Time Gauge Fixing for 3d Loop Quantum Gravity

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THIS THESIS IS SUBMITTED TO THE UNIVERSITY OF GHANA, LEGON IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE AWARD OF MPHIL MATHEMATICAL PHYSICS DEGREE

July, 2015

DECLARATION

This thesis was written in the Department of Mathematics, University of Ghana, Legon and Applied Mathematics Department, University of Waterloo, Canada from September 2014 to July 2015 in partial fulfilment of the requirement for the award of Master of Philosophy degree in Mathematical Physics under the supervision of Dr. Prince Koree Osei of the University of Ghana, Legon and Dr. Maïté Dupuis of the University of Waterloo, Canada.

I hereby declare that except where due acknowledgement is made, this work has never been presented wholly or in part for the award of a degree at the University of Ghana or any other University.

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ACKNOWLEDGEMENTS

First and foremost I give thanks to Allah Almighty for giving me the knowledge and strength to write this thesis.

The next goes to my supervisor, Dr. Prince Koree Osei, who directed me into this interesting topic and guided me through it all. I also thank him for his patience, constructive criticisms and careful reading of the manuscript.

Many thanks goes to Dr. Maite Dupuis, my co-supervisor who choose this interesting topic. Another big thanks goes to her for hosting me for four months at the University of Waterloo, Canada, where part of these work was carried out.

I acknowledge the African Institute for Mathematical sciences, Ghana for supporting me financially through this program. Finally, I want to acknowledge the support of my colleagues Leonard, Justice, Kinsley, Thomas, Christian, Micheal and all staff of the Department of Mathematics, University of Ghana, Legon.

DEDICATION

This work is dedicated to my father, Mr. Awudu Osumanu and my mother, Ms. Emefa Agidi.

Abstract

In this work, we review 3-dimensional gravity by canonically analyzing the Hilbert-Palatini action. We apply a time gauge used in 4-dimensional loop quantum gravity [14] to the simpler case of the 3-dimensional loop quantum gravity. By time gauge fixing this Hilbert-Palatini action it leads to the Gauss constraints, the spatial diffeomorphism constraints, the Hamiltonian constraint and a new constraint $\mathcal{C}$. The gauge symmetries generated by this new constraints are spacetime diffeomorphism and $SO(2)$ gauge transformations. We solve the dynamics of the theory by providing a regularization of the generalized projector operator in terms of the Hamiltonian constraint. We provide the definition of the physical scalar product which can be represented in terms of a sum over finite spinfoam amplitudes. Then we establish a clear-cut link between the canonical quantization of the new theory to the spinfoam model (Ponzano-Regge model) defined in terms of the $SU(2)$ spin networks.

Contents

Chapter 1

Introduction and Motivation

1.1 Quantum Gravity

The task of quantizing gravity is one of the outstanding problems of modern theoretical physics. Attempts to reconcile quantum theory and general relativity (GR) date back to the early part of the twentieth century. Several decades of hard work have yielded an abundance of insights into quantum field theory. But despite this enormous effort, no one has yet succeeded in formulating a complete, self-consistent quantum theory of gravity [5].

Einstein's description of gravity as a curvature of spacetime (geometry of curved spacetime) was seen as a game changer at the time, as this idea totally moved away from the notion of gravity as a force. The consequence of this was the formulation of the theory of GR which gave a deep insight into the geometric structure of the universe at the macroscopic level.

Quantum mechanics (QM), which describes the microscopic world was formulated based on the laws of probability. It allows for a discrete energy spectrum instead of a continuous radiation. QM uses the time evolution of quantum states described by the Schrodinger equation in which the Hamiltonian generates the time evolution. This means that physical quantities can change only in discrete amounts and not in a continuous way [27].

Our understanding of physics, and in general nature have been greatly influenced by QM and GR. Both principles have revolutionized the pivotal structures of Newtonian physics. The notion of absolute time and space ceases to exist, but rather space and time and distances between points of the spacetime manifold, that is, the metric themselves become dynamical objects [27, 28]. Further, the deterministic nature of Newton's equations of motion evaporates at a fundamental level, rather dynamics is reigned by probabilities underlying the Heisenberg uncertainty principle.

A usual phenomenon in fundamental physics is to reconcile two different conceptual theories to find a new synthesis, a satisfactory physical theory must therefore combine both of these fundamental principles, QM and GR, in a consistent way and which will be called quantum gravity (QG). However, despite the success stories of both QM and GR, the quantization of the gravitational field has turned out to be one of the most challenging unsolved problems in theoretical and mathematical physics.

The reason for this incompatibility is that, GR requires spacetime to be a dynamical field, describing the notion of force as a geometry and also requires an independent background structure. On the other hand, QM requires a dynamical field to be quantized, and in addition requires a non-dynamical background spacetime (Hilbert space) for it to take place. In GR, spacetime is dynamical, while in QM, any dynamical entity is made up of quanta and can be in probabilistic superposition states.

Not only are there conceptual differences in quantizing gravity, there are technical obstacles also facing the new synthesis. Einstein's equations which governs the theory of gravity are highly non-linear, and hence very difficult to solve. In addition, conventional field quantization methods which are based on the weak-field perturbation expansion fail in their application to GR since they yield a non-renormalizable theory. GR, hence must be replaced by a more fundamental description that appropriately includes the quantum degrees of freedom of gravity [24].

However, it is clear that while GR has given a very consistent framework for doing classical physics, it has not yet been successful in bringing the quantum realm in its wake [21]. Attempts have been made for decades to formulate a consistent theory of quantum gravity, but with no success so far.

1.2 Loop Quantum Gravity

The past decades have seen an increase in active research in the field of quantum theory of gravity, with some claiming to have found a quantum theory of gravity. Notably among them is string theory, which is based on the physical hypothesis that elementary objects are extended rather than particle-like [27]. A very rich unified theory which includes Yang-Mills fields and gravitons, and free of ultraviolet divergences are a few of string theory's successful story. However, this comes at a cost of additional physics such as supersymmetry, extra dimensions, an infinite number of fields with an arbitrary masses and spins.

In light of this another active area of research in QG is Loop Quantum Gravity (LQG) which takes a different approach to string theory. LQG is a canonical and background independent base approach to the quantization of gravity paying special attention to the conceptual lessons of GR [9, 23, 27, 28]. It proposes that spacetime is actually divided into small chucks, when you zoom out it appears to be a smooth sheet, but when you zoom in, it is a bunch of dots connected by lines or loops. These small fibres, which are woven together offer an explanation of gravity.

LQG is based on a Hamiltonian formulation of GR at the classical level. In this formulation, spacetime is foliated into a family of hypersurfaces of constant time with coordinates on each slice. LQG is a straightforward quantization procedure of GR with the physical inputs of the theory been the well-tested physical theories of QM and GR, there are no major additional physical hypothesis or assumption made in the construction of the theory [27]. All it tries to do is to merge the conceptual ideas of GR into QM. In addition, LQG is based on assumption that even if Einstein's equations are modified at high energy, the general-relativistic notion of space and time are assumed to be true.

The theory is formulated in the more physical four dimensional case. It was initially formulated in terms of the metric variables which was due to Arnowitt, Deser and Misner [28], hence the name ADM formulation. In this formulation, the canonical variables are the metric tensor of three dimensional spatial slices $q_{ab}(x,t)$ (here a, b are 3d space indices) and it's conjugate momentum π^{ab} [23, 28] following an ADM decomposition of the action. The decomposition results in the spatial diffeomorphism $V(q_{ab}, \pi^{ab})$ and Hamiltonian constraints $S(q_{ab}, \pi^{ab})$, however the non-linear dependence on the canonical variables especially by the Hamiltonian constraint makes it mathematically difficult to promote the constraints into quantum operators.

To address the problem of promoting the constraints into operators, a tetrad formulation was introduced. The tetrad is defined as $e^I = e^I_\mu dx^\mu$ such that $g_{\mu\nu} = e^I_\mu e^J_\nu \eta_{IJ}$ (μ, ν are 4d spacetime tangent indices and I, J are 4d internal indices), and η_{IJ} is the Minkowski metric. The extrinsic curvature $\left(K^j_b := \frac{1}{\sqrt{\det E}} K_{ab} E^b_j \delta^{ij}\right)$ and the densitized triad $\left(E^a_i := \frac{1}{2}\epsilon_{ijk}\epsilon^{abc} e^j_b e^k_c\right)$ become the new canonical variables for general relativity after a 3+1 decomposition of spacetime.

In this new formulation, additional degrees of freedom are introduced due to the redundancy exhibited by the new variables: this occurs as there are nine E^a_i

to describe the six components of q^{ab}. As a result an additional constraints- Gauss constraint- in terms of the new variables makes this redundancy manifest. However, the problems similar to using the ADM formulation arise when one tries to quantize the theory.

In the pursuit of solving the non-polynomial associated with the Hamiltonian constraint, Ashtekar introduced a new configuration variable (A_a^i) that behaves as a complex $SU(2)$ connection. The three constraints now become polynomials in the Ashtekar variables i.e the Ashtekar connection A_a^i and the densitized triad E_i^a. This simplification open up ways to quantize the constraint.

However, despite this simplification there are still some issues in properly defining the constraints and solving them. This comes from the fact that the Ashtekar connection is defined as a complex variable, hence in the quantization of the theory it is difficult to ensure one recovers real GR as opposed to complex GR, unless the reality condition is imposed [27]. In addition, there are difficulties of a rather technical nature related to the complete characterization of dynamics and the quantization of the Hamiltonian constraint [23].

Despite these challenges, LQG has been able to predict the discreteness of geometrical operators such as area and volume. This discreteness becomes important at the Planck scale while the geometric operators crowds very rapidly at low energy scale. This property of the geometric operators is consistent with the smooth spacetime picture of classical GR [23].

In carrying out canonical quantization, the guiding principle is mostly that of Dirac. The first is to find a representation of the phase space variables as operators, acting on some kinematical Hilbert space $\mathcal{H}_{kin}$. This representation of the phase space variables should map Poisson brackets into commutators. Next is to define the constraints into self-adjoint operators in $\mathcal{H}_{kin}$, and finally the solutions of these constraints are characterize to define an inner product leading to a physical Hilbert space $\mathcal{H}_{phy}$.

The Poisson bracket between the Ashtekar-Barbero connection and the densitized triad resulted in a delta function (distribution), this leads to a not well-defined function. A regularization of the Poisson bracket is needed to address this problem, and this is done by smearing the phase space variables to obtain the holonomy-flux algebra. The regularization open up ways to quantize these constraints, and such they (constraints) become operators on a kinematical Hilbert space. For more on loop quantum gravity the reader is advice to read [23, 27, 28] and references therein.

1.3 Loops and Spin Networks

The choice of different algebra of basic field functions to promote to quantum operators makes LQG stands out among other different quantization techniques. That choice is a non-canonical algebra based on the holonomies of the gravitational connection [23, 27].

The holonomy in LQG becomes a quantum operator creating "loop states"[1]. In a background independent theory like LQG, the position of a loop state is relevant only with respect to other loops, and not with respect to the background [27]. The space spanned by these loop states are not too singular nor too many, thereby providing a basis for the Hilbert space of LQG. A finite linear combination of some of these loop states form a well defined orthonormal basis in the Hilbert space of a lattice Yang-Mills theory. These orthonormal basis are the spin networks states which eventually form the basis of the mathematical structure of LQG.

As stated earlier, one will like to know what the quanta of space (quantum properties of space) are, and how to describe them, in the pursuit of a quantum theory of spacetime. Interactions of physical systems with other systems may occur in "quanta" and these quanta are determined by the spectral properties of the operators representing the quantities involved in the interaction with the systems. The interactions with the gravitational field (spacetime) are through geometric structures of the physical space resulting in measurements of length, area and volume.

The spectrum of both volume and area turns out to be discrete. From the spectral of the volume, the quanta of space can be thought of as grains of space. These grains of space lie adjacent to each, where in this case adjacency means being contiguous, or being in touch, or being nearby. If one consider a quantum state of space formed by P grains of space, where some are in contact with one another [27].

Then representing this state as an abstract graph Γ with P nodes: these nodes of the graph represent the grains of the space; the links of the graph connect adjacent grains and represent the surfaces separating two adjacent grains. The quantum state is then characterized by the graph Γ, with the labels on the nodes and on the links, and a graph with these labels is called a spin network.

LQG has been able to predict the discreteness of the spectrum of geometrical operators like area and volume. The formulation of LQG in terms of the Ashtekar-

[1] In the loop representation formulation of Maxwell theory, a loop state is a state in which the electric field vanishes everywhere except along a single Faraday line

Barbero variables result in the Gauss constraints, the spatial diffeomorphism constraints and the Hamiltonian constraint. Defining cylindrical functional as quantum states of the Ashtekar connection, spin network states establish themselves as solutions to both Gauss and spatial diffeomorphism constraints. Unfortunately, this can't be said about the Hamiltonian constraint.

1.4 Motivation and Goal of Thesis

General relativity in three dimensions is said to be "simple" but not trivial making it an interesting, and as such a large area of research providing an insight into quantum gravity for the more physical four dimension. Physically three dimensional gravity is a topological field theory, meaning it has no degrees of [5] freedom[2] and its manifold is regarded as a two surface with time.

Conceptually both $(2 + 1)$-dimensional gravity and $(3 + 1)$-dimensional gravity are equivalent and many of the fundamental issues carry over to quantum gravity at the lower dimensional setting. Although $(2+1)$-dimensional model is much simpler, mathematically and physically, and we can write down a quantum theory for the model.

There are however few instances were $(2+1)$-dimensional solutions are quite different physically from those of $(3+1)$-dimensional, and in addition the $(2+1)$-dimensional model does not help in understanding the dynamics of quantum gravity. Based on the analysis of conceptual problems such as nature of time, the role of topology and topology change, the relationship among different approaches to quantization, the model has really proven highly instructive [5].

Canonical analysis of $(2+1)$-dimensional gravity [3] result in the flatness and Gauss constraints, defining a normal to the hypersurface x^0 =const, the Hamiltonian and spatial diffeomorphism constraints are derived. These constraints generate equal time hypersurface which can be deformed and pushed forward in various ways. A geometric algebra reflected in the Poisson algebra of the constraints- known as the Dirac's hypersurface deformation algebra[3] is satisfied by these deformations of the hypersurface. However a quantum representation of this algebra (see [3] and other references therein for more details) has not been reached yet even in the simple case of three dimensional gravity.

[2]There are no gravitational waves in the classical theory, and no propagating gravitons in the quantum theory.

[3]The Dirac algebra is universal, i.e. it is the same algebra for any theory of hypersurfaces embedded in a higher dimensional manifold.

In the 4d decomposition of spacetime, the Ashtekar connection transforms under the local $SU(2)$ or $SO(3)$ transformation with the action being Lorentz invariant. Since three dimensional gravity is simple, one will like to explore if it is possible to time gauge just like in four dimensional gravity, and if successful what can we learn from it with respect to the 4d case. Symmetry plays a crucial role in any gauge theory, and as such performing a time gauge fixing in three dimensional gravity, one expect a break in the original symmetry of the theory.

Description of a quantum space is an issue in quantum gravity, in (3+1)-dimensional LQG and as well in $(2+1)$-dimensional LQG. The quantum space in both cases are describe in terms of $SU(2)$ spin networks, the question then is, if the $2+1$ time gauge fixing of the Palatini action is invariant under $SO(2)$, then can we define a quantum space of this theory in terms of spin networks invariant under $SO(2)$ transformation.

Addressing some of the questions raised above, and also understanding the physics that can be studied as a result of the time gauge fixing in $(2+1)$-dimensional LQG. In addition, how does the time gauge contribute meaningfully in understanding loop quantum gravity, in general motivated this research.

1.5 Organization of Work

We have tried to make this work self-contained and to present all materials needed for understanding the foundations and the theoretical framework of loop quantum gravity and in particular the notion of time gauge fixing in $(2 + 1)$-dimensional loop quantum gravity. Furthermore, we have tried to understand the dynamics of the time gauge action in canonical quantization.

In chapter 2 we discuss the canonical formulation of 3d gravity. We begin with a review on the main ideas of the general theory on canonical formalism. We then discuss the canonical analysis of three dimensional Hilbert-Palatini action as a example of the canonical formalism. A time gauge is introduced and a canonical analysis of the Hilbert-Palatini action is carried out in terms of the time gauge. Subsequent discussions on the time evolution and algebra of constraints are carried out.

In chapter 3 we discuss the classical representation of the phase space variables. To this the holonomies and the flux are introduced which are seen as the basic set phase space function for quantization. Then the Poisson bracket between these phase space variables are computed to give a much simpler expression (algebra) thereby making quantization of these variables feasible.

Chapter 4 looks at the kinematical Hilbert space needed for quantization of our theory. Cylindrical functions as basic observables for quantization are discussed, in addition an inner product of these cylindrical functions are defined leading to the Cauchy completion of the space of the cylindrical functions. Then orthonormal basis of the kinematical Hilbert space are introduced leading to an $SO(2)$ spin networks.

In chapter 5 we discuss the dynamics of the theory in terms of spin foam representation. We begin by introducing a projector operator and then define a regularization of the projector operator. Then the matrix element of this projector operator is constructed. This matrix element is seen as the start of the quantization procedure. Finally, a spin foam representation of the matrix element is carried out.

In chapter 6 a summary of the results of the work is given.

1.6 Conventions and Notations

The notations and conventions mentioned here will be used throughout this work unless otherwise stated. Upper case Latin indices from the middle of the alphabet $I, J, ... = 0, 1, 2$ are $su(2)$ indices. Greek indices from the middle of the alphabet $\mu, \nu, \rho... = 0, 1, 2$ are 3d spacetime indices. Lower case Latin indices from the of the beginning alphabet $a, b, ... = 1, 2$ are 2d space indices. Lower case Latin indices from the middle of the alphabet $i, j, k, ... = 1, 2$ are 2d internal indices. In the case of a $2 + 1$ decomposition of spacetime, timelike indices will be labeled "$t = 0$" in the tangent space and "0" in the internal space. Throughout this work, we use Einstein's summation convention and, raise and lower indices with three dimensional euclidean metric $\delta_{IJ} = dig(1, 1, 1)$.

The $\mathfrak{su}(2)$ generators are denoted by T_I, and in terms of these generators the Lie bracket is

$$[T_I, T_J] \quad = \quad \epsilon_{IJK} T^K, \tag{1.1}$$

where ϵ_{IJK} denotes the fully antisymmetric tensor in three dimensions defined:

$$\epsilon^{IJK} = \begin{cases} 1 & \text{for even permutation of } I, J, K \\ -1 & \text{for odd permutation of } I, J, K \\ 0 & \text{otherwise.} \end{cases}$$

with the conventions $\epsilon_{012} = \epsilon^{012} = 1$.

Chapter 2

Canonical Formulation of 3d Gravity

In $(3+1)$-dimensional LQG with gauge group $G = SL(2, \mathbb{C})$, working with a time gauge normally breaks G into an $SU(2)$ maximal compact subgroup. In this case the canonical analysis simplifies and the phase space is parametrized by an $\mathfrak{su}(2)$-valued connection [6, 14]. A time gauge describes the deviation of the normal to the spacelike hypersurface $\mathrm{t} = 0$ from the time direction. In [14], the time gauge $\chi = (1, 0, 0, 0)$ was introduced in $(3+1)$-dimensional LQG for the parametrization of the phase space.

We will carry out a similar construction for three dimensional gravity in this chapter. Using this time gauge, we obtain at the end of the canonical analysis, a phase space (for the Euclidean signature with $\Lambda = 0$, two copies of three-dimensional Euclidean space as hyperplanes in four-dimensional Euclidean space [5]) parametrized by a gauge connection and it's conjugate momentum, as has been shown in the $(3+1)$-dimensional situation [14].

