UNIVERSITY OF GHANA

LARGE DEVIATION PRINCIPLES FOR EMPIRICAL MEASURES OF SIGNAL TO INTERFERENCE NOISE RATIO (SINR) GRAPH

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DECLARATION

I hereby declare that this submission is my own work towards the award of the PhD degree and that, to the best of my knowledge, it contains no material previously published by another person nor material which had been accepted for the award of any other degree of the university, except where due acknowledgement had been made in the text.

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DEDICATION

This work is dedicated to my late dad Samuel Sakyi-Yeboah.

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ABSTRACT

We obtain a Large Deviation Principle (LDP) and Asymptotic Equipartition Property (AEP) for Critical, Super-critical and Sub-critical telecommunication network modelled as SINR random network. Given devices space D, an intensity measure $\lambda m \in \mathbb{R}_+$ is a transitional kernel $\mathcal{Q}$ from the space D to positive real numbers $\mathbb{R}_+$ and a path loss function. The study defines a Marked Poisson Point Process (MPPP). For a given MPPP and technical constant $\tau_\lambda, \gamma_\lambda : (0, \infty) \to (0, \infty)$; the study defines a Marked SINR Network as a Telecommunication Network and associate it with two empirical measures; the empirical marked measure and the empirical connectivity measure on two different scales as $\lambda^2 a_\lambda$ and λ, on a topological space, where λ is the intensity measure of the PPP which defines a SINR random network. For the class of telecommunication networks, the study proves a joint LDP for the empirical measures of the telecommunication network. Using this joint LDP, the study proves Asymptotic Equipartition Property (AEP) for the stochastic telecommunication network modelled as the marked SINR network. In addition, the study proves a Local Large Deviation Principle (LLDP) and a classical McMillian Theorem for the stochastic SINR network process. Further, for a typical empirical paired measure, we deduce from local large deviation principle a bound on the cardinality of the space of marked SINR network. Note that, the LDP for the empirical measures of this stochastic SINR network modelled as Telecommunication network was derived on space of measures equipped with the $\tau-$ topology, and the LLDP were deduced in the space of the SINR model process without any topological restriction. All our rate function are expressed as relative entropies of the marked SINR on the device space D.

Contents

LIST OF ABBREVIATION

AEP .... Asymptotic Equipartition Property

ATM .... Automated Teller Machine

BSPK .... Binary Phase Shift Keying

CPPC .... China Public Private Partnership Center

CDF .... Cummulative Distribution Function

CLT .... Cramer Limit Theorem

CRG .... Coloured Random Graph

GET .... Gartner Ellis Theorem

HPBW .... Half Power Beam Width

LDT .... Large Deviation Theory

LDP .... Large Deviation Principle

LLDP .... Local Large Deviation Principle

LLN .... Law Large Number

M2M .... Machine to Machine

MCS .... Modulation and Coding Schemes

MGF .... Moment Generation Function

MIMO .... Multiple Input Multiple output

MMSE .... Minimum Mean Square Error

MPPP .... Marked Poisson Point Process

PPP .... Poisson Point Process

PRR .... Packet Reception Rate

QoS .... Quality of Service

SINR .... Signal to Interference Noise Ratio

SNR .... Signal to Noise Ratio

SMB .... Shannon-McMillian Breimen

SWIPT .... Simultaneous Wireless Information and Power Transfer

WLLN .... Weak Law of Large Number

WTN .... Wireless Telecommunication Network

5G .... Fifth Generation

Chapter 1

INTRODUCTION

This chapter consists of an introduction and mathematical preliminary of the large deviation for empirical measures of signal-to-interference noise ratio (SINR) graph model. We first recall in Section (1.1), which provides the background of the study, whilst Section (1.2) contains literature review (development through history). Section 1.3 presents the mathematical preliminaries on the network model, and in the next Section (Section 1.4), we introduce large deviations. Further, Section (1.5) focused on the other tools for the study, Section (1.6) deal with an overview and last Section presents the contribution of large deviation principles (LDP) to the study of the empirical measure of signal-to-interference noise ratio (SINR) graph model.

1.1 Background of the study

Over a billion smartphones are believed to be linked to the complex mobile communication network worldwide. A cellular network, also known as a mobile network, is a communication system in which the connectivity between end nodes is wireless. Wireless communication utilises electromagnetic waves to convey messages across distances, which has become the latest standard in our daily lives (Avin et al., 2012). Wireless communication makes use of electromagnetic waves to send a signal across long distances. As a result, it is a prerequisite for routing across any telecommunication network. A wireless network in telecommunication consists of numerous nodes connected by a wireless channel, which acts as an electrical wave for connecting over long distances and can travel at speeds close to the speed of light. Van-Bosse and Devetak (2006) stated that telecommunication network permits us to talk, send faxes and other information to pretty much anyone on the planet. For instance, mobile networks have changed the way people communicate, and mobile phones have gone from being a luxury commodity to being an indispensable part of daily life. Rather than being hampered by location or technological enhancement, today's cell phone use is determined by social circumstances. Therefore, mobile internet is the next step in the mobile networking revolution, while voice connections fulfil the essential need to communicate, and mobile voice connections begin to filter even deeper into the fabric of everyday life. The mobile internet is on its way to being a popular source of daily information, and simple, flexible mobile access to this data would be expected.

The historical accomplishments of 2G and 3G, as well as the promise of 4G in this decade, have led to agreement on the new fifth generation (5G) of mobile technologies. The fifth-generation (5G) mobile network enables the creation of a new network capable of connecting almost anybody and everything, including computers, objects, and gadgets (Zhang et al., 2015). Thus, wireless connectivity transition from a desirable feature is needed for a large number of items in various industries as a result of fifth-generation. Furthermore, fifth-generation wireless technology intends to give users with peak data rates of tens of gigabits per second (Gbps), ultra-low latency (millisecond level), improved reliability, huge network bandwidth, enhanced availability (one million connects per square kilometer), and a more reliable user experience. New user experiences and industry linkages are enabled by increased performance and productivity. According to Zhang et al. (2015), new applications and use cases implemented in the fifth generation (5G) mobile networks bring unparalleled and complex specifications, one of which is exceptionally high availability. The fifth generation mobile communication technologies are emerging into research fields (Liu, 2016).

In wireless communication system, a wired path between the sender or transmitter and the receiver determines the correct data reception. SINR is used in wireless communication as a way to measure the quality of wireless connection. It serve as a fundamental performance metric for design and analysis. Specifically, SINR denotes metric measurement in the wireless channel status, which affects the bit error rates of the packet (Zhang et al., 2008). The usage of SINR in an integrated wireless network provides a solution in handover optimisation.

Moreover, the SINR is also a quantity that indicates if a given frequency resource is suitable to maintain a communication link properly (Bastidas-Puga et al., 2018). Besides, the SINR is a critical performance measure that influences other metrics related to different layers, modules, and wireless communication systems (for example, spectral efficiency or energy efficiency). More so, SINR serves as an indicator that measures downlink coverage for network planning and optimisation purpose (Hadj-Kacem et al., 2020). Generally, the SINR model is widely accepted ahead of the protocol model due to its accuracy and precision of the network connectivity. In the SINR model, concurrent transmission from the sender and interference is treated as an additive Gaussian white noise. A transmission is deemed successful if and only if the SINR at the receiver end from the nearest base station exceeds a certain threshold. However, the SINR model is dependent on the random nature of the propagation environment, and the power received.

The Signal-to-Interference-plus-Noise Ratio (SINR) relevance stems from the Shannon law in information theory, which provides an explicit formula expressing the maximum possible data throughput. Information theory is a mathematical theory of communication that quantifies information and uses these quantities to model situations and solves communication and data storage optimality problems (Shannon, 1948). It covers the theoretical and practical aspects of data compression and efficient information transmission over noisy networks. The data source entropy provides a lower bound on the compression rate. The ability of the given channel limits rates for accurate information transmission (Nemetz, 2009). The information theory is primarily characterised by a stationary stochastic process and thereby seen as a branch of probability theory because it is dependent on the ergodic theorem. The ergodic theorem states that the empirical mean of function converges in probability to the theoretical or sample mean of function. The ergodic theorem is a very general form of the law of large numbers (LLN), which is vital in probability theory. For instance, the strong law of large number is regarded as Shannon-McMillian-Breiman (SMB) theorem or the Asymptotic Equipartition Property in information theory (AEP).

Large Deviation Theory (LDT) involves the asymptotical computation of probability on an exponential scale. Nemetz (2009) describes the behaviour of a stochastic system when it deviates from its expected behaviour with the theory of Large Deviation.

Further, Jahnel and König (2020) posited that, the LDT is designed to analyse an asymptotic random situation in which a parameter diverges the unlikeliness of the event. Their studies argued that applying the LDT implies that the probability under consideration decays exponentially very fast in this parameter, and the exponential rate (rate function) is derived using the variational optimisation approach.

The Large Deviation Principle (LDP) is the foundation of the LDT, thus a sequence of random variable satisfies the LDP if a rate function determines the exponential decay of the probability of the rare event and can be captured using the variational principle (Touchette, 2018). The LDP allows us to quantify the decay of probability for events always from their ergodic limit on an exponential scale.

According to Borbash and Ephremides (2006), the challenge in computing time routine to meet such connectivity criteria within a wireless network is taken into account. Unless node transmissions and receives are simultaneous, connectivity can be active at the same time. Node transmissions or receipts from more than one node can occur at the same time. A given SINR is exceeded at each receiver when transmitters use optimal transmitter power.

In this study, we will introduce a set of transmitter-receiver pairs in the SINR model located in the plane with an associated SINR requirement. The aim is to satisfy as many of the requirements as possible. The study provides LDP for a wireless network model consisting of Poisson Point Processes (PPP) of transmitters and receivers. The study will associate a family of connectable receivers whose SINR is larger than a certain connectivity threshold to each transmitter.

This study seeks to obtain the LDP for empirical measures of the SINR graph. Specifically; the Local Large Deviation Principle (LLDP), the Large Deviation Principle and information theory for the SINR graph model, Large deviation principle for empirical SINR measure of critical telecommunication. Large deviations, Sharron-McMillian-Breiman(SMB) for super-critical telecommunication network, and Large deviations,and Information theory for sub-critical SINR network model.

1.2 Literature Review

This section presents a review of related literature on the topic forming the basis of this study.

1.2.1 Development through History

In probability theory, the large deviation theory deals with the asymptotic behaviour of tail distribution. Studies from Ikeda (1959) and Jamison et al. (1965) revealed that in 1929, Khintchine worked on the first local limit theorem for large deviation probabilities. The study provided the framework for the fundamental concept of classical statistical mechanics and quantum statistics. The works focused on the sum of independent and identically distributed (iid) random variables but failed to address the probabilities of non -iid random variables. Khinchine theorem is termed as the law of iterated logarithm. Also, some theorem of large deviation probabilities can be attributed to Bernoulli trials, Smirnov, Levy, and Frenchet identities. At a conference in Geneva, Cramér (1944) presented his work on probability theory that deals with the new limit theory, but the theorem can not be generalised. Chernoff et al. (1952) introduced hypothesis testing as measures of information and divergence using the sum of independent observation. Petrov (1954) provided the generalisation and fundamental result of the Cramer limit theorem. However, Petrov generalisation fell short of the sum of independent non-identical random variables. Richter (1957) provided an extended proof for Petrov's generalisation of the Cramer theorem from identical random variables and simultaneously improves to remainder term and growth of the random variable. In 1958, Sanov presented large deviation probabilities on random variables using empirical distribution. Thus, the studies proved the first-order asymptotic behaviour of large deviation probabilities on an empirical distribution (Sanov, 1958). The Sanov theorem served as proof for source coding theorem and pioneered the application of LDT into diverse fields, but generalisation and further extensions weakened the theorem. Blackwell et al. (1959) relied on the argument of the Cramer theorem for large deviation probabilities in the extreme tail of convolution, but the work was limited to iid random variables.

Bahadur and Rao (1960) presented the theoretical framework on the mean deviation, which satisfies Cramer conditions. Linnik (1961) introduces a new technique that yields results whenever the Cramer condition was violated, but the new concept was limited to independent random variables. Rényi et al. (1961) proved the limiting theorem of probability theory focusing on entropy and information measures. The studies presented a simplified version of Linnik information-theoretical proof of Central Limit Theorem (CLT). Nagaev (1965) proved the asymptotic expression that improves Linnik-Petrov outcome relating to large deviation probabilities without the specification of rate function. Besides, Rubin and Sethuraman (1965) sought to improve Linnik's work by estimating the probabilities of moderate deviation. However, moderate deviation turned out to

be less than the least deviation in the spectrum and hence, there was no overlap. In 1966, the novel large deviation outcome for the Wiener measure as a function space was developed. This led to the Schilder theorem, which dealt with some asymptotic approach for wiener integral (Schilder, 1966) . Hoadley et al. (1967) developed a theorem based on an extension of the Sanov theorem, which estimates the large deviation probabilities that statistic can be approximated by empirical CDF. However, the theorem can be weakened to continuity. Also, Varadhan (1966) developed asymptotic probabilities and differential equations in the large deviation theory. Nagaev (1969) introduced the integral local limit theorem involving asymptotics behaviour in large deviation probabilities where there is a violation of Cramer conditions. More so, Saulis focused on the asymptotic expansion for the large deviation probabilities (Saulis and Statulevičius, 1976).
Stone et al. (1974) studies gave a straightforward technique to the Hoadley theorem but under strong conditions. The study proof adapted covers the case of a d-dimensional random variable, but other generalisations are less noticeable. Donsker and Varadhan (1975) adopted the concept of large deviation for occupation time for Markov chains and process. Their studies laid the general foundation that leads to large deviation application in diverse areas. Saulis and Statulevičius (1976) proved the weighted sum of a random variable in large deviation. Groeneboom et al. (1979) extended the Chernoff theorem and proved the large deviation outcome for linear combination function of empirical probability measure. The study technique to large deviations based on multinomial approximations was systematically developed.

In 1984, Varadhan made a fundamental contribution to probability theory and created a unified theory of large deviation (Varadhan, 1984). A year after Varadhan presented his work, Orey developed a relation between Sanov theorem and Shannon-Mcmillian Breiman (SMB) theorem (Bucklew, 1987). Lynch and Sethuraman (1987) evaluated the LDP works, which focused on an induced probability measure by a stochastic process with stationary and independent increments with no Gaussian component. Petz et al. (1988) studies derived a relative entropy using a variational technique. The proof was limited to Newmann algebra.

In 1995, Ellis gave a general framework for the analysis of large deviation in statistical mechanics. The studies emboldened the Cramer theorem, Donsker-Varadhan theorem and others; and provided the proof for large deviation application in Curie-Weiss, Curie-Weiss Pott and Ising model using the sample mean, empirical measures and empirical process (Ellis, 1995). Bryc and Dembo (1996) proved the LDP concerning τ topology holds for the empirical measure of any stationary mixing process with a hyper-exponential mixing rate. O'Connell (1998) deduced LDP for the relative size of the largest connected component in a random graph with edge probability. Even though the rate is not convex, the proof yields an asymptotic process for the probability that the random graph is connected.

Biggins et al. (2004) studies provided a theoretical framework for large deviation for mixtures whereby the study assumed that if a probability measure on a particular space (θ) satisfies the LDP, then the probability measure on θ_n implies that the probability measure also satisfies the LDP. Dembo et al. (2005) proved a large deviation principle in n, with explicit rate function, for the empirical subtree measures of multitype Galton–Watson trees conditioned to have precisely n vertices. Also, they developed single-generation empirical measure LDP for a more general class of random trees, including trees sampled uniformly from the set of all trees with n vertices. Nyrhinen (2005) deduced the proof for necessary and sufficient conditions for the exponential upper bound of the Gartner-Ellis theorem. The proof was limited to the upper bound. In 2007, Vardhan received the Nobel Abel Prize for his contribution in large deviation. His works focused on developing a unified large deviation theory. He posited that large deviation is an integral aspect of probability theory; provides an asymptotic estimate of rare events and the technique is subtle (Ramasubramanian, 2008).

Okamura (2013) was concerned with the asymptotic behaviour of LDP. The study adopted the Hiai-Ohya-Tsukada theorem as a relative entropy and it played the same role as the rate function in LDP using the variation optimisation approach. Doku-Amponsah (2015) provided the theoretical framework for Shannon-MacMillian Breiman (SMB) or Asymptotic Equipartition Property (AEP) for a wireless sensor network. The studies argue that strong law of large number is termed as SMB or AEP in information theory. Also, the studies deduce the joint LDP for the empirical sensor measure and the empirical pair measure of the coloured random geometric graph model. Doku-Amponsah (2018) used a large deviation approach to obtain the asymptotic outcome for multi-type random networks. The study provided an extensive framework on the asymptotic process of evolution and co-evolution and deduced an empirical measure for a multitype random network. Doku-Amponsah (2019) worked on extending the lossy asymptotic property for geometric network structure modelled as Coloured Random Graph (CRG). The study used the large deviation approach to prove the asymptotic equipartition property for network structure. García-Zelada et al. (2019) used the Laplace principle to prove the large deviation lower bound and then applied the technique in singular Gibbs measure on a Polish space. Liu and Dong (2020) gave an insight of large deviation for empirical measures of generalised random graph using asymptotic behaviour. Their study evaluates LDP for empirical pair measure and the empirical neighbourhood measure on a generalised random graph. Chaari (2021) extended the Schilder theorem in large deviation and evaluated the probability measure link on an infinite dimension space. Further, the study determine the family of white noise (positive) distribution using the rate function in LDP.

Presently, large deviation as a branch of probability theory is an attraction for to many researchers based on its numerous applications: finance, communication, insurance, mechanics, engineering, and others. Many works are using the concept of the large deviation techniques but limited in the aspect of the SINR graph model. The study seeks to bridge the gap by providing the theoretical framework for the large deviation for empirical measures of the SINR graph model.

The following section presents the network model (the SINR graph model).

1.3 Mathematical Preliminaries on Network Model

1.3.1 Model Framework

This section is focused on the definitions, theorem, properties and outlined proved for large deviation for empirical measures of the SINR graph model.

1.3.1.1 Point Process

In wireless network, the geographical location of the edge (node) is generally modelled as a Point Process on a plane. Point process is termed as random counting measure. In this study, we fix a dimension $d \in \mathbb{N}$ and a measurable set D is subset of $\mathbb{R}^d$ with respect to the Borel-sigma algebra $\mathcal{B}(\mathbb{R}^d)$.

In D, we assume that the $X = (X_i)_{i \in I}$, with an index set, I is given. We also assume that the point clouds in the set given by

$$\mathcal{S}(D) = \cup_{x \subset D} \Big\{ x : \ |x \cap A| < \infty \text{ ,for any bounded } A \subset D \Big\}, \tag{1.1}$$

where $|A|$ denotes the cardinality of the set A and the element of $\mathcal{S}(D)$ are termed as locally finite set.

1.3.1.2 Poisson Point Process

According to Jahnel and König (2020) stated that the Poisson Point Process (PPP) exists in a finite measurable space and its distribution is characterised by exponential form; a specifically an exponential of the Laplace function form. Note that for PPP, the number of point in a disjoint set are stochastically independent and follows a Poisson distribution.

In this study, we adopted the PPP and used it to model locations' places (that is, their devices), additional boxes (supporting devices) and base station spaces. The study defines PPP as follows:

Definition 1 ***Poison Point Process***

Let λm be a measure in $\mathcal{D}$ that gives finite value for any Radon measure. The random point process $\mathcal{X}$ is termed PPP with intensity measure λm if for all $k \in \mathcal{N}$ and any pairwise disjoint bounded measureable set $A_1, A_2, ..., A_k$ are subset of $\mathcal{D}$., the counting variables $\mathcal{N}_{\mathcal{X}}(A_1), \mathcal{N}_{\mathcal{X}}(A_2)..., \mathcal{N}_{\mathcal{X}}(A_k)$ are independent Poisson distributed random variables with parameter $\lambda m \otimes Q(A_1), \lambda m \otimes Q(A_2), ..., \lambda m \otimes Q(A_k)$

$$\mathbb{P}((\mathcal{N}_{\mathcal{X}}(A_1), \mathcal{N}_{\mathcal{X}}(A_2), \cdots, \mathcal{N}_{\mathcal{X}}(A_k)) = n_1, n_2, \cdots, n_k) := \prod_{i=1}^{k} \frac{e^{-\lambda m \otimes Q(A_i)}[\lambda m \otimes Q(A_i)]^{n_i}}{[n_i]!} \tag{1.2}$$

(Jahnel and Konig, 2003).

Definition 2 ***Marked Poisson Point Process***

Suppose $X = (X_i)_{i\epsilon\mathcal{I}}$ is PPP in $\mathcal{D}$ with intensity measure λm and suppose $(\mathcal{M}, \mathcal{B}(\mathcal{X}))$ is the measurable space, the mark space. Let k be a transition probability from $\mathcal{D}$ to $\mathcal{M}$. Given $\mathcal{I}$, suppose $(\sigma_i)_{i\epsilon\mathcal{I}}$ be an independent collection of $\mathcal{M}$-valued random variable with distribution $\otimes_{i\epsilon\mathcal{I}}\mathcal{Q}(X_i, \cdot)$ then the point process $X_i = (x_i, \sigma_i)_{i\epsilon\mathcal{I}}$ in $\mathcal{D} \times \mathcal{M}$ respectively is termed as k-Marked Poisson Point Process (MPPP). The MPPP is a PPP defined on a Euclidean or topological space (Jahnel and Konig, 2003).

1.3.2 Signal to Interference Noise Ratio Graph Model

The study refers to Signal to Interference Noise Ratio (SINR) graph is defined as follows:

(i) Suppose $X = (X_i)_{i\epsilon\mathcal{I}}$ is a PPP with intensity measure λm .

(ii) X_i has a mark $\sigma(X_i) = \sigma_i$ and this is also known as the battery power or the life span of the power, and this follows an independent exponential distribution with parameter c.

(iii) For any given X, we assigns a mark $\sigma(X_i)$ according to the kernel $Q(\sigma_i, x_i)$.

(iv) For any pair of marked points (X_i, σ_i) and (X_j, σ_j) connection is deemed successful $((x_i, \sigma_i) \to (x_j, \sigma_j))$ or vice vice versa, if $SINR(X_i, X_j, X) \geq \tau_\lambda(\sigma_j, \sigma_i)$ or $SINR(X_j, X_i, X) \geq \tau_\lambda(\sigma_i, \sigma_j)$.

Let us introduce an additional parameter $\gamma_\lambda > 0$, which allows us to tune the interference and write

$$SINR(X_i, X_j, X) := \frac{\sigma_i \ell(\|X_i - X_j\|)}{N_o + \gamma_{(\lambda)}(\sigma_i) \sum_{i\in\mathcal{I}\setminus\{j\}} \sigma_i \ell(\|X_i - X_j\|)} \tag{1.3}$$

where N_o is a constant that denotes the variance of the addictive white noise (ambient Gaussian noise density), $\ell(\|X_i - X_j\|)$ is the path loss function, $\sum_{i\in\mathcal{I}\setminus\{j\}} \sigma_i \ell(\|X_i - X_j\|)$ is the interference and τ and γ are technical constant. The signal attenuation between the transmitter and reception antenna as a function of propagation distance and other factors is referred to as the path loss function. For the path loss function, the large distance implies a low connection and vice-versa. Mathematically, the path loss function is given as;

$$\ell(\|X_i - X_j\|) = \ell(r) = r^{-\alpha} \tag{1.4}$$

where $\alpha \in (0, \infty)$.
We consider $X^\lambda := X^\lambda(\mu, \mathcal{Q}, \ell) = \Big\{[(X_i, \sigma_i), j \in I], E\Big\}$ under the joint law of the MPPP and the SINR graph model. We let E represent the set of edges in the SINR random network. In this thesis, we defines X^λ as an SINR model and $(X_i, \sigma_i) := X_i^\lambda$ as marked site type of i.

1.3.2.1 Type of SINR Graph Model

The study considered three types of SINR graph models: Critical, Super-critical, and Sub-critical SINR network models, defined as follows:

Definition 3 *SINR Network Model*

Consider an SINR graph model X^λ*, if the link probability satisfy*

$$\frac{P_\lambda[(x, \sigma x), (y, \sigma y)]}{a_\lambda} \to g[(x, \sigma_x), (y, \sigma_y)] \tag{1.5}$$

where for all $(x, \sigma_x), (y, \sigma_y) \in X^\lambda$ *and* $g : X^\lambda \times X^\lambda \to [0, \infty)$ *is a nonzero function. The SINR model is said to be*

(i) Critical $\lambda a_\lambda \to 1$.

(ii) Super-critical $\lambda a_\lambda \to \infty$.

(iii) Sub-critical $\lambda a_\lambda \to 0$.

1.4 Large Deviation

In this study, we rely on the theory of large deviation to achieve the study objectives. The rationale of the LDT is to shows the decay rate in an exponential form.

Definition 4 *(Large Deviation Principle)*

Let $\mathcal{X}$ *be a topological space and* $(X_n)_{n\in\mathcal{N}}$ *be a sequence of* $\mathcal{X}-$*value random variables. Furthermore, let* $I : \mathcal{X} \to [0, \infty)$ *be a function. We say that the family* $(X_n)_{n\in\mathcal{N}}$ *satisfies LDP with rate function* I*, if the following conditions are satisfied;*

(i) For every $\alpha \geq 0$*, the set* $\{x : I(x) \geq \alpha\}$ *is compact.*

(ii) For any open set $G \subset \mathcal{X}$.

$$\lim_{n\to\infty} \inf \frac{1}{n} \log \mathbb{P}(X_n \in G) \geq -\inf_{x\in G} I(x)$$

(iii) For a closed set $F \subset \mathcal{X}$.

$$\lim_{n\to\infty} \sup \frac{1}{n} \log \mathbb{P}(X_n \in F) \leq -\inf_{x\in F} I(x)$$

Definition 5 *(Rate function)*

The rate function I is a lower semi-continuous mapping $I : X \to [0, \infty]$ such that for all $\alpha \in [0, \infty]$, the level set $\Psi_I(\alpha) = \{x : I(x) \leq \alpha\}$ is closed. A good rate function is a rate function for which all the level sets $\Psi_I(\alpha)$ are compact subset of X.

Note that if X is a metric space, then I is semi-continuous if and only if

$$\lim_{x_n\to x} \inf I(x_n) \geq I(x) \quad \forall x \in X \tag{1.6}$$

A consequence of a rate function being good is that its infimum is achieved over closed sets.

