q'-1}_{\mathcal{K}(L_{p',q'},\ell_\infty)}}\left[\sum_{j\in\mathbb{Z}}\frac{\|\|\eta\|_{\ell_\infty}\mathbf{1}_{A_j}\|^{p'}_{L_{p'}}}{\|\|\eta\|_{\ell_\infty}\mathbf{1}_{A_j}\|^{p'-q'}_{L_{p'}}}\right]\\
&= \frac{C}{\|\eta\|^{q'-1}_{\mathcal{K}(L_{p',q'},\ell_\infty)}}\left[\sum_{j\in\mathbb{Z}}\|\|\eta\|_{\ell_\infty}\mathbf{1}_{A_j}\|^{q'}_{L_{p'}}\right].
\end{aligned}$$

Therefore

$$\|\Lambda\| \geq \frac{C}{\|\eta\|^{q'-1}_{\mathcal{K}(L_{p',q'},\ell_\infty)}}\|\eta\|^{q'}_{\mathcal{K}(L_{p',q'},\ell_\infty)} = C\|\eta\|_{\mathcal{K}(L_{p',q'},\ell_\infty)}.$$

Thus

$$\|\Lambda\| \geq C\|\eta\|_{\mathcal{K}(L_{p',q'},\ell_\infty)} \tag{5.17}$$

where $C = \frac{1}{2^{k_0+2}}$ completing the proof. □

As $(\mathcal{K}(L_{p,q},\ell_1))^* = \mathcal{K}(L_{p',q'},\ell_\infty)$, it is now evident that the dual of the variation integrable space is the jump bounded space. More rigorously, we prove the following Theorem.

Theorem 5.13. *Let $1 < p, q < \infty$ and let $f = (f_n)_{n\in n} \in \mathcal{M}$. Suppose that*

$$\left\|\sup_{n\in\mathbb{N}} f_n\right\|_{L_{p'}(\mathbb{R})} \leq \frac{p'}{p'-1}\sup_{n\in\mathbb{N}}\|f_n\|_{L_{p'}(\mathbb{R})}. \tag{5.18}$$

Then the dual space of $\mathcal{G}_{p,q}(\mathbb{R})$ is $\mathcal{BD}_{p',q'}(\mathbb{R})$ where $\frac{1}{p}+\frac{1}{p'}=1$ and $\frac{1}{q}+\frac{1}{q'}=1$

Proof. Let $g \in \mathcal{BD}_{p',q'}(\mathbb{R})$. As we saw in the proof of Proposition 5.12, if we set $\epsilon_k = d_kf$ and $\eta_k = d_kg$, then κ defined below is a well defined continuous linear functional on $\mathcal{G}_{p,q}(\mathbb{R})$ for every $f \in \mathcal{G}_{p,q}(\mathbb{R})$ and g exists in $\mathcal{BD}_{p',q'}(\mathbb{R})$ (as we have that η exists in $\mathcal{K}(L_{p',q'},\ell_{r'})$)

$$\kappa_g(f) = \sum_{k=1}^{\infty}\mathbb{E}[d_kfd_kg] \qquad \text{for } f \in \mathcal{G}_{p,q}(\mathbb{R}).$$

Let A_j be the usual subsets of $\mathbb{R}$ define by equation (2.1) such that $A_j \bigcap A_k = \emptyset$ for $j \neq k$

and $\bigcup_{j\in\mathbb{Z}} A_j = \mathbb{R}$. Then

$$\begin{aligned} |\kappa_g(f)| &= \left|\sum_{k=1}^{\infty} \mathbb{E}[d_k f d_k g]\right| \\ &\leq \sum_{k=1}^{\infty} \mathbb{E}[|d_k f||d_k g|] \leq \mathbb{E}\sum_{k=1}^{\infty}[|d_k f| \sup_{k\in\mathbb{N}} |d_k g|] \end{aligned}$$

$$\begin{aligned} |\kappa_g(f)| &\leq \int_{\mathbb{R}} \sum_{k=1}^{\infty} |d_k f| \sup_{k\in\mathbb{N}} |d_k g| \mathrm{d}\mathbb{P} \\ &= \sum_{j\in\mathbb{Z}} \int_{A_j} \sum_{k=1}^{\infty} |d_k f| \sup_{k\in\mathbb{N}} |d_k g| \mathrm{d}\mathbb{P} \\ &\leq \sum_{j\in\mathbb{Z}} \left[\int_{A_j} \left(\sum_{k=1}^{\infty} |d_k f|\right)^p \mathrm{d}\mathbb{P}\right]^{\frac{1}{p}} \left[\int_{A_j} \sup_{k\in\mathbb{N}} |d_k g|^{p'} \mathrm{d}\mathbb{P}\right]^{\frac{1}{p'}} \\ &\leq \left\{\sum_{j\in\mathbb{Z}} \left[\int_{A_j} \left(\sum_{k=1}^{\infty} |d_k f|\right)^p \mathrm{d}\mathbb{P}\right]^{\frac{q}{p}}\right\}^{\frac{1}{q}} \left\{\sum_{j\in\mathbb{Z}} \left[\int_{A_j} \sup_{k\in\mathbb{N}} |d_k g|^{p'} \mathrm{d}\mathbb{P}\right]^{\frac{q'}{p'}}\right\}^{\frac{1}{q'}} \\ &\leq \left\|\sum_{n=0}^{\infty} |d_n f|\right\|_{p,q} \left\|\sup_{n\in\mathbb{N}} |d_n g|\right\|_{p',q'} = \|f\|_{\mathcal{G}_{p,q}(\mathbb{R})} \|g\|_{\mathcal{BD}_{p',q'}(\mathbb{R})}. \end{aligned}$$

Therefore

$$|\kappa_g(f)| \leq C\|g\|_{\mathcal{BD}_{p',q'}(\mathbb{R})}.$$

To prove the converse, we first assume that τ is an arbitrary element in the dual of $\mathcal{G}_{p,q}(\mathbb{R})$ then we show that there exists $g \in \mathcal{BD}_{p',q'}(\mathbb{R})$ such that $\tau = \kappa_g$ and $\|g\|_{\mathcal{BD}_{p',q'}} \leq C\|\tau\|$ for some constant C. By setting $\epsilon_k = d_k f$ for $f \in \mathcal{M}$ we saw earlier that $\mathcal{G}_{p,q}(\mathbb{R}) \subseteq \mathcal{K}(L_{p,q}, \ell_1)$. We also recall that the dual space of $\mathcal{K}(L_{p,q}, \ell_1)$ is $\mathcal{K}(L_{p',q'}, \ell_\infty)$ and τ is a continuous linear functional on $\mathcal{G}_{p,q}(\mathbb{R}) \subseteq \mathcal{K}(L_{p,q}, \ell_1)$. By Hahn-Banach Theorem, τ can be extended to a continuous linear functional on $\mathcal{K}(L_{p,q}, \ell_1)$ having the same operator norm as τ. Let Λ be this extension of τ. Then we have by Proposition 5.12 that there exists $\eta \in \mathcal{K}(L_{p',q'}, \ell_\infty)$ such that

$$\|\Lambda\| = \|\tau\| = \|\eta\|_{\mathcal{K}(L_{p',q'}, \ell_\infty)} \quad \text{and} \quad \Lambda_\eta(\epsilon) = \sum_{k\geq 0} \mathbb{E}(\epsilon_k \eta_k)$$

for $\epsilon \in \mathcal{K}(L_{p,q}, \ell_r)$. Hence

$$\tau(f_n) = \sum_{k=1}^{n} \mathbb{E}[(d_k f)\eta_k] \tag{5.19}$$

is well defined. (We agree for a moment to work with f_n as we will show that $f_n \to f$ in