This chapter is devoted to the canonical formulation of gravity, in particular, Einstein's theory of gravity in three dimensions (two space, one time). We shall begin with a brief description of canonical formulation of gravity of an arbitrary n-dimensional spacetime and later reduce our discussion to the 3-dimensional case. A description of a three dimensional gravity's phase space will be looked at.

Introducing a new time gauge for three dimensional gravity, a decomposition of the triad field via the time gauge will then be carried out. Using this decomposition of the triad, we will apply it to the $(2+1)$-dimensional Hilbert-Palatini action. We look at the new gauge group resulting from the time gauge and the algebra of constraints is computed. Some of the concepts (especially orthonormal frame) discuss in this chapter are extensively used in [16, 22].

2.1 General Theory on Canonical Formalism

A dynamical system in classical mechanics consists of a phase space[1] Γ which is a Poisson (usually symplectic) manifold and a function $H_{cl} : \Gamma \longrightarrow \mathbb{R}$ called the (classical) Hamiltonian of energy function. For example, for a single particle moving along a real line, Γ is the cotangent bundle $T(\mathbb{R})^*$.

In quantum machanics, the phase space Γ is replaced by the set of rays in a complex Hilbert space H, and the space $\mathcal{F}(\Gamma)$ of functions on Γ by the algebra $Op(H)$ of (not necessarily bounded) operators on H. For example, in the case of a single particle moving along the real line, H is the space of square-integrable functions of $x \in \mathbb{R}$.

In section 1.4, three dimensional gravity was discussed as a topological theory with no local degrees of freedom. This feature makes the description of spacetime as a collection of patches (contractible open set of spacetime manifold), solutions of the Einstein equations are isometric to a model spacetime which is determined by the signature and the sign (or vanishing) of the cosmological constant.

For example, in Lorentzian signature with vanishing cosmological constant, the model spacetime is Minkowski space while in the Euclidean signature and positive cosmological constant it is the three-sphere, and so on [5]. This picture leads to the description of $(2 + 1)$-dimensional gravity in terms of "geometric structures."

Depending on the topology of the manifold $\mathcal{M}$, these local solutions may be glued together to obtain non-trivial global solutions of the Einstein equations. This is mathematically executed by the isometry group of the local model spacetime (the 3d Poincare group $ISO(2, 1)$ or the orthogonal group $SO(4)$ in the examples mentioned above). These groups are the local isometries of the model spacetime.

Matter in the form of point particles can be described geometrically in terms of defects of the model spacetime. The space of all these geometric structures, suitably defined, constitutes the phase space of 3d gravity. In addition, the phase space provides us with a symplectic form (Poisson brackets) which is needed for canonical quantization.

Canonical formulation plays a big role in the quantization of gravity. In chapter 1, we discussed about quantum gravity precisely loop quantum gravity and the challenges in quantizing gravity. Some of the conceptual challenges like the meaning

[1] Where solutions of a system lie within a manifold

of observables and what does unitary mean for a closed universe are more noticeable in a canonical analysis. In carrying out a canonical analysis the meaning of quantum gravity state becomes more meaningful.

We now describe in general a canonical formulation of an n-dimensional system and subsequently it quantization. Consider an n-dimensional system where in a canonical analysis the action is written as

$$S = \int L(q_a, \dot{q}_b)dt = \int (p_a\dot{q}_b - H(q_a, p_b))\, dt, \quad (2.1)$$

where the momenta $p_a(t)$ are defined as

$$p_a = \frac{\delta S}{\delta \dot{q}_b} = \frac{\partial L}{\partial \dot{q}_b},$$

where the basic Poisson bracket is

$$\{q_a, p_b\} = \delta_{ab}. \quad (2.2)$$

For general phase space functions f, g this gives

$$\{f, g\} = \frac{\partial f}{\partial q_a}\frac{\partial g}{\partial p_b} - \frac{\partial g}{\partial q_a}\frac{\partial f}{\partial p_b}. \quad (2.3)$$

In a canonical analysis, writing a first order action in the form of the second term in (2.1) one obtains the Hamiltonian. The Hamiltonian defines via its flow the time evolution of the system. By varying the action (2.1) with respect to p and q one obtains respectively the following equations of motions

$$\dot{q} = \frac{\partial H}{\partial p} = \{q, H\}\,, \quad \dot{p} = -\frac{\partial H}{\partial q} = \{p, H\}\,. \quad (2.4)$$

For an arbitrary phase space function, the time evolution is defined by the time evolution of its basic variables thus

$$\dot{f} = \frac{\partial f}{\partial q}\dot{q} + \frac{\partial f}{\partial p}\dot{p}, \quad \text{hence we have} \quad \dot{f} = \{f, H\}. \quad (2.5)$$

The Hamiltonian H (or any other phase space function) generates a flow in a parameter t in the sense that for any phase space function g we can find a one-parameter solution $g(t)$ which satisfies

$$\frac{d}{dt}g(t) = \{g, H\}(t). \quad (2.6)$$

Hence the Poisson bracket $\{g, H\}$ describes the infinitesimal change under a transformation generated by H or any arbitrary phase space function. However the notion of flow generalizes by replacing the Hamiltonian H by any other phase space function F. The Poisson bracket $\{g, F\}$ gives the infinitesimal change of a phase space function g induced by the flow of F.

In the canonical analysis described above, the notion of phase space is provided which is subsequently quantized [10]. This phase space parametrizes the space of solutions with the help of initial data, usually the basic fields and its first time derivatives (momenta). By picking an initial data, one would have to choose an equal time hypersurface[2]. Solutions can then be reconstructed by solving the field equations for the given initial data. One way then to solve this field equations is to use the Hamiltonian to generate time evolution, i.e. the change of initial data under the change of the equal time hypersurface. A particular hypersurface is selected either by putting a restriction [25] on the coordinates $\Phi(x^\alpha)$, or by giving parametrize equations of the form $x^\alpha = x^\alpha(y^i)$, where $y^i (i = 1, 2)$ are coordinates intrinsic to the hypersurfaces.

2.2 Canonical Analysis of 3d Hilbert-Palatini Action

Given the Einstein-Hilbert action in three dimension the basic variable is the metric $g_{\mu\nu}$ which encodes the geometry of the three dimensional manifold $\mathcal{M}$. The metric could be define in terms of a field of n vectors, i.e. orthonormal frame [22]. The orthonormal frame now contains the geometry of spacetime.
We say that a frame $\{e^I\}$, $\{e_I\}$ is orthonormal if in this frame

$$g(e^I, e^J) = e^I{}_\mu e^J{}_\nu g^{\mu\nu} = \delta^{IJ} \tag{2.7}$$

for all $p \in \mathcal{M}$.

Orthonormal frames always exist since at each point $p \in \mathcal{M}$, we may choose for example $e^I = dx^I$, where dx^I are canonical orthonormal basis at the centre of a geodesic coordinate system. For a given spacetime, $(\mathcal{M}, g_{\mu\nu})$, any orthonormal frame yields a new orthonormal frame by transforming the basis through

$$e'^I(x) = L(x)^I{}_J \, e^J(x). \tag{2.8}$$

[2]A hypersurface in an n-dimensional spacetime manifold is an $(n-1)$-dimensional submanifold

If the linear maps $L(x)$ preserve the orthonormality:

$$\delta_{IJ}\, e'^I \otimes e'^J \;=\; \delta_{IJ}\, e^I \otimes e^J,$$

i.e. if

$$L^K{}_I L^P{}_J \delta_{KP} \;=\; \delta_{IJ},$$

which is the defining equation for the Galilean transformations. Thus orthonormal frames are unique up to Local transformations.

We now specify spacetime, through the data $\left(\mathcal{M}, \left\{e^I\right\}\right)$ which is equivalent to $(\mathcal{M}, g_{\mu\nu})$. These are equivalent because for a differentiable manifold with atlas of charts, if we specify e^I, for example through functions $L^J{}_I$, so that

$$e^I(x) \;=\; L^I{}_\mu(x) dx^\mu, \tag{2.9}$$

then

$$g = \delta_{IJ}\, e^I \otimes e^J = \delta_{IJ} L^I{}_\mu L^J{}_\nu dx^\mu \otimes dx^\nu, \tag{2.10}$$

which implies that $\left\{e^I(x)\right\}$ indeed determines $g_{\mu\nu}$

$$g_{\mu\nu} \;=\; \delta_{IJ} L^I{}_\mu(x) L^J{}_\nu(x). \tag{2.11}$$

The theory of interest to us is three dimensional Euclidean gravity in first order formalism. The spacetime $\mathcal{M}$ is a three dimensional oriented smooth manifold. The variable e is the triad, i.e a Lie algebra $\mathfrak{g}$ valued 1-form over $\mathcal{M}$ transforming through the adjoint representation, $F[A]$ is the curvature of the three dimensional spin connection A. In this case the triad and the spin connection are both treated as dynamical variables and therefore varied independently.

For simplicity, we will consider euclidean gravity, hence the spin connection A is defined on a principal $SU(2)$-bundle over $\mathcal{M}$ and the Lie algebra $\mathfrak{g} = \mathfrak{su}(2)$. Assume the spacetime topology to be $\mathcal{M} = \Sigma \times \mathbb{R}$, where Σ is a Riemann surface with arbitrary genus. In addition, we choose a slicing of the spacetime manifold into equal time hypersurface. The materials used in this section can be viewed from [4, 5, 19].
The triad and the spin connection are given as

$$e^I \;=\; e^I_\mu dx^\mu, \quad A^I = \tfrac{1}{2}\epsilon^{IJK} A_{\mu JK} dx^\mu. \tag{2.12}$$

The curvature of A is a 2-form given by $F_I = F_{I\mu\nu} dx^\mu \wedge dx^\nu$. In component form the curvature is:

$$F^I_{\mu\nu} = \partial_\mu A^I_\nu - \partial_\nu A^I_\mu + \epsilon^I{}_{JK} A^J_\mu A^K_\nu. \tag{2.13}$$

The torsion of the spin connection A with the triad e is defined by

$$T^I_{\mu\nu} = \partial_\mu e^I_\nu - \partial_\nu e^I_\mu + \epsilon^{IJK} A_{\mu J} e_{\nu K} - \epsilon^{IJK} A_{\nu J} e_{\mu K}. \tag{2.14}$$

Hence the Hilbert-Palatini action is

$$S[e, A] = \frac{1}{2}\int_{\mathcal{M}} e^I \wedge F_I[A]. \tag{2.15}$$

By varying the action (2.15) with respect to the triad and the spin connection the equations of motion obtained are

$$\begin{aligned} F &= 0 \\ d_A e &= de + [A, e] = 0 \end{aligned} \tag{2.16}$$

the flatness condition and the torsion free condition respectively.

The canonical analysis reproduces the covariant picture [4], while preparing the road to quantization. To see how this happens, we foliate the spacetime manifold into a family of hypersurfaces at constant time t, denoted Σ_t. Without loss of generality, we will denote coordinates now by $t = x^0, x^1, x^2$ where x^0 =const. defines the equal time hypersurface.
Using the component form of the triad (2.12) and the curvature (2.13), the action (2.15) takes the form:

$$\begin{aligned} S &= \frac{1}{2}\int e^I_\mu F_{I\sigma\rho} dx^\mu \wedge dx^\sigma \wedge dx^\rho, \\ &= \frac{1}{2}\int e^I_\mu F_{I\sigma\rho} \epsilon^{\mu\sigma\rho} d^3x. \end{aligned} \tag{2.17}$$

In the second line of (2.17), we made use of the fact that $\epsilon^{\mu\sigma\rho} d^3x = dx^\mu \wedge dx^\sigma \wedge dx^\rho$, where $\epsilon^{\mu\sigma\rho}$ is the three dimensional anti-symmetric tensor.

Splitting only the spacetime indices in (2.17), we have

$$\begin{aligned} S &= \frac{1}{2}\int \left[e^I_0 F_{Iab}\epsilon^{0ab} + e^I_a F_{I0b}\epsilon^{a0b} + e^I_a F_{Ib0}\epsilon^{ab0}\right] dt d^2x \\ &= \frac{1}{2}\int \left[e^I_0 F_{Iab}\epsilon^{ab} - e^I_a F_{I0b}\epsilon^{ab} - e^I_a F_{I0b}\epsilon^{ab}\right] dt d^2x, \end{aligned} \tag{2.18}$$

where we have used the fact that $\epsilon^{0ab} = \epsilon^{ab0} = \epsilon^{ab}$, $\epsilon^{a0b} = -\epsilon^{ab0} = -\epsilon^{ab}$, and

$F_{Ib0} = -F_{I0b}$. Here ϵ^{ab} is the two dimensional anti-symmetric tensor. From (2.18) we have

$$S = \frac{1}{2}\int \left[e_0^I F_{Iab}\epsilon^{ab} - 2e_a^I F_{I0b}\epsilon^{ab}\right] dtd^2x. \tag{2.19}$$

Considering the second term in (2.19), we have

$$\begin{aligned} -e_a^I F_{I0b}\epsilon^{ab} &= -e_a^I \left[\partial_0 A_{Ib} - \partial_b A_{I0} + \epsilon_I^{\ JK} A_{0J} A_{bK}\right]\epsilon^{ab} \\ &= -\epsilon^{ab}\left[e_a^I \partial_0 A_{Ib} - e_a^I \partial_b A_{I0} + e_a^J \epsilon_{JK}^I A_{I0} A_b^K\right]. \end{aligned} \tag{2.20}$$

Substituting (2.20) into (2.19), we obtain

$$\begin{aligned} S &= \frac{1}{2}\int \left[\epsilon^{ab} e_0^I F_{Iab} - 2\epsilon^{ab}\left[e_a^I \partial_0 A_{Ib} - e_a^I \partial_b A_{I0} + e_a^J \epsilon_{JK}^I A_{I0} A_b^K\right]\right] dtd^2x \\ &= \frac{1}{2}\int \left[2\epsilon^{ba} e_a^I \partial_0 A_{Ib} + \epsilon^{ab} e_0^I F_{Iab} + 2A_{I0}\epsilon^{ba}\left(\partial_b e_a^I + \epsilon_{JK}^I A_b^K e_a^J\right)\right] dtd^2x \\ &= \int \left[\epsilon^{ba} e_a^I \partial_0 A_{Ib} + \frac{1}{2}\epsilon^{ab} e_0^I F_{Iab} + A_{I0}\epsilon^{ba}\mathcal{D}_b e_a^I\right] dtd^2x, \end{aligned} \tag{2.21}$$

where $\mathcal{D}_b e_a^I = \partial_b e_a^I + \epsilon_{JK}^I A_b^K e_a^J$ is the covariant derivative of the triad.
The canonical pairs of the spacetime decomposition are then formed by the components of the connection A_{Ib} [3] and their momenta:

$$E^{bI} := \frac{\delta S}{\delta(\partial_0 A_{Ib})} = \epsilon^{ba} e_a^I. \tag{2.22}$$

The the Poisson bracket between the pair is

$$\{A_a^J(x), E_I^b\} = \delta_a^b \delta_I^J \delta^2(x-y), \tag{2.23}$$

the other Poisson brackets involving the canonical variables vanishes

$$\{A_a^J, A_b^K\} = 0, \quad \{E_I^a, E_J^b\} = 0.$$

The components e_0^I and A_{I0} in (2.21) appear without time derivatives and hence effectively act as Lagrange multipliers. Varying (2.21) with respect to these components leads to the following associated equations of motion

$$\begin{aligned} \mathcal{G}^I &:= \epsilon^{ab}\mathcal{D}_a e_b^I = \partial_a E^{aI} + \epsilon_{JK}^I A_a^J E^{aK} = 0 \\ \mathcal{F}^I &:= \frac{1}{2}\epsilon^{ab} F_{ab}^I = \epsilon^{ab}(\partial_a A_b^I + \frac{1}{2}\epsilon_{JK}^I A_a^J A_b^K) = 0. \end{aligned} \tag{2.24}$$

Here, F_{ab}^I is obtained from (2.13). Using the torsion (2.14), then the first term in

[3] This is the spacial part of the spin connection

(2.24) is given by

$$\mathcal{G}^I \;:=\; \frac{1}{2}\epsilon^{ab}T^I_{ab} = \epsilon^{ab}\mathcal{D}_a e^I_b. \qquad (2.25)$$

The associated equations of motions (2.24) constitute constraints on phase space points as they involve only the spin connection A^J_b and the conjugate momenta E^a_I. Hence only phase points in the constraint hypersurface described by (2.24) can lead to a solution of the equations of motion. The appearance of these constraints define a constraint hypersurface in the phase space and they represent some subset of the equations of motion. In addition, a general feature of gauge theories: is the constraints generate gauge symmetries. The constraints $\mathcal{G}_I$ (Gauss constraint) generates infinitesimal $SU(2)$ gauge transformations, and $\mathcal{F}_I$ (flatness constraint), constrains the connection to be flat and generates translation symmetries.

Smearing the constraints in (2.24) with appropriate scalars, these constraints which are first class form the algebra [3]:

$$\begin{aligned}
\{\mathcal{G}[\alpha], \mathcal{G}[\alpha']\} &= \mathcal{G}\left[[\alpha, \alpha']\right] \\
\{\mathcal{F}[N], \mathcal{G}[\alpha]\} &= \mathcal{F}\left[[N, \alpha]\right] \\
\{\mathcal{F}[N], \mathcal{F}[M]\} &= 0,
\end{aligned} \qquad (2.26)$$

where in this case $\mathcal{G}[\alpha] = \int d^2x\ \alpha^I\mathcal{G}_I$ and $\mathcal{F}[N] = \int d^2x\ N^I\mathcal{F}_I$ are the smeared Gauss constraint and the smeared flatness constraint respectively. The algebra in equation (2.26) is the semi direct product of rotation and translation i.e. $ISU(2)$.

In a gauge theory, the momentum and connection variables of the action are redundant: they describe both physical excitations and unobservable 'pure gauge' degrees of freedom that merely represent coordinate changes [5]. Gauge equivalent phase space points are defined by gauge equivalent solutions. Thus the set of physically distinct solutions is given by the set of gauge orbits[4] in the constraint hypersurface. This reduced phase is then given by the constraint hypersurface factored by the gauge transformation.

Alternatively, one can attempt to define this reduced phase space by finding a gauge fixing conditions that would describe a hypersurface that intersect every gauge orbit [10] in the constraint hypersurface in exactly one point. The reduced phase space would then be given by the phase points satisfying the gauge conditions and the set of constraint.

In the next section, we will apply a time gauge to the action (2.15), this time gauge only affects the internal indices. The time gauge breaks the three dimen-

[4]These are sets whose points can be related to each other by gauge transformation.

sional internal rotation to a two dimensional internal rotation, hence our three dimensional phase space [5] will now be parametrized by a two dimensional spin connection and a diad. In this circumstances, the gauge symmetries will be space-time diffeomorphism and an $SO(2)$ gauge transformation.

2.3 Canonical Formalism and Analysis in Time Gauge

In general, canonical analysis in LQG leads to a Gauss, spatial diffeomorphism and Hamiltonian constraints. In three dimensional LQG, the canonical analysis yields the flatness and the Gauss constraints, the algebra between these constraints is the semi-direct product of rotation and translation. An adhoc procedure is introduced to get the spatial diffeomorphism and Hamiltonian constraints [4, 10] in three dimensional LQG.

By time gauge fixing the Hilbert-Palatini action, one hope that the three constraints (Gauss, spatial diffeomorphism and Hamiltonian) of LQG emerge out of the theory, without having to carry out an adhoc procedure to get the spatial diffeomorphism and the Hamiltonian constraints.