Definition 6 *Legendre Transform*

The rate of convergence is characterized by the Fenchel-Legendre transform as stated as:

$$I(a) \triangleq \sup_{\theta}[\theta a - \Lambda(\theta)] \tag{1.7}$$

where $\Lambda(\theta) = \log M_x(\theta)$.

1.4.1 Cramer Theorem

The Cramer theorem deals with large deviation associated with the empirical mean with independent and identical distributed random variables taking values in a finite dimensional space.

Theorem 1.4.1 *Cramer Theorem [Jahnel and König (2020)]*

The Legendre transform is the rate function for LDP of the empirical mean. Thus, we have that

$$\lim_{n\to\infty} \sup \frac{1}{n} \log \mathbb{P}\left(S_n \geq a\right) \leq -I(a) \tag{1.8}$$

This indicate that the bound is tight. Further, we might be interested in more complicated rare events, beyond the interval $[a, \infty]$.

In the Cramer theorem, we controlled the averages of the X_i by changing the measure and then invoked the weak law of large numbers (WLLN).

The Sanov theorem is a useful extension of the Cramer theorem. It provides an LDP for the empirical measure family of an independent and identical distributed random variables.

1.4.2 Sanov Theorem

The Sanov theorem is an extension of the Cramer theorem empirical measure from a class of independent and identical distributed variables that lives on the space of probability measure on $\mathcal{X}$.

Theorem 1.4.2 *Sanov Theorem [Jahnel and König (2020)]*

Let $\mathcal{X}$ be a polish space and $\{X_n\}_{n\geq 1}$ an iid sequence of $\mathcal{X}-$valued random variables with marginal distribution ω. Then their normalized empirical measure $L_\lambda = \frac{1}{n}\sum_{i=1}^{n} \delta_{(X_i)}$ satisfies an LDP in the set of probability measure on $\mathcal{X}$ with rate function;

$$I_\omega(\pi) = H(\pi||\omega) = \sum_{a\in\mathcal{X}} \pi(a) \log \frac{\pi(a)}{\omega(a)} \tag{1.9}$$

We denotes $H(\pi||\omega)$ as the Kullback-Leibler divergence or relative entropy of π with respect to the measure ω.

Lemma 1.4.3 *Relative entropy as Legendre Transform*

For any two probability measure π and ω on a topological space $\mathcal{X}$

$$H(\pi||\omega) = \sup_{g\in B_\nu} [\langle g\,,\,\pi\rangle - \log\langle exp(g)\,,\,\omega\rangle] \tag{1.10}$$

where B_ν represent the sets of bounded and measurable function on $\mathcal{X}$.

Lemma 1.4.4 *Varadhan Lemma*

Suppose that a sequence of random variable $(S_n)_{n\geq 1}$ satifies an LDP with a rate function I, and for every $\theta \in R$

$$\tilde{\Lambda}(\theta) = \lim_{n\to\infty} sup \frac{1}{n} \ln E(e^{\theta S_n}) \tag{1.11}$$

then

(i) The limit above exists and is Fenchel-Legendre Transform of $I(x)$, i.e.

$$\tilde{\Lambda}(\theta) = \sup_{x\in R}[\theta a - I(a)] \tag{1.12}$$

(ii) If the rate function I is convex, then it is the Fenchel-Legendre transform $\tilde{\Lambda}$. i.e.

$$I(a) = \sup_{\theta\in R}[\theta a - \tilde{\Lambda}(\theta)] \tag{1.13}$$

Remark 1 *Varadhan lemma makes precise statement about the exponential decay of the integral of exponential functions of the random variable S_n.*

Note that, neither the central limit theorem (CLT) nor Cramer theorem tell us how fast this convergence is and an analogue for the opposite event. For the purpose of this study, we considered the Gartner-Ellis Theorem (GET) (Zeitouni and Dembo, 1998).

1.4.3 Gartner-Ellis Theorem

Gartner-Ellis Theorem is a powerful result which establishes the existence of LDP for processes. The theorem follows from the existence of a well-behaved limiting cummulant function,which implies not-too-strong dependence between successive value in the sequence.

Theorem 1.4.5 *Gartner-Ellis Theorem[Jahnel and König (2020)]*
Let $\{X_n\}_{n\geq 1}$ be strictly stationary sequence of real-valued random variables satisfying the an LDP. The sequence of probability measures $\{P_n(\cdot) = P\left(\frac{S_n}{n}\right) \in$

$\cdot\}_{n\geq 1}$ *satisfies the LDP with rate* n *and rate function* $\Lambda^*(x)$ *being Fenchel-Legendre transform of the function*

$$\Lambda(\theta) = \lim_{n\to\infty} \frac{1}{n} \ln E(e^{\theta S_n}) \tag{1.14}$$

which implies that

$$\Lambda^*(x) = \sup_{\theta\in R}\{x\theta - \Lambda(\theta)\} \tag{1.15}$$

The Gartner-Ellis Theorem (GET) is the generalization of the Cramer theorem, and also the assumptions for normality can be relaxed. The GET theorem deals with large deviation when the sequence X_n is not necessarily independent. GET defines under which hypothesis sequence $\frac{S_n}{n}$ satisfies an LDP and shows how to calculate the corresponding rate function. This theorem proves large deviation bounds in $\mathbb{R}^d$ by using properties of MGF. This turns out to be quite useful when the random variables of interest are not independent and identically distributed but close to it in some source (Zeitouni and Dembo, 1998).

Remark 2 *The Gartner-Ellis Theorem gives conditions for the existence of a suitable lower bound and, in particular when this is same as the upper bound.*

The GET theorem will be employed in Section(2.4)[i], Section (3.3), Section (4.3)[i] and Section(5.3)[i].

Lemma 1.4.6 (Jahnel and König (2020)) *Let* $(X_i)_{i\in I}$ *be a PPP in a compact set* $D \in \mathbb{R}^d$ *with intensity measure* λm*, where* $\lambda \in \mathbb{R}_+$ *and* m *is a Lebesque measure on* D*. Then the normalized empirical measure* $L_\lambda = \frac{1}{\lambda}\sum_{i\in I}\delta_{(X_i^\lambda)}$ *satisfies an LDP as* $\lambda \to \infty$ *the LDP on the set measures on* D *with rate function given by*

$$\mathbb{P}(L_1 = \omega) \approx exp(-\lambda H(w^n|m^n)) \tag{1.16}$$

where w^n *and* m^n *are the coarsening projection of* w *and* m *in the decomposition of* D*.*

1.5 Preparation

We consider $X^{\lambda}(m, Q, \ell) = \Big\{[(X_i, \sigma_i), i \in I], E\Big\}$ under the joint law of the Marked PPP and the graph. We shall interpret X^{λ} as a marked SINR graph and $(X_i, \sigma_i) := X_i^{\lambda}$ the mark of site i. We write

$$\mathcal{S}(D) = \cup_{x \subset D}\Big\{x : \ |x \cap A| < \infty \ , \text{for any bounded } A \subset D \ \Big\}, \tag{1.17}$$

where $|A|$ denotes the cardinality of the set A. We write $\mathcal{X} = \mathcal{S}(D \times \mathbb{R}_+)$ and by $\mathcal{M}(\mathcal{X})$ we denote the space of positive measures on the space $\mathcal{X}$ equipped with $\tau-$ topology. Henceforth, we shall refer to $\mathcal{X}$ as locally finite subset of the set $D \times \mathbb{R}_+$.

1.5.1 Empirical Measures

In information theory, empirical measures provide a vivid description of a more general event, such as the expected number of connectable receivers per transmitter. Thus, the *empirical measure* is a random variable with values in a measurable space. The study defines the *empirical measure* and *empirical pair measures* as follows:

Definition 7 *Empirical Mark Measure*
For any SINR graph X^{λ}, we define a probability measure, the empirical measure, *$L_1^{\lambda} \epsilon \mathcal{M}(\mathcal{X})$ is given as*

$$L_1^{\lambda}([x, \sigma_x]) := \frac{1}{\lambda}\sum_{i \epsilon \mathcal{I}} \delta_{(\mathcal{X}_i^{\lambda})}([x, \sigma_x]) \tag{1.18}$$

where $L_1^{\lambda}([x, \sigma_x])$ is the proportion of all devices with location and power $[x, \sigma_x]$, $\delta_{(\mathcal{X}_i^{\lambda})}$ denote the random measure concentrated at $\mathcal{X}_i^{\lambda}$ to the distribution of $\mathcal{X}_i$, $\mathcal{X}_i^{\lambda}$ is the location of the device i with the expected number λ, $\mathcal{X}_i$ is the point cloud,and σ_x is the battery power depending of the size of the location (battery power).

Definition 8 *Empirical Pair Measure*

For any SINR graph X^{λ} , we define a symmetric finite probability measure, the empirical pair measure $L_2^{\lambda} \epsilon \mathcal{M}(\mathcal{X} \times \mathcal{X})$ *is given by*

$$L_2^{\lambda}([x, \sigma_x], [y, \sigma_y]) := \frac{1}{a_{\lambda}\lambda^2} \sum_{ij\epsilon E} [\delta_{(\mathcal{X}_i^{\lambda}, \mathcal{X}_j^{\lambda})} + \delta_{(\mathcal{X}_j^{\lambda}, \mathcal{X}_i^{\lambda})}]([x, \sigma_x], [y, \sigma_y]) \tag{1.19}$$

The empirical pair measure counts the number of node connecting any given pair of transmitter.

Note that $\lambda^2 a_{\lambda}$ is the maximum possible number of nodes in the SINR graph. The total mass for the empirical mark $\|L_1^{\lambda}\|$ is 1 and the total mass for the empirical pair measure $\|L_2^{\lambda}\| = \frac{2|E|}{\lambda}$.

Observe that, $\mathcal{M}(\mathcal{X}) \times \mathcal{M}(\mathcal{X} \times \mathcal{X})$ is closed subset of $\mathcal{M}(\mathcal{X}) \times \mathcal{M}(D \times R^{+} \times D \times R^{+})$ and

$$\mathbb{P}((L_1^{\lambda}, L_2^{\lambda}) \in \mathcal{M}(\mathcal{X}) \times \mathcal{M}(\mathcal{X} \times \mathcal{X})) = 1$$

.

The study focused on the LDP for $(L_1^{\lambda}, L_2^{\lambda})$ on the scale λ for the SINR graph with $\frac{1}{a_{\lambda}} = \lambda$ in the subsequent chapters. By exponential equivalence, see (Dembo, 1998), one can deduce the same LDP for any SINR graph model.

1.5.2 The Shannon-McMillian-Breiman Theorem or Asymptotic Equipartition Property

In information theory, the analog of the law of large numbers is the Asymptotic Equipartition property and it is the a direct consequence of the weak law of large numbers.

Remark 3 *The Asymptotic Equipartition Property is the heart of information theory.*

Theorem 1.5.1 *Asymptotic Equipartition Property*

Suppose $X = (X_n)_{n\geq 1}$ are a sequence of independent and identical distributed random variables and satisfy the consequence of weak law of large numbers given by:

$$-\frac{1}{n}\log \mathbb{P}(X = x) \to \mathcal{H}(x) \tag{1.20}$$

where $\mathcal{H}(x)$ is the entropy of x and $\mathbb{P}(X = x)$ is the probability of observing the sequence x, then the probability assigned to an observed sequence is $2^{-n\mathcal{H}(x)}$ Zeitouni and Dembo (1998).

Remark 4 *The number of bits used to describe sequences in the typical set is approximately $n\mathcal{H}(x)$.*

The AEP or SMB will be employed in Section(2.5), Section (4.4) and Section (5.4).

1.5.3 Method of Mixture

The study relies on the method of mixture proposed by Biggins et al. (2004) in large deviations.

For any $\lambda \in \mathbb{R}_+$, we define

$$\mathcal{M}_\lambda(\mathcal{X}) := \big\{\omega \in \mathcal{M}(\mathcal{X}) \,:\, \lambda\omega(a) \in \mathbb{N} \text{ for all } a \in \mathcal{X}\big\},$$
$$\mathcal{M}_\lambda(\mathcal{X}\times\mathcal{X}) := \big\{\pi \in \tilde{\mathcal{M}}(\mathcal{X}\times\mathcal{X}) \,:\, \lambda\,\pi(a,b) \in \mathbb{N}, \text{ for all } a,b \in \mathcal{X}\times\mathcal{X}\big\}.$$

We denote by $\Theta_\lambda := \mathcal{M}_\lambda(\mathcal{X})$ and $\Theta := \mathcal{M}(\mathcal{X})$. With

$$P^{(\lambda)}_{\omega_\lambda}(\eta_\lambda) := \mathbb{P}\big\{L_2^\lambda = \eta_\lambda \,\big|\, L_1^\lambda = \omega_\lambda\big\},$$
$$P^{(\lambda)}(\omega_\lambda) := \mathbb{P}\big\{L_1^\lambda = \omega_\lambda\big\}$$

the joint distribution of L_1^λ and L_2^λ is the mixture of $P^{(\lambda)}_{\omega_\lambda}$ with $P^{(\lambda)}(\omega_\lambda)$ defined as

$$d\tilde{P}^\lambda(\omega_\lambda,\eta_\lambda) := dP^{(\lambda)}_{\omega_\lambda}(\eta_\lambda)\, dP^{(\lambda)}(\omega_\lambda). \tag{1.21}$$

The following lemmas ensure the validity of large deviation principles for the mixtures and for the goodness of the rate function if individual large deviation principles are known. See for example, (Doku-Amponsah et al., 2010, Page 30) and the references therein. We observe that the family of measures $(P^{\lambda}\colon \lambda \in (0,\infty))$ is exponentially tight on Θ. Note that, the method of mixture will be applied in Section(2.4 [ii]),Section(3.4), Section(4.5 [ii]) and Section (5.3[ii]).

1.5.4 Useful Lemma and Corollary

Lemma 1.5.2 *Euler's Lemma*

If $\frac{1}{a_\lambda}P_\lambda[(x_i,\sigma_i),(y_j,\sigma_j)] \to \beta[(x_i,\sigma_i),(y_j,\sigma_j)]$ *for all* $(x_i,\sigma_i),(y_j,\sigma_j)$ *and* $a_\lambda \to 0$

$$\lim_{\lambda\to\infty}[1+\alpha P_\lambda([x_i,\sigma_i],[y_j,\sigma_j])^{\frac{1}{a_\lambda}} = e^{\alpha\beta([x_i,\sigma_i],[y_j,\sigma_j])} \tag{1.22}$$

for all $([x_i,\sigma_i],[y_j,\sigma_j]),\in \mathcal{X}\times\mathcal{X}$ *and* $\alpha \in R$

Proof: *Note that, for any* $\epsilon > 0$ *and for large enough* λ, *we have*

$$[1+a_\lambda(\alpha\beta(x,\sigma_x)(x,\sigma_y))-\epsilon]^{\frac{1}{a_\lambda}} \le [1+\alpha P_\lambda(x,\sigma_x)(y,\sigma_y)]^{\frac{1}{a_\lambda}} \le [1+a_\lambda(\alpha\beta(x,\sigma_x)(y,\sigma_y))+\epsilon]^{\frac{1}{a_\lambda}} \tag{1.23}$$

by the point wise convergence. Hence by the squeeze theorem, we deduce equation 1.22. □

Lemma 1.5.3 *Exponential Tightness*

Suppose the family of measures $\tilde{P}^{\lambda} : \lambda \in \mathbb{R}_+$ *is exponentially tight on* $\Theta \times \mathcal{M}(\mathcal{X}\times\mathcal{X})$ *For every* $\alpha > 0$, $\exists \ell \in \mathcal{N}$ *such that*

$$\lim_{\lambda\to\infty} sup \frac{1}{\lambda^2 a_\lambda} \log \mathbb{P}(|E| > \lambda^2 a_\lambda N) \le -\alpha \tag{1.24}$$

Proof: *Suppose* $\eta > \min_{[(a,b)]\in\mathcal{X}} \mathbb{R}^d[(a,b)] > 0$ *and* $t = \lambda[1-(1-e)e^{-\eta}]$. *Let* $|E|$ *be the number of nodes and using the chebycheff inequality and Euler lemma*

(1.5.1), *we deduce for sufficient large* λ.

$$\begin{aligned}
\mathbb{P}(|E| \geq \lambda^2 a_\lambda \ell) &\leq e^{-\lambda^2 a_\lambda \ell} E\{e^{|E|}\} \\
&\leq e^{-\lambda^2 a_\lambda \ell} \sum_{i=0}^{\infty} \sum_{k=0}^{i} e^k \binom{i}{k} (e^{-\eta})^k (1-e^{-\eta})^{i-k} \frac{e^{-\lambda}\lambda^i}{i!} \\
&= e^{-\lambda^2 a_\lambda \ell} \sum_{i=0}^{\infty} \frac{e^{-\lambda}\lambda^i}{i!} [1 - e^{-\eta} + e^{-\eta} e]^i \\
&= e^{-\lambda^2 a_\lambda \ell} e^{-\lambda} \sum_{i=0}^{\infty} \frac{(\lambda[1 - e^{-\eta} + e^{-\eta} e])^i}{i!} \\
&= e^{-\lambda^2 a_\lambda \ell} e^{-\lambda} e^{\lambda[1 - e^{-\eta} + e^{-\eta} e]} \\
&= e^{-\lambda^2 a_\lambda \ell} e^{-\lambda} e^{\lambda[1-(1-e)e^{-\eta}]} \\
&\leq e^{-\lambda^2 a_\lambda \ell} e^{-\lambda} e^{t} + o(1)) \qquad (1.25)
\end{aligned}$$

observe that, for any α *choose* $\ell \in \mathcal{N}$ *such that* $\ell > q$ *and note that, for sufficiently large such that* λ

$$\mathbb{P}\{|E| \geq \lambda^2 a_\lambda N\} \leq e^{-\lambda^2 a_\lambda \alpha} \qquad (1.26)$$

which complete with the proof. □

Corollary 1.5.4 *Spectral Potential Point*

For $\omega \in \mathcal{P}(\mathcal{X})$ *we define the spectral potential of the marked SINR graph* (X^λ) *conditional on the event* $\{L_1^\lambda = \omega\}$, $U_Q(g, \omega)$ *as*

$$U_Q(g, \omega) = \left\langle g\,,\, e^{-R^D} \omega \otimes \omega \right\rangle. \qquad (1.27)$$

The following remarkable properties holds for U_Q:

(i) It is finite on the space $\mathcal{M}(\mathcal{X} \times \mathcal{X})$.

(ii) It is monotone.

(iii) It is additively homogeneous.

(iv) It is convex .

Lemma 1.5.5 *For $\pi \in \mathcal{M}(\mathcal{X}\times\mathcal{X})$, let $I_\omega(\pi)$ be the Kullback action of the marked SINR graph X^λ. Thus, the following hold for the Kullback action or divergence function $I_\omega(\pi)$:*

(i)

$$I_\omega(\pi) = \sup_{g\in\mathcal{C}} \left\{ \langle g,\, \pi\rangle - \langle g,\, e^{-R^D}\omega \otimes \omega\rangle \right\}$$

(ii) The function $I_\omega(\pi)$ is convex and lower semi-continuous on the space $\mathcal{M}(\mathcal{X}\times\mathcal{X})$.

(iii) For any real α, the set $\left\{\pi \in \mathcal{M}(\mathcal{X}\times\mathcal{X}) : I_\omega(\pi) \le \alpha\right\}$ is weakly compact.

Note that, the spectral potential point will be applied in Section(2.6),Section(4.5) and Section(5.5).

1.6 Overview of the Chapters

The thesis comprises five chapters; the first chapter consists of the introduction, background, literature review, and overview. The second chapter focuses on LLDP, the LDP and information theory for the SINR graph model. The third chapter deals with LPD for empirical SINR measures of critical telecommunication. The fourth chapter examines Large deviations, Asymptotic equipartition property Theorem (AEP) for super-critical SINR networks. The fifth chapter explores Large deviations and Information theory for sub-critical for the SINR Radon Network Models. The sixth chapter presents the conclusion and recommendation of the study.

1.6.1 Chapter One

Chapter one presents a background of the study, development through history, network model, large deviation, preparation, overview and contribution of large deviation principles (LDP) for the empirical measure of signal to interference noise ratio (SINR) graph model.

1.6.2 Chapter Two

Chapter two focus on LLDP, the LDP and information theory for the SINR graph model. The study will claim that for a class of marked SINR graphs and prove a joint LDP for empirical measures, with speed λ in the τ- topology.

Also, from the joint large deviation principle for the marked empirical measures and empirical connectivity measure, we will obtain the Asymptotic Equipartition Property (AEP) for a networked structured data modelled as marked SINR graph. Specifically, the study will prove that for a large dense marked SINR graph, one requires approximately about $\frac{\lambda^2 H(Q\times Q)}{\log 2}$ bits to transmit the information contained in the network with high probability, where $H\left(Q \times Q\right)$ is properly defined entropy for the exponential transition kernel with parameter c.

In addition, the study will prove the LLDP for the class of marked SINR graphs with a speed λ from a potential spectral point of view.

From the LLDP, we will derive a conditional LPD for a marked SINR graph. We will note that, while the joint LDP is established in the $\tau-$ topology, the LLPD assume no topological restriction on the space of the marked SINR graphs.

All rate functions will be expressed in terms of the relative entropy or the kullback action or divergence function of the marked SINR on the devices space D.

1.6.3 Chapter Three

Chapter three deals with LPD for empirical SINR measure of critical telecommunication. For a given Marked Poisson Point Process (MPPP), the study will define SINR and SINR network as a Telecommunication network.

In addition, the study defines the Empirical Measures (marked empirical measures, empirical connectivity measure and empirical SINR measure) of a class of Telecommunication network. For this class of Telecommunication network, the study will prove a joint large deviation for the empirical measure of the Telecommunication Network.

All the rate function will be expressed in terms of relative entropies.

1.6.4 Chapter Four

Chapter four examines Large deviations, asymptotic equipartition property (AEP) for super-critical SINR random network. The study will obtain Large Deviation asymptotics for a super-critical communication network modelled as a SINR network.

To do this, the study will define the empirical power measure and empirical connectivity measure and prove joint LDP for the two empirical measures on two different scales, i.e. λ and $\lambda^2 a_\lambda$ where λ is the intensity measure of the Poisson Point Process (PPP) which defines the SINR random network. Using this joint LDP, the study will prove an AEP for a stochastic telecommunication network modelled as the SINR network.

Further, we will prove the LLDP for the SINR network and from the LLDP, we will prove LDP and a classical McMillian Theorem for the stochastic SINR network processes.

Note, for typical empirical connectivity measure, $h\omega \bigotimes \omega$, we will deduce from the bound on the cardinality of the space of SINR networks to be approximately equal $e^{\lambda^2 a_\lambda |h\omega \bigotimes \omega| H\left(\frac{h\omega \bigotimes \omega}{|h\omega \bigotimes \omega|}\right)}$, where the connectivity probability of the network, P^{x^λ} , satisfies $a_\lambda^{-1} P^{x^\lambda} \to h$.

The LDP for empirical measures of the stochastic SINR network will be observed on spaces of measure equipped with the τ- topology, and the LLDP will obtain in the space of the SINR network process without any topological restrictions.

1.6.5 Chapter Five

The chapter obtains large deviation asymptotic for sub-critical communication networks modelled as signal-interference-noise-ratio (SINR) random networks.

To achieve this, we define the empirical power measure and the empirical connectivity measure, as well as prove joint large deviation principles(LDPs) for the two empirical measures on two different scales. Using the joint LDPs, we prove an Asymptotic equipartition property(AEP) for wireless telecommunication Networks modelled as the subcritical SINR random networks. Further, we prove a Local Large deviation principle(LLDP) for the sub-critical SINR random network. From the LLDPs, we prove the large deviation principle, and a classical McMillan Theorem for the stochastic SINR model processes. Note that, the LDPs for the empirical measures of this stochastic SINR random network model were derived on spaces of measures equipped with the $\tau-$ topology, and the LLDPs were deduced in the space of SINR model process without any topological limitations. We motivate the study by describing a possible anomaly detection test for SINR random networks.

1.6.6 Chapter six

The chapter presents the conclusion and recommendation of the study.

1.7 Contribution

In this study, we defined empirical measures on the SINR network and proved the joint large deviation principle result, including the AEP and the classical McMillian theorem for the critical, super-critical and sub-critical telecommunication networks modelled as the SINR network.

Chapter 2

Local Large Deviation Principle, Large Deviation Principle and Information theory for the SINR graph model

This chapter has already appeared in co-authored paper Sakyi-Yeboah et al. (2020)

2.1 Introduction and Background

2.1.1 Introduction

Wireless ad-hoc and sensor networks have been the topic of much recent research. Now, with the introduction of 5th generation (5G) cellular systems, several techniques including advanced multiple access technology, massive-MIMO, full-duplex, advanced modulation and coding schemes (MCSs), and simultaneous wireless information and power transfer (SWIPT) will constitute the next phase in global telecommunication standard, see (Luo et al., 2019). 5G, a type of communication which is based on parallel processing hardware and artificial intelligence, will play a key role in wireless networks of the next generation, see (Bangerter et al., 2014) . Furthermore, the process of 5G usages will come along with unprecedented and exigent requirement of which connectivity is a vital cornerstone.

In telecommunication, wireless network comprises of a number of nodes which connect over a wireless channel. See (Gupta and Kumar, 2000). The Signal -to -Inference-Plus- Noise Ratio (SINR) determines whether a given pair of nodes can communicate with each other at a given time. Connectivity occurs in wireless network, if two nodes communicate, possibly via intermediate nodes and also, the information transport capacity of the network, See (Ganesh and Torrisi, 2008). In addition, network connectivity is related to various layers, components, and metrics of wireless communication systems; however, one vital performance indicator that strongly affects other metrics as well is the signal-to-interference-plus-noise-ratio (SINR) (Oehmann et al., 2015).

The SINR is of key significance to the analysis and design of wireless networks. In the process of addressing the additional requirement imposed on wireless communication, in particular, a higher availability of a highly accurate modeling of the SINR is required. Grönkvist and Hansson (2001) works on SINR model rely on the assumption that nodes are uniformly distributed in the plane. In contrast, the complexity of solution paves way for computational efficiency. See, (Behzad and Rubin, 2003).