$\mathcal{G}_{p,q}$ as $n \to \infty$) We shall now set

$$g_n \mathbf{1}_{A_j} := \begin{cases} \mathbf{1}_{A_j} \sum_{k=1}^{n} [\mathbb{E}_k \eta_k - \mathbb{E}_{k-1}\eta_k], n \neq 0 \\ g_0 = 0 \end{cases}$$

and show that g_n is a martingale. Indeed we observe that

$$\mathbb{E}(g_n \mathbf{1}_{A_j}) = \mathbf{1}_{A_j} \sum_{k=1}^{n} [\mathbb{E}_k \eta_k - \mathbb{E}_{k-1}\eta_k] = \mathbb{E} \sum_{k=0}^{n} \eta_k \mathbf{1}_{A_j} - \eta_k \mathbf{1}_{A_j} = 0$$

and also by measurability of $\mathbf{1}_{A_j}$, we obtain

$$\mathbf{1}_{A_j} \mathbb{E}_{n-1}(g_n - g_{n-1}) = \mathbf{1}_{A_j} \mathbb{E}_{n-1}(\mathbb{E}_n \eta_n - \mathbb{E}_{n-1}\eta_n) = \mathbf{1}_{A_j}(\mathbb{E}_{n-1}\eta_n - \mathbb{E}_{n-1}\eta_n) = 0.$$

This means that g_n is a martingale. Consequently, since

$$\sup_{k\in\mathbb{N}} |d_k g| \mathbf{1}_{A_j} = \sup_{k\in\mathbb{N}} |(\mathbb{E}_k \eta_k - \mathbb{E}_{k-1}\eta_k|)\mathbf{1}_{A_j} \leq \sup_{k\in\mathbb{N}} (|\mathbb{E}_k|\eta_k| + \mathbb{E}_{k-1}|\eta_k|)\mathbf{1}_{A_j} \leq 2 \sup_{n\in\mathbb{N}} \mathbb{E}_n \left(\sup_{k\in\mathbb{N}} |\eta_k| \mathbf{1}_{A_j} \right),$$

we can invoke the Doob's inequality (5.18) to obtain the following (as in the proof of [65, Theorem 2.32])

$$\begin{aligned} \left\| \sup_{k\in\mathbb{N}} |d_k g| \mathbf{1}_{A_j} \right\|_{p'} &\leq 2 \left\| \sup_{n\in\mathbb{N}} \mathbb{E}_n \left(\sup_{k\in\mathbb{N}} |\eta_k| \mathbf{1}_{A_j} \right) \right\|_{p'} \leq \frac{2p'}{p'-1} \sup_{k\in\mathbb{N}} \left\| \mathbb{E}_n \left(\sup_{k\in\mathbb{N}} |\eta_k| \mathbf{1}_{A_j} \right) \right\|_{p'} \\ &\leq \frac{2p'}{p'-1} \left\| \sup_{k\in\mathbb{N}} |\eta_k| \mathbf{1}_{A_j} \right\|_{p'}. \end{aligned}$$

Hence we have

$$\sum_{j\in\mathbb{Z}} \left\| \sup_{k\in\mathbb{N}} |d_k g| \mathbf{1}_{A_j} \right\|_{p'}^{q'} \leq \left(\frac{2p'}{p'-1} \right)^{q'} \sum_{j\in\mathbb{Z}} \left\| \sup_{k\in\mathbb{N}} |\eta_k| \mathbf{1}_{A_j} \right\|_{p'}^{q'}$$

and by definition

$$\left\| \sup_{k\in\mathbb{N}} |d_k g| \right\|_{p',q'} \leq \frac{2p'}{p'-1} \left\| \sup_{k\in\mathbb{N}} |\eta_k| \right\|_{p',q'} = \frac{2p'}{p'-1} \|\eta\|_{\mathcal{K}(L_{p',q'},\ell_\infty)}.$$

Hence $g \in \mathcal{BD}_{p',q'}(\mathbb{R})$ and $\|g\|_{\mathcal{BD}_{p',q'}(\mathbb{R})} \leq C\|\tau\|$ since $\|\tau\| = \|\eta\|_{\mathcal{K}(L_{p',q'},\ell_\infty)}$. We now show that $f_n \to f$ in $\mathcal{G}_{p,q}$ as $n \to \infty$. We observe that since $f = \sum_{k=0}^{\infty} f_k - f_{k-1}$, we have that

$$f_n - f = f_n - \sum_{k=0}^{\infty} f_k - f_{k-1} = f_n - \sum_{k=0}^{n} f_k - f_{k-1} - \sum_{k=n+1}^{\infty} f_k - f_{k-1} = - \sum_{k=n+1}^{\infty} f_k - f_{k-1}.$$

Hence for $n \to \infty$, $f_n - f \to 0$. Also

$$\sum_{k\geq 0} |d_k(f_n - f)| = \sum_{k\geq 0} |d_k f_n - d_k f| \leq 2\sum_{k\geq 0} ||d_k f_n| + |d_k f|| \leq 4\sum_{k\geq 0} |d_k f| < \infty$$

since $f \in \mathcal{G}_{p,q}$ and thus by the Dominated Convergence Theorem

$$\|f_n - f\|_{\mathcal{G}_{p,q}(\mathbb{R})} = \left\| \sum_{k\geq 0} |d_k(f_n - f)| \right\|_{L_{p,q}(\mathbb{R})} \to 0 \quad \text{for } n \to \infty.$$

That is $f_n \to f$ in $\mathcal{G}_{p,q}(\mathbb{R})$. Consequently, from equation (5.19), as $n \to \infty$, we have that, by setting $\eta_k = d_k g$,

$$\tau(f_n) = \sum_{k=1}^{n} \mathbb{E}[(d_k f)\eta_k] \to \tau(f) = \sum_{k=1}^{\infty} \mathbb{E}[(d_k f)\eta_k] = \kappa_g(f).$$

Hence $\|g\|_{\mathcal{BD}_{p',q'}(\mathbb{R})} \leq C\|\tau\| = C\|\kappa\|$ and the poof is complete. □

5.14 Dual Space Characterization of $H^*_{p,q}$

As an application of the Garsia-type space and the Davis decomposition of martingales in the martingale Hardy-amalgam space $H^*_{p,q}$, the dual space of $H^*_{p,q}$, $(1 \leq p, q \leq 2)$ is characterized in this section. In this section, all the martingales are defined with respect to the stochastic basis $\mathbf{Ps} := (\mathbb{R}, \mathcal{F}, \{\mathcal{F}_n\}_{n\in\mathbb{N}}, \mathbb{P})$ where $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ is the dyadic filtration.

Theorem 5.15. *The dual space of $H^*_{p,q}$ $(1 \leq q \leq p \leq 2)$ can be given with the norm*

$$\|\phi\| := \|\phi\|_{H^s_{p',q'}} + \|\phi\|_{\mathcal{BD}_{p'q'}}$$

where $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1.$

Proof. Let $\phi \in H^s_{p',q'} \cap \mathcal{BD}_{p',q'}$. Then $\phi \in L_2$ since L_2 is dense in $H^s_{p',q'}$ (from the atomic decomposition). Define the functional κ_ϕ by

$$\kappa_\phi(f) = \mathbb{E}(f\phi), \quad (f \in L_2). \tag{5.20}$$

We will show that (5.20) is a bounded linear functional on $H^*_{p,q}$ $(1 \leq p, q \leq 2)$. Linearity follows since the expectation operator is linear. Now let $f \in H^*_{p,q}$ $(1 \leq p, q \leq 2)$. Then by density, $\|f^*\|_2 < \infty$. Thus the martingale $f = \{f_n\}_{n\in\mathbb{N}}$ is L_2 bounded and since $\phi \in L_2$, the functional κ_ϕ defined in (5.20) is well defined. As $f_n \to f$ in L_2 norm (as $n \to \infty$), we have that

