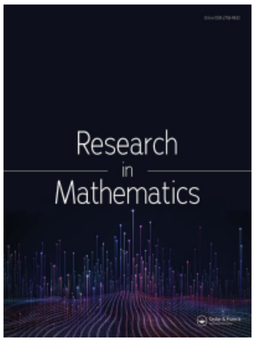

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On the design of paired comparison experiments with application

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ABSTRACT

In practice, paired comparison experiments involving pairs of either full or partial profiles are frequently used. When all attributes have a general common number of levels, the problem of finding optimal designs is considered in the presence of a second-order interaction model. In this setting, the *D*-optimal designs for the second-order interaction model have both types of pairs in which either all attributes have different levels or approximately half of the attributes are different. The proposed optimal designs can be used as a benchmark to compare any design for estimating main effects and two- and three-attribute interactions. A practical situation that incorporates the corresponding second-order interactions is covered.

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full profile; interactions; optimal design; paired comparisons; partial profile; profile strength

1 Introduction

Paired comparisons are related to experiments with two options (alternatives). Such experiments are widely used in many fields of application, including health economics, transportation economics, and marketing, to study individual preferences toward new products or services (Ryan, 2004; Sudhir et al., 2015), where behaviors of interest typically involve quantitative responses (Scheffé, 1952). The present paper draws on this situation of quantitative responses (so-called conjoint analysis where responses are usually assessed on a rating scale) as frequently encountered in marketing research (Green et al., 2004; Rao, 2014; deBekker Grob Ew et al., 2012).

Typically, in paired comparison experiments, respondents trade off one alternative against the other, generated by an experimental design, and are described by several attributes. Usually, one may consider experimental design, which allows estimating all the effects of interest (typically main effects only or main effects plus some higher-order interaction effects) (Hoyos, 2010). Jaynes et al. (2016) noted that the choice of the design for choice experiments is critical because it determines which attributes' effects and their interactions are identifiable. Analyzing higher-order interaction terms can help explain welfare measures' convergence or divergence. However, studies tend to ignore higher-order interactions (Lancsar & Louviere, 2008), modelling only main effects (Mogas et al., 2006). Therefore, the inclusion of interaction terms in design plans should be encouraged (Jaynes et al., 2016). Mandeville et al. (2014) pointed out the importance of identifying main and interaction effects. By combining attributes, design plans can generate a rich source of data to evaluate real-life decision-making processes (Elrod et al., 1992; Kruk et al., 2009; Shah et al., 2015; Soetevent & Kooreman, 2007). The present paper, where any of the three attributes interact, is motivated by the works mentioned above.

Due to the limited cognitive ability to process information in applications, a paired comparison task including many attributes may result in respondent decisions that do not reflect their actual preferences. A way to overcome these behaviors is to simplify the paired comparison task by holding the levels of some of the attributes constant in every pair. The profiles in such a pair are called partial profiles, and the number of attributes allowed with potentially different levels in the partial profiles is called the profile strength (e.g. see Chrzan, 2010; Graßhoff et al., 2003; Green, 1974; Kessels et al., 2011).

In this paper, we mainly introduce an appropriate model for the situation of full and partial profiles and derive optimal designs in the presence of second-order (or three-attribute) interactions. We consider the case when a general common number of level attributes specifies the alternatives. A practical situation of interest incorporating the corresponding second-order interactions is also considered. In the statistical literature, work

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on determining optimal designs for binary attributes has been investigated (Berkum, 1987; Schwabe et al., 2003) in the case of full and/or partial profiles in additive as well as two-attribute interaction setup. Corresponding results when the common number of the attribute levels is larger than two have been obtained (see Graßhoff et al., 2003). Here, we treat the case of three-attribute interactions when the common number of the attribute levels is at least two and provides some detailed insights. The two-level situation has been investigated in the case of both full and partial profiles (Nyarko & Schwabe, 2019).

The remainder of the paper is organized as follows. A general model is introduced in Section 2 for linear paired comparisons related to experiments with two options. Three-attribute interaction model for full and partial profiles is presented in Section 3, and optimal designs are characterized in Section 4. Theoretical design constructions are presented in Section 5, a practical situation of interest is considered in Section 6, and the final Section 7 offers some conclusions.

2 General setting

In any experimental situation, the experiment's outcome depends on some factors (attributes), say, K of influence. In this setting, the dependence can be described by a vector of regression functions $\mathbf{f}$. In what follows, we define a single alternative by $\mathbf{i} = (i_1, \ldots, i_K)$ where i_k is the component of the Kth attribute, $k = 1, \ldots, K$. Any utility (not observe) $\tilde{Y}_n(\mathbf{i})$ of the single alternative $\mathbf{i} = (i_1, \ldots, i_K)$ subject to a block effect μ_n and a random error $\tilde{\varepsilon}_n$, which is assumed to be uncorrelated with constant variance and zero mean, can be formalized by a general linear model

$$\tilde{Y}_n(\mathbf{i}) \quad = \mu_n + \mathbf{f}(\mathbf{i})^{\mathrm{T}}\beta + \tilde{\varepsilon}_n, \tag{1}$$

where the index n denotes the nth presentation, $n = 1, \ldots, N$, and the alternative $\mathbf{i}$ is chosen from a set $\mathcal{I} = \{1, \ldots, v\}^K$. Here, the vector of known regression functions $\mathbf{f}$ describes the form of the functional relationship between the alternative i and the corresponding mean response $E(\tilde{Y}_n(\mathbf{i})) = \mu_n + \mathbf{f}(\mathbf{i})^{\mathrm{T}}\beta$, and β is the unknown parameter vector of interest. Usually, to make statistical inferences on the unknown parameters, several pairs are presented to get rid of the influence of the block effect μ_n due to a variety of unobservable influences. The actual differences of the latent utilities are observed for the alternatives presented in a pair.

More specifically, unlike in standard experimental designs where there is a possibility of only a single or direct observation, in paired comparison experiments, the utilities for the alternatives are usually not directly observed. Only observations $Y_n(\mathbf{i}, \mathbf{j}) = \tilde{Y}_n(\mathbf{i}) - \tilde{Y}_n(\mathbf{j})$ are available for comparing pairs $(\mathbf{i}, \mathbf{j})$ of alternatives $\mathbf{i}$ and $\mathbf{j}$ which are chosen from the design region $\mathcal{X} = \mathcal{I} \times \mathcal{I}$. In that case, the utilities for the alternatives are properly described by the linear paired comparison model

$$Y_n(\mathbf{i}, \mathbf{j}) = (\mathbf{f}(\mathbf{i}) - \mathbf{f}(\mathbf{j}))^{\mathrm{T}}\beta + \varepsilon_n, \tag{2}$$

where $\mathbf{f}(\mathbf{i}) - \mathbf{f}(\mathbf{j})$ is the derived regression function and the random errors $\varepsilon_n(\mathbf{i}, \mathbf{j}) = \tilde{\varepsilon}_n(\mathbf{i}) - \tilde{\varepsilon}_n(\mathbf{j})$ associated with the different pairs $(\mathbf{i}, \mathbf{j})$ are assumed to be uncorrelated with constant variance and zero mean. Here, the block effects μ_n are immaterial.

The performance of the statistical analysis based on a paired comparison experiment depends on the pairs in the presented preference task. The choice of such pairs $(\mathbf{i}_1, \mathbf{j}_1), \ldots, (\mathbf{i}_N, \mathbf{j}_N)$ is called a design of size N. The quality of such a design is measured by its information matrix.

This article considers approximate designs ξ (e.g. see Kiefer, 1959) which are defined as discrete probability measures on the design region $\mathcal{X}$ of all pairs $(\mathbf{i}, \mathbf{j})$. Moreover, every approximate design ξ which assigns only rational weights $\xi(\mathbf{i}, \mathbf{j})$ to all pairs $(\mathbf{i}, \mathbf{j})$ in its support points can be realized as an exact design ξ_N of size N consisting of the pairs $(\mathbf{i}_1, \mathbf{j}_1), \ldots, (\mathbf{i}_N, \mathbf{j}_N)$. Note that for an exact design ξ_N the normalized information matrix $\mathbf{M}(\xi_N)$ coincides with the information matrix $\mathbf{M}(\xi)$ of the corresponding approximate design ξ.

Optimality criteria for approximate designs ξ are functionals of $\mathbf{M}(\xi)$. As a scalar measure of design quality, here we consider the criterion of D-optimality. An approximate design ξ^* is D-optimal if it maximizes the determinant of the information matrix, that is, if $\det\mathbf{M}(\xi^*) \geq \det\mathbf{M}(\xi)$ for every approximate design ξ on $\mathcal{X}$.

3 Second-order interaction model

In applications, one may be interested in the utility estimates of the main effects and interactions between the levels of the attributes. For that setting, optimal designs have been derived (van Graßhoff et al., 2003; Berkum Eem, 1987) in a two-attribute interaction setup. This paper considers a three-attribute interaction model. Corresponding results for the particular case of binary attributes have been obtained (Nyarko & Schwabe, 2019).

Analogous to Nyarko and Schwabe (2019), we first start with the situation of full profiles. In that case, each alternative is represented by level combinations in which all attributes are involved. For such alternatives, we denote by $\mathbf{i} = (i_1, \ldots, i_K)$ and $\mathbf{j} = (j_1, \ldots, j_K)$ the first alternative and the second alternative, respectively, which are both elements of the set $\mathcal{I} = \{1, \ldots, v\}^K$ where 1 and v represent the first and last level of each kth component, $k = 1, \ldots, K$. Here $(\mathbf{i}, \mathbf{j})$ is an ordered pair of alternatives $\mathbf{i}$ and $\mathbf{j}$ which is chosen from the design region $\mathcal{X} = \mathcal{I} \times \mathcal{I}$. Note that for each attribute k, the corresponding regression functions $\mathbf{f}_k = \mathbf{f}$ coincide with the one-way layout (see Graßhoff et al., 2003).