More so, the SINR model can be made a complex model such that each transmission is given a power and then assumes a distance-dependent path loss. A transmission is deemed to be successful if the SINR is more than some specified threshold. See, (Andrews and Dinitz, 2009). In contrast, a lot of recent work has shown that packets are successfully received only when SINR exceeds a given threshold, and assumes that packet reception rate (PRR) is zero below this threshold. See example, (Santi et al., 2009). Further study of the SINR graph model has shown that an SINR model of interference is a more realistic model of interference than the protocol model of interference: a receiver node receives a packet so long as the signal to interference plus noise ratio is above a certain threshold. See, (Bakshi et al., 2017). Furthermore, Manesh and Kaabouch (2017) stated that SINR is successful if the desired receiver surpasses the threshold. This enables the transmitted signal to be decoded with satisfactory root error probability.

The fundamental concept of SINR model determines as transceiver design on communication system that considers interference as noise. Andrews and Dinitz (2009) examine a set of transmitter receiver pairs located in the plane with each having an associated SINR requirement; and satisfies as many of the requirements as possible. In all communication systems, noise generated by circuit component in the receiver is a source of signal interruption. The ratio of the signal power to noise power is termed as SINR. The SINR is a vital indicator of communication link quality. See (Jeske and Sampath, 2004). In the article by Santi et al. (2009) , the wireless link scheduling problem under a graded version of the SINR interference model is revisited. Indeed, the article defines wireless link scheduling problem under the graded SINR model, where they impose an additional constraint on the minimum quality of the usable links.

Li et al. (2005) examined the statistical distribution of the SINR for the Minimum Mean Square Error (MMSE) receiver in multiple-input multiple output (MIMO) wireless communication. Their study decomposed SINR model into two independent random variables; the first part has an exact gamma distribution and the second part was shown to converge in distribution to a Normal distribution and approximate by Generalized Gamma. Also, AlAmmouri et al. (2017) examined the SINR and throughput of dense cellular network with stretched exponential path loss. It was established (in the article) that the area spectral efficiency, which assumes an adaptive SINR threshold, is non-decreasing with the base station density and converges to a constant for high densities.

An accurate SINR estimation provides for both a more efficient system and a higher user−perceived quality of service.

In this chapter, we prove the local large deviation and large deviation principles of the Signal-To-Noise and Interference Ratio graph model (SINR). In the sequel we introduce a Marked Poisson Point Process (MPPP) and the marked SINR graph model. For a class of the marked SINR graph, we define the empirical marked measure and the empirical connectivity measure. Then, we prove a joint Large Deviation Principle (LDP) for the empirical marked measure and the empirical connectivity measure of the marked SINR graph model, with speed λ in the $\tau-$topology. From the joint large deviation principle, we obtain an Asymptotic Equipartition Property (AEP) for network structured data modelled as an SINR graph. See, example, (Doku-Amponsah, 2019) for a generalized version of the AEP for wireless sensor networks.

Further, we prove an LLDP for the SINR graph and deduce weak variant of LDP for the SINR graph models from a spectral potential point. To be specific about this approach, given an empirical marked measure ω, we define the so-called spectral potential $U_{R^D}(\omega, \cdot)$ for the marked SINR graph process, where R^D is a properly defined constant function which depends on the device locations and the marks. And we show that the *Kullback action* or the *divergence function* $I_{\omega}(\pi)$, with respect to the empirical connectivity measure π, is the legendre dual of the spectral potential. See, example (Doku-Amponsah, 2017) for similar results for the critical multitype Galton-Watson process.

2.2 Statement of Results

2.2.1 The Marked SINR Model for Telecommunication Networks.

Fix a dimension $d \in \mathbb{N}$ and a measureable set $D \subset \mathbb{R}^d$ with respect to the Borel-Sigma algebra $\mathcal{B}(\mathbb{R}^d)$. Denote by m the Lebesgues measure on $\mathbb{R}^d$. Given an intensity measure, $\lambda m : D \to [0, 1]$, a probability kernel Q from D to $\mathbb{R}_+$, path loss function $\ell(r) = r^{-\alpha}$, (where $\alpha \in (0, \infty)$), and technical constants $\tau_\lambda, \gamma_\lambda$: $(0\,,\,\infty) \to (0\,,\,\infty)$ we define the marked SINR Graph as follows:

(i) We pick $X = (X_i)_{i\in I}$ as a Poisson Point Process (PPP) with intensity measure $\lambda m : D \to [0, 1]$.

(ii) Given X, we assign each X_i a mark $\sigma(X_i) = \sigma_i$ independently according to the transition kernel $Q(\cdot\,, X_i)$.

(iii) For any two marked points $((X_i,\sigma_i),(X_j,\sigma_j))$ we connect an edge iff

$$SINR(X_i,X_j,X)\geq\tau_\lambda(\sigma_j) \text{ and } SINR(X_j,X_i,X)\geq\tau_\lambda(\sigma_i),$$

where

$$SINR(X_j,X_i,X)=\frac{\sigma_i\ell(\|X_i-X_j\|)}{N_0+\gamma_\lambda(\sigma_j)\sum_{i\in I\setminus\{j\}}\sigma_i\ell(\|X_i-X_j\|)}$$

We consider $X^\lambda(m,Q,\ell)=\Big\{[(X_i,\sigma_i),i\in I],\,E\Big\}$ under the joint law of the Marked PPP and the graph. We shall interpret X^λ as a marked SINR graph and $(X_i,\sigma_i):=X_i^\lambda$ the mark site of type i. We write

$$\mathcal{S}(D)=\cup_{x\subset D}\Big\{x:\ |x\cap A|<\infty \text{ ,for any bounded } A\subset D\Big\},\qquad (2.1)$$

where $|A|$ denotes the cardinality of the set A. We write $\mathcal{X}=\mathcal{S}(D\times\mathbb{R}_+)$ and by $\mathcal{M}(\mathcal{X})$ we denote the space of positive measures on the space $\mathcal{X}$ equipped with $\tau-$ topology. Henceforth, we shall refer to $\mathcal{X}$ as locally finite subset of the set $D\times\mathbb{R}_+$. For any SINR graph X^λ we define a probability measure, the *empirical mark measure*, $L_1^\lambda\in\mathcal{M}(\mathcal{X})$, by

$$L_1^\lambda([x,\sigma_x]):=\frac{1}{\lambda}\sum_{i\in I}\delta_{X_i^\lambda}([x,\sigma_x])$$

and a symmetric finite measure, the *empirical pair measure* $L_2^\lambda\in\mathcal{M}(\mathcal{X}\times\mathcal{X})$, by

$$L_2^\lambda([x,\sigma_x],[y,\sigma_y]):=\frac{1}{\lambda^2}\sum_{(i,j)\in E}[\delta_{(X_i^\lambda,X_j^\lambda)}+\delta_{(X_j^\lambda,X_i^\lambda)}]([x,\sigma_x],[y,\sigma_y]).$$

Note that the total mass $\|L_1^\lambda\|$ of the empirical marked measure is $\mathbb{1}$ and total mass of the empirical pair measure is $2|E|/\lambda^2$. Observe that, $\mathcal{M}(\mathcal{X})\times\mathcal{M}(\mathcal{X}\times\mathcal{X})$ is a closed subset of $\mathcal{M}(\mathcal{X})\times\mathcal{M}(D\times\mathbb{R}_+\times D\times\mathbb{R}_+)$ and

$$\mathbb{P}\Big\{(L_1^\lambda,L_2^\lambda)\in\mathcal{M}(\mathcal{X})\times\mathcal{M}(\mathcal{X}\times\mathcal{X})\Big\}=1.$$

Hence, in view of (Zeitouni and Dembo, 1998, Lemma 4.1.5) it is sufficient to establish Joint LDP for $(L_1^\lambda, L_2^\lambda)$ in the space $\mathcal{M}(\mathcal{X}) \times \mathcal{M}(\mathcal{X} \times \mathcal{X})$. The first theorem in this section, Theorem 2.2.1, is the LDP for the empirical marked measure of the SINR graph models in the space $\mathcal{M}(\mathcal{X})$.

Theorem 2.2.1 *Suppose X^λ is an SINR graph with intensity measure $\lambda m : D \to [0,1]$ and a marked probability kernel Q from D to $\mathbb{R}_+$ and path loss function $\ell(r) = r^{-\alpha}$, for $\alpha > 0$. Then, as $\lambda \to \infty$, L_1^λ satisfies an LDP in the space $\mathcal{M}(\mathcal{X})$ with good rate function*

$$I_1(\omega) = \begin{cases} H(\omega \,|m \otimes Q), & \text{if } \|\omega\| = 1 \\ \infty & \text{otherwise.} \end{cases}$$

We write $R^D([x,\sigma_x],[y,\sigma_y]) := \lim_{\lambda\to\infty} \lambda R_\lambda^D([x,\sigma_x],[y,\sigma_y])$, where

$$R_\lambda^D([x,\sigma_x],[y,\sigma_y]) = \int_D \Big[\frac{\tau_\lambda(\sigma_x)\gamma_\lambda(\sigma_x)}{\tau_\lambda(\sigma_x)\gamma_\lambda(\sigma_x)+(\|z\|^\alpha/\|x-y\|^\alpha)} + \frac{\tau_\lambda(\sigma_y)\gamma_\lambda(\sigma_y)}{\tau_\lambda(\sigma_y)\gamma_\lambda(\sigma_y)+(\|z\|^\alpha/\|y-x\|^\alpha)}\Big] dz.$$

The next theorem, Theorem 2.2.2, is a conditional LDP for the empirical connectivity measure given the empirical marked measure, and joint LDP for the empirical marked measure and empirical connectivity measure of the SINR graph model.

Theorem 2.2.2 *Suppose X^λ is an SINR graph with intensity measure $\lambda m : D \to [0,1]$ and a marked probability kernel Q from D to $\mathbb{R}_+$ and path loss function $\ell(r) = r^{-\alpha}$, for $\alpha > 0$. Let Q be the exponential distribution with parameter c.*

(i) Then, as $\lambda \to \infty$, conditional on the event $L_1^\lambda = \omega$, L_2^λ satisfies an LDP in the space $\mathcal{M}(\mathcal{X} \times \mathcal{X})$ with speed λ and good rate function

$$I_\omega(\pi) = \begin{cases} 0, & \text{if } \pi = e^{-R^D}\omega \otimes \omega \\ \infty & \text{otherwise.} \end{cases} \tag{2.2}$$

(ii) Then as $\lambda \to \infty$, the pair $(L_1^\lambda,\, L_2^\lambda)$ satisfies an LDP in the space $\mathcal{M}(\mathcal{X}) \times \mathcal{M}(\mathcal{X} \times \mathcal{X})$ with speed λ, and good rate function

$$I(\omega,\,\pi) = \begin{cases} H\Big(\omega\,\Big|m \otimes Q\Big), & \text{if } \pi = e^{-R^D}\omega \otimes \omega, \\ \infty & \text{otherwise.} \end{cases} \tag{2.3}$$

where

$$e^{-R^D}\omega \otimes \omega([x,\sigma_x],[y,\sigma_y])) = e^{-R^D([x,\sigma_x],[y,\sigma_y])}\omega([x,\sigma_x])\omega([y,\sigma_y]).$$

In particular, if we assume $\lambda\big[\tau_\lambda(\sigma_x)\gamma_\lambda(\sigma_x)+\tau_\lambda(\sigma_y)\gamma_\lambda(\sigma y)\big] \to \beta(\sigma_x,\sigma_y)$, for $x, y \in D$ and $\sigma_x, \sigma_y \in \mathbb{R}_+$ then we have

$$R^D([x,\sigma_x],[y,\sigma_y]) = q_\alpha \beta(\sigma_x,\sigma_y)\|y-x\|^\alpha,$$

where $q_\alpha := \int_D \|z\|^{-\alpha}dz < \infty$. Note, σ_x and σ_y are iid with common exponential distribution Q, with parameter c and define the so- called Shannon Entropy H by

$$H(Q\times Q) = -\int_{\mathcal{X}}\int_{\mathcal{X}}\Big[e^{-q_\alpha\beta(\sigma_x,\sigma_y)\|y-x\|^\alpha}\log\frac{e^{-q_\alpha\beta(\sigma_x,\sigma_y)\|y-x\|^\alpha}}{(1-e^{-q_\alpha\beta(a,b)\|y-x\|^\alpha})}+\log(1-e^{-q_\alpha\beta(\sigma_x,\sigma_y)\|y-x\|^\alpha})\Big]\times$$
$$Q(d\sigma_x)dx \times Q(d\sigma_y)dy.$$

The next theorem, Theorem 2.2.3, is the Asymptotic Equipartition Theorem or the Shannon-McMillian-Breiman Theorem for the class of SINR graphs

Theorem 2.2.3 *Suppose X^λ is an SINR graph with intensity measure $\lambda m : D \to \mathbb{R}_+$ and a marked probability kernel Q from D to $\mathbb{R}_+$ and path loss function $\ell(r) = r^{-\alpha}$, for $\alpha > 0$. Assume $\lambda\big[\tau_\lambda(a)\gamma_\lambda(a) + \tau_\lambda(b)\gamma_\lambda(b)\big] \to \beta(a,b) \in (0,\infty)$, for all $a, b \in \mathbb{R}_+$. Let Q be the exponential distribution with parameter c. Then,*

$$\lim_{\lambda\to\infty} -\frac{1}{\lambda^2}\log P(X^\lambda) = H(Q\times Q), \qquad \textit{with high probability.}$$

Remark 5 *Theorem 2.2.3 can be interpreted as follows: In order to code or transmit the information contained in a large telecommunication network modelled as SINR graph model, one requires with high probability, approximately* $\lambda^2 H(Q\times Q)/\log 2$ *bits.*

Let $\mathcal{G}_P$ be the set of all marked SINR graphs with intensity measure λm, where $\lambda > 0$. For $\omega \in \mathcal{M}(\mathcal{X})$ we denote by $\mathbb{P}_\omega = \mathbb{P}\Big\{\cdot \,\Big|\, L_1^\lambda = \omega\Big\}$ and write

$$\mathcal{M}_\omega = \Big\{\nu \in \mathcal{M}(\mathcal{X}\times\mathcal{X}) : \|\nu\| = \int_{\mathcal{X}} e^{-q_\alpha \beta(\sigma_x,\sigma_y)\|y-x\|^\alpha}\, \omega(dx, d\sigma_x)\omega(dy, d\sigma_y)\Big\}.$$

Observe that, in this case the rate function $I_\omega(\pi)$ is given by

$$I_\omega(\pi) = \begin{cases} 0, & \text{if } \pi = e^{-R^D}\omega\otimes\omega \\ \infty & \text{otherwise,} \end{cases} \tag{2.4}$$

where

$$R^D([x,\sigma_x],[y,\sigma_y]) = q_\alpha \beta(\sigma_x,\sigma_y)\|y-x\|^\alpha.$$

Next, we state the Local large Deviation Principle for SINR graph model without any topological restriction on the space $\mathcal{G}_P$.

Theorem 2.2.4 *Suppose* X^λ *is an SINR graph with intensity measure* $\lambda m : D \to [0,1]$ *and a marked probability kernel* Q *from* D *to* $\mathbb{R}_+$ *and path loss function* $\ell(r) = r^{-\alpha}$, *for* $\alpha > 0$. *Assume* $\lambda\big[\tau_\lambda(\sigma_x)\gamma_\lambda(\sigma_y) + \tau_\lambda(\sigma_y)\gamma_\lambda(\sigma_x)\big] \to \beta(\sigma_x,\sigma_y) \in (0,\infty)$, *for all* $\sigma_x, \sigma_y \in \mathbb{R}_+$. *Let* Q *be the exponential distribution with parameter* c. *Then,*

(i) for any functional $\nu \in \mathcal{M}_\omega$ *and a number* $\varepsilon > 0$, *there exists a weak neighbourhood* B_ν *such that*

$$\mathbb{P}_\omega\Big\{X^\lambda \in \mathcal{G}_P \,\Big|\, L_2^\lambda \in B_\nu\Big\} \le e^{-\lambda I_\omega(\pi) - \lambda\varepsilon}.$$

(ii) for any $\nu \in \mathcal{M}_\omega$, *a number* $\varepsilon > o$ *and a fine neighbourhood* B_ν, *we have the*

estimate:

$$\mathbb{P}_\omega\Big\{X^\lambda \in \mathcal{G}_P \,\Big|\, L_2^\lambda \in B_\nu\Big\} \geq e^{-\lambda I_\omega(\pi)+\lambda\varepsilon}.$$

The last result, Corollary 2.2.5, is the LDP for the SINR graph model without any topological restriction on the space $\mathcal{G}_P$.

Corollary 2.2.5 *Suppose X^λ is an SINR graph with intensity measure $\lambda m : D \to [0,1]$ and a marked probability kernel Q from D to $\mathbb{R}_+$ and path loss function $\ell(r) = r^{-\alpha}$, for $\alpha > 0$. Assume $\lambda\big[\tau_\lambda(\sigma_x)\gamma_\lambda(\sigma_y) + \tau_\lambda(\sigma_y)\gamma_\lambda(\sigma_x)\big] \to \beta(\sigma_x, \sigma_y)$, for all $\sigma_x, \sigma_y \in \mathbb{R}_+$. Let Q be the exponential distribution with parameter c.*

(i) Let F be closed subset $\mathcal{M}_\omega$. Then we have

$$\limsup_{\lambda\to\infty} \frac{1}{\lambda} \log \mathbb{P}_\omega\Big\{X^\lambda \in \mathcal{G}_P \,\Big|\, L_2^\lambda \in F\Big\} \leq -\inf_{\pi\in F} I_\omega(\pi).$$

(ii) Let O be open subset $\mathcal{M}_\omega$. Then we have

$$\liminf_{\lambda\to\infty} \frac{1}{\lambda} \log \mathbb{P}_\omega\Big\{X^\lambda \in \mathcal{G}_P \,\Big|\, L_2^\lambda \in O\Big\} \geq -\inf_{\pi\in O} I_\omega(\pi).$$

Remark 6 *We observe from Corollary 2.2.5 that*

$$\lim_{\lambda\to\infty} \mathbb{P}_\omega\Big\{X^\lambda \in \mathcal{G}_P \,\Big|\, L_2^\lambda = e^{-R^D}\omega \otimes \omega\Big\} = 1.$$

2.3 Proof of Theorem 2.2.1 by Method of Types

Let $A_1, ..., A_n$ be decomposition of $D \times \mathbb{R}_+ \subset \mathbb{R}^d \times \mathbb{R}_+$. We shall assume henceforth that $n < \lambda$ and note by the locally finite property of the MPPP that we have

$$\sum_{i=1}^{n} \log\Big[\frac{e^{-\lambda m\otimes Q(A_i)}[\lambda m \otimes Q(A_i)]^{\lambda\omega(A_i)}}{[\lambda\omega(A_i)]!}\Big] \leq \log P(L_1^\lambda = \omega)$$
$$\leq \sum_{i=1}^{n} \log\Big[\frac{e^{-\lambda m\otimes Q(A_i)}[\lambda m \otimes Q(A_i)]^{\lambda\omega(A_i)}}{[\lambda\omega(A_i)]!}\Big] + \eta_n$$

where $\lim_{n\to\infty}\lim_{\lambda\to\infty}\frac{1}{\lambda}\eta_n(\lambda, A_1, ..., A_n) = 0$. The proof of Lemma below will use the refined Stirling's formula

$$(2\pi)^{\frac{1}{2}}\lambda^{\lambda+\frac{1}{2}}e^{-\lambda+1/(12\lambda+1)} < \lambda! < (2\pi)^{\frac{1}{2}}\lambda^{\lambda+\frac{1}{2}}e^{-\lambda+1/(12\lambda)}.$$

Lemma 2.3.1 *Suppose X^λ is a marked PPP in a compact set $D\times\mathbb{R}_+$ with intensity measure $\lambda m\otimes Q$ such that m is absolutely continuous measure on D. Then,*

$$e^{-\lambda H\left(\omega^{(n)}\,\middle|\,m^{(n)}\otimes Q^{(n)}\right)+\theta_1(\lambda)} \leq \mathbb{P}\Big\{L_1^\lambda = \omega\Big\} \leq e^{-\lambda H\left(\omega^{(n)}\,\middle|\,m^{(n)}\otimes Q^{(n)}\right)+\theta_2(\lambda)}$$

$$\lim_{\lambda\to\infty}\theta_1(\lambda) = 0,\ \lim_{\lambda\to\infty}\theta_2(\lambda) = \lim_{\lambda\to\infty}\tfrac{1}{\lambda}\eta_n(\lambda, A_1, ..., A_n),$$

where $\omega^{(n)}$ and $m^{(n)}\otimes Q^{(n)}$ are the coarsening projections of ω and $m\otimes Q$ on the decomposition $(A_1, ..., A_n)$.

Proof: For large λ, note that

$$\begin{aligned}\log P(L_1^\lambda = \omega) \leq \sum\{-\lambda m\otimes Q(A_j)\} &- \log[(2\pi)^{\frac{1}{2}}(\lambda m(A_j))^{\lambda\omega(A_j)+\frac{1}{2}}\exp^{-(\lambda\omega(A_j))}] \\ &+ \frac{1}{12(\lambda\omega(A_j)+1} \\ &+ \lambda\omega(A_j)\log[\lambda m\otimes Q(A_j)] \\ &+ \eta_n(\lambda, A_1, ..., A_n)\end{aligned}$$

$$\begin{aligned}\log P(L_1^\lambda = \omega) \leq \sum\{-\lambda m\otimes Q(A_j)\} - \frac{1}{2}\log(2\pi) - [(\lambda\omega(A_j)) + \frac{1}{2}]\log[(\lambda\omega(A_j)] \\ + (\lambda\omega(A_j)) + \frac{1}{12(\lambda\omega(A_j)+1} + \lambda\omega(A_j)\log\{\lambda m\otimes Q(A_j)\} \\ + \eta_n(\lambda, A_1, ..., A_n)\end{aligned}$$

$$\log P(L_1^\lambda = \omega) \leq \sum \Big\{ -\lambda[m \otimes Q(A_j) - \omega(A_j)] - \lambda\omega(A_j)\log\frac{\omega(A_j)}{m \otimes Q(A_j)} - \frac{1}{2}\log[\lambda m(A_j)] + \frac{1}{12[\lambda\omega(A_j)+1]} - \frac{1}{2}\log(2\pi)\Big\} + \eta_n(\lambda, A_1, ..., A_n)$$

$$\log P(L_1^\lambda = \omega) \leq \sum \Big\{ -\lambda[m \otimes Q(A_j) - \omega(A_j)] - \lambda\omega(A_j)\log\frac{\omega(A_j}{m \otimes Q(A_j)} - \lambda[\frac{\log[\lambda\omega(A_j)]}{2\lambda} - \frac{1}{12\lambda^2\lambda\omega(A_j)+\lambda} + \frac{\log(2\pi)}{2\lambda}]\Big\} + \eta_n(\lambda, A_1, ..., A_n)$$

We choose $\theta_2(\lambda)$ as

$$\theta_2(\lambda) = \frac{\log(\lambda\omega(A_j))}{2\lambda} - \frac{1}{12\lambda^2\omega(A_j)+\lambda} + \frac{\log(2\pi)}{2\lambda} + \eta_n(\lambda, A_1, ..., A_n)$$

and observe that

$$\lim_{\lambda\to\infty}\theta_2(\lambda) = \lim_{\lambda\to\infty}\Big[\frac{\log\lambda\omega(A_j)}{2\lambda} - \frac{1}{12\lambda^2\omega(A_j)+\lambda} + \frac{\log(2\pi)}{2\lambda} + \tfrac{1}{\lambda}\eta_n(\lambda, A_1, ..., A_n)\Big]$$
$$= \lim_{\lambda\to\infty}\tfrac{1}{\lambda}\eta_n(\lambda, A_1, ..., A_n)$$

which proves the upper bound in the Lemma 2.3. For large λ, we have the lower bound

$$\log P(L_1^\lambda = \omega) \geq \sum_{j=1}^{n}\{-\lambda m \otimes Q(A_j)\} - \log[(2\pi)^{\frac{1}{2}}(\lambda\omega(A_j))^{\lambda\omega(A_j)+\frac{1}{2}}\exp^{-(\lambda\omega(A_j)}] + \frac{1}{12(\lambda\omega(A_j)+1} + \lambda\omega(A_j)\log[\lambda m \otimes Q(A_j)\}$$

$$\log P(L_1^\lambda = \omega) \geq \sum_{j=1}^{n} \{ -\lambda[m \otimes Q(A_j) - \omega(A_j] - \lambda\omega(A_j)\log[\lambda\omega(A_j)] + \lambda\omega(A_j)\log[\lambda m \otimes Q(A_j)] - \frac{1}{2}\log[\lambda\omega(A_j)] + \frac{1}{12[\lambda\omega(A_j)} - \frac{1}{2}\log(2\pi)\}$$

$$\log P(L_1^\lambda = \omega) \geq \sum_{j=1}^{n} \Big\{ -\lambda[m \otimes Q(A_j) - \omega(A_j)] - \lambda\omega(A_j)\log\frac{\omega(A_j)}{m \otimes Q(A_j)} - \lambda[\frac{\log[\lambda\omega(A_j)]}{2\lambda} - \frac{1}{12\lambda^2\lambda\omega(A_j)} + \frac{\log(2\pi)}{2\lambda}]\Big\}$$

We choose $\theta_1(\lambda)$ as

$$\theta_1(\lambda) = \frac{\log(\lambda\omega(A_j))}{2\lambda} - \frac{1}{12\lambda^2\omega(A_j)} + \frac{\log(2\pi)}{2\lambda},$$

and observe that

$$\lim_{\lambda\to\infty}\theta_1(\lambda) = \lim_{\lambda\to\infty}\Big[\frac{\log(\lambda\omega(A_j))}{2\lambda} - \frac{1}{12\lambda^2\omega(A_j)} + \frac{\log(2\pi)}{2\lambda}\Big] = 0.$$

This proves the lower bound of Lemma 2.3.1

□

Lemma 2.3.2 *Suppose X^λ is an SINR graph with intensity measure $\lambda m : D \to [0,1]$ and a marked probability kernel Q from D to $\mathbb{R}_+$ and path loss function $\ell(r) = r^{-\alpha}$, for $\alpha > 0$. Then, for large λ we have*

$$|I| \leq 2\lambda \ \textit{almost surely.}$$

Proof: Note that $X = (X_i)_{i\in I}$ is a PPP and let $|I| = \sum_{i=1}^{n} I_i$, where $I_i = |X \cap A_i|$ and D is independently decomposed into $A_1, A_2, \cdots, A_n$ and $A_i's$ are disjoint set. We observed that $I_1, I_2, I_3, ..., I_n$ are independent Poisson random variables with parameter $\lambda(A_i)$, where $\lambda(A_i) \leq \lambda + o(n)$ for all $i = 1, 2, ...n$. Note that, there is a real number a_i, such that $|X \cap A_i| < a_i < \infty$.