In this section, we introduce a time gauge which we will use in the decomposition of the triad into its various components. We will then use this decomposition (components of the triad) to carry out a canonical analysis of the Hilbert-Palatini action (2.15). It is necessary that we split both the spacetime and the internal indices into their various components.
Let us choose a gauge (which we will refer to as the time gauge) such that

$$\delta^I \quad = \quad (1,0,0) := n^\mu e^I_\mu, \tag{2.27}$$

where n^μ is the unit normal to the hypersurface. It is convenient to parametrize it as [18]:

$$n^\mu = \left(\frac{1}{N}, -\frac{N^a}{N}\right), \tag{2.28}$$

where N^i is the shift vector and N (normalization constant) is the lapse function. The lapse function describes the proper distance (proper time) by which the hypersurface is deformed in the normal direction, the shift describes the amount of

[5] Two copies of three-dimensional Euclidean space as hyperplanes in four-dimensional Euclidean space.

the tangential deformation (in the coordinates of the hypersurface).
From (2.27), making use of the definition of n^μ, we have

$$\begin{aligned}\delta^I &:= n^\mu e^I_\mu \\ &= n^0 e^I_0 + n^a e^I_a \\ &= \frac{1}{N}\left(e^I_0 - N^a e^I_a\right) \\ &= \frac{1}{N}\left(e^0_0 - N^a e^0_a, e^i_0 - N^a e^i_a\right). \end{aligned} \tag{2.29}$$

Comparing (2.27) and (2.29) we have for the first entries

$$\begin{aligned}\frac{1}{N}\left(e^0_0 - N^a e^0_a\right) &= 1 \\ e^0_0 - N^a e^0_a &= N. \end{aligned} \tag{2.30}$$

For (2.30) to hold implies that $e^0_a = 0$ and $e^0_0 = N$.
Similarly for the second entries of (2.27) and (2.29), we have $e^i_0 - N^a e^i_a = 0 \Rightarrow e^i_0 = N^a e^i_a$. Hence the decomposition of the triad via the time gauge gives:

$$e^0_\mu = (N, 0)\,, \quad e^I_0 = (N, N^a e^i_a). \tag{2.31}$$

The conditions in (2.31) derived are just like those used in [14] for the $(3+1)$-dimensional decomposition of Hilbert-Palatini action. With the choice of the gauge (2.27) and (2.31), our ultimate interest is to describe the phase space of three dimensional gravity, on which the gauge symmetries are spacetime diffeomorphism and $SO(2)$ gauge transformations.

We are now in a position to split the spacetime and internal indices of the three dimensional Palatini action, after which we will apply the result in (2.31) to perform the canonical analysis. Ignoring the integral in (2.15) and carry out the summation with respect to the internal indices, we have

$$e^I \wedge F_I = e^0 \wedge F_0 + e^i \wedge F_i. \tag{2.32}$$

From (2.32), we consider the first term on the right hand side and carrying out the summation as follows:

$$\begin{aligned}e^0 \wedge F_0 &= e^0_\mu dx^\mu \wedge F_{0\nu\rho} dx^\nu \wedge dx^\rho \\ &= e^0_\mu \epsilon^{\mu\nu\rho} F_{0\nu\rho}\, d^3x \\ &= \left[\epsilon^{0bc} e^0_0 F_{0bc} + \epsilon^{a0c} e^0_a F_{00c}\right] dt d^2x \end{aligned}$$

We now substitute the decomposition of the triad (2.31) into (2.33):

$$e^0 \wedge F_0 = \epsilon^{bc} e^0_0 F_{0bc}\, dtd^2x = \epsilon^{bc} N F_{0bc}\, dtd^2x. \tag{2.33}$$

Carrying out the summation on the second term of (2.32) (right hand side), we have

$$\begin{aligned} e^i \wedge F_i &= e^i_\mu dx^\mu \wedge F_{i\nu\rho} dx^\nu \wedge dx^\rho \\ &= e^i_\mu F_{i\nu\rho} \epsilon^{\mu\nu\rho}\, d^3x \\ &= \left[e^i_0 \epsilon^{0\nu\rho} F_{i\nu\rho} + e^i_a \epsilon^{a\nu\rho} F_{i\nu\rho}\right] d^3x \\ &= \left[e^i_0 \epsilon^{0bc} F_{ibc} + e^i_a \epsilon^{a0c} F_{i0c} + e^i_a \epsilon^{ab0} F_{ib0}\right] dtd^2x \\ &= \left[e^i_0 \epsilon^{0bc} F_{ibc} - 2e^i_a \epsilon^{ac0} F_{i0c}\right] dtd^2x \\ &= \left[N^a e^i_a \epsilon^{bc} F_{ibc} - 2e^i_a \epsilon^{ac0} F_{i0c}\right] dtd^2x. \end{aligned} \tag{2.34}$$

Considering the second term in (2.34) and writing the curvature in component form, we have

$$\begin{aligned} -\epsilon^{ac0} e^i_a F_{i0c} &= -\epsilon^{ac0} e^i_a \left[\partial_0 A_{ci} - \partial_c A_{0i} + \epsilon^{IJ}_i A_{0I} A_{cJ}\right] \\ &= -\epsilon^{ac0} e^i_a \left[\partial_0 A_{ci} - \partial_c A_{0i} + \epsilon^{k0}_i A_{0k} A_{c0} + \epsilon^{0j}_i A_{00} A_{cj}\right] \\ &= \epsilon^{ca} e^i_a \left[\partial_0 A_{ci} - \partial_c A_{0i} + \epsilon^{k0}_i A_{0k} A_{c0} + \epsilon^{0j}_i A_{00} A_{cj}\right]. \end{aligned} \tag{2.35}$$

From (2.33), (2.34) and (2.35), the action in (2.13) becomes

$$\begin{aligned} S &= \frac{1}{2} \int \left[2\epsilon^{ca} e^i_a \partial_0 A_{ci} + 2\epsilon^{ca} e^i_a \left(-\partial_c A_{0i} + \epsilon^{k0}_i A_{0k} A_{c0}\right)\right] dtd^2x \\ &+ \frac{1}{2} \int \left[2\epsilon^{ca} e^i_a \epsilon^{0j}_i A_{00} A_{cj} + N\epsilon^{ab} F_{0ab} + \epsilon^{bc} N^a e^i_a F_{ibc}\right] dtd^2x. \end{aligned} \tag{2.36}$$

Carrying out an integration by part on the second term on the right hand side of equation (2.36), we finally get

$$\begin{aligned} S &= \frac{1}{2} \int \left[2\epsilon^{ca} e^i_a \partial_0 A_{ci} + 2\epsilon^{ca} A_{0i} \left(\partial_c e^i_a + \epsilon^{i0}_k e^k_a A_{c0}\right) +\right] dtd^2x \\ &+ \frac{1}{2} \int \left[2A_{00} \epsilon^{ca} e^i_a \epsilon^{0j}_i A_{cj} + N\epsilon^{ab} F_{0ab} + \epsilon^{bc} N^a e^i_a F_{ibc}\right] dtd^2x \\ &= \frac{1}{2} \int \left[2\epsilon^{ca} e^i_a \partial_0 A_{ci} + 2\epsilon^{ca} A_{0i} \tilde{\mathcal{D}}_c e^i_a + 2A_{00} \epsilon^{ca} e^i_a \epsilon^{0j}_i A_{cj}\right] dtd^2x \\ &+ \frac{1}{2} \int \left[N\epsilon^{ab} F_{0ab} + \epsilon^{bc} N^a e^i_a F_{ibc}\right] dtd^2x, \end{aligned} \tag{2.37}$$

where $\tilde{\mathcal{D}}_c e^i_a = \partial_c e^i_a + \epsilon^{i0}_k e^k_a A_{c0}$ is the covariant derivative of the diad. F_{0ab} and F_{ibc} are the curvatures of the spin connections A_{0a} and A_{ib} respectively, and these are given by

$$\begin{aligned} F_{0ab} &= \partial_a A_{0b} - \partial_b A_{0a} + \epsilon_0^{\ IJ} A_{Ia} A_{Jb} \\ &= \partial_a A_{0b} - \partial_b A_{0a} + \epsilon_0^{\ ij} \left(A_{ia} A_{jb} - A_{ja} A_{ib} \right) \\ &:= F_{ab} \\ F_{ibc} &= \partial_b A_{ic} - \partial_c A_{ib} + \epsilon_i^{\ JK} A_{Jb} A_{Kc} \\ &= \partial_b A_{ic} - \partial_c A_{ib} + \epsilon_i^{\ 0k} \left(A_{0b} A_{kc} - A_{kb} A_{0c} \right). \end{aligned}$$

By splitting both the spacetime and internal indices of the action (2.15), the new canonical variables from (2.37) are e_i^a (diad) and A_{ci}. The other variables which do not appear with time derivatives act effectively as Lagrange multipliers i.e A_{00} A_{0i}, N and N^a, thereby making certain equations of motions act as constraints. Thus we find the momenta (the densitized diad with vector densities of weight one)

$$E^{ci} := \frac{\delta S}{\delta(\partial_0 A_{ci})} = \epsilon^{ca} e_a^i \tag{2.38}$$

which is conjugate to A_{ci}. Just like in section 2.2, the Poisson brackets of the canonically conjugated variables can be read off from equation (2.23) as

$$\{A_a^j(x), E_i^b(y)\} = \delta_a^b \delta_i^j \delta^{(2)}(x, y), \tag{2.39}$$

the other Poisson brackets vanish:

$$\{A_b^j, A_a^i\} = 0 \quad \text{and} \quad \{E_i^b, E_j^a\} = 0.$$

Varying the action in (2.37) with respect to the three Lagrange multipliers, we obtain the following associated equations of motion:

$$\begin{aligned} \mathcal{C}^0 &= \epsilon_i^{\ 0j} \epsilon^{ca} e_a^i A_{cj} = 0, \quad \mathcal{G}^i = \epsilon^{ac} \tilde{\mathcal{D}}_a e_c^i = 0 \\ \mathcal{V}_a &= \epsilon^{bc} e_a^i F_{ibc} = 0, \quad \mathcal{H} = \epsilon^{ab} F_{ab} = 0. \end{aligned} \tag{2.40}$$

As equations of motion these do have an unusual feature, that is they do not include any time derivatives. These equations constitute constraints on phase space points as they involve only the canonical variables, that is only phase space points in the constraint hypersurface describe by (2.40) lead to a solution of all the equations of motion. The multipliers A_{00}, A_{0i}, N^a and N enforce the constraints $\mathcal{C}^0$, $\mathcal{G}^i$ (Gauss), $\mathcal{V}_a$ (spatial diffeomorphism) and $\mathcal{H}$ (Hamiltonian) respectively. In subsequent discussions we will see that these constraints generate the symmetries of the theory.

Combining these four constraints in (2.40), the total Hamiltonian is given as

$$H_{tot} = -\int d^2x \left[A_{00}\mathcal{C}^0 + A_{i0}\mathcal{G}^i + N^a\mathcal{V}_a + N\mathcal{H}\right]. \tag{2.41}$$

Hence the system is described by the canonical pair (the diad and the spin connection) and the totally constrained Hamiltonian (2.41).

2.4 Time Evolution of Constraints

How do the dynamical variables change (infinitesimal) under the influence of the constraints (2.40)? This is our topic of discussion in this section. These infinitesimal changes of the dynamical variables are describe by the time evolutions of these constraints. In describing the time evolutions of constraints we follow the literature in [10].

To compute the infinitesimal changes generated by these constraints in equation (2.40), it will be useful to smear them with appropriate scalars in order to deal with the delta functions in the Poisson brackets between the canonical variables:

$$\begin{aligned} \mathcal{C}[\mu] &= \int d^2x\ \mu_0\mathcal{C}^0, \quad \mathcal{G}[\lambda] = \int d^2x\ \lambda^i\mathcal{G}_i \\ \mathcal{V}[\overrightarrow{N}] &= \int d^2x\ N^a\mathcal{V}_a, \quad \mathcal{H}[N] = \int d^2x\ N\mathcal{H}. \end{aligned} \tag{2.42}$$

Computing these infinitesimal changes will help in understanding the physics and in making physical predictions of the theory. These infinitesimal changes will show the Gauss constraints are generators of internal transformations, while the spatial diffeomorphism constraints are generators of diffeomorphism transformations.

Starting with the spatial diffeomorphism constraints, it's infinitesimal changes (i.e acting the spatial diffeomorphism constraints on the densitized diad and the spin connection) are computed below:

$$\begin{aligned} \delta_{\overrightarrow{N}} A^j_b(x) &= \{A^j_b, \mathcal{V}[\overrightarrow{N}]\} \\ &= \left\{A^j_b, \int d^2y\ N^a E^c_i F^i_{ac}\right\} \\ &= \left\{A^j_b, \int d^2y\ N^a E^c_i \left(\partial_a A^i_c - \partial_c A^i_a + \epsilon^i{}_{JK} A^J_a A^K_c\right)\right\} \\ &= \left\{A^j_b, \int d^2y\ N^a E^c_i \left[\partial_a A^i_c - \partial_c A^i_a + \epsilon^i{}_{k0}\left(A^k_a A^0_c - A^0_a A^k_c\right)\right]\right\}. \end{aligned}$$

Applying the distributive property of Poisson bracket to get

$$\begin{aligned}\delta_{\overrightarrow{N}} A_b^j(x) &= \left\{A_b^j, \int d^2y\ (N^a E_i^c)\partial_a A_c^i\right\} - \left\{A_b^j, \int d^2y\ (N^a E_i^c)\partial_c A_a^i\right\} \\ &+ \left\{A_b^j, \int d^2y\ (N^a E_i^c \epsilon^i{}_{k0})\left[A_a^k A_c^0 - A_a^0 A_c^k\right]\right\}.\end{aligned}$$

Integrating the second term by part and using the product rule of Poisson bracket on the third term to get

$$\begin{aligned}\delta_{\overrightarrow{N}} A_b^j(x) &= \int d^2y\ N^a \partial_a A_c^i \left\{A_b^j, E_i^c\right\} + \int d^2y\ A_a^i \partial_c N^a \left\{A_b^j, E_i^c\right\} \\ &+ \int d^2y\ (N^a \epsilon^i{}_{k0}) \left\{A_b^j, E_i^c\right\} \left[A_a^k A_c^0 - A_a^0 A_c^k\right].\end{aligned}$$

Using (2.39) to evaluate the Poisson bracket between the canonical variables above, we have

$$\begin{aligned}\delta_{\overrightarrow{N}} A_b^j(x) &= \int d^2y\ N^a (\partial_a A_c^i) \delta_b^c \delta_i^j \delta^2(x-y) + \int d^2y\ A_a^i (\partial_c N^a) \delta_b^c \delta_i^j \delta^2(x-y) \\ &+ \int d^2y\ (N^a \epsilon^i{}_{k0}) \left[A_a^k A_c^0 - A_a^0 A_c^k\right] \delta_b^c \delta_i^j \delta^2(x-y). \qquad (2.43)\end{aligned}$$

Considering the third term in (2.43), we have

$$\begin{aligned}(N^a \epsilon^i{}_{k0}) \left[A_a^k A_c^0 - A_a^0 A_c^k\right] \delta_b^c \delta_i^j &= (N^a \epsilon^i{}_{k0}) A_a^k A_c^0 \delta_b^c \delta_i^j - (N^a \epsilon^i{}_{k0}) A_a^0 A_c^k \delta_b^c \delta_i^j \\ &= (N^a \epsilon^i{}_{k0}) \left[A_a^k A_c^0 - A_a^0 A_c^k\right] \delta_b^c \delta_i^j \\ &= 0. \qquad (2.44)\end{aligned}$$

Integrating the dirac function in two dimensions in the first two terms in (2.43), we get

$$\begin{aligned}\delta_{\overrightarrow{N}} A_b^j(x) &= \int d^2y\ N^a (\partial_a A_b^j) \delta^2(x-y) + \int d^2y\ A_a^j (\partial_b N^a) \delta^2(x-y) \\ &= N^a \partial_a A_b^j + A_a^j \partial_b N^a \\ &:= \mathcal{L}_{\overrightarrow{N}} A_b^j. \qquad (2.45)\end{aligned}$$

Where $\mathcal{L}_{\overrightarrow{N}} A_b^j$ is the Lie derivative of A_b^j in the direction of the field N^a.

Similarly for the action of the spatial diffeomorphism constraints on the densitized diad we have

$$\begin{aligned}\delta_{\overrightarrow{N}} E_i^a &= \{E_i^a, \mathcal{V}[\overrightarrow{N}]\} \\ &= \left\{E_i^a, \int d^2y\ N^b E_j^c F_{bc}^j\right\}.\end{aligned}$$

Expanding F^j_{bc} from the above equation, we have

$$\begin{aligned}\delta_{\vec{N}}E^a_i &= \left\{E^a_i, \int d^2y\; N^bE^c_j\left(\partial_bA^j_c - \partial_cA^j_b + \epsilon^j{}_{IK}A^I_bA^K_c\right)\right\}\\ &= \left\{E^a_i, \int d^2y\; N^bE^c_j\left[\partial_bA^j_c - \partial_cA^j_b + \epsilon^j{}_{k0}\left(A^k_bA^0_c - A^0_bA^k_c\right)\right]\right\}. \quad (2.46)\end{aligned}$$

By using the properties of Poisson brackets (2.46) becomes

$$\begin{aligned}\delta_{\vec{N}}E^a_i &= \left\{E^a_i, \int d^2y\;(N^bE^c_j)\partial_bA^j_c\right\} - \left\{E^a_i, \int d^2y\;(N^bE^c_j)\partial_cA^j_b\right\}\\ &\quad + \left\{E^a_i, \int d^2y\;(N^bE^c_j\epsilon^j{}_{k0})\left[A^k_bA^0_c - A^0_bA^k_c\right]\right\}.\end{aligned}$$

Integrating the first and second term by part and using the product rule of Poisson bracket on the third term to get

$$\begin{aligned}\delta_{\vec{N}}E^a_i &= -\int d^2y\; N^b\partial_bE^c_j\left\{E^a_i, A^j_c\right\} + \int d^2y\; E^c_j\partial_cN^b\left\{E^a_i, A^j_b\right\}\\ &\quad + \int d^2y\;(N^bE^c_j\epsilon^j{}_{k0})\left(\left\{E^a_i, A^k_b\right\}A^0_c - A^0_b\left\{E^a_i, A^k_c\right\}\right).\end{aligned}$$

By using equation (2.39) to the above, we obtain

$$\begin{aligned}\delta_{\vec{N}}E^a_i &= \int d^2y\; N^b(\partial_bE^c_j)\delta^a_c\delta^j_i\delta^2(x-y) - \int d^2y\; E^c_j(\partial_cN^b)\delta^a_b\delta^j_i\delta^2(x-y)\\ &\quad + \int d^2y\;(N^bE^c_j\epsilon^j{}_{k0})\left(A^0_c\delta^a_b\delta^k_i - A^0_b\delta^a_c\delta^k_i\right)\delta^2(x-y). \quad (2.47)\end{aligned}$$

The third term in (2.47) is computed as

$$\begin{aligned}(N^bE^c_j\epsilon^j{}_{k0})\left(A^0_c\delta^a_b\delta^k_i - A^0_b\delta^a_c\delta^k_i\right) &= (N^bE^c_j\epsilon^j{}_{k0})\left(A^0_c\delta^a_b - A^0_b\delta^a_c\right)\delta^k_i\\ &= 0. \quad (2.48)\end{aligned}$$

Hence integrating the first two terms in (2.47), we obtain

$$\begin{aligned}\delta_{\vec{N}}E^a_i &= \int d^2y\; N^b(\partial_bE^a_i)\delta^2(x-y) - \int d^2y\; E^c_i(\partial_cN^a)\delta^2(x-y)\\ &= N^b\partial_bE^a_i - E^c_i\partial_cN^a\\ &:= \mathcal{L}_{\vec{N}}E^a_i. \quad (2.49)\end{aligned}$$

Where $\mathcal{L}_{\vec{N}}E^a_i$ is the Lie derivative of E^a_i in the direction of the field N^a.