In the presence of up to three-attribute interactions, direct responses $\tilde{Y}_n$ at alternative $\mathbf{i} = (i_1, \ldots, i_K)$ can be modeled as

$$\tilde{Y}_n(\mathbf{i}) = \mu_n + \sum_{k=1}^{K} \mathbf{f}(i_k)^{\mathrm{T}} \beta_k + \sum_{k<\ell} (\mathbf{f}(i_k) \otimes \mathbf{f}(i_\ell))^{\mathrm{T}} \beta_{k\ell} + \sum_{k<\ell<m} (\mathbf{f}(i_k) \otimes \mathbf{f}(i_\ell) \otimes \mathbf{f}(i_m))^{\mathrm{T}} \beta_{k\ell m} + \tilde{\varepsilon}_n, \tag{3}$$

for full profiles, where $\otimes$ denotes the Kronecker product of vectors or matrices, $\beta_k = (\beta_1^k, \ldots, \beta_{v-1}^k)^{\mathrm{T}}$ denotes the main effect of the kth attribute, $\beta_{k\ell} = (\beta_{11}^{(k\ell)} \ldots \beta_{v-1v-1}^{(k\ell)})^{\mathrm{T}}$ is the two-attribute interaction of the kth and ℓth attribute, and $\beta_{k\ell m} = (\beta_{111}^{(k\ell m)} \ldots \beta_{v-1v-1v-1}^{(k\ell m)})^{\mathrm{T}}$ is the three-attribute interaction of the kth, ℓth and mth attribute. The vectors $(\beta_k)_{1\le k\le K}$ of main effects, $(\beta_{k\ell})_{1\le k<\ell\le K}$ of two-attribute interactions and $(\beta_{k\ell m})_{1\le k<\ell<m\le K}$ of three-attribute interactions have dimensions $p_1 = K(v-1)$, $p_2 = (1/2)K(K-1)(v-1)^2$ and $p_3 = (1/6)K(K-1)(K-2)(v-1)^3$, respectively. Hence, the complete parameter vector

$$\beta = ((\beta_k)_{k=1,\ldots,K}^{\mathrm{T}}, (\beta_{k\ell})_{k<\ell}^{\mathrm{T}}, (\beta_{k\ell m})_{k<\ell<m}^{\mathrm{T}})^{\mathrm{T}}, \tag{4}$$

has dimension $p = p_1 + p_2 + p_3 = K(v-1)(1 + (1/6)(K-1)(v-1)(3 + (K-2)(v-1)))$. The corresponding p-dimensional vector $\mathbf{f}$ of regression functions is given by

$$\mathbf{f}(\mathbf{i}) = (\mathbf{f}(i_1)^{\mathrm{T}}, \ldots, \mathbf{f}(i_K)^{\mathrm{T}}, \mathbf{f}(i_1)^{\mathrm{T}} \otimes \mathbf{f}(i_2)^{\mathrm{T}}, \ldots, \mathbf{f}(i_{K-1}) \otimes \mathbf{f}(i_K)^{\mathrm{T}}, \mathbf{f}(i_1)^{\mathrm{T}} \otimes \mathbf{f}(i_2)^{\mathrm{T}} \otimes \mathbf{f}(i_3)^{\mathrm{T}}, \ldots, \mathbf{f}(i_{K-2})^{\mathrm{T}} \otimes \mathbf{f}(i_{K-1})^{\mathrm{T}} \otimes \mathbf{f}(i_K)^{\mathrm{T}})^{\mathrm{T}}. \tag{5}$$

Also here in $\mathbf{f}(\mathbf{i})$, the first K components $\mathbf{f}(i_1), \ldots, \mathbf{f}(i_K)$ are associated with the main effects and have $p_1 = K(v-1)$ parameters, the second components $\mathbf{f}(i_1) \otimes \mathbf{f}(i_2), \ldots, \mathbf{f}(i_{K-1}) \otimes \mathbf{f}(i_K)$ are associated with the two-attribute interactions and have $p_2 = (1/2)K(K-1)(v-1)^2$ parameters, and the remaining components $\mathbf{f}(i_1) \otimes \mathbf{f}(i_2) \otimes \mathbf{f}(i_3), \ldots,$ $\mathbf{f}(i_{K-2}) \otimes \mathbf{f}(i_{K-1}) \otimes \mathbf{f}(i_K)$ are associated with the three-attribute interactions and have $p_3 = (1/6)K(K-1)(K-2)(v-1)^3$ parameters.

As was already pointed out, because of the limited cognitive ability to process information, a preference task including many attributes may enhance respondent decisions that do not reflect their actual preferences. A way to overcome this problem is to use partial profiles. In a partial profile, every pair consists of alternatives described by a predefined number S of attributes. The same attributes are used throughout both alternatives within a pair but with potentially different levels. In contrast, the remaining $K - S$ attributes are not shown and remain thus unspecified. The number S of attributes used in a partial profile is called the profile strength. For instance, in an experimental situation for a total number of attributes $K = 11$ each at two levels, only $S = 4$ of the attributes are shown. In contrast, the remaining $K - S = 7$ attributes are not shown or officially set to 0; the $S = 4$ of the attributes shown in the pairs of alternatives is the profile strength. Green (1974) pointed out that in choosing a profile strength, it has to be taken into account: (a) the number of attributes to vary in each set of alternatives; (b) the number of alternatives to present to respondents for evaluation; and (c) the type of utility (or paired comparison) model to apply in representing the respondent's evaluations. According to Schwabe et al. (2003), typically choosing up to a maximum number of four attributes as representative of the profile strength while there may be 20 or more attributes is enough to reduce cognitive burden as frequently encountered in practice (Großmann, 2017).

For a partial profile, a direct observation may be described by the model (3) when summation is taken only over those S attributes contained in the describing subset. This requires that the profile strength $S \geq 3$ is needed to ensure the identifiability of the three-attribute interactions. In what follows, we introduce an additional level $i_k = 0$, which indicates that the corresponding kth attribute is not present in the partial profile. The corresponding regression functions are extended to $\mathbf{f}(0) = \mathbf{0}$.

With this convention, a direct observation can be described by (3) even for a partial profile i from the set

$$\mathcal{I}^{(S)} = \{\mathbf{i};\ i_k \in \{1,\dots,v\} \text{ for } S \text{ components and } i_k = 0 \text{ for } K-S \text{ components}\}, \quad (6)$$

of alternatives with profile strength S.

For observations in linear paired comparisons, the resulting model is given by

$$\begin{aligned} Y_n(\mathbf{i},\mathbf{j}) &= \sum_{k=1}^{K} (\mathbf{f}(i_k) - \mathbf{f}(j_k))^{\mathrm{T}} \beta_k \\ &+ \sum_{k<\ell} ((\mathbf{f}(i_k) \otimes \mathbf{f}(i_\ell)) - (\mathbf{f}(j_k) \otimes \mathbf{f}(j_\ell)))^{\mathrm{T}} \beta_{k\ell} \\ &+ \sum_{k<\ell<m} ((\mathbf{f}(i_k) \otimes \mathbf{f}(i_\ell) \otimes \mathbf{f}(i_m)) \\ &- (\mathbf{f}(j_k) \otimes \mathbf{f}(j_\ell) \otimes \mathbf{f}(j_m)))^{\mathrm{T}} \beta_{k\ell m} + \varepsilon_n. \end{aligned} \quad (7)$$

The design region, in this case, can be specified as

$$\mathcal{X}^{(S)} = \{(\mathbf{i},\mathbf{j});\ i_k, j_k \in \{1,\dots,v\} \text{ for } S \text{ components and } i_k = j_k = 0 \text{ for exactly } K-S \text{ components}\}, \quad (8)$$

for the set of partial profiles with profile strength S. Notice that for full profiles, each pair in the design region consists of alternatives where all attributes are shown.

4 Optimal designs

In the present setting, we consider optimal designs for the three-attribute interaction paired comparison model (7) with corresponding regression functions $\mathbf{f}(\mathbf{i})$ given by (5). In what follows, we define d as the comparison depth (see Graßhoff et al., 2003), which describes the number of attributes in which the two alternatives presented differ, $d = 1,\dots,S$ (see Nyarko, 2020, p. 9, for example).

For this situation, the design region $\mathcal{X}^{(S)}$ in (8) can be partitioned into disjoint sets

$$\mathcal{X}_d^{(S)} = \{(\mathbf{i},\mathbf{j}) \in \mathcal{X}^{(S)};\ i_k \neq j_k \text{ for exactly } d \text{ components}\}. \quad (9)$$

These sets constitute the orbits with respect to permutations of both the levels $i_k, j_k = 1,\dots,v$ within the attributes as well as among attributes $k = 1,\dots,K$, themselves.

Note that the D-criterion is invariant concerning those permutations, which induce a linear reparameterization (see Schwabe, 1996, p. 17). As a result, it is sufficient to look for optimality in the class of invariant designs (see Kiefer, 1959, p. 282–284, for detailed discussion).