Define $\hat{a} = max(a_1, a_2, \cdots, a_n)$ and observe that $I_i \leq \hat{a}$, for all $i = 1, 2, 3, ..., n$. Write

$$X_i = [I_i - \mathbb{E}(I_i)]$$

Observe the computations of mean and variance are given as:

$$\sum_{i=1}^{n} \mathbb{E}(X_i) = 0$$

Note that, the sum of X_i are also independent and the $\sum_{i=1}^{n} \mathbb{E}(X_i) = 0$ implies that

$$\sum_{i=1}^{n} Var(X_i) = \sum_{i=1}^{n} \mathbb{E}(X_i^2)$$

. Therefore,

$$\sum_{i=1}^{n} Var(X_i) = \sum_{i=1}^{n} \mathbb{E}[I_i - \mathbb{E}(I_i)]^2$$

Hence,

$$\sum_{i=1}^{n} Var(X_i) = \sum_{i=1}^{n} \lambda(A_i) \leq \lambda + o(n) < \infty$$

Hence, by applying the Bennett's inequality to the sequence $X_1, X_2, X_3, ..., X_n$; we have that

$$\mathbb{P}\Big\{I - \mathbb{E}(I) > \lambda\Big\} \leq \exp\{-\tfrac{\lambda}{\hat{a}^2} h(\hat{a})\}, \tag{2.5}$$

where $h(\hat{a}) = (1 + \hat{a}) \log(1 + \hat{a}) - \hat{a}$. Now, we use equation 2.5 to obtain

$$\mathbb{P}\Big\{I \leq \mathbb{E}(I) + \lambda\Big\} \geq 1 - \exp\{-\tfrac{\lambda}{\hat{a}^2} h(\hat{a})\}$$

which gives

$$\lim_{\lambda \to \infty} \mathbb{P}\Big\{I \leq 2\lambda\Big\} \geq 1.$$

This ends the proof of the Lemma.

□

Let $\mathcal{M}_\lambda(\mathcal{X}) := \{\omega \in \mathcal{M}(\mathcal{X}) : \lambda\omega(x) \in \mathbb{N} \text{ for all } x \in \mathcal{X}\}$ and let F be a subset of $\mathcal{M}(\mathcal{X})$. We write $\beta_n := \max(|\mathcal{X} \cap A_1|, |\mathcal{X} \cap A_2|, ..., |\mathcal{X} \cap A_n|)$ and note that $|\mathcal{X} \cap A_i| < \infty$, for all $i = 1, 2, 3, ..., n$ by construction. We use Lemma 2.3.1 and Lemma 2.3.2 to obtain

$$\begin{aligned}
(1+2\lambda)^{-n\beta_n} e^{-\lambda \inf_{\{\omega \in F^o \cap \mathcal{M}_\lambda(\mathcal{X})\}} H\left(\omega^{(n)} \,\middle|\, m^{(n)} \otimes Q^{(n)}\right) + \theta_1(\lambda)} \\
&\leq \sum_{\omega \in F^o \cap \mathcal{M}_\lambda(\mathcal{X})} e^{-\lambda H\left(\omega^{(n)} \,\middle|\, m^{(n)} \otimes Q^{(n)}\right) + \theta_2(\lambda)} \\
&\leq \mathbb{P}\left\{L_1^\lambda \in F\right\} \\
&\leq \sum_{\omega \in cl(F) \cap \mathcal{M}_\lambda(\mathcal{X})} e^{-\lambda H\left(\omega^{(n)} \,\middle|\, m^{(n)} \otimes Q^{(n)}\right) + \theta_2(\lambda)} \\
&\leq (1+2\lambda)^{n\beta_n} e^{-\lambda \inf_{\{\omega \in cl(F) \cap \mathcal{M}_\lambda(\mathcal{X})\}} H\left(\omega^{(n)} \,\middle|\, m^{(n)} \otimes Q^{(n)}\right) + \theta_2(\lambda)}, \qquad (2.6)
\end{aligned}$$

where $\omega^{(n)}$ and $m^{(n)} \otimes Q^{(n)}$ are the coarsening projections of ω and $m \otimes Q$ on the decomposition $(A_1, ..., A_n)$.

Taking limit as $\lambda \to \infty$ we have that

$$\begin{aligned}
\liminf_{\lambda\to\infty} \left\{ - \inf_{\{\omega \in F^o \cap \mathcal{M}_\lambda(\mathcal{X})\}} H\left(\omega^{(n)} \,\middle|\, m^{(n)} \otimes Q^{(n)}\right) \right\} \\
&\leq \lim_{\lambda\to\infty} \frac{1}{\lambda} \log \mathbb{P}\left\{L_1^\lambda \in F\right\} \\
&\leq \limsup_{\lambda\to\infty} \left\{ - \inf_{\{\omega \in cl(F) \cap \mathcal{M}_\lambda(\mathcal{X})\}} H\left(\omega^{(n)} \,\middle|\, m^{(n)} \otimes Q^{(n)}\right) \right\}. \qquad (2.7)
\end{aligned}$$

Now we observe that $cl(F) \cap \mathcal{M}_\lambda(\mathcal{X}) \subset cl(F)$ for all $\lambda \in \mathbb{R}_+$ and hence we have

$$\limsup_{\lambda\to\infty} \left\{ - \inf_{\{\omega \in cl(F) \cap \mathcal{M}_\lambda(\mathcal{X})\}} H\left(\omega^{(n)} \,\middle|\, m^{(n)} \otimes Q^{(n)}\right) \right\} \leq - \inf_{\{\omega \in cl(F)\}} H\left(\omega^{(n)} \,\middle|\, m^{(n)} \otimes Q^{(n)}\right).$$

Using similar arguments as (Zeitouni and Dembo, 1998, Page 17) we obtain

$$\liminf_{\lambda\to\infty}\Big\{-\inf_{\{\omega\in F^o\cap\mathcal{M}_\lambda(\mathcal{X})\}} H\big(\omega^{(n)}\,|\,m^{(n)}\otimes Q^{(n)}\big)\Big\} \geq -\inf_{\{\omega\in F^o\}} H\big(\omega^{(n)}\,|\,m^{(n)}\otimes Q^{(n)}\big)$$

Therefore, we have

$$-\inf_{\{\omega\in F^o\}} H\big(\omega^{(n)}\,|\,m^{(n)}\otimes Q^{(n)}\big) \leq \lim_{\lambda\to\infty}\frac{1}{\lambda}\log\mathbb{P}\Big\{L_1^\lambda\in F\Big\}$$

$$\leq -\inf_{\{\omega\in cl(F)\}} H\big(\omega^{(n)}\,|\,m^{(n)}\otimes Q^{(n)}\big), \quad (2.8)$$

where $\omega^{(n)}$ and $m^{(n)}\otimes Q^{(n)}$ are the coarsening projections of ω and $m\otimes Q$ on the decomposition $(A_1, ..., A_n)$. Now taking limit as $n\to\infty$ we have

$$-\inf_{\{\omega\in F^o\}} H\big(\omega\,|\,m\otimes Q\big) \leq \lim_{\lambda\to\infty}\frac{1}{\lambda}\log\mathbb{P}\Big\{L_1^\lambda\in F\Big\} \leq -\inf_{\{\omega\in cl(F)\}} H\big(\omega\,|\,m\otimes Q\big),$$

which proves the Theorem 2.2.1.

2.4 Proof of Theorem 2.2.2 by Gartner-Ellis Theorem and the Method of Mixing

Let $A_1, ..., A_n$ be the decomposition of the space $D\times\mathbb{R}_+$. Note that, for every $(x, y)\in A_i$, $i = 1, 2, 3, ..., n$, $\lambda L_2^\lambda(x, y)$ given $\lambda L_1^\lambda(x) = \lambda\omega(x)$ is binomial with parameters $\lambda^2\omega(x)\omega(y)/2$ and $p_\lambda(x, y)$. Let Q be the exponential distribution with parameter c.

2.4.1 Proof of Theorem 2.2.2(i) by Gartner-Ellis Theorem

$$R_\lambda^D([x,\sigma_x],[y,\sigma_y]) = \int_D\Big[\tfrac{\tau_\lambda(\sigma_x)\gamma_\lambda(\sigma_x)}{\tau_\lambda(\sigma_x)\gamma_\lambda(\sigma_x)+(\|z\|^\alpha/\|x-y\|^\alpha)} + \tfrac{\tau_\lambda(\sigma_y)\gamma_\lambda(\sigma_y)}{\tau_\lambda(\sigma_y)\gamma_\lambda(\sigma_y)+(\|z\|^\alpha/\|y-x\|^\alpha)}\Big]dz.$$

Lemma 2.4.1 *Suppose X^λ is an SINR graph with intensity measure $\lambda m : D \to [0,1]$ and a marked probability kernel Q from D to $\mathbb{R}_+$ and path loss function $\ell(r) = r^{-\alpha}$, for $\alpha > 0$; then,*

$$p_\lambda([x, \sigma_x], [y, \sigma_y]) = e^{-\lambda R^D_\lambda([x,\sigma_x],[y,\sigma_y])}$$

and

$$\lim_{\lambda \to \infty} \lambda R^D_\lambda([x, \sigma_x], [y, \sigma_y])) = R^D([x, \sigma_x], [y, \sigma_y]).$$

Proof: Calculation of Connectivity Probability by the Laplace Transform: The study observes that the Signal-Interference and Noise Ratio (SINR) is given as

$$SINR(\tilde{X}_j, \tilde{X}_i, \tilde{X}) = \frac{\sigma_i \ell(\|X_i - X_j\|)}{N_0 + \gamma_\lambda(\sigma_j) \sum_{i \in I \setminus \{j\}} \sigma_i \ell(\|X_i - X_j\|)}$$

and the total interference is defined as

$$I_{X,\sigma}(Y) = \sum_{i \epsilon I} \sigma_i I_i,$$

where $I_i = \ell(\|X_i - X_j\|)$.

The probability that $\tilde{X}_i = (x, \sigma_x)$ and $\tilde{X}_j = (y, \sigma_y)$ are connected.

$$P(\tilde{X}_j, \tilde{X}_i) = P\Big[\frac{\sigma_i \ell(\|X_i - X_j\|)}{N_0 + \gamma_\lambda(\sigma_j) \sum_{i \in I \setminus \{j\}} \ell(\|X_i - X_j\|)} \geqslant \tau_\lambda(\sigma_j)\Big] \times$$

$$P\Big[\frac{\sigma_j \ell(\|X_j - X_i\|)}{N_0 + \gamma_\lambda(\sigma_i) \sum_{j \in I \setminus \{i\}} \ell(\|X_j - X_i\|)} \geqslant \tau_\lambda(\sigma_i)\Big]$$

Now, the study deduces that X_i and X_j are independent to enable calculate the converging probabilities. Note that, this converging probabilities can be estimated using:

$$P\Big[\sigma_j \ell(\|X_j - X_i\|) \geqslant \Big[(N_0 + \gamma_\lambda(\sigma_i) \sum_{i \in I \setminus \{j\}} \sigma_i \ell(\|X_j - X_i\|))\tau_\lambda(\sigma_i)\Big]$$

$$P(\tilde{X}_j, \tilde{X}_i) =$$

$$P\Big[\sigma_i \geqslant \frac{(N_0+\gamma_\lambda(\sigma_j) \sum_{i\in I\setminus\{j\}} \sigma_i \ell(\|X_i - X_j\|))\tau_\lambda(\sigma_j)}{\ell(\|X_i - X_j\|)}\Big] \times$$
$$P\Big[\sigma_j \geqslant \frac{(N_0+\gamma_\lambda(\sigma_i) \sum_{j\in I\setminus\{i\}} \sigma_j \ell(\|X_j - X_i\|))\tau_\lambda(\sigma_i)}{\ell(\|X_j - X_i\|)}\Big]$$

Let $X_i = y$, $X_j = x$ and $I_{x,\sigma}(y) = \sum_{j \in I} \ell(\|X_j - y\|)$

$$p_\lambda([x,\sigma_x],[y,\sigma_y]) = \Big[\int_0^\infty P\Big(\sigma \geqslant \frac{\tau_\lambda(\sigma_y)s}{\ell(\|y-x\|)}\Big) P\Big(N_0 + \gamma_\lambda(\sigma_y) I_{x,\sigma}(Y) \in ds\Big)\Big]$$
$$\Big[\int_0^\infty P\Big(\sigma \geqslant \frac{\tau_\lambda(\sigma_x)s}{\ell(\|x-y\|)}\Big) P\Big(N_0 + \gamma_\lambda(\sigma_x) I_{y,\sigma}(X) \in ds\Big)\Big]$$

Assuming that σ follows exponential distribution with parameter (c) , we have

$$p_\lambda([x,\sigma_x],[y,\sigma_y]) = \Big[\int_0^\infty e^{-\frac{c\tau_\lambda(\sigma_y)s}{\ell(\|y-x\|)}} P\Big(N_0 + \gamma_\lambda(\sigma_y) I_{x,\sigma}(Y) \in ds\Big)\Big]$$
$$\Big[\int_0^\infty e^{-\frac{c\tau_\lambda(\sigma_x)s}{\ell(\|x-y\|)}} P\Big(N_0 + \gamma_\lambda(\sigma_x) I_{y,\sigma}(X) \in ds\Big)\Big]$$

Using Laplace Transform gives

$$p_\lambda([x,\sigma_x],[y,\sigma_y]) = \Big[\mathcal{L}_{N_0} + \gamma_\lambda(\sigma_y) I_{Y,\sigma}\Big(\frac{c\tau_\lambda(\sigma_y)s}{\ell(\|y-x\|)}\Big)\Big] \times$$
$$\Big[\mathcal{L}_{N_0} + \gamma_\lambda(\sigma_x) I_{X,\sigma}\Big(\frac{c\tau_\lambda(\sigma_x)s}{\ell(\|x-y\|)}\Big)\Big]$$

Since the exterior noise and interference are independent

$$p_\lambda([x,\sigma_x],[y,\sigma_y]) = \Big[\mathcal{L}_{N_0}\Big(\frac{c\tau_\lambda(\sigma_y)}{\ell(\|y-x\|)}\Big) \mathcal{L}_{I_{(Y,\sigma)}}\Big(\frac{c\tau_\lambda(\sigma_y)\gamma_\lambda(\sigma_y)}{\ell(\|y-x\|)}\Big)\Big] \times$$
$$\Big[\mathcal{L}_{N_0}\Big(\frac{c\tau_\lambda(\sigma_x)}{\ell(\|x-y\|)}\Big) \mathcal{L}_{I_{(X,\sigma)}}\Big(\frac{c\tau_\lambda(\sigma_x)\gamma_\lambda(\sigma_x)}{\ell(\|x-y\|)}\Big)\Big]$$

Assuming there is no external noise ($N = 0$), the study deduces

$$p_\lambda([x,\sigma_x],[y,\sigma_y]) = \Big[\mathcal{L}_{I_{(Y,\sigma)}}\Big(\frac{c\tau_\lambda(\sigma_y)\gamma_\lambda(\sigma_y)}{\ell(\|y-x\|)}\Big)\Big] \times \Big[\mathcal{L}_{I_{(X,\sigma)}}\Big(\frac{c\tau_\lambda(\sigma_x)\gamma_\lambda(\sigma_x)}{\ell(\|x-y\|)}\Big)\Big]$$

Hence, by symmetry, the study deduces that

$$p_\lambda([x,\sigma_x],[y,\sigma_y]) = p([x,\sigma_x],[y,\sigma_y]) =$$

$$\Big[\mathcal{L}_{I_{(Y,\sigma)}}\Big(\frac{c\tau_\lambda(\sigma_y)\gamma_\lambda(\sigma_y)}{\ell(\|y-x\|)}\Big)\Big] \times \Big[\mathcal{L}_{I_{(X,\sigma)}}\Big(\frac{c\tau_\lambda(\sigma_x)\gamma_\lambda(\sigma_x)}{\ell(\|y-x\|)}\Big)\Big]$$

Note that

$$\mathcal{L}_{I_{(X,\sigma)}}(s) = \mathbb{E}(e^{-sI_{(X,\sigma)}}), \text{for } s = \tfrac{c\tau_\lambda(\sigma_x)\gamma_\lambda(\sigma_x)}{\ell(\|y-x\|)}.$$

$$\mathcal{L}_{I_{(X,\sigma)}}(s) = \exp\Big\{\int_D\int_0^\infty [e^{-s\sigma\ell(\|z\|)} - 1]Q(d\sigma,x)\mu(dz)\Big\}$$

Suppose $\mu(dz) = \lambda dz$ and recall that the battery is assumed to be $Q(d\sigma,x) = ce^{-c\sigma}$

$$\mathcal{L}_{I_{(X,\sigma)}}(s) = \exp\Big\{\int_D\int_0^\infty [e^{-s\sigma\ell(\|z\|)} - 1]ce^{-c\sigma}d\sigma\lambda dz\Big\}$$

$$\mathcal{L}_{I_{(X,\sigma)}}(s) = \exp\Big\{\lambda\int_D\int_0^\infty [ce^{-s\sigma\ell(\|z\|)-c\sigma} - ce^{-c\sigma}d\sigma]dz\Big\}$$

$$\mathcal{L}_{I_{(X,\sigma)}} = \exp\Big\{\lambda\int_D [c\int_0^\infty e^{-\sigma[s\ell(\|z\|)+c]} - \int_0^\infty ce^{-c\sigma}d\sigma]dz\Big\}$$

$$\mathcal{L}_{I_{(X,\sigma)}}(s) = \exp\Big\{\lambda\int_D [c\frac{1}{s\ell(\|z\|)+c]} - 1]dz\Big\}$$

$$\mathcal{L}_{I_{(X,\sigma)}}(s) = \exp\Big\{\lambda\int_D \frac{-s\ell(\|z\|)}{s\ell(\|z\|)+c}dz\Big\}$$

$$p_\lambda([x,\sigma_x],[y,\sigma_y]) = \exp\Big\{-\lambda\int_0^\infty \frac{s\ell(\|z\|)}{s\ell(\|z\|)+c}dz - \lambda\int_0^\infty \frac{t\ell(\|z\|)}{t\ell(\|z\|)+c}dz\Big\}$$

By substitution, $s = \frac{c\tau_\lambda(\sigma_x)\gamma_\lambda(\sigma_x)}{\ell(\|x-y\|)}$ and $t = \frac{c\tau_\lambda(\sigma_y)\gamma_\lambda(\sigma_y)}{\ell(\|y-x\|)}$

$$p_\lambda([x,\sigma_x],[y,\sigma_y]) = \exp\Big\{-\lambda\int_D \frac{\frac{c\tau_\lambda(\sigma_x)\gamma_\lambda(\sigma_x)\ell(\|z\|)}{\ell(\|x-y\|)}}{\frac{c\tau_\lambda(\sigma_x)\gamma_\lambda(\sigma_x)}{\ell(\|x-y\|)}\ell(\|z\|)+c}dz - \lambda\int_D \frac{\frac{c\tau_\lambda(\sigma_y)\gamma_\lambda(\sigma_y)\ell(\|z\|)}{\ell(\|y-x\|)}}{\frac{c\tau_\lambda(\sigma_y)\gamma_\lambda(\sigma_y)}{\ell(\|y-x\|)}\ell(\|z\|)+c}dz\Big\}$$

Using $\ell(r) = r^{-\alpha}$, the study derives the expression

$$p_\lambda([x,\sigma_x],[y,\sigma_y]) =$$

$$\exp\Big\{-\lambda\int_D \frac{\tau_\lambda(\sigma_x)\gamma_\lambda(\sigma_x)}{\tau_\lambda(\sigma_x)\gamma_\lambda(\sigma_x)+(\|z\|^\alpha/\|x-y\|^\alpha)}dz - \lambda\int_D \frac{\tau_\lambda(\sigma_y)\gamma_\lambda(\sigma_y)}{\tau_\lambda(\sigma_y)\gamma_\lambda(\sigma_y)+(\|z\|^\alpha/\|y-x\|^\alpha)}dz\Big\}$$

The study expresses

$$R_\lambda^D([x,\sigma_x],[y,\sigma_y]) = \int_D\Big[\frac{\tau_\lambda(\sigma_x)\gamma_\lambda(\sigma_x)}{\tau_\lambda(\sigma_x)\gamma_\lambda(\sigma_x)+(\|z\|^\alpha/\|x-y\|^\alpha)} + \frac{\tau_\lambda(\sigma_y)\gamma_\lambda(\sigma_y)}{\tau_\lambda(\sigma_y)\gamma_\lambda(\sigma_y)+(\|z\|^\alpha/\|y-x\|^\alpha)}\Big]dz$$

and observe that we have

$$p_\lambda([x,\sigma_x],[y,\sigma_y]) = e^{-\lambda R_\lambda^D([x,\sigma_x],[y,\sigma_y])}$$

We therefore deduce that

$$\lim_{\lambda\to\infty}\lambda R_\lambda^D([x,\sigma_x],[y,\sigma_y]) = R^D([x,\sigma_x],[y,\sigma_y])$$

which completes the proof of Lemma 2.4.1. □

Computation of the log moment generation function

Lemma 2.4.2 *Suppose X^λ is an SINR graph with intensity measure $\lambda Leb(x)$: $D \to [0,1]$ and a marked probability kernel Q from D to $\mathbb{R}_+$ and path loss function $\ell(r) = r^{-\alpha}$, for $\alpha > 0$, conditional on the event $L_1^\lambda = \omega$. Let $g : \mathcal{X}\times\mathcal{X} \to \mathbb{R}$ be* *bounded function. Then,*

$$\lim_{\lambda\to\infty}\frac{1}{\lambda}\log\mathbb{E}\Big\{e^{\lambda\langle g,\,L_2^\lambda\rangle/2}\Big|L_1^\lambda=\omega\Big\}$$
$$=\frac{1}{2}\lim_{n\to\infty}\Big[\sum_{j=1}^{n}\sum_{i=1}^{n}\int_{y\in A_j}\int_{x\in A_i}g(x,y)e^{-R^D(x,y)}\omega(dx)\omega(dy)\Big]$$
$$=\frac{1}{2}\int_{\mathcal{X}}\int_{\mathcal{X}}g(x,y)e^{-R^D(x,y)}\omega(dx)\omega(dy). \quad (2.9)$$

Proof: Now we observe that for any measurable partition $(A_1, A_2, \cdots A_n)$ of D that

$$\mathbb{E}\Big\{e^{\int\int\lambda g(x,y)L_2^\lambda(dx,dy)/2}\mid L_1^\lambda=\omega\Big\}=\mathbb{E}\Big\{\prod_{x\in D}\prod_{y\in D}e^{g(x,y)\lambda L_2^\lambda(dx,dy)/2}\Big\}$$

$$\mathbb{E}\Big\{\prod_{x\epsilon D}\prod_{y\epsilon D}e^{g(x,y)\lambda L_2^\lambda(dx,dy/2)}=\prod_{i=1}\prod_{j=1}\prod_{x\epsilon A_i}\prod_{y\epsilon A_j}\mathbb{E}\Big\{e^{\frac{g(x,y)}{\lambda}\lambda^2L_2^\lambda(dx,dy)/2}\Big\}$$

Hence from Lemma 2.4.1 we have

$$\log\mathbb{E}\Big\{e^{\lambda\langle g,L_2^\lambda\rangle/2}\Big|L_1^\lambda=\omega\Big\}=$$

$$\sum_{j=1}^{n}\sum_{i=1}^{n}\int_{A_j}\int_{A_i}\log\Big[(1-p_\lambda(x,y))+p_\lambda(x,y)e^{\frac{g(x,y)}{\lambda}}\Big]^{\lambda^2\omega\otimes\omega(dx,dy)/2}$$

By Euler's Formula, see example (Doku-Amponsah et al., 2010, pp. 1998), the study result shows

$$\frac{1}{\lambda}\log E\{e^{\lambda\langle g,L_2^\lambda\rangle/2}\mid L_1^\lambda=\omega\}=$$
$$\frac{1}{\lambda}\sum_{j=1}\sum_{i=1}\int_{A_i}\int_{A_j}\log\Big[1+\frac{g(x,y)}{\lambda}p_\lambda(x,y)+o(\lambda^2)\Big]^{\lambda^2\omega\otimes\omega(dx,dy)/2}$$

$$\frac{1}{\lambda}\log\mathbb{E}\{e^{\lambda\langle g,L_2^\lambda\rangle/2}\mid L_1^\lambda=\omega\}=$$

$$\lim_{\lambda\to\infty}\sum_{j=1}\sum_{i=1}\int_{A_i}\int_{A_j}\log\Big[1+\frac{g(x,y)}{\lambda}p_\lambda(x,y)+o(\lambda^2)\Big]^{\lambda\omega\otimes\omega(dx,dy)/2}$$

$$\lim_{\lambda\to\infty}\frac{1}{\lambda}\log\mathbb{E}\Big\{e^{\lambda\langle g,L_2^\lambda\rangle/2}\mid L_1^\lambda=\omega\Big\}=$$
$$\frac{1}{2}\sum_{j=1}\sum_{i=1}\int_{A_i}\int_{A_j}\log\Big[e^{g(x,y)e^{-R^D(x,y)}}\Big]\omega\otimes\omega(dx,dy)$$

$$\lim_{\lambda\to\infty}\frac{1}{\lambda}\log\mathbb{E}\{e^{\lambda\langle g,L_2^\lambda\rangle/2}\mid L_1^\lambda=\omega\}=\frac{1}{2}\sum_{j=1}\sum_{i=1}\int_{A_i}\int_{A_j}g(x,y)e^{-R^D(x,y)}\omega\otimes\omega(dx,dy)$$

$$\begin{aligned}\lim_{n\to\infty}\lim_{\lambda\to\infty}\frac{1}{\lambda}\log\mathbb{E}\{e^{\lambda\langle g,L_2^\lambda\rangle/2}\mid L_1^\lambda=\omega\}&=\frac{1}{2}\lim_{n\to\infty}\sum_{j=1}\sum_{i=1}\int_{A_i}\int_{A_j}[g(x,y)e^{-R^D(x,y)}\omega\otimes\omega(dx,dy)]\\&=\frac{1}{2}\int_{\mathcal{X}}\int_{\mathcal{X}}g(x,y)e^{-R^D(x,y)}\omega\otimes\omega(dx,dy)\end{aligned}$$

□

Hence, by Lemma 2.4.2 and the Gartner-Ellis theorem, L_2^λ conditional on $L_1^\lambda=\omega$ obey a large deviation principle with speed λ and a good rate function

$$I_\omega(\pi)=\frac{1}{2}\sup_g\Big\{\int_{\mathcal{X}}\int_{\mathcal{X}}g(x,y)\pi(dx,dy)-\int_{\mathcal{X}}\int_{\mathcal{X}}g(x,y)e^{R^D(x,y)}\omega\otimes\omega(dx,dy)\Big\}$$

which clearly reduces to the rate function given by

$$I_\omega(\pi)=\begin{cases}0 & \text{if } \pi=e^{-R^D}\omega\otimes\omega\\ \infty & \text{otherwise.}\end{cases}\tag{2.10}$$

2.4.2 Proof of Theorem 2.2.1(ii) by Method of Mixtures.

For any $\lambda\in\mathbb{R}_+$, we define

$$\mathcal{M}_\lambda(\mathcal{X}):=\{\omega\in\mathcal{M}(\mathcal{X}):\lambda\omega(a)\in\mathbb{N}\text{ for all }a\in\mathcal{X}\},$$
$$\mathcal{M}_\lambda(\mathcal{X}\times\mathcal{X}):=\big\{\pi\in\tilde{\mathcal{M}}(\mathcal{X}\times\mathcal{X}):\lambda\pi(a,b)\in\mathbb{N},\ \text{ for all }\ a,b\in\mathcal{X}\times\mathcal{X}\big\}.$$

We denote by $\Theta_\lambda := \mathcal{M}_\lambda(\mathcal{X})$ and $\Theta := \mathcal{M}(\mathcal{X})$. With

$$P^{(\lambda)}_{\omega_\lambda}(\eta_\lambda) := \mathbb{P}\big\{L_2^\lambda = \eta_\lambda \,\big|\, L_1^\lambda = \omega_\lambda\big\},$$
$$P^{(\lambda)}(\omega_\lambda) := \mathbb{P}\big\{L_1^\lambda = \omega_\lambda\big\}$$

the joint distribution of L_1^λ and L_2^λ is the mixture of $P^{(\lambda)}_{\omega_\lambda}$ with $P^{(\lambda)}(\omega_\lambda)$ defined as

$$d\tilde{P}^\lambda(\omega_\lambda, \eta_\lambda) := dP^{(\lambda)}_{\omega_\lambda}(\eta_\lambda)\, dP^{(\lambda)}(\omega_\lambda). \tag{2.11}$$

The following lemmas ensure the validity of large deviation principles for the mixtures and for the goodness of the rate function if individual large deviation principles are known. See for example, (Doku-Amponsah et al., 2010, Page 30) and the references therein. We observe that the family of measures $(P^\lambda\colon \lambda \in (0,\infty))$ is exponentially tight on Θ.