The infinitesimal changes of the canonical variables under the influence of the Gaussian constraints are computed as follows, beginning with the action of the

Gauss constraints on the spin connection

$$\begin{aligned}\{A^i_b, \mathcal{G}[\lambda]\} &= \{A^i_b, \int d^2y\ \lambda^j \tilde{\mathcal{D}}_a E^a_j\} \\ &= \left\{A^i_b, \int d^2y\ \left[\tilde{\mathcal{D}}_a(\lambda^j E^a_j) - \left(\tilde{\mathcal{D}}_a \lambda^j\right) E^a_j\right]\right\}.\end{aligned}$$

Applying Stokes theorem to $\int d^2y\ \tilde{\mathcal{D}}_a(\lambda^j E^a_j)$ at the boundary of Σ, the above equation becomes

$$\{A^i_b, \mathcal{G}[\lambda]\} = -\left\{A^i_b, \int d^2y\ \left(\tilde{\mathcal{D}}_a \lambda^j\right) E^a_j\right\}.$$

On applying the Poisson product rule to the above equation, we have

$$\begin{aligned}\{A^i_b, \mathcal{G}[\lambda]\} &= -\tilde{\mathcal{D}}_a \lambda^j \left\{A^i_b, \int d^2y\ E^a_j\right\} - \left\{A^i_b, \int d^2y\ \tilde{\mathcal{D}}_a \lambda^j\right\} E^a_j \\ &= -\tilde{\mathcal{D}}_a \lambda^j \left\{A^i_b, \int d^2y\ E^a_j\right\}, \quad \text{since } \left\{A^i_b, \tilde{\mathcal{D}}_a \lambda^j\right\} = 0 \\ &= -\int d^2y\ \tilde{\mathcal{D}}_a \lambda^j (\delta^a_b \delta^i_j) \delta^2(x-y) \\ &= -\tilde{\mathcal{D}}_b \lambda^i = -\left(\partial_b \lambda^i + \epsilon^{i0}_{\ \ k} \lambda^k A_{b0}\right).\end{aligned} \tag{2.50}$$

Carrying out similar procedure as in (2.50), the action of the Gauss constraints on the momenta is computed as follows

$$\begin{aligned}\{E^a_i, \mathcal{G}[\lambda]\} &= \{E^a_i, \int d^2y\ \lambda^j \tilde{\mathcal{D}}_b E^b_j\} \\ &= \left\{E^a_i, \int d^2y\ \left[\tilde{\mathcal{D}}_b(\lambda^j E^b_j) - \left(\tilde{\mathcal{D}}_b \lambda^j\right) E^b_j\right]\right\}.\end{aligned}$$

Applying Stokes theorem to $\int d^2y\ \tilde{\mathcal{D}}_b(\lambda^j E^b_j)$ at the boundary of Σ, the above equation becomes

$$\begin{aligned}\{E^a_i, \mathcal{G}[\lambda]\} &= -\left\{E^a_i, \int d^2y\ \left(\tilde{\mathcal{D}}_b \lambda^j\right) E^b_j\right\} \\ &= -\left\{E^a_i, \int d^2y\ \tilde{\mathcal{D}}_b \lambda^j\right\} E^b_j - \int d^2y\ \tilde{\mathcal{D}}_b \lambda^j \{E^a_i, E^b_j\}.\end{aligned}$$

But $\{E^a_i, E^b_j\} = 0$. Expanding $\tilde{\mathcal{D}}_b \lambda^j$, we have

$$\begin{aligned}\{E^a_i, \mathcal{G}[\lambda]\} &= -\left\{E^a_i, \int d^2y\ \left(\partial_b \lambda^j + \epsilon^{j0}_{\ \ k} \lambda^k A_{b0}\right)\right\} E^b_j \\ &= -\left\{E^a_i, \int d^2y\ \epsilon^{j0}_{\ \ k} \lambda^k A_{b0}\right\} E^b_j.\end{aligned}$$

Simplifying the above equation, we obtain

$$\begin{aligned} \{E_i^a, \mathcal{G}[\lambda]\} &= -\int d^2y\, \epsilon^{j0}_{\ k} \lambda^k \{E_i^a, A_{b0}\} E_j^b \\ &= 0, \quad \text{since } \{E_i^a, A_{b0}\} = 0. \end{aligned} \tag{2.51}$$

The Hamiltonian constraint is a function of only the spin connection and as such it commute with the spin connection:

$$\{A_b^j, \mathcal{H}[N]\} = 0. \tag{2.52}$$

Next we look at the action of the Hamiltonian constraint on the densitized diad:

$$\begin{aligned} \{E_i^a, \mathcal{H}[N]\} &= \left\{E_i^a, \int d^2y\, N\epsilon^{bc} F_{bc}\right\} \\ &= \left\{E_i^a, \int d^2y\, N\epsilon^{bc}\left[\partial_b A_c^0 - \partial_c A_b^0 + \epsilon^0_{kj}\left(A_b^k A_c^j - A_b^j A_c^k\right)\right]\right\}. \end{aligned}$$

Using the distributive properties of Poisson bracket, we obtain

$$\begin{aligned} \{E_i^a, \mathcal{H}[N]\} &= \left\{E_i^a, \int d^2y\, N\epsilon^{bc}\partial_b A_c^0\right\} - \left\{E_i^a, \int d^2y\, N\epsilon^{bc}\partial_c A_b^0\right\} \\ &+ \left\{E_i^a, \int d^2y\, N\epsilon^{bc}\epsilon^0_{kj}\left(A_b^k A_c^j - A_b^j A_c^k\right)\right\}. \end{aligned}$$

Integrating the first two terms by part and applying the product properties of Poisson bracket to the last term, to obtain

$$\begin{aligned} \{E_i^a, \mathcal{H}[N]\} &= -\int d^2y\, \epsilon^{bc}(\partial_b N)\{E_i^a, A_c^0\} + \int d^2y\, \epsilon^{bc}(\partial_c N)\{E_i^a, A_b^0\} \\ &+ \int d^2y\, N\epsilon^{bc}\epsilon^0_{kj} A_b^k \{E_i^a, A_c^j\} + \int d^2y\, N\epsilon^{bc}\epsilon^0_{kj}\{E_i^a, A_b^k\} A_c^j \\ &- \int d^2y\, N\epsilon^{bc}\epsilon^0_{kj} A_b^j \{E_i^a, A_c^k\} - \int d^2y\, N\epsilon^{bc}\epsilon^0_{kj}\{E_i^a, A_b^j\} A_c^k \end{aligned} \tag{2.53}$$

$\{E_i^a, A_b^0\} = \{E_i^a, A_c^0\} = 0$, since both A_b^0 and A_c^0 are not dynamical variables. Hence equation (2.53) becomes

$$\{E_i^a, \mathcal{H}[N]\} = 0. \tag{2.54}$$

The action of the last constraint on the connection is computed as follows

$$\begin{aligned}
\{A_b^k, \mathcal{C}[\mu]\} &= \left\{A_b^k, \int d^2x\ \mu_0 \epsilon^{0i}_{\ j} E_i^c A_c^j\right\} \\
&= \int d^2x\ \mu_0 \epsilon^{0i}_{\ j} A_c^j \left\{A_b^k, E_i^c\right\} \\
&= \int d^2x\ \mu_0 \epsilon^{0i}_{\ j} A_c^j \delta_i^k \delta_b^c \delta^2(x-y) \\
&= \mu_0 \epsilon^{0i}_{\ j} A_c^j \delta_i^k \delta_b^c = \mu_0 \epsilon^{0k}_{\ j} A_b^j \qquad (2.55)
\end{aligned}$$

Similarly, the action of this constraint on the densitized diad is

$$\begin{aligned}
\{E_k^b, \mathcal{C}[\mu]\} &= \left\{E_k^b, \int d^2x\ \mu_0 \epsilon^{0i}_{\ j} E_i^c A_c^j\right\} \\
&= \int d^2x\ \mu_0 \epsilon^{0i}_{\ j} E_i^c \left\{E_k^b, A_c^j\right\} \\
&= -\int d^2x\ \mu_0 \epsilon^{0i}_{\ j} E_i^c \delta_c^b \delta_k^j \delta^2(x-y) \\
&= -\mu_0 \epsilon^{0i}_{\ j} E_i^c \delta_c^b \delta_k^j = -\mu_0 \epsilon^{0i}_{\ k} E_i^b. \qquad (2.56)
\end{aligned}$$

In summary, the time evolution of the constraints (2.40) are

$$\begin{aligned}
\{A_b^j, \mathcal{V}[\overrightarrow{N}]\} &= N^a \partial_a A_b^j + A_a^j \partial_b N^a = \mathcal{L}_{\overrightarrow{N}} A_b^j \\
\{E_i^a, \mathcal{V}[\overrightarrow{N}]\} &= N^b \partial_b E_i^a - E_i^c \partial_c N^a = \mathcal{L}_{\overrightarrow{N}} E_i^a \\
\{A_a^i, \mathcal{G}[\lambda]\} &= -\mathcal{D}_a \lambda^i = -\left(\partial_a \lambda^i + \epsilon^{i0}_{\ k} \lambda^k A_{a0}\right) \\
\{E_i^a, \mathcal{G}[\lambda]\} &= 0 \\
\{A_b^k, \mathcal{C}[\mu]\} &= \mu_0 \epsilon^{0k}_{\ j} A_b^j \\
\{E_k^b, \mathcal{C}[\mu]\} &= -\mu_0 \epsilon^{0i}_{\ k} E_i^b \\
\{A_b^j, \mathcal{H}[N]\} &= 0 \\
\{E_i^a, \mathcal{H}[N]\} &= 0. \qquad (2.57)
\end{aligned}$$

The first two result in (2.57) shows the spatial diffeomorphism constraints generate diffeomorphisms tangential to the hypersurface with generating vector field N^a. The Gauss constraints generate internal rotations in the indices i (two dimensions) as seen from the third and fourth equations in (2.57). The time evolution themselves are gauge transformation as seen from the various evolution of the constraints. All these transformations exhibited by the constraints comes as no surprise as this was the case when they were no time gauge imposed on the action (2.15). However, the Hamiltonian constraint generate no gauge transformation, since it is expected to describe the dynamics of the new theory.

2.5 Algebra of Constraints

The appearance of constraints, which generate gauge symmetries, is, as one can expert, a general feature of systems with gauge symmetries. The group structure of gauge transformations is reflected in the algebra of constraints, in particular by the fact that the Poisson brackets between constraints is again a combination of constraints. If this is the case for a given set of constraints, it will be called first class [10]. However not all constraints are related by gauge symmetries, in particular there are second class constraints, for which the Poisson brackets do not vanish on the constraint hypersurface. In what follows, we try to examine if the constraints (2.40) are first class constraints, and of particular interest to us of these Poisson bracket algebras, are whether if they completely determine the theory.

We start with the spatial diffeomorphism constraints. A careful look at (2.45) and (2.46), one can write down the action on the smeared spatial diffeomorphism constraints on any arbitrary phase space function. Hence, the action of the smeared spatial diffeomorphism constraints on the spatial diffeomorphism constraints i.e.,

$$\begin{aligned}\{\mathcal{V}_a, \mathcal{V}[\overrightarrow{N}]\} &= \mathcal{L}_{\overrightarrow{N}}\mathcal{V}_a \\ &= \partial_b\left(N^b\mathcal{V}_a\right) + \mathcal{V}_b\partial_a N^b. \end{aligned} \tag{2.58}$$

Hence we have

$$\begin{aligned}\{\int d^2x\ M^a\mathcal{V}_a, \mathcal{V}[\overrightarrow{N}]\} &= \int d^2x\ M^a\mathcal{L}_{\overrightarrow{N}}\mathcal{V}_a \\ &= \int d^2x\ M^a\left[\partial_b\left(N^b\mathcal{V}_a\right) + \mathcal{V}_b\partial_a N^b\right] \\ &= \int d^2x\ [M^c\partial_c N^a - N^a\partial_c M^c]\mathcal{V}_a \\ &= \mathcal{V}[\mathcal{L}_{\overrightarrow{M}}\overrightarrow{N}], \end{aligned} \tag{2.59}$$

where we use integration by parts on the first term of the second line to get the result in the third line. The Poisson bracket of the spatial diffeomorphism constraints is therefore given by

$$\{\mathcal{V}[\overrightarrow{M}], \mathcal{V}[\overrightarrow{N}]\} = \mathcal{V}[\mathcal{L}_{\overrightarrow{M}}\overrightarrow{N}] = \mathcal{V}[[\overrightarrow{M}, \overrightarrow{N}]], \tag{2.60}$$

where $[\overrightarrow{M}, \overrightarrow{N}] = \mathcal{L}_{\overrightarrow{M}}\overrightarrow{N}$ is the commutator of two vector fields, defining the algebra of infinitesimal diffeomorphisms.

The action of the spatial diffeomorphism constraints on the Hamiltonian constraint is computed as

$$\begin{aligned}\{\int d^2x\; N\mathcal{H}, \mathcal{V}[\overrightarrow{M}]\} &= \int d^2x\; N\mathcal{L}_{\overrightarrow{M}}\mathcal{H} \\ &= \int d^2x\; N\left(M^a\partial_a\mathcal{H} + \mathcal{H}\partial_a M^a\right) \\ &= -\int d^2x\; \left(M^a\partial_a N\right)\mathcal{H}. \end{aligned} \tag{2.61}$$

Hence this gives

$$\{\mathcal{H}[N], \mathcal{V}[\overrightarrow{M}]\} = -\mathcal{H}[\mathcal{L}_{\overrightarrow{M}}N] \tag{2.62}$$

for the Poisson bracket between a pseudo-flatness constraint and the spatial diffeomorphism constraints. This gives a Hamiltonian constraint with its smearing field.

Similarly the action of the spatial diffeomorphism constraints on the Gauss constraints is computed as follows

$$\begin{aligned}\{\int d^2x\; \lambda^j\mathcal{G}_j, \mathcal{V}[\overrightarrow{N}]\} &= \int d^2x\; \lambda^j\mathcal{L}_{\overrightarrow{N}}\mathcal{G}_j \\ &= \int d^2x\; \lambda^j\left(N^a\partial_a\mathcal{G}_j + \mathcal{G}_j\partial_b N^b\right) \\ &= -\int d^2x\; \left(\lambda^j\partial_a N^a - N^b\partial_b\lambda^j\right)\mathcal{G}_j \\ &= -\mathcal{G}[\mathcal{L}_{\overrightarrow{N}}\lambda]. \end{aligned} \tag{2.63}$$

Hence this gives

$$\{\mathcal{G}[\lambda], \mathcal{V}[\overrightarrow{N}]\} = -\mathcal{G}[\mathcal{L}_{\overrightarrow{N}}\lambda] \tag{2.64}$$

for the Poisson bracket between the Gauss constraints and the spatial diffeomorphism constraints. Hence this gives Gauss constraint with a shift in its test field λ.

Using the results in (2.50) and (2.51), the Poisson bracket between one Gauss constraints and another is computed as follows:

$$\begin{aligned}\{\mathcal{G}[\alpha], \mathcal{G}[\lambda]\} &= \left\{\mathcal{G}[\alpha], \int d^2y\; \lambda^j\tilde{\mathcal{D}}_a E^a_j\right\} \\ &= -\left\{\mathcal{G}[\alpha], \int d^2y\; \left(\tilde{\mathcal{D}}_a\lambda^j\right)E^a_j\right\} \\ &= -\int d^2y\; \tilde{\mathcal{D}}_a\lambda^j\left\{\mathcal{G}[\alpha], E^a_j\right\} - \int d^2y\; \left\{\mathcal{G}[\alpha], \tilde{\mathcal{D}}_a\lambda^j\right\}E^a_j. \end{aligned}$$

But $\left\{\mathcal{G}[\alpha], E^a_j\right\} = 0$, hence the above equation becomes

$$\begin{aligned}\{\mathcal{G}[\alpha], \mathcal{G}[\lambda]\} &= -\int d^2y \ \left\{\mathcal{G}[\alpha], \partial_a\lambda^j + \epsilon^{j0}_{\ \ k}\lambda^k A_{a0}\right\} E^a_j \\ &= \int d^2y \ \epsilon^{j0}_{\ \ k}\lambda^k \left\{\mathcal{G}[\alpha], A_{a0}\right\} E^a_j \\ &= 0, \quad \text{since}\ \{\mathcal{G}[\alpha], A_{a0}\} = 0. \end{aligned} \tag{2.65}$$

The first term of the third line of (2.65) vanish by applying (2.51). The result in (2.65) mirrors the $so(2)$ algebra commutation relation as

$$[\alpha, \lambda] = [(\alpha)^j \tau_j, (\lambda)^k \tau_k] = 0. \tag{2.66}$$

Hence the Poisson algebra of the Gauss constraints is a representation of the $so(2)$ Lie algebra and we write

$$\{\mathcal{G}[\alpha], \mathcal{G}[\lambda]\} = \mathcal{G}\left[[\alpha, \lambda]\right] = 0. \tag{2.67}$$

The Hamiltonian constraints obviously commutes with each other since its only a function of the spin connection

$$\{\mathcal{H}[M], \mathcal{H}[N]\} = 0. \tag{2.68}$$

We now compute the Poisson algebra between the Gauss constraints and the Hamiltonian constraint

$$\begin{aligned}\{\mathcal{G}[\lambda], \mathcal{H}[N]\} &= \left\{\int d^2x \ \lambda^j(\tilde{\mathcal{D}}_a E^a_j), \mathcal{H}[N]\right\} \\ &= \left\{\int d^2x \ \left[\tilde{\mathcal{D}}_a(\lambda^j E^a_j) - \left(\tilde{\mathcal{D}}_a\lambda^j\right) E^a_j\right], \mathcal{H}[N]\right\} \\ &= -\left\{\int d^2x \ \left(\tilde{\mathcal{D}}_a\lambda^j\right) E^a_j, \mathcal{H}[N]\right\} \\ &= -\int d^2x \ \left\{\tilde{\mathcal{D}}_a\lambda^j, \mathcal{H}[N]\right\} E^a_j - \int d^2x \ \tilde{\mathcal{D}}_a\lambda^j \left\{E^a_j, \mathcal{H}[N]\right\} \\ &= 0, \quad \text{since} \left\{\tilde{\mathcal{D}}_a\lambda^j, \mathcal{H}[N]\right\} = \left\{E^a_j, \mathcal{H}[N]\right\} = 0. \end{aligned} \tag{2.69}$$

We now compute the Poisson bracket between $\mathcal{C}[\mu]$ and the Gauss constraint

$$\begin{aligned}\{\mathcal{C}[\mu], \mathcal{G}[\lambda]\} &= \left\{\int d^2x \ \mu_0 \mathcal{C}^0, \mathcal{G}[\lambda]\right\} \\ &= \left\{\int d^2x \ \left(\mu_0 \epsilon^{0i}_{\ \ j}\right) E^c_i A^j_c, \mathcal{G}[\lambda]\right\}. \end{aligned} \tag{2.70}$$

Using the properties of the Poisson brackets, equation (2.70) becomes

$$\begin{aligned}\{\mathcal{C}[\mu],\mathcal{G}[\lambda]\} &= \int d^2x\ \left(\mu_0\epsilon^{0i}_{\ j}\right) E^c_i \left\{A^j_c,\mathcal{G}[\lambda]\right\} + \int d^2x\ \left(\mu_0\epsilon^{0i}_{\ j}\right) \left\{E^c_i,\mathcal{G}[\lambda]\right\} A^j_c \\ &= -\int d^2x\ \left(\mu_0\epsilon^{0i}_{\ j}E^c_i\right)\tilde{\mathcal{D}}_c\lambda^j, \quad \text{since } \{E^c_i,\mathcal{G}[\lambda]\}=0 \\ &= -\int d^2x\ \left(\mu_0\epsilon^{0i}_{\ j}E^c_i\right)\left[\partial_c\lambda^j + \epsilon^{j0}_{\ k}\lambda^k A_{c0}\right] \qquad (2.71)\end{aligned}$$

Now carrying out an integration by part on the first term of equation (2.71), we arrive at

$$\begin{aligned}\{\mathcal{C}[\mu],\mathcal{G}[\lambda]\} &= -\int d^2x\ \left(\mu_0\epsilon^{0i}_{\ j}\right)\left[-\lambda^j\partial_c E^c_i + E^c_i\epsilon^{j0}_{\ k}\lambda^k A_{c0}\right] \\ &= \int d^2x\ \left(\mu_0\epsilon^{0i}_{\ j}\right)\left[\lambda^j\partial_c E^c_i + \lambda^j E^c_k\epsilon^{k0}_{\ i}A_{c0}\right] \\ &= \int d^2x\ \left(\lambda^j\mu_0\epsilon^{0i}_{\ j}\right)\left[\partial_c E^c_i + \epsilon^{k0}_{\ i}E^c_k A_{c0}\right]. \qquad (2.72)\end{aligned}$$

From the above equation, we have

$$\begin{aligned}\{\mathcal{C}[\mu],\mathcal{G}[\lambda]\} &= \int d^2x\ \left(\lambda^j\mu_0\epsilon^{0i}_{\ j}\right)\tilde{\mathcal{D}}_c E^c_i \\ &= \int d^2x\ \beta^i\tilde{\mathcal{D}}_c E^c_i = \mathcal{G}[\beta], \qquad (2.73)\end{aligned}$$

where $\beta^i = \lambda^j\mu_0\epsilon^{0i}_{\ j}$.