$$\kappa_\phi(f) := \mathbb{E}(f\phi) = \lim_{n\to\infty} \mathbb{E}(f_n\phi).$$

Since $f \in H^*_{p,q}$, we know from Davis decomposition that $f_n = h_n + g_n$ for which $h = \{h_n\}_{n\in\mathbb{N}}$ and $g = \{g_n\}_{n\in\mathbb{N}}$ are martingales where $h \in \mathcal{G}_{p,q}$ and $g \in \mathcal{P}_{p,q}$ such that

$$\|h\|_{\mathcal{G}_{p,q}} \lesssim \|f\|_{H^*_{p,q}} \tag{5.21}$$

and $\|g\|_{\mathcal{P}_{p,q}} \lesssim \|f\|_{H^*_{p,q}}$. We also recall from Corollary 4.15 that

$$\|g\|_{H^s_{p,q}} \lesssim \|f\|_{H^*_{p,q}}. \tag{5.22}$$

Hence we have by linearity of $\mathbb{E}$ that

$$|\mathbb{E}(f_n\phi)| = |\mathbb{E}(h_n\phi + g_n\phi)| \leq |\mathbb{E}(g_n\phi)| + |\mathbb{E}(h_n\phi)| \tag{5.23}$$

From (5.22) we have that $g \in H^s_{p,q}$ since $f \in H^*_{p,q}$. Hence since $\phi \in H^s_{p',q'}$, it follows from Theorem 5.6, for $(q \leq p)$, that

$$|\mathbb{E}(g_n\phi)| \leq \|g_n\|_{H^s_{p,q}} \|\phi\|_{H^s_{p',q'}}. \tag{5.24}$$

Similarly, since $h \in \mathcal{G}_{p,q}$ and $\phi \in \mathcal{BD}_{p',q'}$, we have by Theorem 5.13 that

$$|\mathbb{E}(h_n\phi)| \leq \|h_n\|_{\mathcal{G}_{p,q}} \|\phi\|_{\mathcal{BD}_{p',q'}}. \tag{5.25}$$

Therefore (5.23) becomes

$$|\mathbb{E}(f_n\phi)| \leq \|g_n\|_{H^s_{p,q}} \|\phi\|_{H^s_{p',q'}} + \|h_n\|_{\mathcal{G}_{p,q}} \|\phi\|_{\mathcal{BD}_{p',q'}}$$

and thus

$$|\mathbb{E}(f\phi)| \leq \|g\|_{H^s_{p,q}} \|\phi\|_{H^s_{p',q'}} + \|h\|_{\mathcal{G}_{p,q}} \|\phi\|_{\mathcal{BD}_{p',q'}}.$$

It then follows from (5.21) and (5.22) that

$$|\mathbb{E}(f\phi)| \leq \|f\|_{H^*_{p,q}} \|\phi\|_{H^s_{p',q'}} + \|f\|_{H^*_{p,q}} \|\phi\|_{\mathcal{BD}_{p',q'}}.$$

In other words

$$|\mathbb{E}(f\phi)| \leq \|f\|_{H^*_{p,q}} (\|\phi\|_{H^s_{p',q'}} + \|\phi\|_{\mathcal{BD}_{p',q'}}). \tag{5.26}$$

Thus the functional κ_ϕ is continuous linear functional on $H^*_{p,q}$.

Conversely, assume that κ_ϕ is an arbitrary continuous linear on $H^*_{p,q}$. Then as $H^*_{p,q}$ embeds continuously in H^*_p $(q < p)$, we have by Hahn-Banach theorem that κ_ϕ can be extended to a continuous linear functional $\tilde{\kappa_\phi}$ on H^*_p having the same operator norm as κ_ϕ. It follows from [65, Theorem 2.34] that there exists some $\phi \in L_2$ such that

$$\tilde{\kappa_\phi}(f) = \mathbb{E}(f\phi), \ \ (f \in L_2).$$

In particular

$$\kappa_\phi(f) = \tilde{\kappa_\phi}(f) = \mathbb{E}(f\phi), \ \ (f \in H^*_{p,q}).$$

We observe that since

$$\begin{aligned} f^* := \sup_{n\in\mathbb{N}} |f_n| &= \sup_{n\in\mathbb{N}} \left| \sum_{k=1}^{n} d_k f \right| \\ &\leq \sup_{n\in\mathbb{N}} \sum_{k=1}^{n} |d_k f| \leq \sup_{n\in\mathbb{N}} \sum_{k=1}^{\infty} |d_k f| = \sum_{k=1}^{\infty} |d_k f| \end{aligned}$$

it implies that

$$\|f^*\|_{p,q} \leq \left\| \sum_{k=1}^{\infty} |d_k f| \right\|_{p,q} \qquad (\text{in other words, } \|f\|_{H^*_{p,q}} \leq \|f\|_{\mathcal{G}_{p,q}}).$$

Thus $\mathcal{G}_{p,q} \subseteq H^*_{p,q}$. Therefore κ_ϕ is also a continuous linear functional on $\mathcal{G}_{p,q}$. It follows from Theorem 5.13 that

$$\|\phi\|_{\mathcal{BD}_{p',q'}} \leq C\|\kappa_\phi\| \tag{5.27}$$

We also recall from the embeddings, Theorem 4.14, that $\|f\|_{H^*_{p,q}} \leq C\|f\|_{H^s_{p,q}}$ $(1 \leq p, q \leq 2)$. Therefore κ_ϕ is also a continuous linear functional on $H^s_{p,q}$. Hence by Theorem 5.6, for $(q \leq p)$,

$$\|\phi\|_{H^s_{p',q'}} \leq C\|\kappa_\phi\|. \tag{5.28}$$

From (5.27) and (5.28), we obtain that

$$\|\phi\|_{\mathcal{BD}_{p',q'}} + \|\phi\|_{H^s_{p',q'}} \leq C\|\kappa_\phi\|$$

and the proof is complete. □

Chapter 6

Martingale Transforms Between Martingale Hardy-amalgam Spaces

Burkholder introduced the notion of martingale transforms [6] in the 1960's. Since then, it has become an indispensable tool in the study of some relations between classical martingale Hardy spaces, mostly the predictive spaces $\mathcal{P}_p$ in the classical settings [26, 48]. In Chapter 4, we also saw a role martingale transform played in characterizing the dual space of $H^s_{p,q}$. In this chapter, we will discuss the martingale transforms between the martingale Hardy-amalgam spaces $H^s_{p,q}$, $\mathcal{Q}_{p,q}$ and $\mathcal{P}_{p,q}$. More precisely if $p_1 < p$ and $q_1 < q$ and f is a martingale in $\mathcal{P}_{p_1,q_1}$, then its martingale transforms are the martingales in $\mathcal{P}_{p,q}$ and similarly for $H^s_{p,q}$ and $\mathcal{Q}_{p,q}$. The motivation to look for the various martingale transforms in these spaces comes from the various applications of martingale transforms in general. Especially, with the use of martingale transforms, the upcrossing theorem of martingales was established, the convergence of martingales has also been proved using martingale transforms and L^1-characterization of martingales [6, 48].

Let **Ps** $:= (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_n\}_{n\geq 0})$ be a probability space with the filtration $\{\mathcal{F}_n\}_{n\geq 0}$. Let $\nu = \{\nu_n\}_{n\geq 0}$ be an adapted increasing sequence such that for all n, ν_n is $\mathcal{F}_{n-1}-$measurable, normally referred to as multiplier sequence. If $f = (f_n,\ n \in \mathbb{N})$ is a martingale, then we recall that

$$g_n = \sum_{k=1}^{n} \nu_k d_k f, \quad g_0 = 0$$

is called a martingale transform where $d_k f$ is the usual martingale difference sequence. Burkholder has established the almost everywhere convergence of the martingale transform based on the condition that the maximal function of the multiplier sequence is finite. In fact, we can find the proof of the following convergence result in [6, Theorem 1].