Lemma 2.4.3 *The family of measures* $(\tilde{P}^\lambda\colon \lambda \in \mathbb{R}_+)$ *is exponentially tight on* $\Theta \times \mathcal{M}(\mathcal{X} \times \mathcal{X})$.

Proof: Let $\eta > \min\limits_{a,b} R^D(a,b) > 0$ and $t = 1 - (1 - e^{-1})e^{-\eta}$. Then, we use Chebysheff's inequality and Lemma 2.4.2, to obtain (for sufficiently large λ),

$$\mathbb{P}\Big\{|E| \geq \lambda^2 l\Big\} \leq e^{-\lambda^2 l}\mathbb{E}\{e^{|E|}\} \leq e^{-\lambda^2 l}\sum_{i=0}^{\infty}\sum_{k=0}^{i} e^k \binom{i}{k}\Big(e^{-\eta}\Big)^k\Big(1 - e^{-\eta}\Big)^{i-k}\frac{e^{-\lambda}\lambda^i}{i!}$$
$$\leq e^{-\lambda^2 l}e^{-\lambda}e^{t\lambda}.$$

Given $N \in \mathbb{N}$ we choose $N > q$ and observe that for sufficiently large λ we have

$$\mathbb{P}\Big\{|E| \geq \lambda^2 N\Big\} \leq e^{-\lambda^2 q}.$$

Therefore, we have

$$\mathbb{P}\Big\{\|L_2^\lambda\| \geq \lambda^2 N/2\Big\} \leq e^{-\lambda^2 q/2},$$

which establishes Lemma 2.4.3.

□

Define the function $I\colon \Theta \times \mathcal{M}(\mathcal{X} \times \mathcal{X}) \to [0, \infty]$, by

$$I(\omega, \pi) = \begin{cases} H\Big(\omega \,|m \otimes Q\Big), & \text{if } \pi = e^{-R^D} \omega \otimes \omega \\ \infty & \text{otherwise.} \end{cases} \tag{2.12}$$

Lemma 2.4.4 *I is lower semi-continuous.*

Proof: Let $(\omega, \pi) \in \Theta \times \mathcal{M}(\mathcal{X} \times \mathcal{X})$ and observe that $\pi = e^{-R^D} \omega \otimes \omega$ is closed condition. Further, we note that the relative entropy, $H\Big(\omega \,|m \otimes Q\Big)$, is a lower semi-continuous function on the space $\Theta \times \mathcal{M}(\mathcal{X} \times \mathcal{X})$. As I is a function of a relative entropy, we conclude that I is lower semi-continuous.

□

Using (Biggins et al., 2004, Theorem 5(b)) together with the two previous lemmas and the large deviation principles we have established Theorem 2.2.1 and Theorem 2.2.2(i) ensure that under $(\tilde{P}^\lambda)$ the random variables $(L_1^\lambda, L_2^\lambda)$ satisfy a large deviation principle on $\mathcal{M}(\mathcal{X}) \times \mathcal{M}(\mathcal{X} \times \mathcal{X})$ with good rate function I which ends the proof of Theorem 2.2.2(ii).

2.5 Proof of Theorem 2.2.3 by Large deviation Technique

2.5.1 Proof of Theorem 2.2.3

We begin the proof of the asymptotic equipartition property, by first establishing a weak law of large numbers for the empirical mark measure and the empirical pair measure of the SINR graph.

Lemma 2.5.1 *Suppose X^λ is an SINR graph with intensity measure $\lambda m : D \to [0, 1]$ and a marked probability kernel Q from D to $\mathbb{R}_+$ and path loss function*

$\ell(r) = r^{-\alpha}$, for $\alpha > 0$. Assume $\lambda\big[\tau_\lambda(\sigma_x)\gamma_\lambda(\sigma_y) + \tau_\lambda(\sigma_y)\gamma_\lambda(\sigma_x)\big] \to \beta(\sigma_x, \sigma_y) \in (0, \infty)$, for all $\sigma_x, \sigma_y \in \mathbb{R}_+$.

Let Q be the exponential distribution with parameter c. Then, for $\varepsilon > 0$ we have

$$\lim_{\lambda\to\infty} \mathbb{P}\Big\{ \sup_{(x,\sigma_x)\in\mathcal{X}} \Big| L_1^\lambda(x,\sigma_x) - m\otimes Q(x,\sigma_x)\Big| > \varepsilon\Big\} = 0$$

and

$$\lim_{\lambda\to\infty} \mathbb{P}\Big\{ \sup_{([x,\sigma_x],[y,\sigma_y])\in\mathcal{X}\times\mathcal{X}} \Big| L_2^\lambda([x,\sigma_x],[y,\sigma_y]) - e^{-R^D} m\otimes Q \times m\otimes Q([x,\sigma_x],[y,\sigma_y])\Big| > \varepsilon\Big\} = 0$$

Proof: For any $\epsilon > 0$, let

$$F_1 = \Big\{\omega : \sup_{(x,\sigma_x)\in\mathcal{X}} |\omega(x,\sigma_x) - m\otimes Q(x,\sigma_x)| > \varepsilon\Big\},$$

$$F_2 = \Big\{\pi : \sup_{([x,\sigma_x],[y,\sigma_y])\in\mathcal{X}\times\mathcal{X}} |\varpi([x,\sigma_x],[y,\sigma_y]) - e^{-R^D} m\otimes Q\times m\otimes Q([x,\sigma_x],[y,\sigma_y])| > \varepsilon\Big\}$$

and $F_3 = F_1 \cup F_2$. Now, observe from Theorem 2.2.1(ii) that

$$\limsup_{\lambda\to\infty} \frac{1}{\lambda} \log \mathbb{P}\Big\{(L_1^\lambda, L_2^\lambda) \in F_3^c\Big\} \le - \inf_{(\omega,\pi)\in F_3^c} I(\omega,\pi).$$

It suffices for us to show that I is strictly positive on F_3^c. Suppose there is a sequence that converge to any point $(\omega_\lambda, \pi_\lambda) \to (\omega, \pi)$ such that $I(\omega_\lambda, \pi_\lambda) \downarrow I(\omega,\pi) = 0$. This implies $\omega = m\otimes Q$ and $\pi = e^{-R^D} m\otimes Q \times m\otimes Q$ which contradicts $(\omega,\pi) \in F_3^c$. This ends the proof of the Lemma. □

Now, the distribution of the marked PPP $P_\lambda(x) = \mathbb{P}\Big\{X^\lambda = x\Big\}$ is given by

$$P_\lambda(x) = \prod_{i=1}^{I} m\otimes Q(x_i,\sigma_i) \prod_{(i,j)\in E} \frac{e^{-\lambda R_\lambda^D([x_i,\sigma_i],[y_j,\sigma_j])}}{1 - e^{-\lambda R_\lambda^D([x_i,\sigma_i],[y_j,\sigma_j])}} \times \prod_{(i,j)\in\mathcal{E}} (1 - e^{-\lambda R_\lambda^D([x_i,\sigma_i],[y_j,\sigma_j])}) \prod_{i=1}^{I} (1 - e^{-\lambda R_\lambda^D([x_i,\sigma_i],[y_i,\sigma_i])})$$

$$-\frac{1}{\lambda^2}\log P_\lambda(x) =$$

$$\frac{1}{\lambda}\Big\langle -\log m\otimes Q\,,L_1^\lambda\Big\rangle + \Big\langle -\log\Big(\frac{e^{-\lambda R_\lambda^D}}{1-e^{-\lambda R_\lambda^D}}\Big)\,,L_2^\lambda\Big\rangle$$
$$+\Big\langle -\log(1-e^{-\lambda R_\lambda^D})\,,L_1^\lambda\otimes L_1^\lambda\Big\rangle + \Big\langle -\log(1-e^{-\lambda R_\lambda^D})\,,L_\Delta^\lambda\Big\rangle$$

Notice, $\lim_{\lambda\to\infty}\lambda R_\lambda^D \to R^D$,
$\lim_{\lambda\to\infty}\frac{1}{\lambda}\Big\langle -\log m\otimes Q\,,L_1^\lambda\Big\rangle = \lim_{\lambda\to\infty}\frac{1}{\lambda}\Big\langle -\log(1-e^{-\lambda R_\lambda^D})\,,L_\Delta^\lambda\Big\rangle = 0.$
Using, Lemma 2.5.1 we have

$$\lim_{\lambda\to\infty}\Big\langle -\log\Big(\frac{e^{-\lambda R_\lambda^D}}{1-e^{-\lambda R_\lambda^D}}\Big)\,,L_2^\lambda\Big\rangle = \Big\langle -\log\Big(\frac{e^{-R^D}}{1-e^{-R^D}}\Big)\,,e^{-R^D}m\otimes Q\times m\otimes Q\Big\rangle$$

$$\lim_{\lambda\to\infty}\Big\langle -\log(1-e^{-\lambda R_\lambda^D})\,,L_1^\lambda\otimes L_1^\lambda\Big\rangle = \Big\langle -\log(1-e^{-R^D})\,,m\otimes Q\times m\otimes Q\Big\rangle,$$

which concludes the proof of Theorem 2.2.3.

2.6 Proof of Theorem 2.2.4 and Corollary 2.2.5

For $\omega\in\mathcal{P}(\mathcal{X})$ we define the spectral potential $U_Q(g,\omega)$ of the marked SINR graph (X^λ) conditional on the event $\{L_1^\lambda=\omega\}$, as

$$U_Q(g,\omega) = \Big\langle g\,,\,e^{-R^D}\omega\otimes\omega\Big\rangle. \tag{2.13}$$

The following remarkable properties holds for U_Q:

(i) It is finite on $\mathcal{C}(\omega) := \Big\{g:\mathcal{X}\to\mathbb{R}\,\Big|\,e^{U_Q(g,\omega)}<\infty\Big\}$

(ii) It is monotone.

(iii) It is additively homogeneous.

(iv) It is convex in g.

For $\pi\in\mathcal{M}(\mathcal{X}\times\mathcal{X})$, we observe that $I_\omega(\pi)$ is the Kullback action of the marked SINR graph X^λ.

Lemma 2.6.1 *The following hold for the Kullback action or divergence function $I_\omega(\pi)$:*

(i)

$$I_\omega(\pi)=\sup_{g\in\mathcal{C}}\Big\{\langle g,\,\pi\rangle-\langle g,\,e^{-R^D}\omega\otimes\omega\rangle\Big\}$$

(ii) The function $I_\omega(\pi)$ is convex and lower semi-continuous on the space $\mathcal{M}(\mathcal{X}\times\mathcal{X})$.

(iii) For any real α, the set $\Big\{\pi\in\mathcal{M}(\mathcal{X}\times\mathcal{X}):\,I_\omega(\pi)\le\alpha\Big\}$ is weakly compact.

Interested readers may refer to Doku-Amponsah (2017) for similar proof for empirical measures of the Typed Random Graph Processes, and/or the references therein for proof of the lemma for empirical measures on measurable spaces.

Now note from Lemma 2.6.1, for any $\pi\in\mathcal{M}(\mathcal{X}\times\mathcal{X})$ and $\varepsilon>0$, there exists a function $g\in\mathcal{C}(\omega)$ such that

$$I_\omega(\pi)-\tfrac{\varepsilon}{2}<\langle g\,,\,\pi\rangle-U_Q(g,\omega).$$

Define the probability distribution P_ω by

$$P_\omega(x)=\prod_{(i,j)\in E}e^{g(x_i,x_j)}\prod_{(i,j)\in\mathcal{E}}e^{h_\lambda(x_i,x_j)},$$

where

$$h_\lambda(x,y)=\lambda\log\Big[(1-e^{-\lambda R_\lambda^D(x,y)}+e^{-\lambda R_\lambda^D(x,y)+g(x,y)/\lambda})\Big]$$

Then, observe that

$$\frac{dP_\omega}{d\tilde{P}_\omega}(x)=\prod_{(i,j)\in E}e^{-g(x_i,x_j)/\lambda}\prod_{(i,j)\in\mathcal{E}}e^{-h_\lambda(x_i,x_j)/\lambda}$$
$$=e^{-\lambda(\langle\frac{1}{2}g,L_2^\lambda\rangle-\lambda\langle\frac{1}{2}h_\lambda,L_1^\lambda\otimes L_1^\lambda\rangle)+\langle\frac{1}{2}h_\lambda,L_\Delta^\lambda\rangle}$$

Now, we define the neighbourhood of ν, B_ν by

$$B_\nu := \Big\{\pi \in \mathcal{M}(\mathcal{X} \times \mathcal{X}) : \langle g, \pi\rangle > \langle g, \nu\rangle - \varepsilon/2\Big\}$$

Under the condition $L_2^\lambda \in B_\nu$ we have

$$\frac{dP_\omega}{d\tilde{P}_\omega} < e^{-\langle \frac{1}{2}g,\nu\rangle + U_Q(g,\omega) + \lambda\frac{\varepsilon}{2}} < e^{-\lambda I_\omega(\nu)+\lambda\varepsilon}$$

hence,

$$P_\omega\Big\{x^\lambda \in \mathcal{G}_P \Big| L_2^\lambda \in B_\nu\Big\} \leq \int 1\!\!1_{\{L_2^\lambda \in B_\nu\}} d\tilde{P}_\omega(x^\lambda) \leq \int e^{-\lambda I_{\omega(\nu)} - \lambda\varepsilon} d\tilde{P}_\omega(x^\lambda)$$
$$\leq e^{-\lambda I_\omega(\nu) - \lambda\varepsilon}.$$

.

Observe that $I_\omega(\nu) = \infty$ satisfies the Theorem 2.2.4 (ii). Hence it is sufficient for us to establish it for a probability measure of the form $\nu = ge^{-R^D}\omega \otimes \omega$, where $g = 1$ and for $I_\omega(\nu) = 0$. Fix any number $\varepsilon > 0$ and any neigbourhood $B_\nu \subset \mathcal{M}(\mathcal{X} \times \mathcal{X})$. Now define the sequence of sets

$$\mathcal{G}_P^\lambda = \Big\{y \in \mathcal{G}_P : L_2^\lambda(y) \in B_\nu, \Big|\langle g, L_2^\lambda\rangle - \langle g, \nu\rangle\Big| \leq \tfrac{\varepsilon}{2}\Big\}.$$

Note that for all $y \in \mathcal{G}_P^\lambda$ we have

$$\frac{dP_\omega}{d\tilde{P}_\omega} > e^{-\langle \frac{1}{2}g,\nu\rangle + U_Q(g,\omega) + \lambda\frac{\varepsilon}{2}} > e^{\lambda\varepsilon}.$$

This yields

$$P_\omega(\mathcal{G}_P^\lambda) = \int_{\mathcal{G}_P^\lambda} dP_\omega(y) \geq \int e^{-\langle \frac{1}{2}g,\nu\rangle + U_Q(g,\omega) + \lambda\frac{\varepsilon}{2}} d\tilde{P}_\omega(y) \geq e^{\lambda\varepsilon}\tilde{P}_\omega(\mathcal{G}_P^\lambda).$$

Using the law of large numbers, we have that $\lim_{\lambda\to\infty} \tilde{P}_\omega(\mathcal{G}_P^\lambda) = 1$. This completes of the Theorem.

Proof of Corollary 2.2.5

We observe that, by Lemma 2.4.3 the law of empirical connectivity measure is exponentially tight. Henceforth, without loss of generality we can assume that the set F in Theorem 2.2(ii) above is relatively compact. If we choose any $\varepsilon > 0$; then for each functional $\nu \in F$ we can find a weak neigbourhood such that the estimate of Theorem 2.2(i) above holds. From all these neigbhourhood, we choose a finite cover of $\mathcal{G}_P$ and sum up over the estimate in Theorem 2.2(i) above to obtain

$$\limsup_{\lambda\to\infty} \frac{1}{\lambda} \log \mathbb{P}_\omega \Big\{ X^\lambda \in \mathcal{G}_P \,\Big|\, L_2^\lambda \in F \Big\} \leq -\inf_{\pi\in F} I_\omega(\pi) + \varepsilon.$$

Since ε was arbitrarily chosen and the lower bound in Theorem 2.2(ii) implies the lower bound in Theorem 2.2(i) we have the required results which completes the proof.

2.7 Summary

Given devices space D, an intensity measure $\lambda m \in (0, \infty)$, a transition kernel Q from the space D to positive real numbers $\mathbb{R}_+$, a path-loss function (which depends on the Euclidean distance between the devices and a positive constant α), we define a Marked Poisson Point Process (MPPP). For a given MPPP and technical constants $\tau_\lambda, \gamma_\lambda : (0, \infty) \to (0, \infty)$, we define a Marked Signal-to-Interference and Noise Ratio (SINR) graph, and associate with it two empirical measures; the *empirical marked measure* and the *empirical connectivity measure*.

For a class of marked SINR graphs, we prove a joint *large deviation principle*(LDP) for these empirical measures, with speed λ in the τ-topology. From the joint large deviation principle for the empirical marked measure and the empirical connectivity measure, we obtain an Asymptotic Equipartition Property(AEP) for network structured data modelled as a marked SINR graph.

Specifically, we show that for large dense marked SINR graph one require approximately about $\lambda^2 H(Q \times Q)/\log 2$ bits to transmit the information contained in the network with high probability, where $H(Q \times Q)$ is a properly defined entropy for the exponential transition kernel with parameter c.

Further, we prove a *local large deviation principle* (LLDP) for the class of marked SINR graphs on D, where $\lambda[\tau_\lambda(\sigma_x)\gamma_\lambda(\sigma_y) + \lambda\tau_\lambda(\sigma_y)\gamma_\lambda(\sigma_x)] \to \beta(\sigma_x, \sigma_y)$, $\sigma_x, \sigma_y \in (0, \infty)$, with speed λ from a *spectral potential* point of view. From the LLDP we derive a conditional LDP for the marked SINR graphs.

Note that, while the joint LDP is established in the τ-topology, the LLDP assume no topological restriction on the space of marked SINR graphs. Observe also that all our rate functions are expressed in terms of the *relative entropy* or the *kullback action* or *divergence function* of the marked SINR on the devices space D

Chapter 3

Large Deviation Principle for empirical SINR measure for Critical Telecommunication Network

This chapter has already appeared in co-authored paper Sakyi-Yeboah et al. (2021b)

3.1 Introduction and Background

3.1.1 Introduction

Since the inception of the 19th century, the world has experienced renaissance in the information theory of wireless communication channel and wireless networks. Further, multimedia technologies have seen significant growth in the last two decades. The use of handheld devices and obtaining services offered by the Internet has now become essential in our daily lives. Therefore, the availability of wireless networks and network quality of service (QoS) offer have become vital for mobile users. See, (Hassan et al., 2019) .

Currently, telecommunication is simply an electrical medium of connecting over a distance (location and battery power). Telecommunication was discovered as an electrical waves and it is suggested that they could travel at a speed close to the speed of light. See,Paudel and Bhattarai (2018). The fundamental requirement of routing through any telecommunication network whether it is via voice call or data package; is that each end point on the network has a unique address which enables wireless communication. Cellular systems are now nearly universally deployed and are under ever-increasing pressure to increase the volume of data they can deliver to consumers. Refer to (Andrews et al., 2011) or the references therein.

Now, the advent of multimedia interactive services and the surge in the number of interconnected devices have led to investigation of new approaches that can enhance wireless capacity in 5G networks. See, example (Aravanis et al., 2019). Marvi et al. (2019) posited that, 8.3 billion hand-held devices and 3.3 billion Machine-to-Machine (M2M) devices will be connected by 2021. The number of connected devices would clearly exceed the expected global population of 7.8 billion by that time. The monthly global mobile data traffic is expected to reach 49 exabytes and the annual traffic will exceed half a zettabyte by 2021.

In information theory and telecommunication engineering, wireless consist of nodes which connect over a wireless channel (Gupta and Kumar, 2000). Signal -to- Interference-Plus- Noise Ratio (SINR) is a tool used as rate of information transfer in wireless communication system such as networks. According to Jeske and Sampath (2004), the SINR is an important metric of wireless communication link quality. SINR estimates have several important applications. These include optimizing the transmission power level for a target quality of service, assisting with handoff decisions and dynamically adapting the data rate for wireless internet applications. Communication performance can be improved significantly by adaptive transmissions based on the quality of received signals, i.e., the Signal-to-Interference-plus-Noise Ratio (SINR) (Choi et al., 2013) .

In cellular networks, SINR is a quantity that indicates if a given frequency resource is suitable to properly maintain a communication link. This is the rationale behind the usage in SINR in network to monitor the occurrence of radio link and handover failures, see (Bastidas-Puga et al., 2018) . An accurate SINR estimation provides for both a more efficient system and a higher user

− perceived quality of service. Thus, the SINR is popularly used in wireless connection as a way to measure the quality of wireless connection within the space.

Analogous to the SNR used often in wired communications systems, the SINR is defined as the power of a certain signal of interest divided by the sum of the interference power (from all the other interfering signals) and the power of some background noise. If the power of noise term is zero, then the SINR reduces to the Signal-to-Interference Ratio (SIR). Conversely, zero interference reduces the SINR to the Signal-to-Noise Ratio (SNR), which is used less often when developing mathematical models of wireless networks such as cellular networks.

Agrawal and Kshetrimayum (2017) derived the IPI, ISI and average output SINR expressions for the Binary Phase Shift Keying (BSPK) modulated raised-cosine pulse in the beam forming-based mm-Wave MIMO system. At each receiving antenna, they use the coherent Rake receiver to capture the signal energy, carried by multipath components in the complete IEEE 802.15.3c channel model. Additionally, their paper presents the impacts of pulse duration and Half Power Beam Width (HPBWs) of the transmitting antennas on the average output SINR.

Choi et al. (2013) focused on developing link scheduling schemes that can achieve optimal performance under the SINR model. The underlying argument was to treat an adaptive wireless link as multiple parallel virtual links with different signal quality, building on which they develop throughput-optimal scheduling schemes using a two-stage queuing structure in conjunction with recently developed carrier-sensing techniques. Furthermore, they introduced a novel three-way handshake to ensure, in a distributed manner, that all transmitting links satisfy their SINR requirements.

Keeler et al. (2013) worked on an explicit integral interaction for SINR distribution experienced by a typical user in the downlink channel from the k-th strongest base stations of a cellular network modelled by Poisson point process on the plane. The outcome of their work shows that the whole domain of SINR was valid whenever SINR < 1, where one observes multiple coverage.

Aravanis et al. (2019) employs MGF to provide closed form expressions for the downlink ergodic capacity for the interference limited case, and validated the accuracy of these expressions by the use of extensive Monte Carlo simulations.

Weiss (1995) used large deviations techniques to analyze models of communication networks. It was assumed that the points form a sequence of independent and identical distributed random variables progressing to some powerov processes in discrete or continuous time. Giuliano and Macci (2014) studied the sequences of independent and identical distributed random variables and, under suitable conditions on the (common) distribution function, they proved large deviation principles for sequences of maxima, minima and pairs formed by maxima and minima. They assumed that the independent and identical distributed random variables can either be unbounded or bounded; in the first case maxima and minima have to be suitably normalized.