Denote by $\bar{\xi}_d$ the uniform approximate design which assigns equal weights to each individual distinct (pair) design points in $\mathcal{X}_d^{(S)}$, $\bar{\xi}_d(\mathbf{i},\mathbf{j}) = 1/N_d$ where $N_d = \binom{K}{S}\binom{S}{d} v^S (v-1)^d$ is the number of the distinct design points. Notice that for the associated identical (pair) design points in $\mathcal{X}^{(S)}$, there is no information available. It can be shown that the corresponding uniform design $\bar{\xi}_d$ on the set $\mathcal{X}_d^{(S)}$ has an information matrix with a diagonal block structure. To begin with, we first note that $\mathbf{M} = \frac{2}{v-1}(\mathbf{I}_{v-1} + \mathbf{1}_{v-1}\mathbf{1}_{v-1}^{\mathrm{T}})$ is the information matrix of the one-way layout (e.g. see Graßhoff et al., 2003). Here, $\mathbf{I}_m$ is the identity matrix of order m for every m.

Lemma 1 *Let $d \in \{0,\dots,S\}$. The uniform design $\bar{\xi}_d$ on the set $\mathcal{X}_d^{(S)}$ of comparison depth d has diagonal information matrix*

$$\mathbf{M}(\bar{\xi}_d) = \begin{pmatrix} h_1(d)\mathbf{I}_{p_1} \otimes \mathbf{M} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & h_2(d)\mathbf{I}_{p_2} \otimes \mathbf{M} \otimes \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & h_3(d)\mathbf{I}_{p_3} \otimes \mathbf{M} \otimes \mathbf{M} \otimes \mathbf{M} \end{pmatrix}$$

where

$$\begin{aligned} h_1(d) &= \frac{d}{K},\ h_2(d) = \frac{d}{2vK(K-1)}(2Sv - 2S - dv - v + 2) \text{ and} \\ h_3(d) &= \frac{d}{4v^2K(K-1)(K-2)}(3S^2 + 3S^2v^2 - 6S^2v - 3Sdv^2 \\ &\quad + 3Sdv - 6Sv^2 + 15Sv - 9S + d^2v^2 \\ &\quad + 3dv^2 - 6dv + 2v^2 - 6v + 6). \end{aligned}$$

Proof. of Lemma 1. The quantities $h_1(d)$ and $h_2(d)$ are identical to the terms in Graßhoff et al. (2003). For $h_3(d)$ we proceed by first noting that the auxiliary terms $\sum_{i=1}^{v} \mathbf{f}(i)\mathbf{f}(i)^{\mathrm{T}} = \frac{v-1}{2}\mathbf{M}$ and $\sum_{i\neq j} \mathbf{f}(i)\mathbf{f}(j)^{\mathrm{T}} = -\frac{v-1}{2}\mathbf{M}$.

For the three-attribute interactions, we consider attributes k, ℓ and m, say, and distinguish between pairs in which all three attributes are distinct, pairs in which two of these attributes k and ℓ, say, have distinct levels in the alternatives, while the same level is presented in both alternatives for the remaining attribute and, finally, pairs in which only one of the attributes, say, k has distinct levels in the alternatives, while the same level is presented in both alternatives for the two remaining attributes:

$$\sum_{i_k \neq j_k} \sum_{i_\ell \neq j_\ell} \sum_{i_m \neq j_m} (\mathbf{f}(i_k) \otimes \mathbf{f}(i_\ell) \otimes \mathbf{f}(i_m) - \mathbf{f}(j_k) \otimes \mathbf{f}(j_\ell) \otimes \mathbf{f}(j_m))$$

$$\cdot (\mathbf{f}(i_k) \otimes \mathbf{f}(i_\ell) \otimes \mathbf{f}(i_m) - \mathbf{f}(j_k) \otimes \mathbf{f}(j_\ell) \otimes \mathbf{f}(j_m))^{\mathrm{T}}$$

$$= 2(v-1)^3 \sum_{i_k=1}^{v} \mathbf{f}(i_k)\mathbf{f}(i_k)^{\mathrm{T}} \otimes \sum_{i_\ell=1}^{v} \mathbf{f}(i_\ell)\mathbf{f}(i_\ell)^{\mathrm{T}} \otimes \sum_{i_m=1}^{v} \mathbf{f}(i_m)\mathbf{f}(i_m)^{\mathrm{T}}$$
$$- 2 \sum_{i_k \neq j_k} \mathbf{f}(i_k)\mathbf{f}(j_k)^{\mathrm{T}} \otimes \sum_{i_\ell \neq j_\ell} \mathbf{f}(i_\ell)\mathbf{f}(j_\ell)^{\mathrm{T}} \otimes \sum_{i_m \neq j_m} \mathbf{f}(i_m)\mathbf{f}(j_m)^{\mathrm{T}}$$

$$= \frac{1}{4} v(v-1)^3 (v^2 - 3v + 3)\mathbf{M} \otimes \mathbf{M} \otimes \mathbf{M}, \tag{10}$$

also

$$\sum_{i_k \neq j_k} \sum_{i_\ell \neq j_\ell} \sum_{i_m = j_m} (\mathbf{f}(i_k) \otimes \mathbf{f}(i_\ell) \otimes \mathbf{f}(i_m) - \mathbf{f}(j_k) \otimes \mathbf{f}(j_\ell) \otimes \mathbf{f}(j_m))$$

$$\cdot (\mathbf{f}(i_k) \otimes \mathbf{f}(i_\ell) \otimes \mathbf{f}(i_m) - \mathbf{f}(j_k) \otimes \mathbf{f}(j_\ell) \otimes \mathbf{f}(j_m))^{\mathrm{T}}$$

$$= 2(v-1)^2 \sum_{i_k=1}^{v} \mathbf{f}(i_k)\mathbf{f}(i_k)^{\mathrm{T}} \otimes \sum_{i_\ell=1}^{v} \mathbf{f}(i_\ell)\mathbf{f}(i_\ell)^{\mathrm{T}} \otimes \sum_{i_m=1}^{v} \mathbf{f}(i_m)\mathbf{f}(i_m)^{\mathrm{T}}$$

$$- 2 \sum_{i_k \neq j_k} \mathbf{f}(i_k)\mathbf{f}(j_k)^{\mathrm{T}} \otimes \sum_{i_\ell \neq j_\ell} \mathbf{f}(i_\ell)\mathbf{f}(j_\ell)^{\mathrm{T}} \otimes \sum_{i_m = j_m} \mathbf{f}(i_m)\mathbf{f}(j_m)^{\mathrm{T}}$$

$$= \frac{1}{4} v(v-1)^3 (v-2)\mathbf{M} \otimes \mathbf{M} \otimes \mathbf{M}, \tag{11}$$

and

$$\sum_{i_k \neq j_k} \sum_{i_\ell = j_\ell} \sum_{i_m = j_m} (\mathbf{f}(i_k) \otimes \mathbf{f}(i_\ell) \otimes \mathbf{f}(i_m) - \mathbf{f}(j_k) \otimes \mathbf{f}(j_\ell) \otimes \mathbf{f}(j_m))$$

$$\cdot (\mathbf{f}(i_k) \otimes \mathbf{f}(i_\ell) \otimes \mathbf{f}(i_m) - \mathbf{f}(j_k) \otimes \mathbf{f}(j_\ell) \otimes \mathbf{f}(j_m))^{\mathrm{T}}$$

$$= 2(v-1) \sum_{i_k=1}^{v} \mathbf{f}(i_k)\mathbf{f}(i_k)^{\mathrm{T}} \otimes \sum_{i_\ell=1}^{v} \mathbf{f}(i_\ell)\mathbf{f}(i_\ell)^{\mathrm{T}} \otimes \sum_{i_m=1}^{v} \mathbf{f}(i_m)\mathbf{f}(i_m)^{\mathrm{T}}$$
$$- 2 \sum_{i_k \neq j_k} \mathbf{f}(i_k)\mathbf{f}(j_k)^{\mathrm{T}} \otimes \sum_{i_\ell = j_\ell} \mathbf{f}(i_\ell)\mathbf{f}(j_\ell)^{\mathrm{T}} \otimes \sum_{i_m = j_m} \mathbf{f}(i_m)\mathbf{f}(j_m)^{\mathrm{T}}$$

$$= \frac{1}{4} v(v-1)^3 \mathbf{M} \otimes \mathbf{M} \otimes \mathbf{M}, \tag{12}$$

respectively.

Now for the given attributes, k, ℓ, and m, the pairs with distinct levels in the three attributes occur $\binom{K-3}{S-3}\binom{S-3}{d-3} v^{S-3}(v-1)^{d-3}$ times in $\mathcal{X}_d^{(S)}$, while those which differ in two attributes occur $\binom{3}{2}\binom{K-3}{S-3}\binom{S-3}{d-2} v^{S-3}(v-1)^{d-2}$ times in $\mathcal{X}_d^{(S)}$. Finally, those which differ only in one attribute occur $\binom{3}{1}\binom{K-3}{S-3}\binom{S-3}{d-1} v^{S-3}(v-1)^{d-1}$ times in $\mathcal{X}_d^{(S)}$. As a consequence, the diagonal elements for the three-attribute interactions are given by

$$\frac{1}{N_d}\Big(\frac{1}{4}\binom{K-3}{S-3}\binom{S-3}{d-3} v^{S-2}(v-1)^d (v^2-3v+3)\mathbf{M} \otimes \mathbf{M} \otimes \mathbf{M}$$
$$+ \frac{3}{4}\binom{K-3}{S-3}\binom{S-3}{d-2} v^{S-2}(v-1)^{d+1}(v-2)\mathbf{M} \otimes \mathbf{M} \otimes \mathbf{M}$$
$$+ \frac{3}{4}\binom{K-3}{S-3}\binom{S-3}{d-1} v^{S-2}(v-1)^{d+2}(v-1)^3\mathbf{M} \otimes \mathbf{M} \otimes \mathbf{M}\Big)$$
$$= \frac{d}{4v^2K(K-1)(K-2)}(3S^2 + 3S^2v^2 - 6S^2v - 3Sdv^2 + 3Sdv - 6Sv^2$$
$$+ 15Sv - 9S + d^2v^2 + 3dv^2 - 6dv + 2v^2 - 6v + 6)\mathbf{M} \otimes \mathbf{M} \otimes \mathbf{M},$$

in the information matrix.