Theorem 6.1. *Let $f = (f_n,\ n \in \mathbb{N}),\ f_0 = 0$ be an L^1 bounded martingale and g be f's* *martingale transform with multiplier sequence* $\nu = (\nu_k)_{k\geq 0}$ *defined below;*

$$g_n = \sum_{k=1}^{n} \nu_k d_k f, \quad n \geq 1, \quad g_0 = 0.$$

Then, $g = (g_n)_{n\geq 0}$ *converges almost everywhere on the set* $\{\sup_k |\nu_k| < \infty\}$.

Let us present the theorems that we will discuss in this chapter. We start with the results on $\mathcal{P}_{p,q}$ followed by $H^s_{p,q}$ and finally $\mathcal{Q}_{p,q}$. We also note that all the martingales are defined with respect to $\mathbf{Ps} := (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_n\}_{n\geq 0})$ relative to the filtration $\{\mathcal{F}_n\}_{n\geq 0}$.

6.2 Relations Between $\mathcal{P}_{p,q}$ and $\mathcal{P}_{p_1,q_1}$

Theorem 6.3 (Relation Between $\mathcal{P}_{p,q}$ and $\mathcal{P}_{p_1,q_1}$)**.** *Let* $0 < p < q < \infty,\ 0 < p_1 < p,\ 0 < q_1 = \frac{p_1}{p} q < q$ *and* $\alpha = 1 - \frac{p_1}{p}$. *Let* $f = (f_n,\ n \in \mathbb{N})$ *be a martingale define on* **Ps** *and suppose that* $f \in \mathcal{P}_{p_1,q_1}$. *Let* $\nu = (\nu_k)_{k\geq 0}$ *be the optimal bounded positive increasing adapted process such that* $|f_n| \leq \nu_{n-1}$ *and* $\nu_\infty \in L_{p_1,q_1}$. *Then the process defined by*

$$g_n = \sum_{k=1}^{n} \frac{1}{\nu^\alpha_{k-1}} d_k f, \quad g_0 = 0 \tag{6.1}$$

is a martingale transform of f *and converges almost everywhere.*
Moreover, $g = (g_n,\ n \in \mathbb{N}) \in \mathcal{P}_{p,q}$ *and*

$$\|g\|^q_{\mathcal{P}_{p,q}} \leq \left(\frac{p}{p_1} + 1\right)^q \|f\|^{q_1}_{\mathcal{P}_{p_1,q_1}}.$$

Proof. Let $f \in \mathcal{P}_{p_1,q_1}$. By the hypothesis, $|f_n| \leq \nu_{n-1}$ and $\nu_\infty \in L_{p_1,q_1}$ ($\nu = (\nu_n)_{n\geq 0}$ optimal). From equation (6.1)

$$g_n = \frac{f_n}{\nu^\alpha_{n-1}} + \sum_{k=1}^{n-1} f_k \left(\frac{1}{\nu^\alpha_{k-1}} - \frac{1}{\nu^\alpha_k}\right). \tag{6.2}$$

Indeed

$$\begin{aligned} g_n = \sum_{k=1}^{n} \frac{d_k f}{\nu^\alpha_{k-1}} &= \frac{d_n f}{\nu^\alpha_{n-1}} + \sum_{k=1}^{n-1} \frac{d_k f}{\nu^\alpha_{k-1}} = \frac{f_n}{\nu^\alpha_{n-1}} - \frac{f_{n-1}}{\nu^\alpha_{n-1}} + \sum_{k=1}^{n-1} \frac{d_k f}{\nu^\alpha_{k-1}} \\ &= \frac{f_n}{\nu^\alpha_{n-1}} - \left(\sum_{k=1}^{n-1} \frac{f_k}{\nu^\alpha_k} - \frac{f_{k-1}}{\nu^\alpha_{k-1}}\right) + \left(\sum_{k=1}^{n-1} \frac{f_k}{\nu^\alpha_{k-1}} - \frac{f_{k-1}}{\nu^\alpha_{k-1}}\right) \\ &= \frac{f_n}{\nu^\alpha_{n-1}} - \sum_{k=1}^{n-1} \frac{f_k}{\nu^\alpha_k} + \sum_{k=1}^{n-1} \frac{f_{k-1}}{\nu^\alpha_{k-1}} + \sum_{k=1}^{n-1} \frac{f_k}{\nu^\alpha_{k-1}} - \sum_{k=1}^{n-1} \frac{f_{k-1}}{\nu^\alpha_{k-1}} \end{aligned}$$

$$\begin{aligned} g_n &= \frac{f_n}{\nu_{n-1}^{\alpha}} + \sum_{k=1}^{n-1} \frac{f_k}{\nu_{k-1}^{\alpha}} - \sum_{k=1}^{n-1} \frac{f_k}{\nu_k^{\alpha}} \\ &= \frac{f_n}{\nu_{n-1}^{\alpha}} + \sum_{k=1}^{n-1} f_k \left(\frac{1}{\nu_{k-1}^{\alpha}} - \frac{1}{\nu_k^{\alpha}} \right). \end{aligned}$$

We note that when $\nu_{k-1} \leq t \leq \nu_k$ then $\frac{1}{\nu_k^{\alpha}} \leq \frac{1}{t^{\alpha}}$ which implies that

$$\int_{\nu_{k-1}}^{\nu_k} \frac{1}{\nu_k^{\alpha}} \mathrm{d}t \leq \int_{\nu_{k-1}}^{\nu_k} \frac{1}{t^{\alpha}} \mathrm{d}t \Rightarrow \frac{\nu_k - \nu_{k-1}}{\nu_k^{\alpha}} \leq \int_{\nu_{k-1}}^{\nu_k} \frac{1}{t^{\alpha}} \mathrm{d}t.$$

Therefore, from equation (6.2), we get that

$$|g_n| \leq \int_0^{\nu_{n-1}} \frac{1}{t^{\alpha}} \mathrm{d}t = \left(\frac{1}{1-\alpha} + 1 \right) \nu_{n-1}^{1-\alpha}. \tag{6.3}$$

Indeed since $\frac{1}{\nu_{k-1}^{\alpha}} - \frac{1}{\nu_k^{\alpha}} \geq 0$, we get

$$\begin{aligned} |g_n| &\leq \frac{|f_n|}{\nu_{n-1}^{\alpha}} + \sum_{k=1}^{n-1} |f_k| \left(\frac{1}{\nu_{k-1}^{\alpha}} - \frac{1}{\nu_k^{\alpha}} \right) \\ &\leq \nu_{n-1}^{1-\alpha} + \sum_{k=1}^{n-1} \nu_{k-1} \left(\frac{1}{\nu_{k-1}^{\alpha}} - \frac{1}{\nu_k^{\alpha}} \right) \\ &= \nu_0^{1-\alpha} + \sum_{k=1}^{n-1} (\nu_k^{1-\alpha} - \nu_{k-1}^{1-\alpha}) + \sum_{k=1}^{n-1} \nu_{k-1}^{1-\alpha} - \frac{\nu_{k-1}}{\nu_k^{\alpha}} \\ &= \nu_0^{1-\alpha} + \sum_{k=1}^{n-1} \nu_k^{1-\alpha} + \sum_{k=1}^{n-1} - \frac{\nu_{k-1}}{\nu_k^{\alpha}} \\ &= \nu_0^{1-\alpha} + \sum_{k=1}^{n-1} \frac{\nu_k}{\nu_k^{\alpha}} - \frac{\nu_{k-1}}{\nu_k^{\alpha}} \leq \nu_0^{1-\alpha} + \sum_{k=1}^{n-1} \int_{\nu_{k-1}}^{\nu_k} \frac{1}{t^{\alpha}} \mathrm{d}t \\ &\leq \nu_0^{1-\alpha} + \int_0^{\nu_{n-1}} \frac{1}{t^{\alpha}} \mathrm{d}t \leq \left(\frac{1}{1-\alpha} + 1 \right) \nu_{n-1}^{1-\alpha} \end{aligned}$$

Hence $\sup_n |g_n| < \infty$ since $\{\nu_n\}_{n\geq 0}$ is a bounded sequence. We note that ν_k is $\mathcal{F}_{k-1}$-measurable as it is adapted and so is $\nu_{k-1}^{-\alpha}$. Now the sequence $w_k = \nu_{k-1}^{-\alpha}$ is a positive decreasing sequence which is bounded above by $\nu_0^{-\alpha}$ since $\nu_0 > 0$. Therefore $\sup_k |\nu_k^{-\alpha}| < \infty$ since $\nu_k^{\alpha} \neq 0$ for all k. Thus equation (6.1) is a martingale transform and hence by Theorem 6.1, g_n converges almost everywhere.