Duffy et al. (2011) under suitable assumptions about the large deviation behavior of the state selection and sojourn processes, proved that the empirical laws of the phase process satisfy a sample path large deviation principle. From this large deviation principle, the large deviations behavior of a class of modulated additive processes was deduced and an alternate proof of results for modulated Lévy processes were obtained. With a practical application of the results, they calculated the large deviation rate function for a process that arises as the International Telecommunications Union's standardized stochastic model of two-way conversational speech. Other researchers have also studied large deviation behaviour of the interference in a wireless communication model. See, (Ganesh and Torrisi, 2008) .

In this chapter, we prove a joint large deviation principle for the empirical powered measure, empirical link measure and the empirical SINR measure of critical Telecommunication networks. In this sequel, we prove a joint large deviation principle for the empirical powered measure and empirical link measure of the Critical Telecommunication Network. See, Sakyi-Yeboah et al. (2020) for similar results for the dense Telecommunication Networks. The main techniques used in this chapter are the Gartner-Ellis Theorem, see (Zeitouni and Dembo, 1998, Theorem 2.3.6) and the method of Mixtures as deployed in the PhD Thesis by Doku-Amponsah (2006).

The remaining part of this chapter is organized in the following manner. Section 3.2 contains the statement of the main results; the joint LDP for the empirical powered measure, empirical link measure and empirical SINR measure, the joint LDP for the empirical powered measure and the empirical link measure, and the conditional LDP for the empirical SINR measure given the empirical powered measure and the empirical link measure. In Section 3.3 we give the proofs of Theorem 3.2.2(i) and Theorem 3.2.3. Section 3.4 gives the proofs of Theorem 3.2.2(ii) and Theorem 3.2.1.

3.1.2 Background

For a fix dimension $d \in \mathbb{N}$ and a measureable set $D \subset \mathbb{R}^d$ with respect to the Borel-Sigma algebra $\mathcal{B}(\mathbb{R}^d)$. Denote by m the Lebesgue's measure on $\mathbb{R}^d$. Let $\lambda m : D \to [0, 1]$, be a rate measure, Q a transition kernel from D to $(0, \infty)$ and $\ell(r) = r^{-\alpha}$, where $\alpha \in (0, \infty)$, be a path loss function and $\tau^{(\lambda)}, \gamma^{(\lambda)} : (0\,, \infty) \to (0\,, \infty)$, be technical constants. We define the SINR Graph as follows:

(i) Pick $X = (X_i)_{i \in I}$ a Poisson Point process (PPP) with rate measure $\lambda m : D \to [0, 1]$.

(ii) For X, we assign each X_i a power $\sigma(X_i) = \sigma_i$ independently according to the transition function $Q(\cdot\,, X_i)$.

(iii) For any two powered points $((X_i, \sigma_i), (X_j, \sigma_j))$ we connect a link if and only if

$$SINR(X_i, X_j, X) \geq \tau^{(\lambda)}(\sigma_j) \text{ and } SINR(X_j, X_i, X) \geq \tau^{(\lambda)}(\sigma_i),$$

where

$$SINR(X_j, X_i, X) = \frac{\sigma_i \ell(\|X_i - X_j\|)}{N_0 + \gamma^{(\lambda)}(\sigma_j) \sum_{i \in I \setminus \{j\}} \sigma_i \ell(\|X_i - X_j\|)}$$

We shall consider $X^\lambda(m, Q, \ell) = \Big\{[(X_i, \sigma_i), i \in I],\, E\Big\}$ under the joint law of the powered Poisson Point process and the network. We will interpret X^λ as an SINR network and $(X_i, \sigma_i) := X_i^\lambda$ as the power of device i. We recall Proposition 1 from Sakyi-Yeboah et al. (2020) as follows:

Proposition 1 (Sakyi-Yeboah et al. (2020)) *The link probability of the SINR network, p_λ, is given by*

$$p_\lambda[(x, \sigma_x), (y, \sigma_y)] = e^{-\lambda h_\lambda^D[(x,\sigma_x),(y,\sigma_y)]},$$

$$h_\lambda^D((x, \sigma_x), (y, \sigma_y)) = \int_D \left[\frac{\tau^{(\lambda)}(\sigma_x)\gamma^{(\lambda)}(\sigma_x)}{\tau^{(\lambda)}(\sigma_x)\gamma^{(\lambda)}(\sigma_x)+(\|z\|^\alpha/\|x-y\|^\alpha)} + \frac{\tau^{(\lambda)}(\sigma_y)\gamma^{(\lambda)}(\sigma_y)}{\tau^{(\lambda)}(\sigma_y)\gamma^{(\lambda)}(\sigma_y)+(\|z\|^\alpha/\|y-x\|^\alpha)}\right] m(dz).$$

We assume that there is a sequence of real numbers a_λ and a function $h : \mathcal{D} \times \mathbb{R}_+ \to (0, \infty)$ such that $\lambda^2 a_\lambda \to \infty$ and

$$\lim_{\lambda \uparrow \infty} p_\lambda[(x, \sigma_x), (y, \sigma_y)] = h[(x, \sigma_x), (y, \sigma_y)]$$

We shall call X^λ Critical SINR if $\lambda a_\lambda \to 1$, Sub critical SINR if $\lambda a_\lambda \to 0$ and Super critical SINR if $\lambda a_\lambda \to \infty$. In this chapter, we shall look at Critical SINR network, ie

$$\lim_{\lambda\to\infty} \lambda a_\lambda \to 1$$

Note that from the subsequent chapters, we will denote R_λ^D as h_λ^D.

$$\mathcal{S}(D) = \cup_{x\subset D}\Big\{x:\ |x\cap A| < \infty \text{ , for any bounded } A\subset D\ \Big\}. \tag{3.1}$$

Write $\mathcal{X} = \mathcal{S}(D\times\mathbb{R}_+)$ and denote by $\mathcal{M}(\mathcal{X})$, the space of positive measures on the space $\mathcal{X}$ equipped with $\tau-$ topology. Henceforth, we shall call $\mathcal{X}$ a locally finite subset of the set $\mathcal{S}(D)$.

Empirical measures of the SINR Networks: For any SINR graph X^λ we define a probability measure, the *empirical power measure*, $L_1^\lambda \in \mathcal{M}(\mathcal{X})$, by

$$L_1^\lambda\big((x,\sigma_x)\big) := \frac{1}{\lambda}\sum_{i\in I}\delta_{X_i^\lambda}\big((x,\sigma_x)\big)$$

and a symmetric finite measure, the *empirical pair measure* $L_2^\lambda \in \mathcal{M}(\mathcal{X}\times\mathcal{X})$, by

$$L_2^\lambda\big((x,\sigma_x),(y,\sigma_y)\big) := \frac{1}{\lambda}\sum_{(i,j)\in E}[\delta_{(X_i^\lambda,X_j^\lambda)} + \delta_{(X_j^\lambda,X_i^\lambda)}]\big((x,\sigma_x),(y,\sigma_y)\big).$$

Note that the total mass $\|L_1^\lambda\|$ of the empirical power measure is $\mathbb{1}$ and total mass of the empirical link measure is $2|E|/\lambda^2$.

Theorem 3.1.1 (Sakyi-Yeboah et al. (2020)) *Suppose X^λ is an SINR network with rate measure $\lambda m : D \to [0,1]$ and a power probability function q from D to $(0,\infty)$ and path loss function $\ell(r) = r^{-\alpha}$, for $\alpha > 0$. Then, as $\lambda\to\infty$, L_1^λ satisfies an LDP in the space $\mathcal{M}(\mathcal{X})$ with good rate function*

$$I_1(\omega) = \begin{cases} H(\omega\,|m\otimes q), & \textit{if } \|\omega\| = 1 \\ \infty & \textit{otherwise.} \end{cases}$$

3.2 Statement of Main Results

We, define the empirical SINR measure $\mathcal{L}^{1,2}_{(x,\sigma_x)}$ by

$$\mathcal{L}^{1,2}_{(x,\sigma_x)}(a) := \frac{1}{\|N_\lambda((x,\sigma_x))\|} \sum_{i \in N_\lambda([x,\sigma])} \delta_{SINR(X_i^\lambda,(x,\sigma_x),L_1^\lambda)}(a),$$

where

$$N_\lambda(x) = \Big\{ k \in I : (k,j) \in E,\, X_j^\lambda = x \Big\}$$

We observe that we have

$$\|N_\lambda((x,\sigma_x))\| = \lambda \int_{\mathcal{X}} L_2^\lambda((x,\sigma_x),(dy,d\sigma_y))$$

and

$$\mathcal{L}^{1,2}_{(x,\sigma_x)}(a) = \frac{1}{\|N^\lambda((x,\sigma_x))\|} \int_{\mathcal{X}} \Phi_a^\lambda\Big((x,\sigma_x)\,,\,(y,\sigma_y)\,,\,L_1^\lambda\Big) L_2^\lambda((dy,d\sigma_y),(dx,d\sigma_x)),$$

where

$$\Phi_a^\lambda\Big((x,\sigma_x)\,,\,(y,\sigma_y)\,,\,\omega\Big) = 1\!\!1_{\Big\{\tau^{(\lambda)}(\sigma_y) \le Sinr\big((y,\sigma_y),(x,\sigma_x)\,,\omega\big) \le a(\sigma_x)\Big\}}$$

We write

$$\mathcal{L}^{1,2} := \Big(\mathcal{L}^{1,2}_{(x,\sigma_x)},\,(x,\sigma_x) \in \mathcal{X}\Big).$$

Theorem 3.2.1, is a Joint Large deviation principle for the empirical measures of the Sinr network models. We recall from Subsection 3.1.2 the definition of h_λ^D as

$$h_\lambda^D((x,\sigma_x),(y,\sigma_y)) = \int_D \Big[\tfrac{\tau^{(\lambda)}(\sigma_x)\gamma^{(\lambda)}(\sigma_x)}{\tau^\lambda(\sigma_x)\gamma^\lambda(\sigma_x)+(\|z\|^\alpha/\|x-y\|^\alpha)} + \tfrac{\tau^{(\lambda)}(\sigma_y)\gamma^{(\lambda)}(\sigma_y)}{\tau^{(\lambda)}(\sigma_y)\gamma^{(\lambda)}(\sigma_y)+(\|z\|^\alpha/\|y-x\|^\alpha)}\Big] m(dz)$$

and write

$$h_*\omega \otimes \omega((x,\sigma_x),(y,\sigma_y))) := h_*((x,\sigma_x),(y,\sigma_y))\omega((x,\sigma_x))\omega((y,\sigma_y)).$$

We define $\Big\langle f_\omega,\, \pi\Big\rangle$ by

$$\Big\langle f_\omega,\, \pi\Big\rangle_{(x,\sigma_x)}(a) = \frac{1}{\pi_2((x,\sigma_x))}\int_{\mathcal{X}} f_a((x,\sigma_x),(y,\sigma_y),\omega)\pi(dz\,,\,(x,\sigma)).$$

Observe that, for a finite probability measure π

$$\Big\langle \Phi_\omega^\lambda,\, \pi\Big\rangle_{(x\,,\,\sigma_x)}(a) = \frac{1}{\pi_2((x,\sigma_x))}\int_{\mathcal{X}} \Phi_a^\lambda\Big((y,\sigma_y)\,,\,(x,\sigma_x)\,,\,\omega\Big)\pi((dy,d\sigma_y)\,,\,(dx,d\sigma_x)), \tag{3.2}$$

is a probability measure.

We write $\lim_{\lambda\to\infty} \Phi_a^\lambda = \Phi_a$ and note that

$$\lim_{\lambda\to\infty}\Big\langle \Phi_\omega^\lambda,\, \pi\Big\rangle_{(x,\sigma_x)}(a) = \Big\langle \Phi_\omega,\, \pi\Big\rangle_{(x,\sigma_x)}(a), \quad \text{for all } a\in[\tau,\infty),$$

by he dominated convergence theorem.

Theorem 3.2.1 *Let X^λ be a critical powered SINR network with rate measure $\lambda m : D \to [0,1]$ and a power probability function q from D to $(0,\,\infty)$ and path loss function $\ell(r) = r^{-\alpha}$, for $\alpha > 0$. Suppose q is an exponential distribution with mean $1/c$. Then, as $\lambda\to\infty$, the triplet $(L_1^\lambda, L_2^\lambda, \mathcal{L}^{1,2})$ satisfies a large deviation principle in the space $\mathcal{M}(\mathcal{X})\times\mathcal{M}(\mathcal{X}\times\mathcal{X})\times\mathcal{M}([\tau,\,\infty))$ with speed λ and good rate function*

$$\begin{aligned} J_*(\omega,\pi,\nu) = H\Big(\omega\Big|m\otimes q\Big) + \frac{1}{2}\mathcal{H}\Big(\pi\|h_*\omega\otimes\omega\Big) \\ + \frac{1}{2}\int_{\mathcal{X}} H\Big(\nu_{(x,\sigma_x)}\Big\|\Big\langle \Phi_\omega,\pi\Big\rangle_{[x\,,\,\sigma_x]}\Big)\pi_2((dx,d\sigma_x)) \end{aligned} \tag{3.3}$$

Remark 7 Interpretation of the Rate Function: *The rate function can be regarded as the cost of having a powered SINR network corresponding to the empirical measures triplet* (ω, π, ν)*. This cost may be divided into three separate costs:*

(i) *The first term is the cost of having the empirical powered measure* ω *is known. This cost is non-negative and it is zero iff* $\omega = m \otimes q$*.*

(ii) *The second term is the cost of obtaining the empirical link measure* π *given the empirical powered measure* ω*. This cost is also non-negative and it is zero iff* $\pi = h_*\omega \otimes \omega$

(iii) *The last term is the cost of obtaining the empirical SINR measure given the empirical powered measure* ω *and the empirical link measure* π*. This cost is also non-negative and it is zero iff* $\nu = \Big\langle \Phi_\omega, \pi \Big\rangle$*.*

This implies the cost $J_*\big(\omega, \pi, \nu\big) = 0$ *iff* $\omega = m \otimes q$*,* $\pi = h_*(m \otimes q) \otimes (m \otimes q)$ *and*

$$\nu = \Big\langle \Phi_{m\otimes q}, h_*(m \otimes q) \otimes (m \otimes q) \Big\rangle$$

We write $B_y(s) := \{x : \|x - y\| < s\}$, $\mathcal{B}^t_{(\sigma_x,\sigma_y)} := B_y\Big(\Big[\frac{c\sigma_y}{\tau(\sigma_x)\gamma(\sigma_x)}\Big]^{1/\alpha}\Big[\int_D \|z - x\|^{-\alpha} m(dz)\Big]^{-1/\alpha}\Big)$ and note that the typical behaviour of the empirical SINR measure is as

$$\begin{aligned}
\nu_{(x,\sigma_x)}(a) &= \Big\langle \Phi_{m\otimes q}, h_*(m \otimes q) \otimes (m \otimes q) \Big\rangle_{(x,\sigma_x)}(a) \\
&= \int_{\mathcal{X}} \Phi_a\Big((x, \sigma_x), (y, \sigma_y), m \otimes q\Big) \frac{h_*((x, \sigma_x), (y, \sigma_y))m(dy)q(d\sigma_y)}{\int_{\mathcal{X}} h_*((x, \sigma_x), (y, \sigma_y))m(dy)q(d\sigma_y)} \\
&= \int_{\mathbb{R}_+} \int_D \mathbb{1}_{\mathcal{B}^\tau_{(\sigma_x,\sigma_y)} \setminus \mathcal{B}^a_{(\sigma_x,\sigma_y)}}(x) \frac{e^{-c\sigma_y} h_*((x, \sigma_x), (y, \sigma_y))m(dy)d\sigma_y}{\int_{\mathbb{R}_+} \int_D e^{-c\sigma_y} h_*((x, \sigma_x), (y, \sigma_y))m(dy)d\sigma_y}
\end{aligned}$$

where $B \setminus A = B \cap A^c$ and $\mathbb{1}_\Gamma(x)$ denote the indicator function on the set Γ.
Theorem 3.2.2 below is a conditional large deviation principle for the empirical link measure given the empirical power measure, and joint LDP for the empirical

power measure and empirical link measure of the SINR network model. We define a relative entropy $\mathcal{H}$ by

$$\mathcal{H}(\pi\|h_{*}\omega\otimes\omega):=\begin{cases} H(\pi\,\|\,h_{*}\omega\otimes\omega)+\Big(\|h_{x}\omega\otimes\omega\|-\|\pi\|\Big), & \text{if } \|\pi\|>0.\\ \infty & \text{otherwise.}\end{cases} \tag{3.4}$$

Theorem 3.2.2 *Let X^{λ} is an SINR network with rate measure $\lambda m : D\to[0,1]$ and a powered probability function q from D to $(0,\infty)$ and path loss function $\ell(r)=r^{-\alpha}$, for $\alpha>0$. Suppose q is an exponential distribution with parameter c.*

(i) Then, as $\lambda\to\infty$, conditional on the event $L_1^{\lambda}=\omega$, L_2^{λ} satisfies a large deviation principle in the space $\mathcal{M}(\mathcal{X}\times\mathcal{X})$ with speed λ and good rate function

$$I_{\omega}(\pi)=\frac{1}{2}\mathcal{H}(\pi\|h_{*}\omega\otimes\omega) \tag{3.5}$$

(ii) Then as $\lambda\to\infty$, the pair $(L_1^{\lambda},L_2^{\lambda})$ satisfies a large deviation principle in the space $\mathcal{M}(\mathcal{X})\times\mathcal{M}(\mathcal{X}\times\mathcal{X})$ with speed λ, and good rate function

$$I(\omega,\pi)=H\Big(\omega\,\Big|\,m\otimes q\Big)+\frac{1}{2}\mathcal{H}(\pi\|h_{*}\omega\otimes\omega), \tag{3.6}$$

where

$$h_{*}\omega\otimes\omega((x,\sigma_x),(y,\sigma_y))=h_{*}((x,\sigma_x),(y,\sigma_y))\omega((x,\sigma_x))\omega((y,\sigma_y)).$$

Theorem 3.2.3 *Let X^{λ} is a critical powered SINR network with rate measure $\lambda m : D\to[0,1]$ and a powered probability function q from D to $(0,\infty)$ and path loss function $\ell(r)=r^{-\alpha}$, for $\alpha>0$. Suppose X is an SINR network conditional on the event $\Big\{(L_1^{\lambda},L_2^{\lambda})=(\omega,\,\pi)\Big\}$. Then, as $\lambda\to\infty$, the empirical SINR measure*

$\mathcal{L}^{1,2}$ satisfies an LDP in the space $\mathcal{M}([\tau,\,\infty))$ with speed λ and good rate function

$$\tilde{J}(\nu) = \frac{1}{2}\int_{\mathcal{X}} H\Big(\nu_{(x,\sigma_x)}\Big\|\,\langle\Phi_\omega,\pi\rangle_{(x,\sigma_x)}\Big)\pi_2((dx,d\sigma_x)),$$

where π_2 denote second marginal of the finite measure π.

3.3 Proof of Theorem 3.2.2 and Theorem 3.2.1

3.3.1 Proof of Theorem 3.2.2(i) by Gartner-Ellis Theorem

Suppose $A_1, ..., A_n$ is a decomposition of the space $D\times\mathbb{R}_+$. Observe that, for every $(x, y) \in A_i \times A_j$, $i, j = 1, 2, 3, ..., n$, $\lambda L_2^{\lambda}(x, y)$ given $\lambda L_1^{\lambda}(x) = \lambda\omega(x)$ is binomial distributed with parameters $\lambda^2\omega(x)\omega(y)/2$ and $p_\lambda(x, y)$. Let Q be the exponential distribution with parameter c. We recall the function R_λ^D from the previous sections as follows:

$$h_\lambda^D((x,\sigma_x),(y,\sigma_y)) = \int_D \Big[\frac{\tau^{(\lambda)}(\sigma_x)\gamma^{(\lambda)}(\sigma_x)}{\tau^{(\lambda)}(\sigma_x)\gamma^{(\lambda)}(\sigma_x)+(\|z\|^\alpha/\|x-y\|^\alpha)}+\frac{\tau^{(\lambda)}(\sigma_y)\gamma^{(\lambda)}(\sigma_y)}{\tau^{(\lambda)}(\sigma_y)\gamma^{(\lambda)}(\sigma_y)+(\|z\|^\alpha/\|y-x\|^\alpha)}\Big]m(dz).$$

Lemma 3.3.1 is key component in the application of the Gartner-Ellis Theorem.

Lemma 3.3.1 *Let X^λ be an SINR network with rate measure $\lambda m : D \to [0, 1]$ and a powered probability function q from D to $(0,\infty)$ and path loss function $\ell(r) = r^{-\alpha}$, for $\alpha > 0$, conditional on the event $L_1^\lambda = \omega$. Let $g : \mathcal{X}\times\mathcal{X} \to \mathbb{R}$ be bounded function. Then,*

$$\begin{aligned}\lim_{\lambda\to\infty}\frac{1}{\lambda}\log\mathbb{E}\Big\{e^{\lambda\langle g,\,L_2^\lambda\rangle}\Big|L_1^\lambda=\omega\Big\} &= \frac{1}{2}\lim_{n\to\infty}\sum_{j=1}^{n}\sum_{i=1}^{n}\Big\langle 1-e^g,\,h_*\omega\otimes\omega\Big\rangle_{A_i\times A_j}\\ &= \frac{1}{2}\Big\langle 1-e^g,\,h_*\omega\otimes\omega\Big\rangle_{\mathcal{X}\times\mathcal{X}}.\end{aligned}$$

Proof: Now we observe that

$$\mathbb{E}\Big\{e^{\int\int \lambda g(x,y)L_2^\lambda(dx,dy)/2}\Big|L_1^\lambda=\omega\Big\}=\mathbb{E}\Big\{\prod_{x\in\mathcal{X}}\prod_{y\in\mathcal{X}}e^{\lambda g(x,y)L_2^\lambda(dx,dy)/2}\Big\}$$

$$\mathbb{E}\Big\{\prod_{x\in\mathcal{X}}\prod_{y\in\mathcal{X}}e^{g(x,y)\lambda L_2^\lambda(dx,dy/2)}=\prod_{i=1}\prod_{j=1}\prod_{x\in A_i}\prod_{y\in A_j}\mathbb{E}\Big\{e^{g(x,y)\lambda L_2^\lambda(dx,dy)/2}\Big\}$$

$$\log\Big\{e^{\lambda\langle g,L_2^\lambda\rangle/2}\Big|L_1^\lambda=\omega\Big\}=\sum_{j=1}^{n}\sum_{i=1}^{n}\int_{A_j}\int_{A_i}\log\Big[1-p(x,y)+p(x,y)e^{g(x,y)}\Big]^{\lambda^2\omega\otimes\omega(dx,dy)/2}$$

By the dominated convergence theorem

$$\frac{1}{\lambda}\log E\{e^{\lambda\langle g,L_2^\lambda\rangle/2}\mid L_1^\lambda=\omega\}=$$

$$\frac{1}{\lambda}\sum_{j=1}\sum_{i=1}\int_{A_i}\int_{A_j}\log\Big[1-(1-e^{g(x,y)})p_\lambda(x,y)+o(\lambda)\Big]^{\lambda^2\omega\otimes\omega(dx,dy)/2}$$

$$\frac{1}{\lambda}\log E\{e^{\lambda\langle g,L_2^\lambda\rangle/2}\mid L_1^\lambda=\omega\}=$$

$$\frac{1}{\lambda}\sum_{j=1}\sum_{i=1}\int_{A_i}\int_{A_j}\log\Big[1-(1-e^{g(x,y)})p_\lambda(x,y)+o(\lambda)\Big]^{\lambda^2\omega\otimes\omega(dx,dy)/2}$$

$$\frac{1}{\lambda}\log\mathbb{E}\{e^{\lambda\langle g,L_2^\lambda\rangle/2}\|L_1^\lambda=\omega\}=$$

$$\lim_{\lambda\to\infty}\sum_{j=1}\sum_{i=1}\int_{A_i}\int_{A_j}\log\Big[1-(1-e^{g(x,y)})p_\lambda(x,y)+o(\lambda)\Big]^{\lambda\omega\otimes\omega(dx,dy)/2}$$

$$\lim_{\lambda\to\infty}\frac{1}{\lambda}\log\mathbb{E}\Big\{e^{\lambda\langle g,L_2^\lambda\rangle/2}\|L_1^\lambda=\omega\Big\}=\frac{1}{2}\sum_{j=1}\sum_{i=1}\int_{A_i}\int_{A_j}\Big[(1-e^{g(x,y)})h_*(x,y)\omega\otimes\omega(dx,dy)\Big]$$

$$\lim_{\lambda\to\infty}\frac{1}{\lambda}\log\mathbb{E}\{e^{\lambda\langle g,L_2^\lambda\rangle/2}\mid L_1^\lambda=\omega\}=\frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n}\Big\langle 1-e^g,\,h_*\omega\otimes\omega\Big\rangle_{A_i\times A_j}$$

$$\lim_{n\to\infty}\lim_{\lambda\to\infty}\frac{1}{\lambda}\log\mathbb{E}\{e^{\lambda\langle g,L_2^\lambda\rangle/2}\Big|L_1^\lambda=\omega\} = \frac{1}{2}\lim_{n\to\infty}\sum_{j=1}^{n}\sum_{i=1}^{n}\Big\langle 1-e^g,\, h_*\omega\otimes\omega\Big\rangle_{A_i\times A_j}$$
$$=\frac{1}{2}\Big\langle 1-e^g,\, h_*\omega\otimes\omega\Big\rangle_{\mathcal{X}\times\mathcal{X}}$$

Hence, by Gartner-Ellis theorem, conditional on the event $\Big\{L_1^\lambda=\omega\Big\}$, L_2^λ obey a large deviation principle with speed λ and rate function

$$I_\omega(\pi)=\frac{1}{2}\sup_g\Big\{\Big\langle g,\,\pi\Big\rangle_{\mathcal{X}\times\mathcal{X}}+\Big\langle 1-e^g,\, h_*\omega\otimes\omega\Big\rangle_{\mathcal{X}\times\mathcal{X}}\Big\}$$

which when solved, see example Doku-Amponsah (2006), would clearly reduces to the good rate function given by

$$I_\omega(\pi)=\frac{1}{2}\mathcal{H}(\pi\|h_*\omega\otimes\omega). \tag{3.7}$$

□

3.3.2 Proof of Theorem 3.2.3 by Gartner-Ellis Theorem

The first step in proof of Theorem 3.2.1 is a large deviation principle for the sequence of measures $\Big(\mathcal{L}^{1,2}_{(x,\sigma_x)},(x,\sigma_x)\in\mathcal{X}\Big)$ conditional on the set

$$\Big\{(L_1^\lambda,L_2^\lambda)=(\omega,\,\pi)\Big\}.$$

Lemma 3.3.2 *Let X^λ be a critical powered SINR network with rate measure $\lambda m: D\to[0,1]$ and a powered probability function q from D to $(0,\infty)$ and path loss function $\ell(r)=r^{-\alpha}$, for $\alpha>0$. Then, for every (x,σ_x), we have*

$$\lim_{\lambda\to\infty}\mathbb{P}\Big\{\tau^{(\lambda)}(\sigma_x)\le SINR((X_i,\sigma_i),(x,\sigma_x),\,L_1^\lambda)\le a(\sigma_x),\,(x,\sigma_x)\in\mathcal{X}\Big|(L_1^\lambda,\,L_2^\lambda)=(\omega,\pi)\Big\}$$

$$=\Big\langle\Phi_\omega\,,\,\pi\Big\rangle(a) \tag{3.8}$$

Proof: We compute the probability

$$\mathbb{P}\Big\{\tau^{(\lambda)}(\sigma_x) \le SINR((X_i,\sigma_i),(x,\sigma_x),\, L_1^{\lambda}) \le a(\sigma_x)\Big|(L_1^{\lambda},\, L_2^{\lambda}) = (\omega,\pi)\Big\}$$
$$= \frac{1}{\pi_2((x,\sigma_x))}\int_{\mathcal{X}} \Phi_a^{\lambda}\Big((x,\sigma_x)\,,\,(y,\sigma_y)\,,\,\omega\Big)\pi([dy,\, d\sigma_y],(dx,d\sigma_x))$$
$$= \Big\langle \Phi_{\omega}\,,\,\pi\Big\rangle_{(x\,,\,\sigma_x)}(a)$$

□ Taking limits as $\lambda \to \infty$ on both sides we have (3.8) which ends the proof of Lemma 3.3.2.