Note that for $d = 0$, all pairs have identical attributes $(\mathbf{i} = \mathbf{j})$, $h_r(0) = 0$ for $r = 1, 2, 3$ and the information is zero. Hence, the comparison depth $d = 0$ can be neglected. Further, invariant designs $\bar{\xi}$ can be written as a convex combination of uniform designs on the comparison depths d with positive weights $w_d \geq 0$, $\sum_{d=1}^{S} w_d = 1$. In this case, the information matrix of the corresponding invariant design $\bar{\xi}$ also has the following diagonal structure:

Lemma 2 *Let $\bar{\xi}$ be an invariant design on $\mathcal{X}^{(S)}$. Then, $\bar{\xi}$ has diagonal information matrix*

$$\mathbf{M}(\bar{\xi}) = \begin{pmatrix} h_1(\bar{\xi})\mathbf{I}_{p_1} \otimes \mathbf{M} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & h_2(\bar{\xi})\mathbf{I}_{p_2} \otimes \mathbf{M} \otimes \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & h_3(\bar{\xi})\mathbf{I}_{p_3} \otimes \mathbf{M} \otimes \mathbf{M} \otimes \mathbf{M} \end{pmatrix}$$

where $h_r(\bar{\xi}) = \sum_{d=1}^{S} w_d h_r(d)$, $r = 1, 2, 3$.

We now consider optimal designs for the three-attribute interaction terms in the corresponding information matrix $\mathbf{M}(\bar{\xi}_d)$ by maximizing the diagonal entries $h_3(d)$. Results for main effects and two-attribute interactions can be found (see Graßhoff et al., 2003). However, the result obtained for the three-attribute interactions is novel. The resulting designs optimize every invariant design criterion including D_A-optimality (see Nyarko & Schwabe, 2019). Notice that D-optimality, which is discussed in (Kiefer, 1959, Section 4), refers to minimizing the determinant $\det \mathbf{M}(\bar{\xi}_d)^{-1}$ of the parameter estimates.

Table 1. Values of the optimal comparison depths d_2^* of the D-optimal uniform designs $\bar{\xi}_{d_2^*}$ for the three-attribute interactions in the case of full profiles ($S = K$) and v-levels

	v									
K	2	3	4	5	6	7	8	9	10	20
3	1	1	1	1	1	1	1	1	1	1
4	4	1	2	2	2	2	2	2	2	2
5	5	2	2	3	3	3	3	3	3	3
6	6	6	3	3	3	4	4	4	4	4
7	7	7	7	4	4	4	4	5	5	5
8	8	8	8	5	5	5	5	5	6	6
9	9	9	9	9	6	6	6	6	6	7
10	10	10	10	10	6	7	7	7	7	8

Source: The author.

The numerical results presented in Table 1 mean that those pairs of alternatives should be used which differ in the comparison depth d_2^* subject to the profile strength S. It is worth pointing out that using a single comparison depth makes it possible to identify or estimate all the parameters of the corresponding three-attribute interaction model according to some criterion like the D-criterion. According to the Kiefer-Wolfowitz equivalence theorem (Kiefer & Wolfowitz, 1960), a design ξ^* is D-optimal if the variance function defined as $V((\mathbf{i},\mathbf{j}),\xi) = (\mathbf{f}(\mathbf{i}) - \mathbf{f}(\mathbf{j}))^{\mathrm{T}}\mathbf{M}(\xi)^{-1}(\mathbf{f}(\mathbf{i}) - \mathbf{f}(\mathbf{j}))$ is bounded by the number of model parameters $p = p_1 + p_2 + p_3$, $V((\mathbf{i},\mathbf{j}),\xi^*) \le p$. This variance function plays an important role in using the D-criterion.

In Table 1, we note that the values of d_2^* were obtained by first calculating the values of $h_3(d)$ and determining the maximum. It is worthwhile mentioning that for very moderate values of v ($v = 2$, for example) the optimal comparison depth $d_2^* = S$ but this is not true for the case when $S = K = 3$. Moreover, for sufficiently large values of v ($v = 20$, for example) the optimal comparison depth $d_2^* = S - 2$.

Theorem 1 (a) For $S = 3$ the uniform design $\bar{\xi}_1$ is D-optimal for the vector of three-attribute interaction effects $(\beta_{k\ell m})^{\mathrm{T}}_{k<\ell<m}$.

(b) *For $S \ge 4$ the uniform design $\bar{\xi}_{d_2^*}$ is D-optimal for the vector of three-attribute interaction effects $(\beta_{k\ell m})^{\mathrm{T}}_{k<\ell<m}$.*

Proof of Theorem 1. (a) Optimality is achieved when h_3 is maximized. For $S = 3$ we get $h_3(1) = (v^2 - 2v + 1)/4v^2 p_3$ which establishes the result in this case.

(b) For $S \ge 4$, note that h_3 is a polynomial of degree 3 in the comparison depth d with a positive leading coefficient. If we extend h_3 to a function defined on the real line, then it is point symmetric with respect to $((Sv - S - v + 2)/v, h_3((Sv - S - v + 2)/v))$ and attains its local minimum at $d_{3,\min} = (Sv - S - v + 2)/v + \sqrt{9Sv + 3v^2 - 9S - 18v + 18}/(3v)$. The result of $h_3(d_{3,\min})$ is equal to $c(3Sv^2 - 3Sv - 3S - d_{3,\min}v^2 - 3v^2 + 6v)$ for some suitable positive constant c depending on both K and S. Inserting the solution for $d_{3,\min}$ into the last factor yields $3Sv^2 - 3Sv - 3S - d_{3,\min}v^2 - 3v^2 + 6v = 2Sv^2 - 2Sv - 3S - v\sqrt{Sv + v^2/3 - S - 2v + 2} - 2v^2 + 4v > 0$ for $S \ge 4$ and $v \ge 2$. Hence, $h_3(d_{3,\max}) > 0$ and, by symmetry, the value of h_3 at the local maximum $d_{3,\max} = (Sv - S - v + 2)/v - \sqrt{9Sv + 3v^2 - 9S - 18v + 18}/(3v)$ is equal to $h_3(d_{3,\max}) < h_3(S)$ which proves that the global maximum of h_3 is attained at $d \le S$ for $0 \le d \le S$.

For invariant designs $\bar{\xi}$, the value of the variance function evaluated at comparison depth d may be denoted by $V(d,\bar{\xi})$, say, where $V(d,\bar{\xi}) = V((\mathbf{i},\mathbf{j}),\bar{\xi})$ on $\mathcal{X}_d^{(S)}$. It can be shown that in this case, the variance function $V(d,\bar{\xi})$ of the invariant design $\bar{\xi}$ has the following structure.

Theorem 2 For every invariant design $\bar{\xi}$, the variance function $V(d,\bar{\xi})$ is given by

$$V(d,\bar{\xi}) = d(v-1)\left(\frac{1}{h_1(\bar{\xi})} + \frac{v-1}{4vh_2(\bar{\xi})}(2Sv - 2S - dv - v + 2) + \frac{(v-1)^2}{24v^2 h_3(\bar{\xi})}\lambda(d)\right),$$

where

$$\lambda(d) = 3S^2 + 3S^2v^2 - 6S^2v - 3Sdv^2 + 3Sdv - 6Sv^2 + 15Sv - 9S + d^2v^2 + 3dv^2 - 6dv + 2v^2 - 6v + 6.$$

Proof of Theorem 2. First, we note that

$$\mathbf{M}(\bar{\xi})^{-1} = \begin{pmatrix} \frac{1}{h_1(\bar{\xi})}\mathbf{I}_{p_1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{1}{h_2(\bar{\xi})}\mathbf{I}_{p_2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{1}{h_3(\bar{\xi})}\mathbf{I}_{p_3} \end{pmatrix},$$

for the inverse of the information matrix of the design $\bar{\xi}$.