We observe that as $(1-\alpha)p = p_1$, we also have that $\|\nu_{\infty}^{1-\alpha} \mathbf{1}_{\Omega_j}\|_p = \|\nu_{\infty} \mathbf{1}_{\Omega_j}\|_{p_1}^{\frac{p_1}{p}}$. Therefore

$$\|\nu_{\infty}^{1-\alpha}\|_{p,q} = \|\nu_{\infty}\|_{p_1,q_1}^{\frac{q_1}{q}} := \|f\|_{\mathcal{P}_{p_1,q_1}}^{\frac{q_1}{q}}. \tag{6.4}$$

Let $\beta_{n-1} = \left(\frac{1}{1-\alpha}+1\right)\nu_{n-1}^{1-\alpha}$. Then the sequence $\beta = (\beta_n)_{n\geq 0}$ is also an increasing positive adapted process. Hence by definition

$$\|g\|_{\mathcal{P}_{p,q}} \leq \left\| \left(\frac{1}{1-\alpha}+1\right)\nu_\infty^{1-\alpha} \right\|_{p,q} = \left(\frac{1}{1-\alpha}+1\right)\|\nu_\infty^{1-\alpha}\|_{p,q}.$$

Equation (6.4) then gives us

$$\|g\|_{\mathcal{P}_{p,q}} \leq \left(\frac{p}{p_1}+1\right)\|f\|_{\mathcal{P}_{p_1,q_1}}^{\frac{q_1}{q}}$$

establishing the result. ☐

The next Theorem states that if $g \in \mathcal{P}_{p,q}$ is the martingale transform of f, then $f \in \mathcal{P}_{p_1,q_1}$ which in turn is the martingale transform of g.

Theorem 6.4 (Relation Between $\mathcal{P}_{p,q}$ and $\mathcal{P}_{p_1,q_1}$)**.** *Let* $0 < p < q < \infty,\ 0 < p_1 < p,\ 0 < q_1 = \frac{p_1}{p}q < q$ *and* $\alpha = 1 - \frac{p_1}{p}$. *Let* $f = (f_n,\ n \in \mathbb{N})$ *be a martingale define on* **Ps** *and let* $\nu = (\nu_k)_{k\geq 0}$ *be a bounded positive increasing adapted process such that* $\nu_\infty \in L_{p_1,q_1}$. *Let* $g = (g_n,\ n \in \mathbb{N}) \in \mathcal{P}_{p,q}$ *be a martingale transform of* f *defined by*

$$g_n = \sum_{k=1}^{n} \frac{1}{\nu_{k-1}^{\alpha}} d_k f, \qquad g_0 = 0$$

such that $\mathbb{E}(g_n) < \infty$. *Then*

(a) $f_n = \sum_{k=1}^{n} \nu_{k-1}^{\alpha} d_k g$ *and converges almost everywhere and*

(b) $f \in \mathcal{P}_{p_1,q_1}$ *and moreover,*

$$\|f\|_{\mathcal{P}_{p_1,q_1}} \lesssim \|g\|_{\mathcal{P}_{p,q}} \|\nu_\infty\|_{p_1,q_1}^{1-\frac{p_1}{p}}.$$

Proof. Let $g \in \mathcal{P}_{p,q}$. Then there exists an optimal positive increasing adapted sequence $u = (u_n)_{n\geq 0}$, such that $|g_n| \leq u_{n-1}$ and $u_\infty \in L_{p,q}$.

Part (a) follows by observing that $d_n g = g_n - g_{n-1} = \frac{d_n f}{\nu_{n-1}^{\alpha}} \Longrightarrow d_n f = \nu_{n-1}^{\alpha} d_n g$. Therefore

$$f_n = \sum_{k=1}^{n} \nu_{k-1}^{\alpha} d_k g.$$

Since g is a martingale and ν is a bounded positive increasing and adapted and for all n, ν_n is $\mathcal{F}_{n-1}-$measurable, the convergence of f_n follows from Theorem 6.1.

For Part (b), since $f_n = \sum_{k=1}^{n} \nu_{k-1}^{\alpha} d_k g$, proceeding as in the proof of equation (6.2), we obtain

$$f_n = g_n \nu_{n-1}^{\alpha} - \sum_{k=1}^{n-1} g_k d_k \nu^{\alpha}.$$

Hence since $(u_n)_{n\geq 0}$ is increasing and $d_k\nu^{\alpha}$ is positive, we obtain that

$$\begin{aligned} |f_n| &\leq |g_n|\nu_{n-1}^{\alpha} + \sum_{k=1}^{n-1} |g_k| d_k \nu^{\alpha} \\ &\leq u_{n-1}\nu_{n-1}^{\alpha} + \sum_{k=1}^{n-1} u_{k-1} d_k \nu^{\alpha} \leq u_{n-1} v_{n-1}^{\alpha} + u_{n-1} \sum_{k=1}^{n-1} d_k \nu^{\alpha} \\ &\leq 2u_{n-1}\nu_{n-1}^{\alpha} \leq 2u_{\infty}\nu_{n-1}^{\alpha}. \end{aligned}$$

Let $\gamma_{n-1} = 2u_{\infty}\nu_{n-1}^{\alpha}$. Then, the sequence $\gamma = (\gamma_n)_{n\geq 0}$ is also positive increasing and bounded adapted process. Hence by definition,

$$\|f\|_{\mathcal{P}_{p_1,q_1}} \leq \|\gamma_{\infty}\|_{p_1,q_1} = 2\|u_{\infty}\nu_{\infty}^{\alpha}\|_{p_1,q_1}.$$

Hence

$$\|f\|_{\mathcal{P}_{p_1,q_1}} \leq 2\|u_{\infty}\nu_{\infty}^{\alpha}\|_{p_1,q_1}. \tag{6.5}$$

By definition,

$$\|u_{\infty}\nu_{\infty}^{\alpha}\|_{p_1,q_1} = \left(\sum_{j\in\mathbb{Z}} \|u_{\infty}\nu_{\infty}^{\alpha}\mathbf{1}_{\Omega_j}\|_{p_1}^{q_1}\right)^{\frac{1}{q_1}}.$$

With the choice of α, and also observing that $q_1 = (1-\alpha)q$, we apply Hölder's inequality to obtain

$$\|u_{\infty}\nu_{\infty}^{\alpha}\|_{p_1,q_1} \leq \|u_{\infty}\|_{p,q}\|\nu_{\infty}\|_{p_1,q_1}^{1-\frac{p_1}{p}}.$$

Inequality (6.5) then becomes

$$\|f\|_{\mathcal{P}_{p_1,q_1}} \leq 2\|u_{\infty}\|_{p,q}\|\nu_{\infty}\|_{p_1,q_1}^{1-\frac{p_1}{p}} := 2\|g\|_{\mathcal{P}_{p,q}}\|\nu_{\infty}\|_{p_1,q_1}^{1-\frac{p_1}{p}}.$$

and the Theorem is proved. □

6.5 Relations Between $H_{p,q}^s$ and H_{p_1,q_1}^s

In this section, we will discuss the boundedness of the martingale transforms on the martingale Hardy-amalgam spaces $H_{p,q}^s$ and H_{p_1,q_1}^s.