Lemma 3.3.3 *Let X^{λ} is a critical powered SINR graph with rate measure $\lambda m : D \to [0,1]$ and a powered probability function q from D to $(0,\infty)$ and path loss function $\ell(r) = r^{-\alpha}$, for $\alpha > 0$. Suppose q is an exponential distribution with parameter c. Then, for every (x,σ_x), we have*

$$\lim_{\lambda\to\infty}\frac{1}{\lambda}\log\mathbb{E}\Big\{e^{\int_{\mathcal{X}} N_{\lambda}((dx,d\sigma_x))\langle g,\,\mathcal{L}^{1,2}_{(x,\sigma_x)}\rangle}\Big|(L_1^{\lambda},\, L_2^{\lambda}) = (\omega,\,\pi)\Big\} =$$
$$\frac{1}{2}\Big\langle\, \log\Big\langle e^{g}\,,\,\Big\langle\Phi_{\omega}\,,\,\pi\Big\rangle_{\cdot}\Big\rangle_{[\tau,\infty)}\,,\,\pi_2\Big\rangle_{\mathcal{X}} \tag{3.9}$$

Proof: We observe that $\Big(\mathcal{L}^{1,2}_{(x,\sigma_x)}|N((x,\sigma_x)),\,(x,\sigma_x)\in\mathcal{X}\Big)$ are independent distributed as

$$\Big(\Big\langle\Phi_{\omega}^{\lambda}\,,\,\pi\Big\rangle_{(x\,,\,\sigma_x)},\,(x,\sigma_x)\in\mathcal{X}\Big).$$

$$\mathbb{E}\Big\{e^{\int_{\mathcal{X}}\Big\langle g,\,\mathcal{L}^{1,2}_{(x,\sigma_x)}\Big\rangle N_{\lambda}((dx,d\sigma_x))/2}\Big|(L_1^{\lambda},\, L_2^{\lambda}) = (\omega,\,\pi)\Big\}$$
$$= \prod_{(dx,d\sigma_x)\in\mathcal{X}}\mathbb{E}_{\Big\langle\Phi_{\omega}^{\lambda},\pi\Big\rangle_{[x\,,\,\sigma_x]}}\Big[\prod_{i\in N_{\lambda}([dx,dx])/2} e^{g(SINR(X_i^{\lambda},(x,\sigma_x),\omega))}\Big]$$
$$= \prod_{(dx,d\sigma_x)\in\mathcal{X}}\Big(\mathbb{E}_{\Big\langle\Phi_{\omega}^{\lambda},\pi\Big\rangle_{[x\,,\,\sigma_x]}}\Big[e^{g(SINR(X_i^{\lambda},(x,\sigma_x),\omega))}\Big]\Big)^{N_{\lambda}((dx,d\sigma_x))/2}$$
$$= \prod_{(dx,d\sigma_x)\in\mathcal{X}}\Big(\int_{\tau}^{\infty} e^{g(a)}\Big\langle\Phi_{\omega}^{\lambda}\,,\,\pi\Big\rangle_{(x\,,\,\sigma_x)}(da)\Big)^{N_{\lambda}((dx,d\sigma_x))/2} \tag{3.10}$$

Now taking limit of normalized logarithm of (3.9) using DCT and observed that

$N_\lambda((dx, d\sigma_x))/\lambda \to \pi_2((dx, d\sigma_x))$, $\Big\langle \Phi^\lambda_\omega\,,\,\pi\Big\rangle_{(x\,,\sigma_x)} \to \Big\langle \Phi_\omega\,,\,\pi\Big\rangle_{(x\,,\sigma_x)}$ as $\lambda \to \infty$ we have (3.9), which ends the proof of Lemma 3.3.3 $\square$

Now, by the Gartner-Ellis Theorem, Conditional on the event $\Big\{(L^\lambda_2, L^\lambda_1) = (\omega, \pi)\Big\}$,the probability measure $\mathcal{L}^{1,2}$ obeys an LDP with speed λ and rate function

$$\tilde{J}(\nu) = \frac{1}{2}\sup_g \Big\{\Big\langle\Big\langle g,\,\nu.\Big\rangle_{[\tau,\infty]},\,\pi_2\Big\rangle_{\mathcal{X}} - \Big\langle \log\Big\langle e^g\,,\,\Big\langle\Phi_\omega\,,\,\pi\Big\rangle_{.}\Big\rangle_{[\tau,\infty)}\,,\,\pi_2\Big\rangle_{\mathcal{X}}\Big\}.$$

Using the variational formulation of relative entropy we have that

$$\tilde{J}(\nu) = \frac{1}{2}\int_{\mathcal{X}} H\Big(\nu_{(x,\sigma_x)}\Big\|\,\Big\langle\Phi_\omega\,,\,\pi\Big\rangle_{(x\,,\sigma_x)}\Big)\pi_2((dx\,,\,d\sigma_x)),$$

which proves Theorem 3.2.1.

3.4 Proof of Theorem 3.1.1(ii) and Theorem 3.2.1 by Method of Mixtures

3.5 Proof of Theorem 3.1.1

For any $\lambda \in (0, \infty)$, we define

$$\mathcal{M}_\lambda(\mathcal{X}) := \Big\{\omega \in \mathcal{M}(\mathcal{X})\;:\;\lambda\omega(x) \in \mathbb{N} \text{ for all } x \in \mathcal{X}\Big\},$$
$$\tilde{\mathcal{M}}_\lambda(\mathcal{X}\times\mathcal{X}) := \Big\{\pi \in \tilde{\mathcal{M}}_*(\mathcal{X}\times\mathcal{X})\;:\;\lambda\,\pi(x,y) \in \mathbb{N},\;\text{ for all }\; x,y \in \mathcal{X}\times\mathcal{X}\Big\}.$$

We denote by $\Theta_\lambda := \mathcal{M}_\lambda(\mathcal{X})$ and $\Theta := \mathcal{M}(\mathcal{X})$. With

$$P^{(\lambda)}_{\omega_\lambda}(\eta_\lambda) := \mathbb{P}\big\{L^\lambda_2 = \eta_\lambda\,\big|\,L^\lambda_1 = \omega_\lambda\big\}\,,$$
$$P^{(\lambda)}(\omega_\lambda) := \mathbb{P}\big\{L^\lambda_1 = \omega_\lambda\big\}$$

$$P^{(\lambda)}_{(\omega_\lambda,\pi_\lambda)}(\nu_{(x,\sigma_x)}) := \mathbb{P}\Big\{\mathcal{L}^{1.2}_{(x,\sigma_x)} = \nu_{(x,\sigma_x)}\big|(L^\lambda_1, L^\lambda_2) = (\omega_\lambda, \pi_\lambda)\Big\}$$

the joint distribution of L^λ_1 and L^λ_2 is the mixture of $P^{(\lambda)}_{\omega_\lambda}$ with $P^{(\lambda)}(\omega_\lambda)$, and the joint distribution of $\mathcal{L}^{1,2}$, L_1^λ and L_2^λ is a mixture of $\tilde{P}^\lambda$ with $P^{(\lambda)}_{(\omega_\lambda,\pi_\lambda)}$ as follows:

$$d\tilde{P}^\lambda(\omega_\lambda,\eta_\lambda) := dP^{(\lambda)}_{\omega_\lambda}(\eta_\lambda)\,dP^{(\lambda)}(\omega_\lambda). \tag{3.11}$$

$$dP_\lambda(\nu,\omega_\lambda,\eta_\lambda) := dP^{(\lambda)}_{(\omega_\lambda,\pi_\lambda)}(\nu)d\tilde{P}^\lambda(\omega_\lambda,\eta_\lambda).$$

(Biggins et al., 2004, Theorem 5(b)) gives criteria for the validity of large deviation principles for the mixtures and for the goodness of the rate function if individual large deviation principles are known. The following three lemmas ensure validity of these conditions.

Observe that the family of measures $(P^{(\lambda)}\colon \lambda\in(0,\infty))$ is exponentially tight on Θ.

Lemma 3.5.1 *(i) The family of measures $(\tilde{P}^\lambda\colon \lambda\in(0,\infty))$ is exponentially tight on $\Theta\times\tilde{\mathcal{M}}_*(\mathcal{X}\times\mathcal{X})$.*

(ii) The family of measures $(P^\lambda\colon \lambda\in(0,\infty))$ is exponentially tight on $\Theta\times\tilde{\mathcal{M}}_*(\mathcal{X}\times\mathcal{X})\times\mathcal{M}([\tau,\infty)$.

Define the function $I\colon \Theta\times\mathcal{M}_*(\mathcal{X}\times\mathcal{X})\to[0,\infty]$, by

$$I(\omega,\pi) = H\Big(\omega\,\Big|m\otimes q\Big) + \mathcal{H}\Big(\pi\|h_*\omega\otimes\omega\Big) \tag{3.12}$$

and recall from Theorem 3.2.3 that

$$\tilde{J}(\nu) = \frac{1}{2}\int_{\mathcal{X}} H\Big(\nu_{(x,\sigma_x)}\,\Big\|\,\Big\langle\Phi_\omega\,,\,\pi\Big\rangle_{[x\,,\,\sigma_x]}\Big)\pi_2((dx,d\sigma_x)).$$

Lemma 3.5.2 *(i) I is lower semi-continuous.*

(ii) $\tilde{J}$ is lower semi-continuous.

By (Biggins et al., 2004, Theorem 5(b)) the two previous lemmas and the large deviation principles we have established in Theorem 3.2.2 and Theorem 3.2.3 ensure that under $(\tilde{P}^\lambda)$ and P_λ the random variables $(\omega_\lambda,\pi_\lambda)$ and $(\nu,\omega_\lambda,\pi_\lambda)$

satisfy a large deviation principle on $\mathcal{M}(\mathcal{X}) \times \tilde{\mathcal{M}}(\mathcal{X} \times \mathcal{X})$ and $\Theta \times \tilde{\mathcal{M}}_*(\mathcal{X} \times \mathcal{X}) \times \mathcal{M}([\tau, \infty)$ with good rate function I and $\tilde{J}$ respectively, which ends the proof of Theorem 3.2.2.

3.6 Summary

For a *powered Poisson process*, we define *Signal-to-Interference-plus-Noise Ratio* (SINR) and the SINR network as a Telecommunication Network. We define the Empirical Measures (*empirical powered measure*, *empirical link measure* and *empirical SINR measure*) of a class of Telecommunication Networks. For this class of Telecommunication Network we prove a joint large deviation principle for the empirical measures of the Telecommunication Networks. All our rate functions are expressed in terms of relative entropies.

Chapter 4

Large Deviations, Shannon-McMillan-Breiman Theorem for Super-Critical Telecommunication Networks.

This chapter has already appeared in co-authored paper Sakyi-Yeboah et al. (2021c).

4.1 Introduction and Background

4.1.1 Introduction

Databases are now becoming very complex in structure and efficient mathematical tools are necessary for studying the databases.Example, in the database of China Public Private Partnership Center(CPPPC), enterprise' status and between enterprises may be seen to form stochastic networks (Wang, 2018). To code or transmit data from sources such as the databases of CPPPC, one require an efficient approximate pattern machine schemes and a coding algorithms for such data source, and large deviations through the asymptotic equipartition property for the SINR random network will be crucial in this regards.

Large deviations may be regarded as a group of efficient mathematical techniques (stochastic methods) often used to estimate asymptotic properties of increasingly rare events such as their empirical measures and most likely manner of occurrence. See, for example, (Weiss, 1995). There are many applications of large deviation techniques to SINR networks as a model for Telecommunication networks. Some of these applications include, but are not limited to, the analysis of bi-stability in networks, example, notorious bi-stability in multiple access protocols such as the Aloha, and the stochastic behaviour of Automated Teller Machine (ATM) such as the admission control, sizing of internal buffers, and the simulation of ATM models. See, (Weiss, 1995).

The Shannon-MacMillian-Breiman (SMB) Theorem or the Asymptotic Equipartition Property (AEP) may be regarded as the strong law of large numbers in information theory. It says output source of a stochastic data source may be partitioned into two sets, namely the set of typical events and the set of atypical events. The SMB is the foundation of all approximate pattern matching and coding algorithms.
Researchers over the last two decades have given some large deviation analysis for telecommunication networks modelled as a sequence of i.i.d random variables and/or markov chains in discrete and continuous times. See, (Weiss, 1995) and reference therein. Sakyi-Yeboah et al. (2020) and Sakyi-Yeboah et al. (2021b) defined empirical measures on the SINR network and proved some joint LDP results including the SMB and the classical MacMillian theorem for the dense or critical telecommunication networks modelled as the SINR network.

In this chapter, we prove joint large deviation principles on the scales λ and $\lambda^2 a_\lambda$, where λ is the intensity measure of the underlining PPP of the SINR network. Further See, Doku-Amponsah (2006) or Doku-Amponsah et al. (2010) for similar results for the colored random graph models. From these LDPs, we prove an asymptotic equipartition property, see example (Doku-Amponsah (2012), for the SINR networks .

Further, we prove a local LDP for the SINR networks. See for example, Doku-Amponsah (2017) or Doku-Amponsah (2016) and reference therein. From the local LDP we deduce asymptotic bounds on the cardinality of the set of SINR networks for a given typical empirical power measure. We also prove from the local LDP an LDP for the SINR network processes.

The remaining part of the chapter is organized in this manner: Section 4.2 contains the main results; Theorem 4.2.1, Theorem 4.2.2 Theorem 4.2.3, Theorem 4.2.4, Corollary 4.2.5 and Corollary 4.2.6. In Section 4.3 we presents the proof of the main results of the article, Theorem 4.2.1. Section 4.3.3 contains proof of the SBM result, see Theorem 4.2.3 and Section 4.5; Proof of Theorem 4.2.4, Corollary 4.2.5 and Corollary 4.2.6. Finally, we give the conclusion of the article in Section 4.6

4.1.2 Background

We fix dimension $d \in \mathbb{N}$ and some measurable set $\mathcal{D} \subset \mathbb{R}^d$ with respect to the Borel-Sigma algebra $\mathcal{B}(\mathbb{R}^d)$. For an intensity function, $\lambda m : \mathcal{D} \to [0,1]$, a transition kernel from $\mathcal{D}$ to $(0,\infty)$, $\mathcal{Q}$ and a path loss function, $\ell(r) = r^{-\alpha}$, where $\alpha \in (0,\infty)$, and some technical constants; $\tau^{(\lambda)}, \gamma^{(\lambda)} : (0\,,\,\infty) \to (0\,,\,\infty)$, we define the SINR network model as follows:

(i) We pick $X = (X_i)_{i\in I}$ a Poisson Point process (PPP) with rate measure $\lambda m : D \to [0,1]$.

(ii) Given X, we assign each X_i a power $\sigma(X_i) = \sigma_i$ independently according to the transition function $\mathcal{Q}(\cdot\,,\,X_i)$.

(iii) For any two powered points $((X_i,\sigma_i),(X_j,\sigma_j))$ we connect a link iff

$$SINR(X_i,X_j,X) \geq \tau^{(\lambda)}(\sigma_j) \text{ and } SINR(X_j,X_i,X) \geq \tau^{(\lambda)}(\sigma_i),$$

where

$$SINR(X_j,X_i,X) = \frac{\sigma_i \ell(\|X_i - X_j\|)}{N_0 + \gamma^{(\lambda)}(\sigma_j)\sum_{i\in I\setminus\{j\}} \sigma_i \ell(\|X_i - X_j\|)}$$

We denote by E is the set of links in the SINR network and shall consider $X^\lambda := X^\lambda(m, \mathcal{Q}, \ell) = \Big\{[(X_i, \sigma_i), j \in I],\ E\Big\}$ under the joint law of the powered Poisson Point Process and the Network. We will interpret X^λ as an SINR Network and $(X_i, \sigma_i) := X_i^\lambda$ as the power type of device i. We recall from Sakyi-Yeboah et al. (2020) that the link/connectivity probability of the SINR network, P^{x^λ}, is given by $P^{x^\lambda}((x,\sigma_x),(y,\sigma_y)) = e^{-\lambda h_\lambda^{\mathcal{D}}((x,\sigma_x),(y,\sigma_y)))}$, where

$$h_\lambda^{\mathcal{D}}((x,\sigma_x),(y,\sigma_y)) = \int_D \Big[\frac{\tau^{(\lambda)}(\sigma_x)\gamma^{(\lambda)}(\sigma_x)}{\tau^{(\lambda)}(\sigma_x)\gamma^{(\lambda)}(\sigma_x)+(\|z\|^\alpha/\|x-y\|^\alpha)} + \frac{\tau^{(\lambda)}(\sigma_y)\gamma^{(\lambda)}(\sigma_y)}{\tau^{(\lambda)}(\sigma_y)\gamma^{(\lambda)}(\sigma_y)+(\|z\|^\alpha/\|y-x\|^\alpha)}\Big] m(dz).$$

We have assumed there exists a sequence of real numbers,a_λ and a function $h : \mathcal{D} \times \mathbb{R}_+ \to (0, \infty)$ such that $\lambda^2 a_\lambda \to \infty$ and $\lim_{\lambda\uparrow\infty} a_\lambda^{-1} P^{x^\lambda}((x,\sigma_x),(y,\sigma_y)) = h((x,\sigma_x),(y,\sigma_y))$.

Sakyi-Yeboah et al. (2021b) studied the critical SINR Networks (i.e. $\lambda a_\lambda \to 1$). In this chapter, we shall look at Sup-critical SINR Networks.(i.e.$\lim_{\lambda\to\infty} \lambda a_\lambda \to \infty$).

We define the set $\mathcal{S}(\mathcal{D})$ by

$$\mathcal{S}(\mathcal{D}) = \cup_{x\subset\mathcal{D}}\Big\{x :\ |x \cap A| < \infty \text{ ,for any bounded } A \subset \mathcal{D}\ \Big\}. \tag{4.1}$$

where card A means the cardinality of the set A.

Write $\mathcal{X} = \mathcal{S}(\mathcal{D} \times \mathbb{R}_+)$ and $\mathcal{M}(\mathcal{X})$, denote the space of positive measures on the space $\mathcal{X}$ equipped with $\tau-$ topology. Note, $\mathcal{X}$ a locally finite subset of the set $\mathcal{D} \times \mathbb{R}$. See, example, Sakyi-Yeboah et al. (2021b) and Jahnel and Konig (2003) For any SINR Network X^λ we define a probability measure, the *empirical power measure*, $L_1^\lambda \in \mathcal{M}(\mathcal{X})$, by

$$L_1^\lambda\big((x,\sigma_x)\big) := \frac{1}{\lambda}\sum_{i\in I} \delta_{X_i^\lambda}\big((x,\sigma_x)\big)$$

and a finite measure, the *empirical connectivity measure* $L_2^\lambda \in \mathcal{M}(\mathcal{X} \times \mathcal{X})$, by

$$L_2^\lambda\big((x,\sigma_x),(y,\sigma_y)\big) := \frac{1}{\lambda^2 a_\lambda} \sum_{(i,j)\in E} [\delta_{(X_i^\lambda,X_j^\lambda)} + \delta_{(X_j^\lambda,X_i^\lambda)}]\big((x,\sigma_x),(y,\sigma_y)\big).$$

Note that the total mass $\|L_1^\lambda\|$ of the empirical power measure is 1 and total mass of the empirical link measure is $2|E|/\lambda^2 a_\lambda$.

4.2 Main Results

Theorem 4.2.1, is a Joint Large deviation principle for the empirical measures of the SINR network models.We recall from Subsection 4.1.2 the definition of $h_\lambda^{\mathcal{D}}$ as

$$h_\lambda^{\mathcal{D}}((x,\sigma_x),(y,\sigma_y)) = \int_D \Big[\tfrac{\tau^{(\lambda)}(\sigma_x)\gamma^{(\lambda)}(\sigma_x)}{\tau^{(\lambda)}(\sigma_x)\gamma^{(\lambda)}(\sigma_x)+(\|z\|^\alpha/\|x-y\|^\alpha)} + \tfrac{\tau^{(\lambda)}(\sigma_y)\gamma^{(\lambda)}(\sigma_y)}{\tau^{(\lambda)}(\sigma_y)\gamma^{(\lambda)}(\sigma_y)+(\|z\|^\alpha/\|y-x\|^\alpha)}\Big] m(dz)$$

and write

$$h\omega \otimes \omega((x,\sigma_x),(y,\sigma_y))) := h((x,\sigma_x),(y,\sigma_y))\omega((x,\sigma_x))\omega((y,\sigma_y)).$$

We write

$$\langle \pi, f\rangle_{\mathcal{X}} = \int_{\mathcal{X}} f(x)\pi(dx)$$

and define the relative entropy of the probability measure π with respective to another probability measure ω by

$$H(\pi|\omega) = \Big\langle \pi, \log\frac{\pi}{\omega}\Big\rangle_{\mathcal{X}}$$

Theorem 4.2.1 *Let X^λ be a super critical powered SINR network with rate measure $\lambda m : D \to [0,1]$ and a power probability function $\mathcal{Q}(\cdot,\sigma) = ce^{-c\sigma}$, $\sigma \geq 0$ and path loss function $\ell(r) = r^{-\alpha}$, for $\alpha > 0$. Thus, the connectivity probability P^{x^λ} of X^λ satisfies $a_\lambda^{-1}P^{X^\lambda} \to h$ and $\lambda a_\lambda \to \infty$. Then, as $\lambda \to \infty$,* *the pair of measures* $(L_1^\lambda, L_2^\lambda)$ *satisfies a large deviation principle in the space* $\mathcal{M}(\mathcal{X}) \times \mathcal{M}(\mathcal{X} \times \mathcal{X})$

(i) with speed λ *and good rate function*

$$I_{Sc}^1(\omega, \pi) = \begin{cases} H\Big(\omega \Big| m \otimes q\Big) & \text{if } \pi = h\omega \otimes \omega \\ \infty & \text{otherwise.} \end{cases} \tag{4.2}$$

(ii) with speed $\lambda^2 a_\lambda$ *and good rate function*

$$I_{Sc}^2(\omega, \pi) = \frac{1}{2}\mathcal{H}\Big(\pi \| h\omega \otimes \omega\Big) \tag{4.3}$$

where

$$\mathcal{H}(\pi \| h\omega \otimes \omega) := \begin{cases} H(\pi \parallel h\omega \otimes \omega) + \Big(\|h\omega \otimes \omega\| - \|\pi\|\Big), & \text{if } \|\pi\| > 0. \\ \infty & \text{otherwise.} \end{cases} \tag{4.4}$$

and

$$h\omega \otimes \omega((x, \sigma_x), (y, \sigma_y)) = h((x, \sigma_x), (y, \sigma_y))\omega(x, \sigma_x)\omega(y, \sigma_y).$$

Theorem 4.2.2 below is a key step in the proof of Theorem 4.2.1. See section 4.3.3.