Now, by Lemma 2 in Graßhoff et al. (2003), it is sufficient to note that for the k-th main effects, the variance function is given by

$$(\mathbf{f}(i_k) - \mathbf{f}(j_k))^{\mathrm{T}}\mathbf{M}^{-1}(\mathbf{f}(i_k) - \mathbf{f}(j_k)) = v - 1. \quad (13)$$

Further, for the regression function associated with the two-attribute interactions of the attributes k and ℓ, say, we obtain

$$(\mathbf{f}(i_k)\otimes\mathbf{f}(i_\ell)-\mathbf{f}(j_k)\otimes\mathbf{f}(j_\ell))^{\mathrm{T}}\mathbf{M}^{-1}\otimes\mathbf{M}^{-1}(\mathbf{f}(i_k)\otimes\mathbf{f}(i_\ell)-\mathbf{f}(j_k)\otimes\mathbf{f}(j_\ell))$$

$$=\mathbf{f}(i_k)^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{f}(i_k)\cdot\mathbf{f}(i_\ell)^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{f}(i_\ell)+\mathbf{f}(j_k)^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{f}(j_k)\cdot\mathbf{f}(j_\ell)^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{f}(j_\ell)$$

$$-\mathbf{f}(i_k)^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{f}(j_k)\cdot\mathbf{f}(i_\ell)^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{f}(j_\ell)-\mathbf{f}(j_k)^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{f}(i_k)\cdot\mathbf{f}(j_\ell)^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{f}(i_\ell)$$

$$=\begin{cases}\frac{(v-1)^2(v-2)}{2v} & \text{for } i_k\neq j_k, i_\ell\neq j_\ell\\ \frac{(v-1)^3}{2v} & \text{for } i_k\neq j_k, i_\ell=j_\ell \text{ or } i_k=j_k, i_\ell\neq j_\ell.\end{cases}\tag{14}$$

Accordingly, for the regression function associated with the interaction of the attributes k, ℓ and m, say, we obtain

$$(\mathbf{f}(i_k)\otimes\mathbf{f}(i_\ell)\otimes\mathbf{f}(i_m)-\mathbf{f}(j_k)\otimes\mathbf{f}(j_\ell)\otimes\mathbf{f}(j_m))^{\mathrm{T}}\mathbf{M}^{-1}\otimes\mathbf{M}^{-1}\otimes\mathbf{M}^{-1}$$

$$\cdot(\mathbf{f}(i_k)\otimes\mathbf{f}(i_\ell)\otimes\mathbf{f}(i_m)-\mathbf{f}(j_k)\otimes\mathbf{f}(j_\ell)\otimes\mathbf{f}(j_m))$$

$$=\mathbf{f}(i_k)^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{f}(i_k)\cdot\mathbf{f}(i_\ell)^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{f}(i_\ell)\cdot\mathbf{f}(i_m)^{\mathrm{T}}\mathbf{M}^{-1}f(i_m)$$

$$+\mathbf{f}(j_k)^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{f}(j_k)\cdot\mathbf{f}(j_\ell)^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{f}(j_\ell)\cdot\mathbf{f}(j_m)^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{f}(j_m)$$

$$-\mathbf{f}(i_k)^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{f}(j_k)\cdot\mathbf{f}(i_\ell)^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{f}(j_\ell)\cdot\mathbf{f}(i_m)^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{f}(j_m)$$

$$-\mathbf{f}(j_k)^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{f}(i_k)\cdot\mathbf{f}(j_\ell)^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{f}(i_\ell)\cdot\mathbf{f}(j_m)^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{f}(i_m)$$

$$=\begin{cases}\frac{(v-1)^3(v^2-3v+3)}{4v^2} & \text{for } i_k\neq j_k, i_\ell\neq j_\ell, i_m\neq j_m\\ \frac{(v-1)^4(v-2)}{4v^2} & \text{for } i_k\neq j_k, i_\ell\neq j_\ell, i_m=j_m\\ \frac{(v-1)^5}{4v^2} & \text{for } i_k\neq j_k, i_\ell=j_\ell, i_m=j_m.\end{cases}\tag{15}$$

Now for a pair of alternatives $(\mathbf{i},\mathbf{j})\in\mathcal{X}_d^{(S)}$ of comparison depth d: there are exactly d attributes of the main effects for which i_k and j_k differ, there are $\frac{1}{2}d(d-1)$ two-attribute interaction terms for which $(i_k i_\ell)$ and $(j_k j_\ell)$ differ in all two attributes k and ℓ, there are $d(S-d)$ two-attribute interaction terms for which $(i_k i_\ell)$ and $(j_k j_\ell)$ differ in exactly one attribute k or ℓ, there are $\frac{1}{6}d(d-1)(d-2)$ three-attribute interaction terms for which $(i_k i_\ell i_m)$ and $(j_k j_\ell j_m)$ differ in all three attributes k, ℓ and m, there are $\frac{1}{2}(S-d)d(d-1)$ three-attribute interaction terms for which $(i_k i_\ell i_m)$ and $(j_k j_\ell j_m)$ differ in exactly two of the associated three attributes and finally, there are $\frac{1}{2}(S-d)(S-d-1)d$ three-attribute interaction terms for which $(i_k i_\ell i_m)$ and $(j_k j_\ell j_m)$ differ in exactly one of the associated three attributes. As a consequence, we obtain

$$\begin{aligned}V(d,\bar{\xi})&=(\mathbf{f}(\mathbf{i})-\mathbf{f}(\mathbf{j}))^{\mathrm{T}}\mathbf{M}(\bar{\xi})^{-1}(\mathbf{f}(\mathbf{i})-\mathbf{f}(\mathbf{j}))\\ &=\frac{d(v-1)}{h_1(\bar{\xi})}+\frac{d(d-1)}{2}\frac{(v-1)^2(v-2)}{2vh_2(\bar{\xi})}\\ &\quad+d(S-d)\frac{(v-1)^3}{2vh_2(\bar{\xi})}\\ &\quad+\frac{d(d-1)(d-2)}{6}\frac{(v-1)^3(v^2-3v+3)}{4v^2h_3(\bar{\xi})}\\ &\quad+\frac{(S-d)d(d-1)}{2}\frac{(v-1)^4(v-2)}{4v^2h_3(\bar{\xi})}\\ &\quad+\frac{(S-d)(S-d-1)d}{2}\frac{(v-1)^5}{4v^2h_3(\bar{\xi})}\\ &=\frac{d(v-1)}{h_1(\bar{\xi})}+\frac{d(v-1)^2}{4vh_2(\bar{\xi})}(2Sv-2S-dv-v+2)\\ &\quad+\frac{d(v-1)^3}{24v^2h_3(\bar{\xi})}(3S^2v^2-6S^2v-6Sv^2+3S^2-3Sdv^2+3Sdv\\ &\qquad+3dv^2+15Sv-9S+d^2v^2-6dv\\ &\qquad+2v^2-6v+6),\end{aligned}$$

for $(\mathbf{i},\mathbf{j})\in\mathcal{X}_d^{(S)}$ which proves the proposed formula.

In the case of a single comparison depth, it can be shown that the corresponding invariant design $\bar{\xi}$ has the following structure. It is worth noting that if $d=d'$, then by the Kiefer-Wolfowitz equivalence theorem (Kiefer & Wolfowitz, 1960) the variance function $V(d,\bar{\xi}_{d'})$ will be equal to the total number of the model parameters, $p=p_1+p_2+p_3$.

Corollary 1 *For a uniform design $\bar{\xi}_{d'}$ on a single comparison depth d' the variance function is given by*

$$V(d,\bar{\xi}_{d'})=\frac{d}{d'}\left(p_1+p_2\frac{2Sv-2S-dv-v+2}{2Sv-2S-d'v-v+2}+p_3\frac{\lambda(d)}{\lambda(d')}\right).$$

The following result gives an upper bound on the number of comparison depths required for a D-optimal design.

Theorem 3 The D-optimal design $\bar{\xi}$ for the three-attribute interaction model is supported on, at most, three different comparison depths S, d^*, and d^*+1.

Proof of Theorem 3. According to a corollary of Kiefer-Wolfowitz equivalence theorem (Kiefer & Wolfowitz, 1960, p. 364) for the D-optimal design

$\bar{\xi}^*$, the variance function $V(d, \bar{\xi}^*)$ is equal to the number of parameters p for all D. By Theorem 2, the variance function is a cubic polynomial in the comparison depth D with a positive leading coefficient. The variance function $V(d, \bar{\xi}^*)$ may thus be equal to p for, at most, three different values $d_1 < d_2 < d_3$ of D, say. Now, by the Kiefer-Wolfowitz equivalence theorem (Kiefer & Wolfowitz, 1960) itself $V(d, \bar{\xi}^*) \leq p$ for all $d = 0, 1, \ldots, S$.

We now make use of the analytic tool of the Kiefer-Wolfowitz equivalence theorem to establish the D-optimality of the design $\bar{\xi}^*$ by direct maximization of $\ln(\det(\mathbf{M}(w^*_{d^*}\bar{\xi}^*{}_{d^*} + (1 - w^*_{d^*})\bar{\xi}_S)))$ (e.g. see Kiefer & Wolfowitz, 1960).

By Lemma 1 the entries of the information matrix $\mathbf{M}(\bar{\xi}^*, w^*_S)$ are specified by

$$h_1(\bar{\xi}^*, w^*_S) = w^*_S h_1(S) + (1 - w^*_S)h_1(d^*) = \frac{Sv - S + Sw^*_S + 2w^*_S v - 3w^*_S - 2v + 3}{Kv},$$

$$h_2(\bar{\xi}^*, w^*_S) = w^*_S h_2(K) + (1 - w^*_S)h_2(d^*) = \frac{\lambda_1}{2K(K-1)v^2},$$

where

$$\lambda_1 = S^2v^2 - 2S^2v - Sv^2 - S^2w^*_S - Sw^*_S v + 2w^*_S v^2 + S^2 + 3Sv + 2Sw^*_S - 2v^2 - 5w^*_S v - 2S + 5v + 3w^*_S - 3,$$

and

$$h_3(\bar{\xi}^*, w^*_S) = w^*_S h_3(S) + (1 - w^*_S)h_3(d^*) = \frac{\lambda_2}{4K(K-1)(K-2)v^3},$$

$$\lambda_2 = S^3v^3 - 3S^3v^2 - 3S^2v^3 + 3S^3v + S^3w^*_S + 9S^2v^2 + 2Sv^3 - 4Sw^*_S v^2 - S^3 - 9S^2v - 3S^2w^*_S - 2Sv^2 + 9Sw^*_S v + 6w^*_S v^2 + 3S^2 - 3Sv - 3Sw^*_S - 6v^2 - 15w^*_S v + 3S + 15v + 9w^*_S - 9.$$