Theorem 6.6 (Relation Between $H_{p,q}^s$ and H_{p_1,q_1}^s). *Let* $0 < p < q < \infty,\ 0 < p_1 < p,\ 0 <$

$q_1 = \frac{p_1}{p}q < q$ and $\alpha = 1 - \frac{p_1}{p}$. Let $f = (f_n,\ n \in \mathbb{N})$ be a martingale define on **Ps**. *Let $s(f)$ be the conditional quadratic variation operator which we assume it is bounded and non-zero. Then the process defined by*

$$g_n = \sum_{k=1}^{n} \frac{1}{s_k^{\alpha}(f)} d_k f, \quad g_0 = 0 \tag{6.6}$$

is the martingale transform of f and converges almost everywhere. Moreover, if $f \in H^s_{p_1,q_1}$, then $g \in H^s_{p,q}$ and

$$\|g\|^q_{H^s_{p,q}} \leq \left(\frac{p}{p_1}\right)^{\frac{q}{2}} \|f\|^{q_1}_{H^s_{p_1,q_1}}$$

Proof. Let $s(f)$ be the conditional quadratic variation operator. Then by definition $s_k(f)$ is $\mathcal{F}_{k-1}-$measurable. Thus $s_k^{-\alpha}(f)$ is $\mathcal{F}_{k-1}-$measurable and positive and bounded adapted decreasing process. Hence g_n is a martingale transform. Since $s_k(f)$ is positive and bounded, $s_k^{-\alpha}(f)$ is also positive and bounded. That is $\sup_k |s_k^{-\alpha}(f)| < \infty$. Thus by Theorem 6.1, g_n converges almost everywhere. Suppose that $f \in H^s_{p_1,q_1}$. Then $\|s(f)\|_{p_1,q_1} < \infty$. From equation (6.6), we obtain by measurability that

$$d_k^2 g = \frac{d_k^2 f}{s_k^{2\alpha}(f)} \Longrightarrow \mathbb{E}_{k-1} d_k^2 g = \frac{\mathbb{E}_{k-1} d_k^2 f}{s_k^{2\alpha}(f)}.$$

Since $\mathbb{E}_{k-1} d_k^2 f = s_k^2(f) - s_{k-1}^2(f)$, we sum both sides of the last equality to obtain

$$s_n^2(g) = \sum_{k=1}^{n} \frac{s_k^2(f) - s_{k-1}^2(f)}{s_k^{2\alpha}(f)}. \tag{6.7}$$

Since $s_k(f)$ is increasing, we observe that for $t > 0$, $s_{k-1}^2(f) \leq t \leq s_k^2(f)$ and thus

$$\frac{1}{s_k^{2\alpha}(f)} \leq \frac{1}{t^{\alpha}}.$$

It follows that

$$\int_{s_{k-1}^2(f)}^{s_k^2(f)} \frac{1}{s_k^{2\alpha}(f)} \mathrm{d}t \leq \int_{s_{k-1}^2(f)}^{s_k^2(f)} \frac{1}{t^{\alpha}} \mathrm{d}t.$$

That is

$$\frac{s_k^2(f) - s_{k-1}^2(f)}{s_k^{2\alpha}(f)} \leq \int_{s_{k-1}^2(f)}^{s_k^2(f)} \frac{1}{t^{\alpha}} \mathrm{d}t.$$

We deduce from this and (6.7) that

$$s_n^2(g) \leq \sum_{k=1}^{n} \int_{s_{k-1}^2(f)}^{s_k^2(f)} \frac{1}{t^{\alpha}} \mathrm{d}t$$

and hence we obtain that

$$s_n^2(g) \leq \int_0^{s_n^2(f)} \frac{1}{t^\alpha}\mathrm{d}t \leq \frac{1}{1-\alpha}(s(f))^{2-2\alpha}$$

which gives us

$$s(g) \leq \left(\frac{1}{1-\alpha}\right)^{\frac{1}{2}} s^{1-\alpha}(f).$$

Now since

$$\|s^{1-\alpha}(f)\|_{p,q} = \|s(f)\|_{p_1,q_1}^{\frac{q_1}{q}},$$

we have that

$$\|s(g)\|_{p,q} \leq \left(\frac{p}{p_1}\right)^{\frac{1}{2}} \|s(f)\|_{p_1,q_1}^{\frac{q_1}{q}}.$$

Thus by definition

$$\|g\|_{H^s_{p,q}} \leq \left(\frac{p}{p_1}\right)^{\frac{1}{2}} \|f\|_{H^s_{p_1,q_1}}^{\frac{q_1}{q}}.$$

and the Theorem is proved. □

The converse of the above Theorem is the following Theorem. That is given that $g \in H^s_{p,q}$ is the martingale transform of f, then f is the martingale transform of g and $f \in H^s_{p_1,q_1}$.

Theorem 6.7 (Relation Between $H^s_{p,q}$ and $H^s_{p_1,q_1}$)**.** *Let* $0 < p < q < \infty,\ 0 < p_1 < p,\ 0 < q_1 = \frac{p_1}{p}q < q$ *and* $\alpha = 1 - \frac{p_1}{p}$. *Let* $f = (f_n,\ n \in \mathbb{N})$ *be a martingale define on* **Ps** *and* $s(f)$ *be the conditional quadratic variation operator which we assume it is bounded and non-zero. Let* $g = (g_n,\ n \in \mathbb{N}) \in H^s_{p,q}$ *be a martingale transform of* f *defined by*

$$g_n = \sum_{k=1}^{n} \frac{1}{s_k^\alpha(f)} d_k f, \quad g_0 = 0$$

such that $\mathbb{E}(g_n) < \infty$. *Then*

(a) $f_n = \sum_{k=1}^n s_k^\alpha(f) d_k g$ *and converges almost everywhere and*

(b) $f \in H^s_{p_1,q_1}$ *and moreover,*

$$\|f\|_{H^s_{p_1,q_1}}^{\frac{q_1}{q}} \leq \|g\|_{H^s_{p,q}}.$$

Proof. Part (a) follows by observing that $d_n g = g_n - g_{n-1} = \frac{d_n f}{s_n^\alpha(f)} \implies d_n f = s_n^\alpha(f) d_n g$. Therefore

$$f_n = \sum_{k=1}^{n} s_k^\alpha(f) d_k g.$$

Since g is a martingale and $s(f)$ is a bounded positive increasing and adapted and for all n, $s_n(f)$ is $\mathcal{F}_{n-1}-$measurable, the convergence of f_n follows from Theorem 6.1. For

Part (b), we have from equation (6.6) that $d_k f = d_k g s_k^{\alpha}(f)$ and by measurability and the increasing property of $s(\cdot)$, we get that

$$\mathbb{E}_{k-1} d_k^2 f \leq s^{2\alpha}(f) \mathbb{E}_{k-1} d_k^2 g.$$

Summing both sides, we shall obtain

$$s^2(f) \leq s^{2\alpha}(f) s^2(g)$$

and thus

$$s^{1-\alpha}(f) \leq s(g).$$

But

$$\|s^{1-\alpha}(f)\|_{p,q} = \|s(f)\|_{p_1,q_1}^{\frac{q_1}{q}}.$$

Therefore

$$\|s(f)\|_{p_1,q_1}^{\frac{q_1}{q}} \leq \|s(g)\|_{p,q}.$$

The proof is complete. □

6.8 Relation Between $\mathcal{Q}_{p,q}$ and $\mathcal{Q}_{p_1,q_1}$

This section is devoted to the discussion of the boundedness of the martingale transforms on the martingale Hardy-amalgam spaces $\mathcal{Q}_{p,q}$ and $\mathcal{Q}_{p_1,q_1}$. We start by showing that if $f \in \mathcal{Q}_{p_1,q_1}$, then the martingale transform of f is in the space $\mathcal{Q}_{p,q}$ and converges almost everywhere.