Poisson bracket between $\mathcal{C}[\mu]$ and the diffeomorphism constraint is computed as follows

$$\begin{aligned}\{\mathcal{C}[\mu],\mathcal{V}[\overrightarrow{N}]\} &= \left\{\int d^2x\ \left(\mu_0\epsilon^{0i}_{\ j}\right)E^c_i A^j_c, \mathcal{V}[\overrightarrow{N}]\right\} \\ &= \int d^2x\ \left(\mu_0\epsilon^{0i}_{\ j}\right)\left[E^c_i\{A^j_c,\mathcal{V}[\overrightarrow{N}]\} + \{E^c_i,\mathcal{V}[\overrightarrow{N}]\}A^j_c\right] \\ &= \int d^2x\ \left(\mu_0\epsilon^{0i}_{\ j}\right)\left[E^c_i\mathcal{L}_{\overrightarrow{N}}A^j_c + A^j_c\mathcal{L}_{\overrightarrow{N}}E^c_i\right]. \qquad (2.74)\end{aligned}$$

Expanding the Lie derivatives of the dynamical variables, we obtain from equation (2.74)

$$\begin{aligned}\{\mathcal{C}[\mu],\mathcal{V}[\overrightarrow{N}]\} &= \int d^2x\ \left(\mu_0\epsilon^{0i}_{\ j}\right)\left[E^c_i\left(N^b\partial_b A^j_c + A^j_b\partial_c N^b\right) + A^j_c\left(N^b\partial_b E^c_i - E^b_i\partial_b N^c\right)\right] \\ &= \int d^2x\ \left(\mu_0\epsilon^{0i}_{\ j}\right)\left[E^c_i\left(N^b\partial_b A^j_c\right) + A^j_c\left(N^b\partial_b E^c_i\right)\right] \\ &= \int d^2x\ \left(\mu_0\epsilon^{0i}_{\ j}\right)\left[N^b\partial_b\left(A^j_c E^c_i\right)\right]. \qquad (2.75)\end{aligned}$$

Integrating by parts the last line of equation (2.75), we obtain

$$\begin{aligned}
\{\mathcal{C}[\mu], \mathcal{V}[\overrightarrow{N}]\} &= -\int d^2x \ \left(\mu_0 \epsilon^{0i}_{\ j}\right) \left[\left(A^j_c E^c_i\right) \partial_b N^b\right] \\
&= -\int d^2x \ \left(\mu_0 \epsilon^{0i}_{\ j}\right) \left(A^j_c E^c_i\right) \partial_b N^b = -\int d^2x \ \mu_0 \mathcal{C}^0 \partial_b N^b \\
&= -\int d^2x \ \left(\mu_0 \partial_b N^b\right) \mathcal{C}^0 \\
&= \mathcal{C}[\mathcal{L}_{\overrightarrow{N}} \mu],
\end{aligned} \tag{2.76}$$

where $\mathcal{L}_{\overrightarrow{N}} \mu = -\mu_0 \partial_b N^b$.

The Poisson bracket between $\mathcal{C}[\beta]$ and $\mathcal{C}[\alpha]$ is computed as follows

$$\begin{aligned}
\{\mathcal{C}[\mu], \mathcal{C}[\alpha]\} &= \{\int d^2x \ \left(\mu_0 \epsilon^{0i}_{\ j}\right) E^c_i A^j_c, \mathcal{C}[\alpha]\} \\
&= \int d^2x \ \left(\mu_0 \epsilon^{0i}_{\ j}\right) \left[E^c_i \{A^j_c, \mathcal{C}[\alpha]\} + \{E^c_i, \mathcal{C}[\alpha]\} A^j_c\right] \\
&= \int d^2x \ \left(\mu_0 \epsilon^{0i}_{\ j}\right) \left[(\alpha_0 \epsilon^{0j}_{\ k}) E^c_i A^k_c - (\alpha_0 \epsilon^{0k}_{\ i}) A^j_c E^c_k\right] \\
&= \int d^2x \ \left(\mu_0 \epsilon^{0i}_{\ j}\right) \left[\alpha_0 \mathcal{C}^0 - \alpha_0 \mathcal{C}^0\right] \\
&= 0.
\end{aligned} \tag{2.77}$$

Poisson bracket between $\mathcal{C}[\mu]$ and the Hamiltonian constraint is computed as follows

$$\begin{aligned}
\{\mathcal{C}[\mu], \mathcal{H}[N]\} &= \{\int d^2x \ \left(\mu_0 \epsilon^{0i}_{\ j}\right) E^b_i A^j_b, \mathcal{H}[N]\} \\
&= \int d^2x \ \left(\mu_0 \epsilon^{0i}_{\ j}\right) \left[E^b_i \{A^j_b, \mathcal{H}[N]\} + \{E^b_i, \mathcal{H}[N]\} A^j_b\right] \\
&= 0.
\end{aligned} \tag{2.78}$$

In summary, the algebra of constraints associated with the Gauss constraints resulting from the time gauge of the action (2.15) are

$$\begin{aligned}
\{\mathcal{G}[\lambda], \mathcal{V}[\overrightarrow{N}]\} &= -\mathcal{G}[\mathcal{L}_{\overrightarrow{N}} \lambda], \qquad \{\mathcal{G}[\alpha], \mathcal{G}[\lambda]\} = 0 \\
\{\mathcal{G}[\lambda], \mathcal{H}[N]\} &= 0, \qquad \{\mathcal{C}[\mu], \mathcal{G}[\lambda]\} = \mathcal{G}[\beta] \\
\{\mathcal{C}[\mu], \mathcal{C}[\alpha]\} &= 0, \qquad \{\mathcal{C}[\mu], \mathcal{V}[\overrightarrow{N}]\} = \mathcal{C}[\mathcal{L}_{\overrightarrow{N}} \mu] \\
\{\mathcal{C}[\mu], \mathcal{H}[N]\} &= 0.
\end{aligned} \tag{2.79}$$

As stated earlier, the group structure of gauge transformations is reflected in the algebra of constraints. The second equation of (2.79), shows the result of the Poisson bracket of a Gauss constraint with itself vanishes. This shows the group structure of the theory is an $SO(2)$ group structure. Physically, the time gauge breaks the $SU(2)$ symmetry of the initial theory to an $SO(2)$ symmetry of the new dynamical variables. This confirms our earlier assertion of $SO(2)$ gauge transformations as one of the gauge symmetries of the new theory.

The other algebra of constraints are

$$\begin{aligned} \{\mathcal{V}[\overrightarrow{M}], \mathcal{V}[\overrightarrow{N}]\} &= \mathcal{V}[\mathcal{L}_{\overrightarrow{M}}\overrightarrow{N}] = \mathcal{V}[[\overrightarrow{M}, \overrightarrow{N}]] \\ \{\mathcal{H}[N], \mathcal{V}[\overrightarrow{M}]\} &= -\mathcal{H}[\mathcal{L}_{\overrightarrow{M}}N] \\ \{\mathcal{H}[M], \mathcal{H}[N]\} &= 0. \end{aligned} \tag{2.80}$$

From (2.79) and (2.80), one can see that the constraints are closed and consistent, since the various Poisson bracket of the constraints computed vanish on the constraint hypersurface. Acting any constraint on the spatial difffeomorphism constraint yields the other constraint with a shift in the smearing field. The reason for this is that the smearing functions are not functions of the canonical variables and so the spatial diffeomorphism does not generate diffeomorphisms on them. They do however generate diffeomorphims on everything else. This is equivalent to leaving everything else fixed while shifting the smearing field.

Chapter 3

Classical Representation of Phase Space Variables

The choice of a basic set of phase space function $\{f\}$ is an integral part of canonical quantization. This set is promoted into operators $\{\hat{f}\}$ on some kinematical Hilbert space. By promoting the set into operators, they should map Poisson brackets into commutators, i.e

$$[\hat{f}, \hat{g}] \quad = \quad i\hbar\widehat{\{f, g\}}. \tag{3.1}$$

However, this basic set should be sufficiently large, so as to distinguish between phase space points.

The result of the Poisson bracket between the diad and the spin connection at the classical level gives a dirac delta function. At the quantum level, this delta function has no equivalent proper operators. Hence one must smear the canonical variables with an appropriate smooth and fast decaying function, in order to select a proper set of the basic variables at the quantum level.

In selecting a proper set of the basic variable, the Poisson bracket between them should result in a simple expression forming an algebra. By forming an algebra, quantization of this basic set become feasible. Furthermore, one would like that the phase space functions transform in a simple way under gauge transformations, so that these can be implemented as gauge transformations in a kinematical Hilbert space.

Subsequent discussions in this chapter will focus on the descretized form of the spin connection and the diad, these form a basic set of the phase space function to be quantized. Computation of the Poisson bracket between these basic set will be carried out. The result will show a much simpler expression compared to the non-descretized version of the canonical variables.

3.1 Holonomies

Any kind of spin connection on a manifold gives rise, through its parallel transport maps, to some notion of holonomy. The holonomy of a connection on a smooth manifold is a general geometrical consequence of the curvature of the connection measuring the extent to which parallel transport around closed loops fails to preserve the geometrical data being transported. The holonomy of a connection is closely related to the curvature of the connection, and represent finite parallel transporters. The materials discussed in this section can be viewed from [3, 10].

Let the curve γ be a continuous, piecewise smooth map:

$$\begin{aligned} \gamma : [0, 1] &\longmapsto \mathcal{M} \\ s &\longmapsto x^a(s) \end{aligned}$$

such that $\dot{\gamma}^a \equiv \frac{dx^a(s)}{ds}$ is the tangent vector to γ.
By parallel transporting an internal vector $V = V^j \tau_j$ (τ_j is the generator of $SO(2)$) along γ, V transforms in the adjoint representation of the group $SO(2)$ defined by

$$V(s) \;=\; h(s)V(0)h(s)^{-1}. \tag{3.2}$$

The holonomy $h(s)$ along the curve γ is such that:

$$\dot{\gamma}^a \mathcal{D}_a V(s) \;=\; 0, \quad V(0) = V, \tag{3.3}$$

where $\mathcal{D}_a V = \partial_a V + [A_a, V]$.

Computing for the holonomy, we have from (3.3):

$$\begin{aligned} \dot{\gamma}^a \mathcal{D}_a V(s) &= 0 \\ &= \left(\frac{d}{ds}h(s)\right) V h(s)^{-1} + h(s) V \left(\frac{d}{ds}h(s)^{-1}\right) \\ &+ \dot{\gamma}^a A_a h(s) V h(s)^{-1} - h(s) V h(s)^{-1} \dot{\gamma}^a A. \end{aligned} \tag{3.4}$$

The above equation is satisfied if:

$$\frac{d}{ds}h(s) \;=\; -\dot{\gamma}^a A_a(\gamma(s))h(s), \quad h(0) = 1. \tag{3.5}$$

Integrating (3.5) along γ:

$$\begin{aligned}\int_0^s \frac{dh(s)}{ds}ds &= -\int_0^s \dot{\gamma}^a A_a(\gamma(s'))h(s')ds + C \\ h(s) &= -\int_0^s \dot{\gamma}^a A_a(\gamma(s'))h(s')ds + C. \end{aligned} \tag{3.6}$$

Applying the condition $h(0) = 1$ into (3.6), we get $C = 1$.
Hence the holonomy is given as

$$h(s) = 1 - \int_0^s \dot{\gamma}^a A_a(\gamma(s'))h(s')ds'. \tag{3.7}$$

Carrying out an iteration of (3.7), the first two expressions are:

$$\begin{aligned} h(s) &= 1 - \int_0^s \dot{\gamma}^a(s_1)A_a(\gamma(s_1))\left(1 - \int_0^s \dot{\gamma}^a(s_2)A_a(\gamma(s_2))h(s_2)ds_2\right)ds_1 \\ &= 1 - \int_0^s \dot{\gamma}^a(s_1)A_a(\gamma(s_1))ds_1 - \int_0^s\int_0^{s_1} X - \cdots \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_0^s\cdots\int_0^{s_n}\dot{\gamma}^a(s_1)A_a(\gamma(s_1))\cdots\dot{\gamma}^a(s_{n+1})A_a(\gamma(s_{n+1}))h(s_{n+1}) \\ &= \mathcal{P}\exp\left(-\int_\gamma \dot{\gamma}^a(s)A_a(\gamma(s))\right) = \mathcal{P}\exp\left(-\int_\gamma A\right), \end{aligned} \tag{3.8}$$

where $X = \dot{\gamma}^a(s_1)A_a(\gamma(s_1))\dot{\gamma}^a(s_2)A_a(\gamma(s_2))h(s_2)ds_1ds_2$, $A = \dot{\gamma}^a(s)A_a(\gamma(s))$ and $\mathcal{P}$ is the path ordering, and it is from left to right i.e $\gamma(s_1) < \gamma(s_2) < \ldots\ldots\ldots < \gamma(s_{n+1})$.

3.2 Fluxes

The holonomy (3.8) constituent half of the basic set phase space function to be quantized, thus we need the other half to have a complete basic set of phase space function. Smearing the diad over a one-dimensional submanifold we obtain the other half of the basic set of phase space function. Given a piecewise analytic curve γ^*, intersecting another curve γ transversely at one point, with the tangent vectors of γ, γ^* at this intersection point positively oriented, the geometric flux variables is defined [10]:

$$E_\gamma = \int_{\gamma^*} h_{\gamma,\gamma^*}(s)e_a(\gamma^*(s))\dot{\gamma^*}^a(s)(h_{\gamma,\gamma^*}(s))^{-1}\, ds, \tag{3.9}$$

where h_{γ,γ^*} is a parallel transport matrix from the point $\gamma^*(s)$ on the curve γ^* to the beginning point of γ. The holonomies[1] are inserted in (3.9) to ensure a local transformation behaviour under internal rotations, i.e.

$$E_\gamma \longrightarrow g(\gamma(0))E_\gamma g(\gamma(0))^{-1} = E'_\gamma. \tag{3.10}$$

The starting point of γ is used instead of γ^*, so that the transformation behaviour of E_γ and h_γ are related. The parallel transport $h_{\gamma,\gamma^*}(s)$ is defined by choosing a parametrization of the curves γ, γ^* such that the intersection point in both curves corresponds to $s = \frac{1}{2}$. Then we have

$$h_{\gamma,\gamma^*} = \left(h_\gamma\left(\frac{1}{2}\right)\right)^{-1} h_{\gamma^*}\left(\frac{1}{2}, s\right), \tag{3.11}$$

where $h_{\gamma^*}\left(\frac{1}{2}, s\right)$ is the parallel transport from the parameter s to the parameter $\frac{1}{2}$ on the curve γ^*.

Let us write the component form E^i_γ of the flux (3.9). Define

$$\begin{aligned} E_\gamma &= E^i_\gamma \tau_i \\ -2\,\mathrm{tr}(\tau_i\tau_j) &= \delta_{ij}. \end{aligned} \tag{3.12}$$

Combing the two equations in (3.12), the component form of the flux:

$$\begin{aligned} \delta_{ij} &= -2\,\mathrm{tr}\,(\tau_i\tau_j) \\ \delta_{ij}E^j &= -2\,\mathrm{tr}\left(\tau_i\tau_j E^j\right) \\ E_i &= -2\,\mathrm{tr}\,(\tau_i E) = -2\,\mathrm{tr}\,(E\tau_i)\,. \end{aligned} \tag{3.13}$$

Using (3.13) the component form of the flux (3.9) along γ is

$$\begin{aligned} (E_\gamma)_i &= \int_{\gamma^*} (-2)\,\mathrm{tr}\left[\tau_i h_{\gamma,\gamma^*}(s) e^j_b \tau_j (h_{\gamma,\gamma^*}(s))^{-1}\right] \dot{\gamma^*}^b ds \\ &= \int_{\gamma^*} (-2)\,\mathrm{tr}\left[\tau_i h_{\gamma,\gamma^*}(s) \tilde{\epsilon_{ba}} E^{aj} \tau_j (h_{\gamma,\gamma^*}(s))^{-1}\right] \dot{\gamma^*}^b ds \\ &= \int_{\gamma^*} (-2)\,\mathrm{tr}\left[\tau_i h_{\gamma,\gamma^*}(s) \tau_j (h_{\gamma,\gamma^*}(s))^{-1}\right] \dot{\gamma^*}^b \tilde{\epsilon_{ba}} E^{aj}(\gamma^*(s)) ds \\ &= \int_{\gamma^*} f_{ij}(s) \dot{\gamma^*}^b \tilde{\epsilon_{ba}} E^{aj}(\gamma^*(s)) ds, \end{aligned} \tag{3.14}$$

where $f_{ij}(s) = (-2)\,\mathrm{tr}\,[\tau_i h_{\gamma,\gamma^*}(s)\tau_j(h_{\gamma,\gamma^*}(s))^{-1}]$.

[1] This differs from standard LQG flux variables which are defined without the parallel transport with holonomies.

3.3 Poisson Brackets

The basic set of phase space function are now formed by the matrix components of the holonomies (3.8) and by the (vector) components of the fluxes (3.9). We now find the Poisson brackets between these phase space functions.