Now, since the determinant of the information matrix $M(\bar{\xi}^*, w^*_S)$ is proportional to $h_1(\bar{\xi}^*, w^*_S)^{p_1} h_2(\bar{\xi}^*, w^*_S)^{p_2} h_3(\bar{\xi}^*, w^*_S)^{p_3}$, we thus obtain

$$\ln\det(\mathbf{M}(\bar{\xi}^*, w^*_S)) = c + K(v-1)\cdot\ln(Sv - S + Sw^*_S + 2w^*_S v - 3w^*_S - 2v + 3) + \frac{K(K-1)(v-1)^2\cdot\ln\lambda_1}{2} + \frac{K(K-1)(K-2)(v-1)^3\cdot\ln\lambda_2}{6},$$

where c is a constant independent of the weight w^*_S. Taking derivatives with respect to w^*_S, we obtain

$$\frac{\partial}{\partial w^*_S}\ln\det(\mathbf{M}(\bar{\xi}^*, w^*_S)) = K(v-1)\cdot\left(\frac{S + 2v - 3}{Sv - S + Sw^*_S + 2w^*_S v - 3w^*_S - 2v + 3}\right) + \frac{K(K-1)(v-1)^2}{2\lambda_1}\cdot(-S^2 - Sv + 2S + 2v^2 - 5v + 3) + \frac{K(K-1)(K-2)(v-1)^3}{6\lambda_2}\cdot(S^3 - 4Sv^2 - 3S^2 - 3S + 9Sv + 6v^2 - 15v + 9)$$

which has root $w^*_S = 1 - w^*_{d^*}$. This root gives a maximum for the determinant. The design $\bar{\xi}^*$ is thus D-optimal when we consider the particular case of the reduced design region $\mathcal{X}_S \cup \mathcal{X}_{d^*}$.

Again, by inserting the corresponding functions $h_1(\bar{\xi}^*, w^*_S), h_2(\bar{\xi}^*, w^*_S)$ and $h_3(\bar{\xi}^*, w^*_S)$ into the representation of the variance function $V(S, d, \bar{\xi}^*, w^*_S)$ in Theorem 2, we obtain

$$V(S, d^*, \bar{\xi}^*, w^*_S) = V(S, d^*, d^* + 1, \bar{\xi}^*, w^*_S) \leq p.$$

Hence, S, d^* and $d^* + 1$ are integer solutions for the maximum of the variance function, which shows the D-optimality of the design because of the equivalence theorem by Kiefer and Wolfowitz (1960).

For the results on parts of the parameter vector involving main effects up to three-attribute interaction, the D-optimal design for the complete parameter vector may depend on the profile strength S as can be seen by the numerical examples for the case of arbitrary levels, $v \geq 2$ presented in Table 2. We note that for the case

Table 2. Optimal designs with intermediate comparison depths d^* in boldface and optimal weights $w^*_{d^*}$ of the form $(d^*, w^*_{d^*})$ for the case of full profiles ($S = K$) and v-levels

	v						
K	2	3	4	5	6	7	8
4	(**2**, 0.857)	**2**	**2**	**2**	**2**	**2**	**2**
5	(**2**, 0.833)	(**2**, 0.667)	**3**	**3**	**3**	**3**	**3**
6	(**3**, 0.732)	(**3**, 0.789)	**3**	**4**	**4**	**4**	**4**
7	(**3**, 0.697)	(**4**, 0.322)	**4**	**4**	**4**	**5**	**5**
8	(**3**, 0.644)	**4**	(**5**, 0.425)	**5**	**5**	**5**	**5**
9	(**4**, 0.577)	**5**	**5**	**6**	**6**	**6**	**6**
10	(**4**, 0.538)	**5**	**6**	**6**	**7**	**7**	**7**

Source: The author.

$v = 2$, $S = K = 3$ of full profiles and complete interactions, the D-optimal design given explicitly in Nyarko and Schwabe (2019) indicates that all three comparison depths are needed for D-optimality.

For $S \geq 4$, numerical computations indicate that at most two different comparison depths S and d^* may be required for D-optimality. Because it is generally difficult to specify an explicit formula for calculating d^*, the following Table 2 shows the corresponding optimal designs with their optimal comparison depths d^* in boldface and their corresponding weights $w^*_{d^*}$ for various choices of attributes K between 4 and 10 and levels $v = 2, \ldots, 8$. Entries of the form $(d^*, w^*_{d^*})$ indicate that invariant designs $\bar{\xi}^* = w^*_{d^*} \bar{\xi}^*{}_{d^*} + (1 - w^*_{d^*}) \bar{\xi}_S$ have to be

Table 3. Values of the variance function $V(d, \bar{\xi}^*)$ for $\bar{\xi}^*$ from Table 2 in the case of full profiles ($S = K$) and v-levels (boldface 1 corresponds to the optimal comparison depths d^*)

		d									
K	v	1	2	3	4	5	6	7	8	9	10
4	2	0.875	**1**	0.875	**1**						
	3	0.813	**1**	0.938	1						
	4	0.793	**1**	0.953	0.983						
	5	0.783	**1**	0.962	0.980						
	6	0.777	**1**	0.968	0.980						
	7	0.773	**1**	0.973	0.981						
	8	0.770	**1**	0.976	0.982						
5	2	0.760	**1**	0.960	0.880	**1**					
	3	0.723	**1**	1	0.954	**1**					
	4	0.689	0.967	**1**	0.952	0.987					
	5	0.666	0.951	**1**	0.961	0.981					
	6	0.653	0.941	**1**	0.968	0.980					
	7	0.644	0.934	**1**	0.972	0.981					
	8	0.638	0.929	**1**	0.976	0.982					
6	2	0.701	0.983	**1**	0.906	0.855	**1**				
	3	0.624	0.921	**1**	0.968	0.932	**1**				
	4	0.591	0.895	**1**	0.993	0.963	0.997				
	5	0.576	0.882	0.997	**1**	0.972	0.992				
	6	0.560	0.865	0.987	**1**	0.976	0.989				
	7	0.550	0.854	0.981	**1**	0.979	0.988				
	8	0.543	0.846	0.977	**1**	0.982	0.988				
7	2	0.615	0.917	**1**	0.956	0.879	0.863	**1**			
	3	0.553	0.860	0.988	**1**	0.963	0.941	**1**			
	4	0.519	0.822	0.965	**1**	0.981	0.962	0.997			
	5	0.498	0.800	0.952	**1**	0.992	0.974	0.993			
	6	0.487	0.787	0.944	**1**	0.999	0.983	0.995			
	7	0.479	0.777	0.937	0.997	**1**	0.985	0.994			
	8	0.471	0.768	0.929	0.994	**1**	0.987	0.993			
8	2	0.559	0.872	**1**	1	0.945	0.884	0.884	**1**		
	3	0.490	0.792	0.948	**1**	0.990	0.958	0.948	1		
	4	0.462	0.759	0.924	0.993	**1**	0.980	0.969	**1**		
	5	0.442	0.732	0.902	0.981	**1**	0.988	0.977	0.995		
	6	0.429	0.716	0.889	0.974	**1**	0.994	0.982	0.994		
	7	0.421	0.706	0.880	0.970	**1**	0.997	0.987	0.995		
	8	0.415	0.698	0.874	0.960	**1**	1	0.991	0.996		
9	2	0.504	0.811	0.962	**1**	0.969	0.910	0.868	0.883	**1**	
	3	0.437	0.726	0.894	0.972	**1**	0.969	0.946	0.946	1	
	4	0.414	0.696	0.872	0.965	**1**	0.994	0.977	0.971	1	
	5	0.397	0.674	0.853	0.953	0.995	**1**	0.989	0.981	1	
	6	0.384	0.657	0.836	0.940	0.989	**1**	0.992	0.985	0.996	
	7	0.376	0.645	0.825	0.932	0.985	**1**	0.995	0.988	0.995	
	8	0.370	0.637	0.817	0.927	0.982	**1**	0.997	0.990	0.996	
10	2	0.462	0.763	0.932	**1**	0.997	0.956	0.905	0.874	0.896	**1**
	3	0.395	0.669	0.843	0.938	**1**	0.972	0.953	0.938	0.947	1
	4	0.374	0.642	0.822	0.929	0.981	**1**	0.987	0.974	0.972	1
	5	0.359	0.622	0.803	0.917	0.977	**1**	1	0.989	0.985	1
	6	0.348	0.606	0.786	0.903	0.968	0.996	**1**	0.993	0.988	0.998
	7	0.340	0.594	0.774	0.892	0.961	0.993	**1**	0.995	0.990	0.997
	8	0.335	0.586	0.765	0.885	0.956	0.990	**1**	0.996	0.991	0.996

Source: author.

As was already pointed out, the corresponding designs possess many comparisons. For the case of binary attributes ($v = 2$) the designs in Table 2 when $K = 4, 5$ and 6, for instance, consist of 96, 320 and 1280 pairs, respectively. In this situation, it is possible to construct a design for estimating main effects and two and three-attribute interactions. For example, if $K = 3$, $v = 2$ and $d^* = 1$, the paired comparison design consists of 24 pairs. In the next section, algorithms for generating such designs are presented. It is worth noting that the design contains repeated pairs, which is a result of a large number of comparisons. This design is used as an example to assess university students' satisfaction with online teaching and psychological pressure on learning during the COVID-19 pandemic later on.

considered, while for single entries d^* the optimal design $\bar{\xi}^* = \bar{\xi}^*_{d^*}$ has to be considered which is uniform on the optimal comparison depth d^*. In particular, for the case $S = K = 4$ of full profiles where $v = 2$ the corresponding invariant design depends on the optimal comparison depths $d^* = 2$ and $S = 4$ with optimal weights $w^*_{d^*} = 0.857$ and $w^*_{d^*} = 0.143$, respectively. The optimality of the designs presented in Table 2 has been checked numerically under the equivalence theorem. The values of the normalized variance function $V(d, \bar{\xi}^*)/p$ are recorded in Table 3, where maximal values less than or equal to 1 establish optimality.