Theorem 6.9 (Relation Between $\mathcal{Q}_{p,q}$ and $\mathcal{Q}_{p_1,q_1}$). *Let $0 < p < q < \infty$, $0 < p_1 < p$, $0 < q_1 = \frac{p_1}{p} q < q$ and $\alpha = 1 - \frac{p_1}{p}$. Let $f = (f_n,\ n \in \mathbb{N})$ be a martingale define on* **Ps** *and suppose that $f \in \mathcal{Q}_{p_1,q_1}$. Let $\nu = (\nu_k)_{k\geq 0}$ be the optimal bounded positive increasing adapted process such that $S_n(f) \leq \nu_{n-1}$ and $\nu_\infty \in L_{p_1,q_1}$. Then the process defined by*

$$g_n = \sum_{k=1}^{n} \frac{1}{\nu_{k-1}^{\alpha}} d_k f, \quad g_0 = 0 \tag{6.8}$$

is a martingale transform of f and converges almost everywhere.
Moreover, $g = (g_n,\ n \in \mathbb{N}) \in \mathcal{Q}_{p,q}$ and

$$\|g\|_{\mathcal{Q}_{p,q}}^q \leq \left(\frac{p}{p_1} + 1\right) \|f\|_{\mathcal{Q}_{p_1,q_1}}^{q_1}$$

Proof. Let $f \in \mathcal{Q}_{p_1,q_1}$. By hypothesis, $S_n(f) \leq \nu_{n-1}$ and $\nu_\infty \in L_{p_1,q_1}$ $(\nu = (\nu_n)_{n\geq 0}$

optimal). Considering equation (6.8), we have that

$$|d_n g|^2 = \nu_{n-1}^{-2\alpha}|d_n f|^2. \tag{6.9}$$

Since $\nu = (\nu_k)_{k\geq 0}$ is increasing and $S_k(f) \leq \nu_{k-1}$, we observe from equation (6.9) that, (as $S_0(f) = 0$),

$$\begin{aligned}
S_n^2(g) &= \sum_{k=1}^{n} \frac{S_k^2(f) - S_{k-1}^2(f)}{\nu_{k-1}^{2\alpha}} \\
&= \frac{S_n^2(f)}{\nu_{n-1}^{2\alpha}} - \frac{S_{n-1}^2(f)}{\nu_{n-1}^{2\alpha}} + \sum_{k=1}^{n-1} \frac{S_k^2(f) - S_{k-1}^2(f)}{\nu_{k-1}^{2\alpha}} \\
&= \frac{S_n^2(f)}{\nu_{n-1}^{2\alpha}} - \left(\sum_{k=1}^{n-1} \frac{S_k^2(f)}{\nu_k^{2\alpha}} - \frac{S_{k-1}^2(f)}{\nu_{k-1}^{2\alpha}}\right) + \sum_{k=1}^{n-1} \frac{S_k^2(f) - S_{k-1}^2(f)}{\nu_{k-1}^{2\alpha}} \\
&= \frac{S_n^2(f)}{\nu_{n-1}^{2\alpha}} + \sum_{k=1}^{n-1} \frac{S_k^2(f)}{\nu_{k-1}^{2\alpha}} - \frac{S_k^2(f)}{\nu_k^{2\alpha}} \\
&\leq \nu_{n-1}^{2-2\alpha} + \sum_{k=1}^{n-1} S_k^2(f)\left(\frac{1}{\nu_{k-1}^{2\alpha}} - \frac{1}{\nu_k^{2\alpha}}\right) \\
&\leq \nu_{n-1}^{2-2\alpha} + \sum_{k=1}^{n-1} \nu_{k-1}^2\left(\frac{1}{\nu_{k-1}^{2\alpha}} - \frac{1}{\nu_k^{2\alpha}}\right) \\
&= \nu_0^{2-2\alpha} + \sum_{k=1}^{n-1}(\nu_k^{2-2\alpha} - \nu_{k-1}^{2-2\alpha}) + \sum_{k=1}^{n-1} \nu_{k-1}^{2-2\alpha} - \frac{\nu_{k-1}^2}{\nu_k^{2\alpha}} \\
&= \nu_0^{2-2\alpha} + \sum_{k=1}^{n-1} \nu_k^{2-2\alpha} + \sum_{k=1}^{n-1} -\frac{\nu_{k-1}^2}{\nu_k^{2\alpha}} \\
&= \nu_0^{2-2\alpha} + \sum_{k=1}^{n-1} \frac{\nu_k^2}{\nu_k^{2\alpha}} - \frac{\nu_{k-1}^2}{\nu_k^{2\alpha}} \leq \nu_0^{2-2\alpha} + \sum_{k=1}^{n-1} \int_{\nu_{k-1}^2}^{\nu_k^2} \frac{1}{t^\alpha}\mathrm{d}t \\
&\leq \nu_0^{2-2\alpha} + \int_0^{\nu_{n-1}^2} \frac{1}{t^\alpha}\mathrm{d}t \leq \left(\frac{1}{1-\alpha} + 1\right)\nu_{n-1}^{2-2\alpha}.
\end{aligned}$$

Hence we get that

$$S_n(g) \leq \left(\frac{1}{1-\alpha} + 1\right)^{\frac{1}{2}} \nu_{n-1}^{1-\alpha}.$$

This implies that $S(g) \leq \left(\frac{p}{p_1} + 1\right)^{\frac{1}{2}} \nu_\infty^{\frac{p_1}{p}} < \infty$. Also $\nu_{k-1}^{-\alpha}$ is adapted. Thus g_n is a martingale transform and by Theorem 6.1, g_n converges almost everywhere since $\sup_k |\nu_k^{-\alpha}| < \infty$ (as $\nu_0 > 0$) and f is a martingale.

Let $\beta_{n-1} = \sqrt{\frac{1}{1-\alpha} + 1\nu_{n-1}^{1-\alpha}}$. Then the sequence $\beta = (\beta_n)_{n\geq 0}$ is also an increasing positive and bounded adapted process. Hence by definition

$$\|g\|_{\mathcal{Q}_{p,q}} \leq \|\beta_\infty\|_{p,q} = \left\| \sqrt{\frac{1}{1-\alpha} + 1\nu_{n-1}^{1-\alpha}} \right\|_{p,q} = \sqrt{\frac{1}{1-\alpha} + 1}\|\nu_\infty^{1-\alpha}\|_{p,q}.$$

In the same manner as we obtained equation (6.4), we also have that

$$\|\nu_\infty^{1-\alpha}\|_{p,q} = \|\nu_\infty\|_{p_1,q_1}^{\frac{q_1}{q}} := \|f\|_{\mathcal{Q}_{p_1,q_1}}^{\frac{q_1}{q}}.$$

Therefore

$$\|g\|_{\mathcal{Q}_{p,q}}^q \leq \left(\frac{1}{1-\alpha} + 1\right)^{\frac{q}{2}} \|f\|_{\mathcal{Q}_{p_1,q_1}}^{q_1}.$$

In other words

$$\|g\|_{\mathcal{Q}_{p,q}}^q \leq \left(\frac{p}{p_1} + 1\right)^{\frac{q}{2}} \|f\|_{\mathcal{Q}_{p_1,q_1}}^{q_1}$$

and the theorem is proved.