Consider the holonomy h_γ, we make a split, as we require only the part near the intersection of γ with γ^*. Hence we have

$$h_\gamma = h_\gamma\left(1, \frac{1}{2}+\frac{\varepsilon}{2}\right) h_\gamma\left(\frac{1}{2}+\frac{\varepsilon}{2}, \frac{1}{2}-\frac{\varepsilon}{2}\right) h_\gamma\left(\frac{1}{2}-\frac{\varepsilon}{2}, 0\right) \tag{3.15}$$

for ε small. Approximate the middle term in (3.15) to get

$$h_\gamma\left(\frac{1}{2}+\frac{\varepsilon}{2}, \frac{1}{2}-\frac{\varepsilon}{2}\right) = 1 - \int_{\frac{1}{2}-\frac{\varepsilon}{2}}^{\frac{1}{2}+\frac{\varepsilon}{2}} \dot{\gamma}^b(s') A_b^j(\gamma(s'))\tau_j \, ds' + O(\epsilon^2). \tag{3.16}$$

Substituting (3.16) into (3.15), we obtain

$$\begin{aligned} h_\gamma &= h_\gamma\left(1, \frac{1}{2}+\frac{\varepsilon}{2}\right) h_\gamma\left(\frac{1}{2}-\frac{\varepsilon}{2}, 0\right) + h_\gamma\left(1, \frac{1}{2}+\frac{\varepsilon}{2}\right) O(\epsilon^2) h_\gamma\left(\frac{1}{2}-\frac{\varepsilon}{2}, 0\right) \\ &\quad - h_\gamma\left(1, \frac{1}{2}+\frac{\varepsilon}{2}\right)\left[\int_{\frac{1}{2}-\frac{\varepsilon}{2}}^{\frac{1}{2}+\frac{\varepsilon}{2}} \dot{\gamma}^b(s') A_b^j(\gamma(s'))\tau_j \, ds'\right] h_\gamma\left(\frac{1}{2}-\frac{\varepsilon}{2}, 0\right). \end{aligned} \tag{3.17}$$

Using (3.14) and (3.17) we compute the Poisson bracket between the flux and the holonomy as follows

$$\begin{aligned} \{(E_\gamma)_i, h_\gamma\} &= -h_\gamma\left(1, \frac{1}{2}+\frac{\varepsilon}{2}\right) \tau_i h_\gamma\left(\frac{1}{2}-\frac{\varepsilon}{2}, 0\right) \int_{\frac{1}{2}-\frac{\varepsilon}{2}}^{\frac{1}{2}+\frac{\varepsilon}{2}} \dot{\gamma}^b(s')\{(E_\gamma)_i, A_b^i(\gamma(s'))\} ds' \\ &= -h_\gamma\left(1, \frac{1}{2}+\frac{\varepsilon}{2}\right) \tau_k h_\gamma\left(\frac{1}{2}-\frac{\varepsilon}{2}, 0\right) \int_{\gamma^*} \int_{\frac{1}{2}-\frac{\varepsilon}{2}}^{\frac{1}{2}+\frac{\varepsilon}{2}} f_{ij}(s) \epsilon_{\tilde{ba}} \dot{\gamma^*}^b \dot{\gamma}^c \\ &\quad \times \{E^{aj}(\gamma^*(s)), A_c^k(\gamma(s))\} ds ds'. \end{aligned} \tag{3.18}$$

Making use of (2.39), we have (3.18) to be

$$\begin{aligned} \{(E_\gamma)_i, h_\gamma\} &= h_\gamma\left(1, \frac{1}{2}+\frac{\varepsilon}{2}\right) \tau_k h_\gamma\left(\frac{1}{2}-\frac{\varepsilon}{2}, 0\right) \\ &\quad \times \int_{\gamma^*} \int_{\frac{1}{2}-\frac{\varepsilon}{2}}^{\frac{1}{2}+\frac{\varepsilon}{2}} f_{ij}(s) \epsilon_{\tilde{ba}} \dot{\gamma^*}^b \dot{\gamma}^c \delta^{jk} \delta_c^a \delta^{2D}(s-s') ds ds' \\ &= h_\gamma\left(1, \frac{1}{2}+\frac{\varepsilon}{2}\right) \tau_k h_\gamma\left(\frac{1}{2}-\frac{\varepsilon}{2}, 0\right) \\ &\quad \times \int_{\gamma^*} \int_{\frac{1}{2}-\frac{\varepsilon}{2}}^{\frac{1}{2}+\frac{\varepsilon}{2}} f_i^k(s) \epsilon_{\tilde{ba}} \dot{\gamma^*}^b \dot{\gamma}^a \delta^{2D}(s-s') ds ds'. \end{aligned} \tag{3.19}$$

On the assumption that γ, γ^* are positively oriented at the intersection, we have $\tilde{\epsilon}_{ba}\dot{\gamma^*}^b\dot{\gamma}^a > 0$ near the intersection point. Solving the integral of the delta function in (3.19), and taking the limit $\epsilon \longrightarrow 0$ then we have $s \longrightarrow \frac{1}{2}$ and (3.19) becomes

$$\{(E_\gamma)_i, h_\gamma\} = h_\gamma\left(1, \frac{1}{2}\right)\tau_k h_\gamma\left(\frac{1}{2}, 0\right) f_i^k\left(\frac{1}{2}\right). \tag{3.20}$$

Now we solve for the last term on the right hand side of (3.20) to get

$$\begin{aligned} f_i^k\left(\frac{1}{2}\right) &= -(2)\,\mathrm{tr}\left[\tau_i h_{\gamma,\gamma^*}\left(\frac{1}{2}\right)\tau^k\left(h_{\gamma,\gamma^*}\left(\frac{1}{2}\right)\right)^{-1}\right] \\ &= (-2)\,\mathrm{tr}\left[\tau^k\left(h_{\gamma,\gamma^*}\left(\frac{1}{2}\right)\right)^{-1}\tau_i h_{\gamma,\gamma^*}\left(\frac{1}{2}\right)\right] \\ &= (-2)\,\mathrm{tr}\left[\tau^k h_\gamma\left(\frac{1}{2}, 0\right)\tau_i\left(h_\gamma\left(\frac{1}{2}, 0\right)\right)^{-1}\right]. \end{aligned} \tag{3.21}$$

To get the second line of (3.21), we perform a cyclic permutation of the first line of (3.21). Applying (3.11), the third line of (3.21) then follows.

Performing a cyclic permutation on the last line of $f_i^k\left(\frac{1}{2}\right)$ in (3.21), equation (3.20) becomes

$$\begin{aligned} \{(E_\gamma)_i, h_\gamma\} &= h_\gamma\left(1, \frac{1}{2}\right)\tau_k h_\gamma\left(\frac{1}{2}, 0\right)(-2)\,\mathrm{tr}\left[\tau_i\left(h_\gamma\left(\frac{1}{2}, 0\right)\right)^{-1}\tau^k h_\gamma\left(\frac{1}{2}, 0\right)\right] \\ &= h_\gamma\left(1, \frac{1}{2}\right)h_\gamma\left(\frac{1}{2}, 0\right)\tau_k(-2)\,\mathrm{tr}(\tau_i\tau^k)\left(h_\gamma\left(\frac{1}{2}, 0\right)\right)^{-1}h_\gamma\left(\frac{1}{2}, 0\right). \end{aligned}$$

Making use of (3.12), the above becomes

$$\begin{aligned} \{(E_\gamma)_i, h_\gamma\} &= h_\gamma\left(1, \frac{1}{2}\right)h_\gamma\left(\frac{1}{2}, 0\right)\tau_k\delta_i^k\left(h_\gamma\left(\frac{1}{2}, 0\right)\right)^{-1}h_\gamma\left(\frac{1}{2}, 0\right) \\ &= h_\gamma\left(1, \frac{1}{2}\right)h_\gamma\left(\frac{1}{2}, 0\right)\tau_i\left(h_\gamma\left(\frac{1}{2}, 0\right)\right)^{-1}h_\gamma\left(\frac{1}{2}, 0\right) \\ &= h_\gamma\left(1, \frac{1}{2}\right)h_\gamma\left(\frac{1}{2}, 0\right)\tau_i \\ &= h_\gamma\tau_i. \end{aligned} \tag{3.22}$$

Finally, we obtain the Poisson brackets between a flux and a holonomy both associated to the curve γ. The last line of (3.22) can be read in terms of matrix component wise as

$$\left\{E^i_\gamma, (h_\gamma)_{mn}\right\} = (h_\gamma\tau^i)_{mn}. \tag{3.23}$$

The Poisson brackets between holonomies and holonomies will vanish since the holonomies are functionals of the spin connection only:

$$\{(h_\gamma)_{mn}, (h_\gamma)_{m'n'}\} = 0. \tag{3.24}$$

However the Poisson brackets between fluxes and fluxes initially appears not to vanish due to the fact that the fluxes involve the diad variables and the holonomies, but this is not the case. Here we demonstrate this, using the requirement that the Jacobi identity has to hold for the following Poisson brackets:

$$\left\{\left\{E^i_\gamma, E^j_\gamma\right\}, h_\gamma\right\} + \left\{\left\{E^j_\gamma, h_\gamma\right\}, E^i_\gamma\right\} + \left\{\left\{h_\gamma, E^i_\gamma\right\}, E^j_\gamma\right\} = 0.$$

From which we get:

$$\begin{aligned}\left\{\left\{E^i_\gamma, E^j_\gamma\right\}, h_\gamma\right\} &= -\left\{\left\{E^j_\gamma, h_\gamma\right\}, E^i_\gamma\right\} - \left\{\left\{h_\gamma, E^i_\gamma\right\}, E^j_\gamma\right\} \\ &= \left\{\left\{E^i_\gamma, h_\gamma\right\}, E^j_\gamma\right\} - \left\{\left\{E^j_\gamma, h_\gamma\right\}, E^i_\gamma\right\}.\end{aligned} \tag{3.25}$$

Applying the result of (**??**) to (3.25) we have

$$\begin{aligned}\left\{\left\{E^i_\gamma, E^j_\gamma\right\}, h_\gamma\right\} &= \left\{h_\gamma \tau^i, E^j_\gamma\right\} - \left\{h_\gamma \tau^j, E^i_\gamma\right\} \\ &= -h_\gamma \tau^j \tau^i + h_\gamma \tau^i \tau^j \\ &= h_\gamma [\tau^j, \tau^i] \\ &= h_\gamma[0] = 0.\end{aligned} \tag{3.26}$$

This can be satisfy if

$$\left\{E^i_\gamma, E^j_\gamma\right\} = 0. \tag{3.27}$$

The last line in (3.26) was possible because of the commutation relation of the $so(2)$ algebra.
In summary the Poisson brackets of the basic set of the phase space function are

$$\begin{aligned}\{(E_\gamma)_i, h_\gamma\} &= h_\gamma \tau_i \\ \{(h_\gamma)_{mn}, (h_\gamma)_{m'n'}\} &= 0 \\ \left\{E^i_\gamma, E^j_\gamma\right\} &= 0.\end{aligned} \tag{3.28}$$

The first equation of (3.28), we have the holonomy appearing, hence the Poisson bracket leads to a phase space function in our basic set selected for quantization. In addition it involves only objects associated to one and the same curve. The results in (3.28) form an algebra and this coincides with the Poisson algebra of functions on the co-tangent space $T^*SO(2)$ space of $SO(2)$ which can be equipped with a phase space structure.

Chapter 4

Kinematical Hilbert Space

The choice of the canonical variables have been found. We will now aim at a canonical (Dirac) quantization of the system, this will also provide the tools to derive a (covariant) path integral quantization. In defining a kinematical Hilbert space of the system, we first find a representation of the phase space variables as operators on some space of functions, introduce the notion of a cylindrical functions (space of functions of operators), after which we equip this cylindrical functions with an inner product. This allows us to find an orthonormal basis for the cylindrical functions.

4.1 The quantization of edges(Quantum Operators)

In this section we will follow the excellent literature in [10]. Consider a fixed pair of curves γ, γ^* and rename the label γ to an edge label e. Thus we can quantize this edge as a subsystem from which we generalize to the whole system. The holonomies as stated already are $SO(2)$ group elements. The phase space functions which are to be quantized are functions of these holonomies, i.e., functions on the group $SO(2)$. This means to quantize the functions of the holonomies as multiplication operators on some space of functions

$$\psi_e : g_e \longrightarrow f_e(g_e)$$

on the group $SO(2)$. Thus for a state $\mid \psi_e >$ in the holonomy representation $< g_e \mid \psi_e >$, we define the the action of a function of the holonomy $f_e(h_e)$ as

$$\hat{f} \mid \psi_e > \;=\; \mid f_e \psi_e > . \tag{4.1}$$

By using (4.1), these multiplication operators obviously commute by using the fact that the Poisson brackets should be mapped into commutators i.e.

$$\begin{aligned} i\hbar\widehat{\{f_e, g_e\}} \mid \psi_e > &= [\hat{f}_e, \hat{g}_e] \mid \psi_e > \\ &= \left(\hat{f}_e\hat{g}_e \mid \psi_e > -\hat{g}_e\hat{f}_e \mid \psi_e\right) \\ &= \left(\hat{f}_e \mid g_e\psi_e > -\hat{g}_e \mid f_e\psi_e >\right) \\ &= (\mid f_e g_e\psi_e > - \mid g_e f_e\psi_e >) \\ &= 0. \end{aligned} \tag{4.2}$$

This then satisfy the commutation relation as in the classical case, that is the Poisson bracket between the holonomies in (3.24).

Next we compute the commutation relation between a flux and a holonomy, let first consider the Poisson brackets between these two variables in terms of the matrix elements of the holonomy i.e.,

$$\begin{aligned} \left\{E^i_e, (h_e)_{mn}\right\} &= \left(h_e\tau^i\right)_{mn} \\ &= \frac{d}{dt}_{|t=0}\left(h_e \exp(t\tau^i)\right)_{mn} \\ &=: L^i_e(h_e)_{mn}, \end{aligned} \tag{4.3}$$

where L^i_e is the left invariant derivative (this is a Lie derivative on the group manifold). The matrix elements can be understood as functions of the group and this generalizes to functions of the holonomy

$$\begin{aligned} \left\{E^i_e, f_e(h_e)\right\} &= \left(L^i_e f_e\right)(h_e) \\ &= f_e\left(h_e\tau^i\right) \\ &=: \frac{d}{dt}_{|t=0} f\left(h_e \exp(t\tau^i)\right). \end{aligned} \tag{4.4}$$

Applying this result in (4.4) to the state $\mid 1 >$ and mapping the Poisson brackets into commutators, we arrive at

$$\begin{aligned} i\hbar\widehat{(L^i_e f_e)} \mid 1 > &= i\hbar\widehat{\{E^i, f_e(h_e)\}} \mid 1 > \\ &= [\hat{E}^i, \hat{f}_e] \mid 1 > \\ &= \hat{E}^i\hat{f}_e \mid 1 > -\hat{f}_e\hat{E}^i \mid 1 > . \end{aligned} \tag{4.5}$$

From the last line of (4.5) we get:

$$i\hbar\widehat{(L^i_e f_e)} \mid 1 > \;=\; \hat{E}^i \mid f_e >, \tag{4.6}$$

the right handside of (4.6) was possible of since we take $\hat{E}^i$ to act as a derivative operator.

Let us consider the state $|\,\psi_e >= \hat{\psi}_e|1>$, then (4.6) becomes

$$\begin{aligned} i\hbar\widehat{(L_e^i\psi_e)}|1> &= \hat{E}_e^i|\psi_e> \\ i\hbar|L_e^i\psi_e> &= \hat{E}_e^i|\psi_e> . \end{aligned} \tag{4.7}$$

Hence the action of the fluxes E_e^i on a state $|\psi_e>$ is given by

$$\hat{E}_e^i|\psi_e> \;=\; i\hbar|L_e^i\psi_e> . \tag{4.8}$$

Using (4.8) we compute the the commutation relation between two fluxes on the state $|\psi_e>$:

$$\begin{aligned} [\hat{E}_e^i, \hat{E}_e^j]|\psi_e> &= \left(\hat{E}_e^i\hat{E}_e^j - \hat{E}_e^j\hat{E}_e^i\right)|\psi_e> \\ &= (i\hbar)^2|(L_e^i \circ L_e^j - L_e^j \circ L_e^i)\psi_e> . \end{aligned} \tag{4.9}$$

Evaluating the commutator of the left invariant derivative of the left handside of (4.9) on a matrix element $(h_e)_{mn}$ amounts to

$$\begin{aligned} \left(L_e^i \circ L_e^j - L_e^j \circ L_e^i\right)(h_e)_{mn} &= \frac{d}{ds}_{|s=0}\frac{d}{dt}_{|t=0}\left(\left(h_e\mathbf{e}^{(s\tau^i)}\mathbf{e}^{(t\tau^j)}\right)_{mn} - \left(h_e\mathbf{e}^{(s\tau^j)}\mathbf{e}^{(t\tau^i)}\right)_{mn}\right) \\ &= \frac{d}{ds}_{|s=0}\frac{d}{dt}_{|t=0}\left(h_e\left(\mathbf{e}^{(s\tau^i)}\mathbf{e}^{(t\tau^j)} - \mathbf{e}^{(s\tau^j)}\mathbf{e}^{(t\tau^i)}\right)_{mn}\right) \\ &= \left(h_e[\tau^i, \tau^j]\right)_{mn} \\ &= (h_e \times 0) = 0. \end{aligned} \tag{4.10}$$

Hence this generalizes to functions of matrix elements and we have

$$\begin{aligned} [\hat{E}_e^i, \hat{E}_e^j] &= i\hbar\left\{\widehat{E_e^i, E_e^j}\right\} \\ &= 0. \end{aligned} \tag{4.11}$$

Confirming our result from the classical point of view thus mapping the Poisson brackets into commutators. In addition, we have managed to find a representation for the fluxes and holonomies as operators on the space of functions of the group $SO(2)$.

4.2 Cylindrical Functions

In (2 + 1)-dimensional LQG, one usually works on a two dimensional space-like hypersurface Σ and build states of the spin connection, which gives the geometry of Σ. These states of spin connection are the cylindrical functions and they depend on the spin connection through the holonomies of the edges of a graph. Hence when defining a kinematical Hilbert space of the theory, the spin connection is considered as the configuration variable.

Let $\mathcal{G}$ be a space of the smooth $2d$ real spin connection defined everywhere on Σ, except, possibly, at isolated points. Fix the topology of Σ, say to a disk. In section 3.1, we noted that an oriented curve in Σ and a spin connection determine a group element $h_\gamma[A]$, called the holonomy (fundamental variable) of the spin connection along the path γ.
The holonomy transforms under gauge invariance of the gauge group $G = SO(2)$:

$$h_\gamma \longrightarrow g h_\gamma g^{-1}, \quad \text{where } g \in G. \tag{4.12}$$

For a given γ, the holonomy $h_\gamma[A]$ is a functional on $\mathcal{G}$.
Let us consider an ordered collection Γ of smooth oriented paths γ_e with $e = 1, 2, ..., N$ edges and a smooth function $f(h_1, ..., h_N)$

$$f : SO(2)^{N_\gamma} \longrightarrow \mathbb{C}$$

of N group elements.
The couple (Γ, f) defines a functional of the spin connection:

$$\varphi_[A] \;\; := \;\; f\left(h_{\gamma_1}[A], h_{\gamma_2}[A], ..., h_{\gamma_N}\right). \tag{4.13}$$

Defining a space of all functionals $\varphi[A]$, for all Γ and f, these functionals are called "cylindrical functions", and denote the space of these functionals as $\mathcal{S}$. This is the algebra of basic observables upon which the definition of $\mathcal{H}_{kin}$ will be based upon. Similarly, the gauge symmetry of the cylindrical functions φ can be read off from (4.12):

$$\varphi\left(h_1, h_2, ..., h_N\right) \;\; = \;\; \varphi\left(g_{s(1)} h_1 g_{t(1)}^{-1}, ..., g_{s(N)} h_N g_{t(N)}^{-1}\right), \tag{4.14}$$

$\forall g_i \in G$. Where $s(i)$ and $t(i)$ are the source vertex and target vertex respectively.

Throughout this work we will be dealing with $SO(2)$ gauge invariant function of the spin connection. The simplest of these gauge invariant function of the spin connection is the Wilson loop. It is a function of the form:

$$W_\gamma[A] \;=\; \mathrm{tr}\left(\rho(h_\gamma[A])\right) \tag{4.15}$$

for some loop in Σ and some representation ρ of the group $SO(2)$. The invariance of (4.15) is possible due to the invariance of the trace and of the holonomy from (4.12). This makes the Wilson loop an element of φ according to the definition in (4.13). The graph of the Wilson loop consists just of a single close edge ($e = \gamma$).