5 Algorithmic design approach

This section provides a theoretical construction method for the corresponding optimal designs. The construction uses an algorithmic design approach (Großmann et al., 2012; Nyarko & Doku-Amponsah, 2022) to create an exact design with N pairs. Let $\xi_{N,d}$ be an exact design with attributes K that differ in d (so-called comparison depth), allowing estimation of main effects and two and three attribute interactions. The algorithm generates two $N \times K$ matrices $\mathbf{I}$ and $\mathbf{J}$ from the design region $\mathcal{X}_d$ with treatment combinations $\mathbf{i}_1, \ldots, \mathbf{i}_N$ and $\mathbf{j}_1, \ldots, \mathbf{j}_N$, respectively.

For given K and d, the method requires three building blocks involving an $m \times (K - d)$ matrix $\mathbf{F}$ which represents a regular two-level factorial design of resolution *III* or higher for orthogonal coded (i.e. ± 1) $K - d$ binary attributes, a Hadamard matrix $\mathbf{H}$ of order $t \geq d$, and a matrix $\mathbf{B}$ of dimension a $d \times b$, which represents a balanced incomplete block design for K treatments $k = 1, \ldots, K$ in b blocks of size d. These building blocks are used to construct designs $\xi_{N,d}$ with $N = bmt$ treatment combinations selected from $\mathcal{X}_d$. The following is an illustration of the construction:

Step 1: Let $\mathbf{A}$ be a $t \times d$ matrix obtained by selecting d columns from $\mathbf{H}$ and let $\mathbf{F}$ be an $m \times (K - d)$ matrix representing the regular factorial design.

Step 2: Combining the rows of $\mathbf{A}$ and $\mathbf{F}$ yields the $mt \times K$ matrix $\mathbf{I}$. The matrix $\mathbf{J}$ is obtained similarly by using $-\mathbf{A}$.

Step 3: Rearrange the columns of $\mathbf{I}$ and $\mathbf{J}$ based on a permutation derived from the first b blocks of $\mathbf{B}$. More specifically, the design's mt pairs are obtained by combining every row of the permuted matrix of $\mathbf{I}$ with the same row of the permuted matrix of $\mathbf{J}$ (see Großmann & Schwabe, 2015; Großmann et al., 2012). This procedure is repeated for each of the remaining b columns of $\mathbf{B}$. The final design has $N = bmt$ treatment combinations.

For illustrative purposes, we construct a design to compare products with $K = 3$ attributes, where the choice task to be presented to subjects differs in only one attribute ($d = 1$). In an experimental situation, suppose a researcher is interested in constructing a design $\xi_{N,d}$ with $N = 24$ pairs that differ in only one attribute to compare products with $K = 3$ attributes. For this situation, the pairs $(\mathbf{i}_n, \mathbf{j}_n)$ for $n = 1, \ldots, 24$ with orthogonal coded levels ± 1, where the orthogonal coded first and last level of each attribute are assigned with the actual levels 1 and 2, respectively (see Table 4). A Hadamard matrix of order $t = 1$, a regular 2^3 full factorial design, and an incomplete block design with blocks $\{1\}, \{2\}$, and $\{3\}$ were used to generate the design. The levels of attribute 1 in each alternative are determined by a column from the combined rows of the Hadamard matrix and the regular 2^3 full factorial design in the $1 - 8$ pairs. The levels of the attributes in columns 2 and 3 are determined by the corresponding column of the combined rows of the Hadamard matrix and the regular factorial design for pairs $9 - 16$ and $17 - 24$, respectively, while the levels of the other remaining attributes are the same in both alternatives and depend on the regular 2^3 full factorial design.

Table 4. Design for estimating main effects and two- and three-attribute interactions of $K = 3$ two-level attributes

n	Pair $(\mathbf{i}_n, \mathbf{j}_n)$
1	((1,1,1,1,1,1,1), (2,1,1,2,2,1,2))
2	((1,2,1,2,1,2,2), (2,2,1,1,2,2,1))
3	((1,1,1,1,1,1,1), (2,1,1,2,2,1,2))
4	((1,2,1,2,1,2,2), (2,2,1,1,2,2,1))
5	((1,1,2,1,2,2,2), (2,1,2,2,1,2,1))
6	((1,2,2,2,2,1,1), (2,2,2,1,1,1,2))
7	((1,1,2,1,2,2,2), (2,1,2,2,1,2,1))
8	((1,2,2,2,2,1,1), (2,2,2,1,1,1,2))
9	((1,1,1,1,1,1,1), (1,2,1,2,1,2,2))
10	((1,1,2,1,2,2,2), (1,2,2,2,2,1,1))
11	((1,1,1,1,1,1,1), (1,2,1,2,1,2,2))
12	((1,1,2,1,2,2,2), (1,2,2,2,2,1,1))
13	((2,1,1,2,2,1,2), (2,2,1,1,2,2,1))
14	((2,1,2,2,1,2,1), (2,2,2,1,1,1,2))
15	((2,1,1,2,2,1,2), (2,2,1,1,2,2,1))
16	((2,1,2,2,1,2,1), (2,2,2,1,1,1,2))
17	((1,1,1,1,1,1,1), (1,1,2,1,2,2,2))
18	((2,1,1,2,2,1,2), (2,1,2,2,1,2,1))
19	((1,1,1,1,1,1,1), (1,1,2,1,2,2,2))
20	((2,1,1,2,2,1,2), (2,1,2,2,1,2,1))
21	((1,2,1,2,1,2,2), (1,2,2,2,2,1,1))
22	((2,2,1,1,2,2,1), (2,2,2,1,1,1,2))
23	((1,2,1,2,1,2,2), (1,2,2,2,2,1,1))
24	((2,2,1,1,2,2,1), (2,2,2,1,1,1,2))

Source: author.

It is worth noting that for given values of K and d, and by selecting a regular two-level full factorial design $\mathbf{F}$ with m treatment combinations and a Hadamard matrix $\mathbf{H}$ of appropriate order d, similar designs can be constructed with $N = bmd$ pairs of the aforementioned type by performing a computer search (e.g. see Großmann & Schwabe, 2015) over appropriately selected balanced incomplete block design $\mathbf{B}$ with K treatments $k = 1, \ldots, K$ in b blocks of size d.

Source	LogWorth	PValue
Timey access to teaching schedule information	20.297	0.00000
Teaching methods and arrangements	12.104	0.00000
Teaching methods and arrangements*Instructors preparation*Timely access to teaching schedule information	4.945	0.00001
Teaching methods and arrangements*Instructors preparation	3.942	0.00011
Teaching methods and arrangements*Timely access to teaching schedule information	2.291	0.00511
Instructors' preparation	1.975	0.01059
Instructors' preparation*Timely access to teaching schedule information	0.201	0.62998

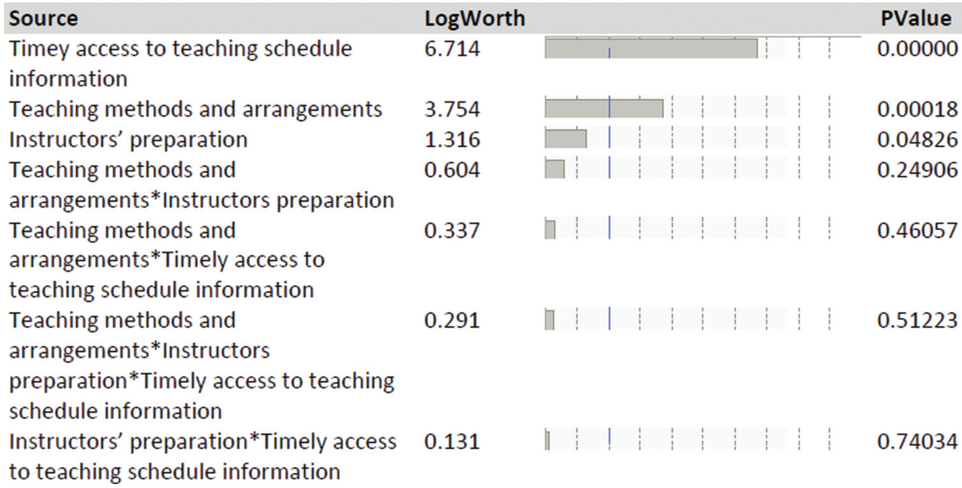
Figure 1. Effect summary of students' satisfaction with online teaching during the COVID-19 pandemic.

Source	LogWorth	PValue
Teaching methods and arrangements	3.547	0.00028
Teaching methods and arrangements*Instructors preparation*Timely access to teaching schedule information	3.042	0.00091
Timey access to teaching schedule information	2.184	0.00655
Teaching methods and arrangements*Timely access to teaching schedule information	1.726	0.01880
Teaching methods and arrangements*Instructors preparation	0.958	0.11020
Instructors' preparation	0.594	0.25495
Instructors' preparation*Timely access to teaching schedule information	0.452	0.35296

Figure 2. Effect summary of students experiencing little psychological pressure during COVID-19 pandemic.