$\square$

Similarly, if $g \in \mathcal{Q}_{p,q}$ is the martingale transform of f, then $f \in \mathcal{Q}_{p_1,q_1}$ and moreover, f is the martingale transform of g. This is the statement below.

Theorem 6.10 (Relation Between $\mathcal{Q}_{p,q}$ and $\mathcal{Q}_{p_1,q_1}$)**.** *Let* $0 < p < q < \infty$, $0 < p_1 < p$, $0 < q_1 = \frac{p_1}{p}q < q$ *and* $\alpha = 1 - \frac{p_1}{p}$. *Let* $f = (f_n,\ n \in \mathbb{N})$ *be a martingale define on* **Ps** *and let* $\nu = (\nu_k)_{k\geq 0}$ *be a bounded positive increasing adapted process such that* $\nu_\infty \in L_{p_1,q_1}$. *Let* $g = (g_n,\ n \in \mathbb{N}) \in \mathcal{Q}_{p,q}$ *be a martingale transform of* f *defined by*

$$g_n = \sum_{k=1}^{n} \frac{1}{\nu_{k-1}^{\alpha}} d_k f, \quad g_0 = 0$$

such that $\mathbb{E}(g_n) < \infty$. *Then*

(a) $f_n = \sum_{k=1}^{n} \nu_{k-1}^{\alpha} d_k g$ *and converges almost everywhere and*

(b) $f \in \mathcal{Q}_{p_1,q_1}$ *and moreover,*

$$\|f\|_{\mathcal{Q}_{p_1,q_1}} \lesssim \|g\|_{\mathcal{Q}_{p,q}} \|\nu_\infty\|_{p_1,q_1}^{1-\frac{p_1}{p}}.$$

Proof. Similar to the proof of Theorem 6.4, Part (a) is established. For Part (b), let $g \in \mathcal{Q}_{p,q}$. Then there exists an optimal increasing positive adapted process $u = (u_k)_{k\geq 0}$

such that $S_n(g) \leq u_{n-1}$ and $u_\infty \in L_{p,q}$. From equation (6.9), we have that

$$|d_n f|^2 = \nu_{n-1}^{2\alpha} |d_n g|^2 \Longrightarrow S_n^2(f) = \sum_{k=1}^{n} \nu_{k-1}^{2\alpha} [S_k^2(g) - S_{k-1}^2(g)].$$

We also observe from the proof of Theorem 6.4 that

$$S_n^2(f) = S_n^2(g)\nu_{n-1}^{2\alpha} - \sum_{k=1}^{n-1} S_k^2(g) d_k \nu^{2\alpha}$$

and therefore

$$S_n(f) \leq \sqrt{2} u_\infty \nu_{n-1}^{\alpha}.$$

Let $\gamma_{n-1} = \sqrt{2} u_\infty \nu_{n-1}^{\alpha}$. Then the sequence $\gamma = (\gamma_n)_{n \geq 0}$ is also positive increasing and bounded adapted process. Hence by definition,

$$\|f\|_{\mathcal{Q}_{p_1,q_1}} \leq \|\gamma_\infty\|_{p_1,q_1} = \sqrt{2} \|u_\infty \nu_\infty^{\alpha}\|_{p_1,q_1}.$$

In other words

$$\|f\|_{\mathcal{Q}_{p_1,q_1}} \leq \sqrt{2} \|u_\infty \nu_\infty^{\alpha}\|_{p_1,q_1}. \tag{6.10}$$

By definition,

$$\|u_\infty \nu_\infty^{\alpha}\|_{p_1,q_1} = \left(\sum_{j \in \mathbb{Z}} \|u_\infty \nu_\infty^{\alpha} \mathbf{1}_{\Omega_j}\|_{p_1}^{q_1} \right)^{\frac{1}{q_1}}.$$

With the choice of α, and noting that $q_1 = (1-\alpha)q$, we apply Hölder's inequality to get

$$\|u_\infty \nu_\infty^{\alpha}\|_{p_1,q_1} \leq \|u_\infty\|_{p,q} \|\nu_\infty\|_{p_1,q_1}^{1-\frac{p_1}{p}}.$$

Inequality (6.10) then becomes

$$\|f\|_{\mathcal{Q}_{p_1,q_1}} \leq \sqrt{2} \|u_\infty\|_{p,q} \|\nu_\infty\|_{p_1,q_1}^{1-\frac{p_1}{p}} = \sqrt{2} \|g\|_{\mathcal{Q}_{p,q}} \|\nu_\infty\|_{p_1,q_1}^{1-\frac{p_1}{p}}.$$

and the theorem is proved. □

Chapter 7

Conclusion and Recommendations

In this study, we have introduced the martingale Hardy-amalgam spaces and provided a theory on some of the properties of martingales in these spaces. The martingale Hardy-amalgam spaces are generalizations of the classical martingale Hardy spaces. The martingale Hardy-amalgam spaces are also considered as proper substitute to the classical martingale Hardy spaces as the amalgam spaces are good substitutes to the Lebesgue spaces.

We have studied two atomic decompositions of the new martingale Hardy-amalgam spaces introduced. We have also characterized the duality of a Garsia-type spaces. As applications of the atomic decompositions and the dual of the Garsia-type space, we were able to characterize the dual spaces of the martingale Hardy-amalgam spaces. We have also extended on the Burkholder-Davis-Gundy inequality involving martingales in the martingale Hardy-amalgam spaces. Other classical martingale inequalities were also extended to martingales in the martingale Hardy-amalgam spaces. In addition, the theory on the extension of the martingale embeddings for martingale in the martingale Hardy-amalgam space were also discussed. The Davis decompositions of martingales in the martingale Hardy-amalgam spaces was also studied and as applications of these decompositions, the dual space of $H_{p,q}^*$ was established. Finally, we discussed the martingale transforms that exist between martingale Hard-amalgam spaces. These transforms, as we saw in the study, were very useful as we applied these martingale transform techniques in the dual space characterizations. The results obtained in this study will be relevant for further studies especially when one wants to describe other properties of these newly introduced spaces. For instance, the results obtained can serve as a foundation if one wants to determine whether the constants that appear in the martingale inequalities are sharp or not. It will also help other researchers who are interested in research in this area of mathematics.

As a limitation to this study, one realizes that the atomic decompositions of $H^*_{p,q}$ and $H^S_{p,q}$ were not discussed. This limitation is due to the fact that the atomic decompositions of the classical spaces H^*_p and H^S_p do not exist yet. In fact the stopping time argument, which is the method employed to characterize the atoms in the classical spaces $H^s_p, \mathcal{P}_p, \mathcal{Q}_p$ cannot be applied to the martingales in the spaces H^*_p, H^S_p. So at the moment one is not able to describe the atoms of martingales in these spaces. Hence the available knowledge does not permit one to make further generalizations of these martingale Hardy-amalgam spaces, $H^*_{p,q}$ and $H^S_{p,q}$, as far as atomic decompositions are concerned. However, this problem is partially solved in the sense that if one assumes that the stochastic basis is regular, then by Theorem 4.17, the spaces $H^S_{p,q}$ and $H^*_{p,q}$ are equivalent to the other three spaces $H^s_{p,q}$, $\mathcal{Q}_{p,q}$ and $\mathcal{P}_{p,q}$. Thus $H^S_{p,q}$ and $H^*_{p,q}$ assume the same properties as that of $H^s_{p,q}$, $\mathcal{Q}_{p,q}$ and $\mathcal{P}_{p,q}$. Thus, in particular, all the five martingale Hardy-amalgam spaces will have the same atomic decomposition when the stochastic basis is regular. This problem of characterizing the atoms in the classical spaces H^*_p and H^S_p and that of $H^*_{p,q}$ and $H^S_{p,q}$ is challenging and it is not harmful to suggest that the solution will require the development of new set of tools instead of the stopping time argument. Hence as part of my future research, it is in my interest to find solutions to these open problems.

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