4.3 Representation of Cylindrical Functions (Inner Product)

These cylindrical functions defined in section 4.2 provide the possible states for a Hilbert space structure and for quantization. In this section we will define a kinematical Hilbert space $\mathcal{H}_{kin}$ for the space of cylindrical functions and give a representation of these cylindrical functions. To this effect we will introduce a positive normalized state μ_{AL}- the Ashtekar-Lewandowski measure-on the cylindrical functions and thus we obtain a definition of a kinematical inner product. This inner product must be invariant under left and right translations and this is given by the Haar measure of $SO(2)$.

Given a functional $\varphi[A]$ in the space $\mathcal{S}$, $\mu_{AL}(\varphi_{\gamma,f})$ is defined as:

$$\mu_{AL}(\varphi_{\gamma,f}) \;=\; \int_{SO(2)} \prod_e dh_e \; f\left(h_{e_1}, h_{e_2}, ..., h_{e_N}\right), \tag{4.16}$$

where $h_e \in SO(2)$ and dh is the normalized Haar measure of $SO(2)$.
Using the properties of μ_{AL}, we endow the space of cylindrical functions with an inner product:

$$\begin{aligned} \langle \varphi_{\gamma,f} | \varphi_{\gamma',g} \rangle \;&:=\; \mu_{AL}\left(\overline{\varphi_{\gamma,f}} \varphi_{\gamma',g}\right) \\ &=\; \int_{SO(2)} \prod_e dh_e \; \overline{f\left(h_{e_1}, ..., h_{e_N}\right)} g\left(h_{e_1}, ..., h_{e_N}\right). \end{aligned} \tag{4.17}$$

Given the Haar measure of the group $SO(2)$ to be $dh = (2\pi)^{-1} d\theta$ for $\theta \in [0, 2\pi]$, then μ_{AL} is obviously normalized as $\mu_{AL}(1) = 1$, and positive i.e.,

$$\mu_{AL}\left(\overline{\varphi_{\gamma,f}} \varphi_{\gamma',f}\right) \;=\; \int_{SO(2)} \prod_e dh_e \; \overline{f\left(h_{e_1}, ..., h_{e_N}\right)} f\left(h_{e_1}, ..., h_{e_N}\right) \geq 0. \tag{4.18}$$

The state μ_{AL} is known as the Ashtekar-Lewandowski (AL) measure. Starting with the space of smooth functions on the group we can complete it to a Hilbert space, thus $\mathcal{H}_{kin} = \mathcal{L}^2(S^1)$ (space of complex-valued square-integrable function on the circle) is the Cauchy completion of the space of cylindrical functions on the AL measure. Next, we define spin networks as basis for the resulting $\mathcal{L}^2$ space.

4.4 Orthonormal Basis of $\mathcal{H}_{kin}$

Introducing an orthonormal basis for $\mathcal{H}_{kin}$ requires that one uses the Peter-Weyl decomposition theorem. The theorem states that given a compact Lie group, a complete orthogonal basis is given by the set of all matrix elements of all the unitary irreducible representation of the compact group. The Peter-Weyl decomposition theorem can be viewed as a generalization of the Fourier theorem for functions on the unit circle.

To introduce the orthonormal basis of $\mathcal{H}_{kin}$, we will like to first redefine the cylindrical functions in equation (4.13). The holonomy $h_\gamma[A]$ been an element of $SO(2)$, can be written as

$$h_{\gamma_N} = \exp(in_N\theta), \tag{4.19}$$

where N is the total number of edges within a particular graph.
Hence from (4.13) the cylindrical function is rewritten as

$$\varphi_{f,\gamma}[\theta] := f\left(h_{\gamma_1}[\theta], \ldots\ldots, h_{\gamma_N}[\theta]\right). \tag{4.20}$$

From the Peter-Weyl decomposition, it states that given a function $f \in \mathcal{L}^2[SO(2)]$, the function can be expressed as a sum over unitary irreducible representation of $SO(2)$:

$$f(\theta) = \sum_n f_n \rho_n(h(\theta)), \tag{4.21}$$

where $\rho_n(h(\theta)) = \exp(in\theta)$ are the unitary one dimensional irreducible representation of $SO(2)$ for $n \in \mathbb{Z}$ and

$$f_n = \frac{1}{2\pi}\int_{SO(2)} d\theta \; \overline{\rho_n(h(\theta))} f(\theta) \tag{4.22}$$

are the Fourier components.

The orthogonality relation for unitary representation of $SO(2)$ is

$$\int_{SO(2)} d\theta \; \overline{\rho_m(h(\theta))}\rho_n(h(\theta)) \;=\; 2\pi\delta_{m,n} \tag{4.23}$$

where $\rho_n(h(\theta)) = \exp(in\theta)$ for $n \in \mathbb{Z}$. These are orthonormal and hence provide a basis for the space $\mathcal{L}^2[SO(2)]$, a basis that corresponds to the decomposition into irreducible of $\mathcal{L}^2[SO(2)]$ as a representation of $SO(2)$. One then has from the Peter-Weyl theorem

$$\left(\rho, \mathcal{L}^2[SO(2)]\right) \;=\; \bigoplus_{n\in\mathbb{Z}} (\rho_n, \mathbb{C})\,. \tag{4.24}$$

A consequence of the Peter-Weyl decomposition theorem to functions $f : SO(2)^{N_\gamma} \longrightarrow \mathbb{C}$, is it allows us to write any cylindrical function (4.20) as

$$\varphi_{\gamma,f}[\theta] \;=\; \sum_{n_1\ldots.n_{N_l}} f_{n_1\ldots.n_{N_l}}\rho_{n_1}\left(h_{\gamma_1}[\theta]\right)\ldots..\rho_{n_{N_l}}\left(h_{\gamma_N}[\theta]\right), \tag{4.25}$$

where $f_{n_1\ldots.n_{N_l}}$ are the Fourier components.

Gauge transformations in equation (4.12) induces a gauge action on Fourier modes which simply reads:

$$\rho_n(h) \longrightarrow \rho_n(g_s)\rho_n(h)\rho_n(g_t^{-1}). \tag{4.26}$$

A basis of gauge invariant functions is then constructed by contracting the tensor product of representation matrices in (4.25) with a $so(2)$-invariant tensors or $so(2)$-intertwiners. By selecting an orthonormal basis of intertwiners i_v, where v labels the elements of the basis, one can write a basis of gauge invariant elements of $\mathcal{S}$ called the spin network basis. Each spin network is labelled by a graph $\Gamma \subset \Sigma$, set of irreducible representations n_e labelling links (edges) of the graph Γ and a set of intertwiners i_v labelling the vertices (nodes) of the graph Γ.
The intertwiner i_v lives in the tensor product of the $SO(2)$ irreducible representation coming in and going out of the vertex v, precisely:

$$i_v : \bigotimes_{e|s(e)=v} V^{n_e^s} \longrightarrow \bigotimes_{e|t(e)=v} V^{n_e^t}, \tag{4.27}$$

where V^n is the representation space of $SO(2)$.

Hence the spin network function is defined as

$$\psi_{n_e,i_v}[\theta] \;=\; \bigotimes_v i_v \bigotimes_e \rho_{n_e}(h_e[\theta]). \tag{4.28}$$

The inner product between any two of these $SO(2)$ spin networks is computed as:

$$\begin{aligned} \langle \psi_{n_e,i_v} | \psi_{\tilde{n}_e,\tilde{i}_v} \rangle &= \frac{1}{2\pi} \int_0^{2\pi} d\theta \; \psi_{n_e,i_v} \overline{\psi_{\tilde{n}_e,\tilde{i}_v}} \\ &= \prod_e \delta_{n_e,\tilde{n}_e} \prod_v \langle i_v | \tilde{i}_v \rangle. \end{aligned} \tag{4.29}$$

Chapter 5

Dynamics

The spinfoam approach of quantum gravity aims at constructing a rigorous well defined mathematical notion of path integral for LQG as a tool for solving the dynamics of the theory [19]. Characterization of the kernel of the quantum constraints of the theory given by quantization of the classical constraints of general relativity defines the dynamics. The aim of these spinfoam models is to explicitly construct a generalized projection operator from $\mathcal{H}_{kin}$ into the kernel of quantum scalar constraint.

In [19], the authors establish a clear cut link between the spin foam approach and the canonical quantization of three dimensional gravity. This they achieved by constructing a physical scalar product (matrix element) of a regularized projector operator which can be represented in terms of sum of finite spin foam amplitudes.

In this present work, we make a link between these approaches (i.e. spin foam and canonical quantization), and to understand how the time gauge procedure from $SU(2)$ to $SO(2)$ works at the quantum level. Essentially, we want to solve the dynamics of our theory, by first constructing the matrix element of the projector P on the physical Hilbert space.

In chapter 4, a kinematical Hilbert space of the theory in terms of the holonomies was defined. And this can be seen as a first step towards understanding the quantum geometry [17] of spin foams and its link to canonical formalism. In the long run, the goal is to have a spin foam approach whose boundary states have an $SO(2)$ invariant spin networks.

5.1 Canonical Quantization

Solving for the dynamics of the theory, we will provide a regularization of the projector operator P-which is defined in terms of the Hamiltonian constraint. This allows for the construction of a physical scalar operator (or matrix element) of P. With the matrix element constructed, an illustration on how it admit a spin foam state representation independent of the regularization [19, 26] is carried out. The matrix elements of the projector P are called the transition amplitudes. The projector constructed on the physical Hilbert space of the theory is formally defined as

$$P = \prod_{x\in\Sigma} \delta(H(x)) = \int D[N] \exp\left(i\int_\Sigma NH(A)\right). \tag{5.1}$$

The expression (5.1) can be seen as a starting point for a spacetime representation of quantum gravity. We will introduce a regularization of (5.1) which subsequently will serve as a regularization of its matrix element $< P\psi, \psi' >$ for any pair of spin network states $\psi, \psi' \in \mathcal{H}_{kin}$. We will follow the excellent work of [19] in carrying out this task.

Let us define Γ and Γ' to be graphs on which ψ and ψ' are defined respectively. An arbitrary cellular decomposition of Σ denoted by $\Sigma_\epsilon^{\Gamma\Gamma'}$, where $\epsilon \in \mathbb{R}$, is defined such that:

1. The graphs of Γ and Γ' are both contained in the graph defined by the union of 0-cells and 1-cells in $\Sigma_\epsilon^{\Gamma\Gamma'}$.

2. For each two-cells (plaquette) p there exists a ball $\mathcal{B}_\epsilon$ of radius ϵ such that $p \subset \mathcal{B}_\epsilon$. Hence a consequence of this is that all 2-cells shrink to zero when $\epsilon \longrightarrow 0$.

Let us now consider a local patch $U \subset \Sigma$ where we choose the cellular decomposition to be a square with cells of coordinate length ϵ (regulator). The integral in the exponential in (5.1) can be written as a Riemann sum in the patch U as

$$\begin{aligned} H[N] &= \int_U \mathrm{tr}[NH(A)] \\ &= \lim_{\epsilon\longrightarrow 0} \sum_{p^i} \epsilon^2\, \mathrm{tr}[N_{p^i} H_{p^i}] \end{aligned} \tag{5.2}$$

where p^i labels i^{th} plaquette in the path U, $N_{p^i} \in so(2)$, and $H_{p^i} \in so(2)$ are values at some interior point of the 2-cells.

Since the gauge group $SO(2)$ is abelian, by using Stokes theorem the holonomy can be define as

$$h_\gamma \;=\; \mathcal{P}\exp\left(\int_{S(\gamma)} H\right), \tag{5.3}$$

where $S(\gamma)$ is the surface bounded by γ. Expanding (5.3), the holonomy $h_{p^i} \in SO(2)$ around the 2-cells p^i can be written as

$$h_{p^i}[A] \;=\; \mathbb{I} + \epsilon^2 H_{p^i}(A) + O(\epsilon^4). \tag{5.4}$$

Equation (5.2) can now be written as

$$H[N] \;=\; \lim_{\epsilon\longrightarrow 0} \sum_{p^i} \mathrm{tr}[N_{p^i} h_{p^i}[A]]. \tag{5.5}$$

Here we notice that the explicit dependence of ϵ on the right hand side of (5.5) has dropped from the sum, a sign that we can remove it upon quantization. Hence the projector (5.1) takes the form

$$P \;=\; \int \prod_{p_i} dN_{p^i} \exp\left(i \lim_{\epsilon\longrightarrow 0} \sum_{p^i} \mathrm{tr}[N_{p^i} h_{p^i}]\right). \tag{5.6}$$

We now define the physical scalar product of ψ and ψ' as:

$$\begin{aligned} \langle \psi, \psi' \rangle_{phy} &= \langle P\psi, \psi' \rangle \\ &:= \langle \int \prod_{p_i} dN_{p^i} \exp\left(i \lim_{\epsilon\longrightarrow 0} \sum_{p^i} \mathrm{tr}[N_{p^i} h_{p^i}]\right) \psi, \psi' \rangle \\ &= \lim_{\epsilon\longrightarrow 0} \langle \int \prod_{p_i} dN_{p^i} \exp\left(i \sum_{p^i} \mathrm{tr}[N_{p^i} h_{p^i}]\right) \psi, \psi' \rangle. \end{aligned} \tag{5.7}$$

Since the exponential function is a continuous function, the limit can come out of the scalar product in second line of (5.7).
Integrating (5.7) with respect to N_{p^i} we get

$$\langle P\psi, \psi' \rangle \;=\; \lim_{\epsilon\longrightarrow 0} \langle \prod_{p^i} \delta(h_{p^i})\psi, \psi' \rangle, \tag{5.8}$$

where $\delta(h_{p^i})$ is the delta function on the $SO(2)$ irreducible representation. These irreducible representation of $SO(2)$ are one-dimensional, are labelled by an integer n, and their character is $\chi^n(g) = e^{in\phi}$, where $g \in SO(2)$.

Expanding the delta function, we have

$$\delta(h_{p^i}) = \sum_n \chi^n(h_{p^i}). \tag{5.9}$$

Hence (5.7) becomes

$$\langle P\psi, \psi' \rangle = \lim_{\epsilon \longrightarrow 0} \langle \prod_{p^i} \delta(h_{p^i})\psi, \psi' \rangle \tag{5.10}$$

We now illustrate using the result in equation (5.10), on how the spin foam representation arises. The spin foam representation of the generalized projection operator (5.6) follows from inserting resolutions of the identity in $\mathcal{H}_{kin}$ [19] between consecutive delta distribution in (5.10). The resolutions of the identity in $\mathcal{H}_{kin}$ is given by

$$\mathbf{1} = \sum_{\Gamma \subset \Sigma, \{n\}_\Gamma} |\Gamma, \{n\} >< \Gamma, \{n\}|. \tag{5.11}$$

Inserting this resolution of identities between the delta distribution into the definition of $\langle P\psi, \psi' \rangle$, from the large set of graphs in Σ, only a finite number of intermediate graphs survive in the computation of $\langle P\psi, \psi' \rangle$.

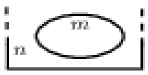
Considering the ordering of the plaquettes such that the first delta distribution in the regularization of P is evaluated on the lower central plaquette with respect to the loop l. Then the first delta function acts on the initial state $|n\rangle$ as

$$\delta(h_p)|n\rangle = \sum_m \chi^m(h_p)$$


, (5.12)

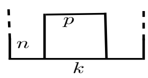
where $\chi^m(h_p)$ is the character or the trace of the one dimensional m-representation matrix of $U \in SO(2)$. Equation (5.12) becomes

$$\delta(h_p)|n\rangle = \sum_m \mathrm{tr}(\rho_m(h_p))$$


$$= \sum_m$$


(5.13)

where also $\mathrm{tr}(\rho_m(h_p))$ is known as the Wilson loop operator. The operator acts on the initial state $|n\rangle$ in terms of spin network states as

$$\delta(h_p)|n\rangle = \sum_{p,m} N_{n,k,p}\delta_{p,m}$$


. (5.14)

From (5.14), it implies that when inserting the resolution of the identity (5.11) between the first and the second delta distributions, the corresponding term in $\langle P\psi, \psi' \rangle$ only the intermediate states on the right hand side of (5.14) will survive. The delta function (5.14) acts on the initial state by attaching new infinitesimal loop. Subsequent delta function have the same effect by creating new infinitesimal loop associated to each corresponding plaquettes. Hence in this way each term in $\langle P\psi, \psi' \rangle$ explores the set of intermediate spin network states based only on those graphs that are contained in the regulating cellular decomposition. Finally, when contracted with the final state ψ' only the sequence which represent consistent histories remain.

Chapter 6

Conclusion and Outlook

In this thesis, we time gauge fix the three dimensional Hilbert-Palatini action and got some new physics resulting from the canonical analysis.
The triad is decomposed into its various components via the time gauge (2.27). In addition, we applied the triad decomposition to carry out a canonical analysis on the Hilbert-Palatini action. The diad and the spin connection (two dimensions) emerge as the new dynamical variables of the new theory.

By time gauge fixing the Hilbert-Palatini action, the usual constraints associated to an action upon a canonical analysis is carried are obtain, thus the Gauss, the spatial diffeomorphism and the Hamiltonian constraints. However, a new constraint, *i.e.*, $\mathcal{C}$ also emerges. In computing the time evolution associated with the Gauss constraint $\mathcal{G}$, the new theory generates an internal rotation in two dimensions. Additionally, diffeomorphism invariance is achieved from the time evolution associated with the spatial diffeomorphism constraint $\mathcal{V}$.

In computing the algebra of constraints for the constraints of the new theory, we found out the new theory was consistent and closed, in the sense that all the Poisson brackets between the different constraints vanish on the constraint hypersurface. The gauge transformation of the new system is an $SO(2)$ gauge transformation.

In chapter 3, a descretized form of the new dynamical variables is defined. These form a basic set of the phase space function to be quantized. The Poisson bracket between these basic set result in a much simpler expression compared to the non-descretized will be carried out. The result shows a much simpler expression compared to the non-descretized version of the canonical variables.

A kinematical Hilbert space is constructed for the new theory with an $SO(2)$ spin network. In carry out quantization the new theory we follow a similar approach in [19]. We constructed a rigorous definition of a generalized projection operator from the kinematical Hilbert space. In addition, we provided the definition of the physical scalar product (matrix element) of the projection operator. This matrix element represents a sum over finite spin-foam amplitudes. As a result a very trivial spinfoam is obtained as compared to the Ponzano-Regge spinfoam model. This trivial spinfoam model has $SO(2)$ spin networks as it boundaries. However in the Ponzano-Regge spinfoam model [27], there is a contraction of four $3j$-symbols (associated to a trivalent spin network) which results in a function of six spins $\{j_1, j_2, j_3, j_4, j_5, j_6\}$ with the faces bounded by the vertex of a tetrahedron. This new function is the Wigner $6j$-symbol.

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Appendix A

ADM Decomposition of $2+1$ Gravity

The unification of space and time play a crucial role in general relativity. Physically, general relativity is the discovery that spacetime and gravitational field are the same entity. What we call "spacetime" is itself a physical object in many respects similar to the electromagnetic field. This led to the discovery that there is no spacetime at all, and so what Newton called "space" and Minkowski called "spacetime" is nothing but a dynamical object called the gravitational field [27].

The Arnowitt-Deser-Misner (ADM) formalism however introduces an explicit- although largely arbitrary division of spacetime into spatial and temporal divisions. The decomposition of spacetime into space and time furnishes a natural setting for the initial value problem, and it underlies Wheeler's 'geometrodynamical' picture of classical general relativity as the dynamics of evolving spatial geometries. By providing a canonical description of the gravitational phase space, it leads to a Hamiltonian version of classical gravity, and suggests a useful approach to canonical quantization [5].