Source	LogWorth	PValue
Timey access to teaching schedule information	13.083	0.00000
Teaching methods and arrangements	6.308	0.00000
Teaching methods and arrangements*Instructors preparation*Timely access to teaching schedule information	3.398	0.00040
Teaching methods and arrangements*Instructors preparation	3.282	0.00052
Teaching methods and arrangements*Timely access to teaching schedule information	1.226	0.05937
Instructors' preparation	0.787	0.16315
Instructors' preparation*Timely access to teaching schedule information	0.117	0.76428

Figure 3. Effect summary of students experiencing lots of psychological pressure during COVID-19 pandemic.



Source	LogWorth		PValue
Timey access to teaching schedule information	6.714		0.00000
Teaching methods and arrangements	3.754		0.00018
Instructors' preparation	1.316		0.04826
Teaching methods and arrangements*Instructors preparation	0.604		0.24906
Teaching methods and arrangements*Timely access to teaching schedule information	0.337		0.46057
Teaching methods and arrangements*Instructors preparation*Timely access to teaching schedule information	0.291		0.51223
Instructors' preparation*Timely access to teaching schedule information	0.131		0.74034

Figure 4. Effect summary of students experiencing no psychological pressure during COVID-19 pandemic.

6 Application

This section considers a practical situation where up to three-factor interaction is interesting. In particular, we employ the design presented in Table 4 to assess university students' satisfaction with online teaching and psychological pressure on learning during the COVID-19 pandemic. The JMP Pro (Version 16.0) statistical software was used to analyze the responses of 150 students of the University of Ghana who were intercepted on the university campus. The full sample results are presented (Figure 1). These results were further classified according to levels of psychological pressure (Figures 1–4). The Log-Worth and the corresponding P-values of the various factors (or attributes) are also reported in ascending order of importance. It can be observed that, in most cases, the three-factor interactions perform well and are important.

7 Discussion

For paired comparisons where the alternatives are described by an analysis of variance model with three-attribute interactions, optimal designs require that pairs should be considered in which either all attributes have distinct levels or approximately a portion of the attributes are distinct, and the remaining portion of the attributes coincides to obtain a D-optimal design for the whole parameter vector. Optimal designs may be concentrated on one, two, or three different comparison depths depending on the number of levels and attributes. The so obtained approximate and exact designs can serve as a benchmark to judge the efficiency of any competing design. A practical situation of interest is explored where the design identifies main effects and two- and three-attribute interactions.

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In modern scientific experiments, paired comparison experiments involving pairs of either full or partial profiles are frequently used. Typically, one may be interested in the main effects as well as interactions between the attributes. Accordingly, we introduce an appropriate model for full and partial profiles and derive optimal designs that allow estimating all the main effects plus two plus three attribute interaction effects of interest with real-life applications.

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References

Berkumvan, E. E. M. (1987). *Optimal Paired Comparison Designs for Factorial Experiments CWI Tract 31*. Centrum voor Wiskunde en Informatica.

Chrzan, K. (2010). Using partial profile choice experiments to handle large numbers of attributes. *International Journal of Market Research*, *52*, 827–840. https://doi.org/10.2501/S1470785310201673

deBekker Grob Ew, Ryan, M., Gerard, K., & de Bekker-Grob, E. W. (2012). Discrete choice experiments in health economics: A review of the literature. *Health Economics*, *21*(2), 145–172. https://doi.org/10.1002/hec.1697

Elrod, T., Louviere, J. J., & Davey, K. S. (1992). An empirical comparison of ratings-based and choice-based conjoint models. *Journal of Marketing Research*, *29*, 368–377. https://doi.org/10.1177/002224379202900307

Graßhoff, U., Großmann, H., Holling, H., & Schwabe, R. (2003). Optimal paired comparison designs for first-order interactions. *Statistics*, *37*, 373–386. https://doi.org/10.1080/0233188031000154812

Green, P. E. (1974). On the design of choice experiments involving multifactor alternatives. *The Journal of Consumer Research*, *1*, 61–68. https://doi.org/10.1086/208592

Green, P. E., Krieger, A. M., & Wind, Y. (2004). Thirty years of conjoint analysis: Reflections and prospects. In Wind Y, Green P, editors. *Marketing research and modeling: Progress and prospects* (pp. 117–139). Springer.

Großmann, H. (2017). Partial-profile choice designs for estimating main effects and interactions of two-level attributes from paired comparison data. *Journal of Statistical Theory and Practice*, *11*, 236–253. https://doi.org/10.1080/15598608.2016.1197868

Großmann, H., & Schwabe, R. (2015). Design for discrete choice experiments. In A. Dean, M. Morris, J. Stufken, & D. Bingham (Eds.), *Handbook of Design and Analysis of Experiments* (pp. 787–831). Chapman and Hall/CRC.

Großmann, H., Schwabe, R., & Gilmour, S. G. (2012). Designs for first-order interactions in paired comparison experiments with two-level factors. *Journal of Statistical Planning and Inference*, *142*, 2395–2401. https://doi.org/10.1016/j.jspi.2012.02.033

Hoyos, D. (2010). The state of the art of environmental valuation with discrete choice experiments. *Ecological Economics*, *69*, 1595–1603. https://doi.org/10.1016/j.ecolecon.2010.04.011

Jaynes, J., Wong, W. K., & Xu, H. (2016). Using blocked fractional factorial designs to construct discrete choice experiments for healthcare studies. *Statistics in Medicine*, *35*(15), 2543–2560. https://doi.org/10.1002/sim.6882

Kessels, R., Jones, B., & Goos, P. (2011). Bayesian optimal designs for discrete choice experiments with partial profiles. *Journal of Choice Modelling*, *4*, 52–74. https://doi.org/10.1016/S1755-5345(13)70042-3

Kiefer, J. (1959). Optimum experimental designs. *Journal of the Royal Statistical Society, Series B*, *21*(2), 272–304. https://doi.org/10.1111/j.2517-6161.1959.tb00338.x

Kiefer, J., & Wolfowitz, J. (1960). The equivalence of two extremum problems. *Canadian Journal of Mathematics*, *12*, 363–366. https://doi.org/10.4153/CJM-1960-030-4

Kruk, M. E., Paczkowski, M., Mbaruku, G., De Pinho, H., & Galea, S. (2009). Women's preferences for place of delivery in rural Tanzania: A population-based discrete choice experiment. *American Journal of Public Health*, *99*(9), 1666–1672. https://doi.org/10.2105/AJPH.2008.146209

Lancsar, E., & Louviere, J. (2008). Conducting discrete choice experiments to inform healthcare decision making. *Pharmacoeconomics*, *26*, 661–677. https://doi.org/10.2165/00019053-200826080-00004

Mandeville, K. L., Lagarde, M., & Hanson, K. (2014). The use of discrete choice experiments to inform health workforce policy: A systematic review. *BMC Health Services Research*, *14*, 1–14. https://doi.org/10.1186/1472-6963-14-367

Mogas, J., Riera, P., & Bennett, J. (2006). A comparison of contingent valuation and choice modelling with second-order interactions. *Journal of Forest Economics*, *12*, 5–30. https://doi.org/10.1016/j.jfe.2005.11.001

Nyarko, E. (2020) Optimal 2^K paired comparison designs for third-order interactions. Statistical Papers https://doi.org/10.1007/s00362-020-01174-4

Nyarko, E., & Doku-Amponsah, K. (2022). Approximate and exact optimal designs for paired comparison experiments. *Calcutta Statistical Association Bulletin*, *74*(1), 00080683221079965. https://doi.org/10.1177/00080683221079965

Nyarko, E., & Schwabe, R. (2019). Optimal designs for second-order interactions in paired comparison experiments with binary attributes. *Journal of Statistical Theory and Practice 13*, *13*(4). https://doi.org/10.1007/s42519-019-0051-x

Rao, V. R. (2014). *Applied Conjoint Analysis*. Springer. https://doi.org/10.1007/978-3-540-87753-0

Ryan, M. (2004). Discrete choice experiments in health care. *The BMJ*, *328*(7436), 360–361. https://doi.org/10.1136/bmj.328.7436.360

Scheffé, H. (1952). An analysis of variance for paired comparisons. *Journal of the American Statistical Association*, *47*, 381–400. https://doi.org/10.1080/01621459.1952.10501179

Schwabe, R. (1996). *Optimum designs for multi-factor models*. Springer.

Schwabe, R., Graßhoff, U., Großmann, H., & Holling, H. (2003) Optimal 2^K paired comparison designs for partial profiles. In: PROBASTAT 2002, Proceedings of the Fourth International Conference on Mathematical Statistics, Smolenice, PROBASTAT 2002, Proceedings of the Fourth International Conference on Mathematical Statistics, Smolenice. Slovakia, 26:79–86

Shah, K. K., Tsuchiya, A., & Wailoo, A. J. (2015). Valuing health at the end of life: A stated preference discrete choice experiment. *Social Science & Medicine*, *124*, 48–56. https://doi.org/10.1016/j.socscimed.2014.11.022

Soetevent, A. R., & Kooreman, P. (2007). A discrete-choice model with social interactions: With an application to high school teen behavior. *Journal of Applied Econometrics*, *22*, 599–624. https://doi.org/10.1002/jae.924

Sudhir, K., Priester, J., Shum, M., Atkin, D., Foster, A., Iyer, G., Jin, G., Keniston, D., Kitayama, S., Mobarak, M., Qian, Y., Tewari, I., & Wood, W. (2015). Research opportunities in emerging markets: An inter-disciplinary perspective from marketing, economics, and psychology. *Customer Needs and Solutions*, *2*(4), 264–276. https://doi.org/10.1007/s40547-015-0044-1