The ADM approach to $2+1$-dimensional general relativity start with a slicing of the spacetime manifold M into constant-time surfaces Σ_t, each provided with a coordinate system $\{x^i\}$ and an induced metric $g_{ij}(t,x^i)$. To obtain the full three-geometry, we must describe the way nearby time slice Σ_t and Σ_{t+dt} fit together. To do so, we start with a point on Σ_t with coordinates x^i, and displace it infinitesimally in the direction normal to Σ_t as illustrated in the diagram below.

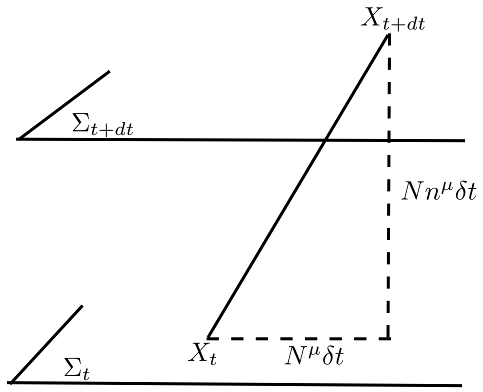


Figure A.1: The ADM decomposition of spactime

From figure A.1, define a time flow vector

$$\begin{aligned} \tau^{\mu} &= Nn^{\mu} + N^{\mu} \\ &:= \frac{\partial x^{\mu}}{\partial t} = (1, 0, 0), \end{aligned} \tag{A.1}$$

where n^{μ} is the unit normal to the hypersurface, N^{μ} is the unit tangential to the hypersurface, and N is the lapse function serving as a normalization constant. It is convenient to parametrize both n^{μ} and N^{μ} as

$$\begin{aligned} n^{\mu} &= \left(\frac{1}{N}, -\frac{N^i}{N}\right) \\ N^{\mu} &= \left(0, N^i\right). \end{aligned} \tag{A.2}$$

N^i is the shift vector and describes the amount of the tangential deformation (in the coordinates of the hypersurface). The spacetime metric is $g_{\mu\nu}$ which encodes all the information of the geometry of the three dimensional manifold. It also serves a lowering and raising tensor that is

$$g_{\mu\nu} n^{\mu} n^{\nu} = -1 \quad \Longrightarrow \quad n^{\mu} n^{\mu} = -1,$$

where the negative one shows n^{μ} is timelike.

Now we compute the ADM form of the metric:

$$\begin{aligned} g_{\mu\nu}\tau^{\mu}\tau^{\nu} &= g_{00}\tau^0\tau^0 \\ &= g_{00}. \end{aligned} \tag{A.3}$$

Here we apply Einstein's summation convention in line one. Subsequently, we use the second line of (A.1), thus $\tau^{\mu} = (1, 0, 0)$ to get the result in the second line of

(A.3). Use the first line of (A.1), we have

$$\begin{aligned} g_{\mu\nu}\tau^{\mu}\tau^{\nu} &= g_{\mu\nu}\left(Nn^{\mu}+N^{\mu}\right)\left(Nn^{\nu}+N^{\nu}\right) \\ &= g_{\mu\nu}N^{2}n^{\mu}n^{\nu}+g_{\mu\nu}N^{\mu}N^{\nu}+g_{\mu\nu}Nn^{\mu}N^{\nu}+g_{\mu\nu}Nn^{\nu}N^{\mu} \\ &= -N^{2}+g_{ij}N^{i}N^{j}. \end{aligned} \tag{A.4}$$

Here $g_{\mu\nu}Nn^{\mu}N^{\nu} = g_{\mu\nu}Nn^{\nu}N^{\mu} = 0$, since n^{μ} and N^{ν} are perpendicular to each other. For the second term in line two of (A.4) we made use of $N^{\mu} = (0, N^{i})$ to get the result in the next line of (A.4). From (A.3) and (A.4) we have

$$g_{00} = -N^{2}+g_{ij}N^{i}N^{j}. \tag{A.5}$$

Carrying out with the calculations we have:

$$\begin{aligned} g_{\mu\nu}\tau^{\mu}N^{\nu} &= g_{\mu\nu}\left(Nn^{\mu}+N^{\mu}\right)N^{\nu} \\ &= g_{\mu\nu}Nn^{\mu}N^{\nu}+g_{\mu\nu}N^{\mu}N^{\nu} \\ &= g_{ij}N^{i}N^{j}. \end{aligned} \tag{A.6}$$

Repeating the same calculations as in (A.6) but using Einstein's summation convention to expand, and making use of $\tau^{\mu} = (1, 0, 0)$ to get

$$\begin{aligned} g_{\mu\nu}\tau^{\mu}N^{\nu} &= g_{00}\tau^{0}N^{0}+g_{01}\tau^{0}N^{1}+g_{02}\tau^{0}N^{2}+g_{10}\tau^{1}N^{0}+g_{11}\tau^{1}N^{1} \\ &\quad +g_{12}\tau^{1}N^{2}+g_{20}\tau^{2}N^{0}+g_{21}\tau^{2}N^{1}+g_{22}\tau^{2}N^{2} \\ &= g_{00}\tau^{0}N^{0}+g_{01}\tau^{0}N^{1}+g_{02}\tau^{0}N^{2} \\ &= g_{0j}N^{j}. \end{aligned} \tag{A.7}$$

From (A.6) and (A.7) we have

$$g_{0j}N^{j} = g_{ij}N^{i}N^{j}. \tag{A.8}$$

Expanding the terms in (A.8)

$$\begin{aligned} g_{01}N^{1}+g_{02}N^{2} &= g_{11}N^{1}N^{1}+g_{12}N^{1}N^{2}+g_{21}N^{2}N^{1}+g_{22}N^{2}N^{2} \\ &= \left(g_{11}N^{1}+g_{12}N^{2}\right)N^{1}+\left(g_{21}N^{1}+g_{22}N^{2}\right)N^{2}. \end{aligned} \tag{A.9}$$

Equating corresponding terms we have: $g_{01}N^{1} = \left(g_{11}N^{1}+g_{12}N^{2}\right)N^{1} \implies g_{01} = g_{1j}N^{j}$. Similarly, $g_{02} = g_{2j}N^{j}$, hence we have

$$\begin{aligned} g_{01}+g_{02} &= g_{1j}N^{j}+g_{2j}N^{j} \\ g_{0i} &= g_{ij}N^{j}. \end{aligned} \tag{A.10}$$

Since the spacetime metric is symmetric, from (A.10) we have $g_{i0} = g_{ij}N^j$. The line element for three-dimensional spacetime is given as

$$\begin{aligned} dS^2 &= g_{\mu\nu}dx^\mu dx^\nu \\ &= g_{00}dx^0dx^0 + g_{01}dx^0dx^1 + g_{02}dx^0dx^2 + g_{10}dx^1dx^0 + g_{11}dx^1dx^1 \\ &\quad +g_{12}dx^1dx^2 + g_{20}dx^2dx^0 + g_{21}dx^2dx^1 + g_{22}dx^2dx^2 \\ &= \left(-N^2 + g_{ij}N^iN^j\right)dt^2 + g_{ij}N^jdtdx^i + g_{ij}N^idx^jdt + g_{ij}dx^idx^j \\ &= -N^2dt^2 + g_{ij}\left(dx^i + N^idt\right)\left(dx^j + N^jdt\right). \end{aligned} \tag{A.11}$$

Here to get to line three, we applied the results in (A.5) and (A.10) to the second line. The result in (A.11) is the ADM form of the the metric. Expressing the metric in a matrix form we have

$$g_{\mu\nu} = \begin{pmatrix} N^iN_i - N^2 & N_i \\ N_j & g_{ij} \end{pmatrix}. \tag{A.12}$$

The inverse of the matrix is

$$g^{\mu\nu} = \begin{pmatrix} -\frac{1}{N^2} & \frac{N^j}{N^2} \\ \frac{N^i}{N^2} & \left(gij - \frac{N^jN^j}{N^2}\right) \end{pmatrix}. \tag{A.13}$$

The six components of the spacetime metric are now replaced by the three components of the induced Riemannian metric g_{ij} of Σ plus the two components of the shift vector N^i and the lapse function.

Appendix B

Representation Theory

B.1 Group Representation

Definition B.1.1. A finite dimensional representation of a Lie group G on a vector space over a field K is a group homomorphism from G to $GL(V)$, the general linear group on V. That is, a representation is a map [1]

$$\phi : G \longrightarrow GL(V) \tag{B.1}$$

such that $\phi(g_1 g_2) = \phi(g_1)\phi(g_2)$, for all $g_1, g_2 \in G$. The dimension of the representation is the dimension of the vector space V. Denote the representation of G in V by (G, V).

Definition B.1.2. Let (G, V) be a representation. A subspace U of V is called invariant or G-invariant if $g.U \subset U$ for $g \in G$.

A representation (G, V) has always at least two invariant subspaces, namely $\{0\}$ and V.

Definition B.1.3. A representation is called irreducible if the only invariant subspaces are $\{0\}$ and V.

B.2 Peter-Weyl Decomposition Theorem

Definition B.2.1. A matrix coefficient of a group G is a complex-valued function φ on G given as the composition

$$\varphi = L \circ \pi$$

where π is a finite-dimensional group representation of G, and L is a linear functional on the vector space of the endomorphisms of V which contains $GL(V)$ as an open subset.

The Peter-Weyl decomposition theorem is a theory of harmonic analysis, applying to topological groups that are compact but not necessarily abelian. The theorem is a collection of results generalizing the significant facts about the decomposition of the regular representation of any finite group.
The theorem has three parts: (i) The set of matrix coefficients of a group G is dense in the space of continuous complex functions on G equipped with the uniform norm. (ii) Let ρ be a unitary representation of a compact group G on a complex Hilbert space $\mathcal{H}$. Then $\mathcal{H}$ splits into an orthogonal direct sum of irreducible finite dimensional unitary representation of G. (iii) The matrix coefficients for G, suitably renormalized, are an orthonormal basis of the square integrable functions of G ($\mathcal{L}^2(G)$). In particular, $\mathcal{L}^2(G)$ decomposes into an orthogonal direct sum of all the irreducible unitary representations.

B.3 $SU(2)$ Representation

Commutation relation: The generators of the $su(2)$ algebra are given as

$$T_i = \frac{\sigma_i}{2}, \quad \text{where } i = \{x, y, z\} \ . \tag{B.2}$$

Where σ_i's are the Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_y = \begin{pmatrix} 0 & -\imath \\ \imath & 0 \end{pmatrix}, \ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{B.3}$$

The generators satisfy the following commutation relations:

$$[T_i, T_j] = \imath \epsilon_{ijk} T_k \tag{B.4}$$

Representation: Define the Casimir operator (commutes with all the T_i, $i.e$, $[T^2, T_i] = 0$)

$$T^2 = T_x^2 + T_y^2 + T_z^2.$$

By convention choose T_z to be the observable to be diagonalized simultaneously with T^2. Therefore we have

$$T^2|jm\rangle = j(j+1)|jm\rangle, \quad \text{and} \quad T_z|jm\rangle = m|jm\rangle \tag{B.5}$$

Define the non-Hermitian operators (raising and lowering operators):

$$T_{\pm} = T_x \pm \imath T_y \tag{B.6}$$

The following relations exist between the non-Hermitian and the diagonalized operators:

$$\left[T^2, T_{\pm}\right] = 0, \quad [T_z, T_{\pm}] = \pm T_{\pm}, \quad [T_+, T_-] = 2T_z \ . \tag{B.7}$$

The eigenvalues of the non-Hermitian operators are given as follows

$$T_{\pm}|jm\rangle = \sqrt{(j \mp m)\,(j \pm m + 1)}|jm \pm 1\rangle, \tag{B.8}$$

where j ranges from $-m$ to m.

Appendix C

Lie Derivative

Functions, tensor fields and one-forms can be differentiated with respect to a vector field. Lie derivative evaluates the change of a tensor field (including scalar function, vector field and one-form), along the flow of another vector field. This change is coordinate invariant and therefore the Lie derivative is defined on any differentiable manifold. Considering vector fields as infinitesimal generators on a manifold $\mathcal{M}$, then the Lie derivatives are infinitesimal representation of the diffeomorphism group on tensor fields.

C.1 Lie derivative of a function

Given a vector field V defined on $\mathcal{M}$, there exist a smooth map $\sigma : \mathbb{R} \times \mathcal{M} \longrightarrow \mathcal{M}$ called the flow of V, such that $\sigma_0(p) = p$, $\sigma_t(\sigma_s(p)) = \sigma_{t+s}(p)$ and

$$\frac{d}{dt}\sigma_t = V(\sigma_t(p)) \tag{C.1}$$

are satisfied. The follow of V is denoted by $\sigma_t(p)$, where $t \in \mathbb{R}$ and $p \in \mathcal{M}$. Writing the component of the flow of V with respect to the local coordinates x^μ we have

$$\sigma_t^\mu(p) = x^\mu(p) + tV^\mu(p) + O(t^2) \tag{C.2}$$

Given a function $f : \mathcal{M} \longrightarrow \mathbb{R}$, the Lie derivative acting on this function with respect to a vector field V changes along the flow of V. Therefore defining it using the difference $f(\sigma_t(p)) - f(p)$, i.e. the difference between f in the point p and the translated point $\sigma_t(p)$. In that way we can get a measure of the change of f in the direction of the flow of V. Hence, we define the Lie derivative $\mathcal{L}_V f$ of the function

f along the vector field V as

$$\mathcal{L}_V f)(p) \quad = \quad \lim_{t \longrightarrow 0} \frac{f(\sigma_t(p)) - f(p)}{t}. \tag{C.3}$$

Using the expression in (C.2), we have for small t

$$f(\sigma_t(p)) \quad = \quad f(p) + tV^{\mu}\frac{\partial f(p)}{\partial x^{\mu}} + O(t^2). \tag{C.4}$$

In local coordinate system x^{μ}, equation (C.3) is given as

$$\mathcal{L}_V f \quad = \quad V^{\mu}\frac{\partial f}{\partial x^{\mu}}. \tag{C.5}$$

C.2 Lie derivative of a vector field

We now want to define the Lie derivative of a vector field W with respect to a vector field V, however let us first define an induced map. Let $\Phi : \mathcal{M} \longrightarrow \mathcal{M}$ be a map. For a given point $p \in \mathcal{M}$ we can define the induced map $D_p\Phi : T_p(\mathcal{M}) \longrightarrow T_{\Phi(p)}(\mathcal{M})$ as follows. Writing

$$V' \quad = \quad D_p\Phi(V) \tag{C.6}$$

we define the induced map as

$$V'[f](\Phi(p)) \quad = \quad V[f \circ \Phi](p). \tag{C.7}$$

In terms of coordinates we have that

$$V'[f] \quad = \quad V^{\mu}\frac{\partial \Phi^{\nu}}{\partial x^{\mu}}\frac{\partial f}{\partial x^{\nu}}. \tag{C.8}$$

The Lie derivative acting on W quantifies how much W changes along the flow of V. Taking the difference $W(\sigma_t(p)) - W(p)$ i.e. the difference between W at p and at $\sigma_t(p)$, and then look at the limit $t \longrightarrow 0$. This however is not well defined since $W(\sigma_t(p)) \in T_{\sigma_t(p)}(\mathcal{M})$ and $W(p) \in T_p(M)$ are vectors in two different tangent spaces, hence one cannot subtract them from each other. Thus we need to find a way to take $W(\sigma_t(p))$ and map it into a vector in the tangent space $T_p(M)$. This can be done with the induced map $D_{\sigma_t(p)}\sigma_{-t} : T_{\sigma_t(p)}(M) \longrightarrow T_p(M)$.

We now define the Lie derivative $\mathcal{L}_V W$ of the vector field W along the vector field

V as

$$\mathcal{L}_V W = \lim_{t \longrightarrow 0} \frac{D_{\sigma_t(p)}\sigma_{-t}(W) - W(p)}{t}, \tag{C.9}$$

for any point $p \in \mathcal{M}$. The induced map in coordinates is given as

$$D_{\sigma_t(p)}\sigma_{-t}(W)[f] = W^\nu(\sigma_t(p)) \frac{\partial \sigma^\mu_{-t}}{\partial x^\nu}(\sigma_t(p)) \frac{\partial f}{\partial x^\mu}(p). \tag{C.10}$$

Using the expression in (C.2), we have for small t

$$\begin{aligned} W^\nu(\sigma_t(p)) &= W^\nu(p) + tV^\rho \frac{\partial W^\nu}{\partial x^\rho}(p) + O(t^2) \\ \frac{\partial \sigma^\mu_{-t}}{\partial x^\nu}(\sigma_t(p)) &= \delta^\mu_\nu - t\frac{\partial V^\mu}{\partial x^\nu}(p) + O(t^2). \end{aligned} \tag{C.11}$$

Computing from this the components of the Lie derivative $\mathcal{L}_V W$ we have

$$(\mathcal{L}_V W)^\mu = V^\nu \frac{\partial W^\mu}{\partial x^\nu} - W^\nu \frac{\partial V^\mu}{\partial x^\nu}. \tag{C.12}$$

We can see that this is nothing but the components of the commutator $[V, W]$ of V and W, thus we have

$$\mathcal{L}_V W = [V, W]. \tag{C.13}$$

Appendix D

SO(2) Haar Measure and Useful Identities

A Haar measure is a way to assign an "invariant volume" to subsets of locally compact topological groups and subsequently define an integral functions on those groups.

Definition D.0.1. A Haar on a group G is a measure $\mu : \Sigma \longrightarrow [0, \infty)$, with Σ a σ-algebra containing all Borel subsets of G, such that

- $\mu(G) = 1$
- $\mu(\gamma S) = \mu(S) \quad \forall \gamma \in G, S \in \Sigma.$

Here $\gamma S = \{\gamma\alpha | \alpha \in S\}$

We may associate to any measure μ on G a bounded linear functional

$$E : L(G, \Sigma, \mu) \longrightarrow \mathbb{C}$$

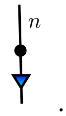
by

$$E(f) = \int_G f(\gamma) d\mu(\gamma). \tag{D.1}$$

Hence in this case for the group $SO(2)$ we have

$$\int_{SO(2)} f(\gamma) d\mu(\gamma) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) d\theta. \tag{D.2}$$

We now present here the various identities used in section 5.1. The identities are for the gauge group $SO(2)$, for $SU(2)$ and general treatment of these identities see [19, 20]. We start by recalling the graphical notation that is commonly used in dealing with group representation theory. An n unitary irreducible $SO(2)$ representation matrix ρ_n is represented by an oriented line pointed downwards. The group element $g \in SO(2)$ on which the representation matrix is evaluated is depicted by a dark box in the middle. Namely,

$$\rho_n(g) \quad = $$  . (D.3)

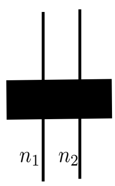
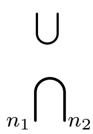
The tensor product of representation matrices is simply represented by a set of parallel lines carrying the corresponding representation labels and orientation. An important object is the integral of the tensor product of unitary irreducible representations. We denote the Haar measure integration by a dark box overlapping the different representation lines as follows:

$$I \quad = \quad \int dg \; \rho_{n_1}(g)\rho_{n_2}(g).......\rho_{n_N}(g) = $$  . (D.4)

The invariance of the Haar measure implies $I^2 = I$ and the invariance of I under right and left action of the group; therefore, I defines the projection operator $I^{n_1 n_2 .... n_N} : n_1 \otimes n_2 \otimes .... \otimes n_N \longrightarrow Inv[n_1 \otimes n_2 \otimes .... \otimes n_N]$. With this, it is easy to write the basic identities that follow from the properties of the Haar measure, namely

 $$= \quad \delta_{n,0}$$ (D.5)

and

 $$= \quad \delta_{n_1,n_2}$$  (D.6)