Theorem 4.2.2 *Let* X^λ *be a super critical powered SINR network with rate measure* $\lambda m : D \to [0, 1]$ *and a power probability function* $\mathcal{Q}(\cdot, \sigma) = ce^{-c\sigma}$, $\sigma \geq 0$ *and path loss function* $\ell(r) = r^{-\alpha}$, *for* $\alpha > 0$. *Thus, the connectivity probability* P^{x^λ} *of* X^λ *satisfies* $a_\lambda^{-1} P^{z^\lambda} \to h$ *and* $\lambda a_\lambda \to \infty$. *Let* X^λ *be a super critical powered SINR network conditional on the event* $\{L_1^\lambda = \omega\}$. *Then, as* $\lambda \to \infty$, *the pair of measures* $(L_1^\lambda, L_2^\lambda)$ *satisfies a large deviation principle in the space* $\mathcal{M}(\mathcal{X}) \times \mathcal{M}(\mathcal{X} \times \mathcal{X})$

(i) with speed λ *and good rate function*

$$I_\omega^1(\pi) = \begin{cases} 0 & \textit{if } \pi = h\omega \otimes \omega \\ \infty & \textit{otherwise.} \end{cases} \tag{4.5}$$

(ii) with speed $\lambda^2 a_\lambda$ and good rate function

$$I_\omega^2(\pi) = \frac{1}{2}\mathcal{H}\Big(\pi \| h\omega \otimes \omega\Big). \tag{4.6}$$

The next theorem is the Asymptotic equipartition property (AEP) for SINR networks. See (Dembo et al., 2005) for AEP for the random fields on $\mathbb{R}^d$

Theorem 4.2.3 *Let X^λ is a super critical powered SINR network with rate measure $\lambda m : D \to [0,1]$ and a power probability function $\mathcal{Q}(\sigma) = ce^{-c\sigma}, \sigma > 0$ and path loss function $\ell(r) = r^{-\alpha}$, for $\alpha > 0$. Thus, the connectivity probability P^{x^λ} of X^λ satisfies $a_\lambda^{-1} P^{x^\lambda} \to h$ and $\lambda a_\lambda \to \infty$. Suppose the sequence a_λ of X^λ is such that $\lambda a_\lambda \log \lambda \to \infty$ and $a_\lambda / \log \lambda \to -1$. Then, we have*

$$\lim_{\lambda \to \infty} \mathbb{P}\Big\{\Big| -\frac{1}{a_\lambda \lambda^2 \log \lambda} \log P(X^\lambda) - \int_{\mathcal{X}\times\mathcal{X}} h((x,\sigma_x),(y,\sigma_y))Q(d\sigma_x)Q(d\sigma_y)dxdy\Big| \geq \varepsilon\Big\} = 0.$$

We can deduce from Theorem 4.2.3 the following vital information : To transmit the information contained in a large supercritical SINR random network one needs with high probability, about

$$\frac{\lambda^2 a_\lambda \log \lambda}{2\log 2}\Big[\int_{\mathcal{X}\times\mathcal{X}} h((x,\sigma_x),(y,\sigma_y))Q(d\sigma_x)Q(d\sigma_y)dxdy\Big] \textit{bits.}$$

We denote by $\mathcal{G}_p$ be the space of all SINR networks with intensity measure $\lambda m : D \to (0,1)$ and state the local large deviation principle, Theorem 4.2.4 below:

Theorem 4.2.4 *Let X^λ be a super critical powered SINR network with rate measure $\lambda m : D \to [0,1]$ and a power probability function $\mathcal{Q}(\sigma) = ce^{-c\sigma}, \sigma > 0$*

and path loss function $\ell(r) = r^{-\alpha}$, for $\alpha > 0$. Thus, the connectivity probability P^{x^λ} of X^λ satisfies $a_\lambda^{-1}P^{x^\lambda} \to h$ and $\lambda a_\lambda \to \infty$. Then,

(i) for any functional $\pi \in \mathcal{M}(\mathcal{X} \times \mathcal{X})$ and a number $\varepsilon > 0$, there exists a weak neighbourhood B_ν such that

$$\mathbb{P}_\omega\Big\{X^\lambda \in \mathcal{G}_P \,\Big|\, L_2^\lambda \in B_\nu\Big\} \le e^{-\frac{1}{2}\lambda^2 a_\lambda \mathcal{H}(\pi\|h\omega\otimes\omega) - \lambda a_\lambda \varepsilon}.$$

(ii) for any $\pi \in \mathcal{M}(\mathcal{X} \times \mathcal{X})$, a number $\varepsilon > 0$ and a fine neighbourhood B_ν, we have the estimate:

$$\mathbb{P}_\omega\Big\{X^\lambda \in \mathcal{G}_P \,\Big|\, L_2^\lambda \in B_\nu\Big\} \ge e^{-\frac{1}{2}\lambda^2 a_\lambda \mathcal{H}(\pi\|h\omega\otimes\omega) + \lambda a_\lambda \varepsilon}.$$

We define for telecommunication networks an entropy $h : \mathcal{M}(\mathcal{X} \times \mathcal{X}) \to [0.\infty]$ by

$$h(\pi) := \Big(\|\pi\| - \|\lambda\omega \otimes \omega\| - \Big\langle \pi\,,\, \log \frac{\pi}{\|h\omega\otimes\omega\|}\Big\rangle\Big)/2. \tag{4.7}$$

Corollary 4.2.5 below provide the bound on the number of SINR Networks give in a given neigbourhood.

Corollary 4.2.5 (McMillian Theorem) *Let $\mathcal{G}_p$ be a super critical powered SINR network with rate measure $\lambda m : D \to [0,1]$ and a power probability function $\mathcal{Q}(\sigma) = ce^{-c\sigma}, \sigma > 0$ and path loss function $\ell(r) = r^{-\alpha}$, for $\alpha > 0$. Thus, the connectivity probability P^{x^λ} of every $x^\lambda \in \mathcal{G}_p$ satisfies $a_\lambda^{-1}P^{x^\lambda} \to h$ and $\lambda a_\lambda \to \infty$.*

(i) For any empirical connectivity measure ν on $\mathcal{M}(\mathcal{X} \times \mathcal{X})$ and $\varepsilon > 0$, there exists a neighborhood B_ν such that

$$Card\Big(\big\{x^\lambda \in \mathcal{G}_p \,|\, L_2^\lambda \in B_\nu\big\}\Big) \ge e^{\lambda^2 a_\lambda (h(\nu) - \varepsilon)}.$$

(ii) for any neighborhood B_ν and $\varepsilon > 0$, we have

$$Card\Big(\big\{x^\lambda \in \mathcal{G}_p \,|\, L_2^\lambda \in B_\nu\big\}\Big) \le e^{\lambda^2 a_\lambda (h(\nu) + \varepsilon)},$$

where $Card(A)$ means the cardinality of A.

Remark 8 *For* $\pi = h\omega \otimes \omega$, *we have* $Card\Big(\Big\{x \in \mathcal{G}_p\Big\}\Big) \approx e^{\lambda^2 a_\lambda \|h\omega\otimes\omega\| h\big(h\omega\otimes\omega/\|h\omega\otimes\omega\|\big)}$.

Corollary 4.2.6 *Let* X^λ *be a super critical powered SINR network with rate measure* $\lambda m : D \to [0,1]$ *and a power probability function* $\mathcal{Q}(\sigma) = ce^{-c\sigma}, \sigma > 0$ *and path loss function* $\ell(r) = r^{-\alpha}$, *for* $\alpha > 0$. *Thus, the connectivity probability* P^{x^λ} *of* X^λ *satisfies* $a_\lambda^{-1}P^{x^\lambda} \to h$ *and* $\lambda a_\lambda \to \infty$.

(i) Let F *be closed subset* $\mathcal{M}_\omega$. *Then we have*

$$\limsup_{\lambda\to\infty} \frac{1}{\lambda^2 a_\lambda} \log \mathbb{P}_\omega\Big\{X^\lambda \in \mathcal{G}_P \,\Big|\, L_2^\lambda \in F\Big\} \leq -\tfrac{1}{2} \inf_{\omega \in F} \Big\{\mathcal{H}(\pi\|h\omega\otimes\omega)\Big\}.$$

(ii) Let O *be open subset* $\mathcal{M}_\omega$. *Then we have*

$$\liminf_{\lambda\to\infty} \frac{1}{\lambda^2 a_\lambda} \log \mathbb{P}_\omega\Big\{X^\lambda \in \mathcal{G}_p \,\Big|\, L_2^\lambda \in O\Big\} \geq -\tfrac{1}{2} \inf_{\pi \in O} \Big\{\mathcal{H}(\pi\|h\omega\otimes\omega)\Big\}.$$

4.3 Proof of Theorem 4.2.1 by Gartner-Ellis Theorem and Method of Mixtures

4.3.1 Proof of Theorem 4.2.2(i)

Suppose $A_1, ..., A_n$ is a decomposition of the space $\mathcal{D} \times \mathbb{R}_+$. Observe that, for every $(x, y) \in A_i \times A_j$, $i, j = 1, 2, 3, ..., n$, $\lambda L_2^\lambda(x, y)$ given $\lambda L_1^\lambda(x) = \lambda\omega(x)$ is binomial with parameters $\lambda^2\omega(x)\omega(y)/2$ and $P^{x^\lambda}(x, y)$. Let Q be the exponential distribution with parameter c. We recall the function $h_\lambda^{\mathcal{D}}$ from the previous sections and note that Lemma 4.3.1 is a key component in the application of the Gartner-Ellis Theorem. See, for example, (Zeitouni and Dembo, 1998).

Lemma 4.3.1 *Let* X^λ *is a super critical powered SINR network with rate measure* $\lambda m : D \to [0,1]$ *and a power probability function* $\mathcal{Q}(\sigma) = ce^{-c\sigma}, \sigma > 0$

and path loss function $\ell(r) = r^{-\alpha}$, for $\alpha > 0$. Thus, the connectivity probability P^{x^λ} of X^λ satisfies $a_\lambda^{-1}P^{x^\lambda} \to h$ and $\lambda a_\lambda \to \infty$. Let X^λ be a supercritical SINR network, conditional on the event $L_1^\lambda = \omega$. Let $g : \mathcal{X}\times\mathcal{X} \to \mathbb{R}$ be bounded function. Then,

$$\lim_{\lambda\to\infty}\frac{1}{\lambda}\log\mathbb{E}\Big\{e^{\lambda\langle g,\, L_2^\lambda\rangle}\Big|L_1^\lambda = \omega\Big\} = \frac{1}{2}\lim_{n\to\infty}\sum_{j=1}^{n}\sum_{i=1}^{n}\Big\langle g,\, h\omega\otimes\omega\Big\rangle_{A_i\times A_j}$$
$$= \frac{1}{2}\Big\langle g,\, h\omega\otimes\omega\Big\rangle_{\mathcal{X}\times\mathcal{X}}.$$

Proof: Now we observe that

$$\mathbb{E}\Big\{e^{\int\int\lambda g(x,y)L_2^\lambda(dx,dy)/2}\Big|L_1^\lambda = \omega\Big\} = \mathbb{E}\Big\{\prod_{x\in\mathcal{X}}\prod_{y\in\mathcal{X}}e^{\lambda g(x,y)L_2^\lambda(dx,dy)/2}\Big\}$$

$$\mathbb{E}\Big\{\prod_{x\in\mathcal{X}}\prod_{y\in\mathcal{X}}e^{g(x,y)\lambda L_2^\lambda(dx,dy/2)}\Big\} = \prod_{i=1}\prod_{j=1}\prod_{x\in A_i}\prod_{y\in A_j}\mathbb{E}\Big\{e^{g(x,y)\lambda L_2^\lambda(dx,dy)/2}\Big\}$$

$$\log\Big\{e^{\lambda\langle g,L_2^\lambda\rangle/2}\Big|L_1^\lambda = \omega\Big\} =$$
$$\sum_{j=1}^{n}\sum_{i=1}^{n}\int_{A_j}\int_{A_i}\log\Big[1 - P^{x^\lambda}(x,y) + P^{x^\lambda}(x,y)e^{g(x,y)/\lambda a_\lambda}\Big]^{\lambda^2\omega\otimes\omega(dx,dy)/2} + o(n)$$

By the dominated convergence theorem

$$\frac{1}{\lambda}\log E\{e^{\lambda\langle g,L_2^\lambda\rangle/2} \mid L_1^\lambda = \omega\} =$$
$$\frac{1}{\lambda}\sum_{j=1}\sum_{i=1}\int_{A_i}\int_{A_j}\log\Big[1 - (1 - e^{g(x,y)/\lambda a_\lambda})P^{x^\lambda}(x,y)\Big]^{\lambda^2\omega\otimes\omega(dx,dy)/2} + o(n)/\lambda$$

$$\frac{1}{\lambda}\log\mathbb{E}\{e^{\lambda\langle g,L_2^\lambda\rangle/2}|L_1^\lambda = \omega\} =$$
$$\lim_{\lambda\to\infty}\sum_{j=1}\sum_{i=1}\int_{A_i}\int_{A_j}\log\Big[1 + g(x,y)h(x,y)/\lambda + o(\lambda)/\lambda\Big]^{\lambda\omega\otimes\omega(dx,dy)/2} + o(n)/\lambda$$

$$\lim_{\lambda\to\infty}\frac{1}{\lambda}\log\mathbb{E}\{e^{\lambda\langle g,L_2^\lambda\rangle/2} \mid L_1^\lambda = \pi\} = \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n}\Big\langle g,\, h\omega\otimes\omega\Big\rangle_{A_i\times A_j}$$

$$\lim_{\lambda\to\infty}\frac{1}{\lambda}\log\mathbb{E}\{e^{\lambda\langle g,L_2^\lambda\rangle/2}\Big|L_1^\lambda=\omega\}=\frac{1}{2}\lim_{n\to\infty}\sum_{j=1}^{n}\sum_{i=1}^{n}\Big\langle g,\,h\omega\otimes\omega\Big\rangle_{A_i\times A_j}$$
$$=\frac{1}{2}\Big\langle g,\,h\omega\otimes\omega\Big\rangle_{\mathcal{X}\times\mathcal{X}}.$$

Hence, by Gartner-Ellis theorem, conditional on the event $\Big\{L_1^\lambda=\omega\Big\}$, L_2^λ obey a large deviation principle with speed λ and variational formulation of the rate function

$$I_\omega(\pi)=\frac{1}{2}\sup_g\Big\{\Big\langle g,\,\pi\Big\rangle_{\mathcal{X}\times\mathcal{X}}-\Big\langle g,\,h\omega\otimes\omega\Big\rangle_{\mathcal{X}\times\mathcal{X}}\Big\}$$

which when solved, see example (Doku-Amponsah, 2006), would clearly reduces to the good rate function given by

$$I_\omega^1(\pi)=0.$$

if

$$\pi=h\omega\otimes\omega$$

□

4.3.2 Proof of Theorem 4.2.2(ii)

Similarly, we take $A_1,...,A_n$ as a decomposition of the space $\mathcal{D}\times\mathbb{R}_+$. We recall the function h_λ^D from the previous sections and state the following Lemma. Lemma 4.3.2 is a key component in the application of the Gartner-Ellis Theorem. See, Zeitouni and Dembo (1998).

Lemma 4.3.2 *Let X^λ be a super critical powered SINR network with rate measure $\lambda m: D\to[0,1]$ and a power probability function $\mathcal{Q}(\sigma)=ce^{-c\sigma},\sigma>0$ and path loss function $\ell(r)=r^{-\alpha}$, for $\alpha>0$. Thus, the connectivity probability P^{x^λ} of X^λ satisfies $a_\lambda^{-1}P^{x^\lambda}\to h$ and $\lambda a_\lambda\to\infty$. Let X^λ be a supercritical SINR network, conditional on the event $L_1^\lambda=\omega$. Let $g:\mathcal{X}\times\mathcal{X}\to\mathbb{R}$ be bounded function. Then,*

$$\lim_{\lambda\to\infty}\frac{1}{\lambda^2 a_\lambda}\log\mathbb{E}\Big\{e^{\lambda^2 a_\lambda\langle g,\,L_2^\lambda\rangle}\Big|L_1^\lambda=\omega\Big\} = -\frac{1}{2}\lim_{n\to\infty}\sum_{j=1}^{n}\sum_{i=1}^{n}\Big\langle 1-e^g,\,h\omega\otimes\omega\Big\rangle_{A_i\times A_j}$$
$$= -\frac{1}{2}\Big\langle 1-e^g,\,h\omega\otimes\omega\Big\rangle_{\mathcal{X}\times\mathcal{X}}.$$

Proof: Now we observe that

$$\mathbb{E}\Big\{e^{\int\int\lambda^2 a_\lambda g(x,y)L_2^\lambda(dx,dy)/2}\Big|L_1^\lambda=\omega\Big\} = \mathbb{E}\Big\{\prod_{x\in\mathcal{X}}\prod_{y\in\mathcal{X}}e^{\lambda^2 a_\lambda g(x,y)L_2^\lambda(dx,dy)/2}\Big\}$$

$$\mathbb{E}\Big\{\prod_{x\in\mathcal{X}}\prod_{y\in\mathcal{X}}e^{g(x,y)\lambda L_2^\lambda(dx,dy/2)} = \prod_{i=1}\prod_{j=1}\prod_{x\in A_i}\prod_{y\in A_j}\mathbb{E}\Big\{e^{\lambda^2 a_\lambda g(x,y)L_2^\lambda(dx,dy)/2}\Big\}\times e^{o(n)}$$

$$\log\Big\{e^{\lambda^2 a_\lambda\langle g,L_2^\lambda\rangle/2}\Big|L_1^\lambda=\omega\Big\} =$$
$$\sum_{j=1}^{n}\sum_{i=1}^{n}\int_{A_j}\int_{A_i}\log\Big[1-P^{x^\lambda}(x,y))+P^{x^\lambda}(x,y)e^{g(x,y)}\Big]^{\lambda^2\omega\otimes\omega(dx,dy)/2}+o(n)$$

By the dominated convergence theorem

$$\frac{1}{\lambda^2 a_\lambda}\log E\{e^{\lambda\langle g,L_2^\lambda\rangle/2}\mid L_1^\lambda=\omega\} =$$
$$\frac{1}{\lambda^2 a_\lambda}\sum_{j=1}\sum_{i=1}\int_{A_i}\int_{A_j}\log\Big[1-(1-e^{g(x,y)})P^{x^\lambda}(x,y)\Big]^{\lambda^2\omega\otimes\omega(dx,dy)/2}+o(n)/\lambda^2 a_\lambda$$

$$\frac{1}{\lambda^2 a_\lambda}\log\mathbb{E}\{e^{\lambda\langle g,L_2^\lambda\rangle/2}\|L_1^\lambda=\omega\} =$$
$$\lim_{\lambda\to\infty}\sum_{j=1}\sum_{i=1}\int_{A_i}\int_{A_j}\log\Big[1-(1-e^{g(x,y)})P^{x^\lambda}(x,y)\Big]^{\lambda\omega\otimes\omega(dx,dy)/2}+o(n)/\lambda^2 a_\lambda$$

$$\lim_{\lambda\to\infty}\frac{1}{\lambda^2 a_\lambda}\log\mathbb{E}\Big\{e^{\lambda\langle g,L_2^\lambda\rangle/2}\Big|L_1^\lambda=\omega\Big\} = -\frac{1}{2}\sum_{j=1}\sum_{i=1}\int_{A_i}\int_{A_j}\Big[(1-e^{g(x,y)})h(x,y)\omega\otimes\omega(dx,dy)\Big]$$

$$\lim_{\lambda\to\infty}\frac{1}{\lambda^2 a_\lambda}\log\mathbb{E}\{e^{\lambda\langle g,L_2^\lambda\rangle/2}\mid L_1^\lambda=\omega\}=-\frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n}\Big\langle 1-e^g,\,h\omega\otimes\omega\Big\rangle_{A_i\times A_j}$$

$$\begin{aligned}\lim_{\lambda\to\infty}\frac{1}{\lambda^2 a_\lambda}\log\mathbb{E}\{e^{\lambda\langle g,L_2^\lambda\rangle/2}\Big|L_1^\lambda=\omega\}&=-\frac{1}{2}\lim_{n\to\infty}\sum_{j=1}^{n}\sum_{i=1}^{n}\Big\langle 1-e^g,\,h\omega\otimes\omega\Big\rangle_{A_i\times A_j}\\&=-\frac{1}{2}\Big\langle 1-e^g,\,h\omega\otimes\omega\Big\rangle_{\mathcal{X}\times\mathcal{X}}\end{aligned}$$

Hence, by Gartner-Ellis theorem, conditional on the event $\Big\{L_1^\lambda=\omega\Big\}$, L_2^λ obey a large deviation principle with speed $\lambda^2 a_\lambda$ and variational formulation of the rate function

$$I_\omega(\pi)=\frac{1}{2}\sup_g\Big\{\Big\langle g,\,\pi\Big\rangle_{\mathcal{X}\times\mathcal{X}}+\Big\langle 1-e^g,\,h\omega\otimes\omega\Big\rangle_{\mathcal{X}\times\mathcal{X}}\Big\}$$

which when solved, see example Doku-Amponsah (2006), would clearly reduces to the good rate function given by

$$I_\omega^2(\pi)=\frac{1}{2}\mathcal{H}(\pi\|h\omega\otimes\omega). \tag{4.8}$$

□

4.3.3 Proof of Theorem 4.2.1 by Method of Mixtures.

For any $\lambda\in(0,\infty)$, we define

$$\begin{aligned}\mathcal{M}_\lambda(\mathcal{X})&:=\Big\{\omega\in\mathcal{M}(\mathcal{X})\,:\,\lambda\omega(x)\in\mathbb{N}\text{ for all }x\in\mathcal{X}\Big\},\\\tilde{\mathcal{M}}_\lambda(\mathcal{X}\times\mathcal{X})&:=\Big\{\pi\in\tilde{\mathcal{M}}(\mathcal{X}\times\mathcal{X})\,:\,\lambda\,\omega(x,y)\in\mathbb{N},\ \text{ for all }\ x,y\in\mathcal{X}\times\mathcal{X}\Big\}.\end{aligned}$$

We denote by $\Theta_\lambda:=\mathcal{M}_\lambda(\mathcal{X})$ and $\Theta:=\mathcal{M}(\mathcal{X})$. We write

$$\begin{aligned}P_{\omega_\lambda}^{(\lambda)}(\eta_\lambda)&:=\mathbb{P}\big\{L_2^\lambda=\eta_\lambda\,\big|\,L_1^\lambda=\omega_\lambda\big\}\,,\\P^{(\lambda)}(\omega_\lambda)&:=\mathbb{P}\big\{L_1^\lambda=\omega_\lambda\big\}\end{aligned}$$

The joint distribution of L_1^λ and L_2^λ is the mixture of $P_{\omega_\lambda}^{(\lambda)}$ with $P^{(\lambda)}(\omega_\lambda)$, as follows:

$$d\tilde{P}^\lambda(\omega_\lambda, \eta_\lambda) := dP_{\omega_\lambda}^{(\lambda)}(\eta_\lambda)\, dP^{(\lambda)}(\omega_\lambda). \tag{4.9}$$

(Biggins et al., 2004, Theorem 5 (b)) gives criteria for the validity of large deviation principles for the mixtures and for the goodness of the rate function if individual large deviation principles are known. The following three lemmas ensure validity of these conditions.

Observe that the family of measures $(P^{(\lambda)} \colon \lambda \in (0,\infty))$ is exponentially tight on Θ.

Lemma 4.3.3 *(i) The family of measures $(\tilde{P}^\lambda \colon \lambda \in (0,\infty))$ is exponentially tight on $\Theta \times \tilde{\mathcal{M}}_*(\mathcal{X} \times \mathcal{X})$.*

(ii) The family measures $(P^{x^\lambda} \colon \lambda \in (0,\infty))$ is exponentially tight on $\Theta \times \tilde{\mathcal{M}}_*(\mathcal{X} \times \mathcal{X})$.

We refer to (Sakyi-Yeboah et al., 2020, Lemma 4.3) for similar proof for Large deviation Principle on the scale λ^2

Define the function $I_{Sc}^2, I_{Sc}^1 \colon \Theta \times \mathcal{M}_*(\mathcal{X} \times \mathcal{X}) \to [0,\infty]$, by

$$I_{Sc}^1\big(\omega,\pi\big) = \begin{cases} H\Big(\omega \Big| m \otimes \mathcal{Q}\Big) & \text{if } \pi = h\omega \otimes \omega \\ \infty & \text{otherwise.} \end{cases} \tag{4.10}$$

$$I_{Sc}^2\big(\omega,\pi\big) = \frac{1}{2}\mathcal{H}\Big(\pi \| h\omega \otimes \omega\Big). \tag{4.11}$$

Lemma 4.3.4 *(i) I_{Sc}^1 is lower semi-continuous.*

(ii) I_{Sc}^2 is lower semi-continuous.

By (Biggins et al., 2004, Theorem 5(b)) the two previous lemmas, the LDP for the empirical power measure, see, (Sakyi-Yeboah et al., 2020, Theorem 2.1) and the large deviation principles we have established Theorem 4.2.2 ensure that under

$(\tilde{P}^{\lambda})$ and P^{x^λ} the random variables $(\omega_\lambda, \pi_\lambda)$ satisfy a large deviation principle on $\mathcal{M}(\mathcal{X}) \times \tilde{\mathcal{M}}(\mathcal{X} \times \mathcal{X})$ and $\Theta \times \tilde{\mathcal{M}}_\lambda(\mathcal{X} \times \mathcal{X})$ on the speeds λ and $\lambda^2 a_\lambda$ with good rate functions I^1_{Sc} and I^2_{Sc} respectively, which ends the proof of Theorem 4.2.1.

4.4 Proof of Theorem 4.2.3 by Large deviations

4.4.1 Proof of Theorem 4.2.3

In order to establish the asymptotic equipartition property, we first prove a weak law of large numbers for the empirical powered measure and the empirical connectivity measure of the SINR network.

Lemma 4.4.1 *Let X^λ be a super critical powered SINR network with rate measure $\lambda m : D \to [0,1]$ and a power probability function $\mathcal{Q}(\sigma) = ce^{-c\sigma}, \sigma > 0$ and path loss function $\ell(r) = r^{-\alpha}$, for $\alpha > 0$. Thus, the connectivity probability P^{x^λ} of X^λ satisfies $a_\lambda^{-1} P^{x^\lambda} \to h$ and $\lambda a_\lambda \to \infty$. Then,*

$$\lim_{\lambda \to \infty} \mathbb{P}\Big\{ \sup_{(x,\sigma_x) \in \mathcal{X}} \Big| L_1^\lambda(x,\sigma_x) - m \otimes \mathcal{Q}(x,\sigma_x) \Big| > \varepsilon \Big\} = 0$$

and

$$\lim_{\lambda \to \infty} \mathbb{P}\Big\{ \sup_{([x,\sigma_x],[y,\sigma_y]) \in \mathcal{X} \times \mathcal{X}} \Big| L_2^\lambda([x,\sigma_x],[y,\sigma_y]) - hm \otimes \mathcal{Q} \times m \otimes \mathcal{Q}([x,\sigma_x],[y,\sigma_y]) \Big| > \varepsilon \Big\} = 0$$

Proof: Let

$$F_{1,\mathcal{X}} = \Big\{ \omega : \sup_{(x,\sigma_x) \in \mathcal{X}} |\omega(x,\sigma_x) - m \otimes \mathcal{Q}(x,\sigma_x)| > \varepsilon \Big\},$$

$$F_{2,\mathcal{X}} = \Big\{ \pi : \sup_{([x,\sigma_x],[y,\sigma_y]) \in \mathcal{X} \times \mathcal{X}} |\pi([x,\sigma_x],[y,\sigma_y]) - hm \otimes \mathcal{Q} \times m \otimes \mathcal{Q}([x,\sigma_x],[y,\sigma_y])| > \varepsilon \Big\}$$

and $F_{3,\mathcal{X}} = F_{1,\mathcal{X}} \cup F_{2,\mathcal{X}}$. Now, observe from Theorem 4.2.1 that

$$\limsup_{\lambda \to \infty} \frac{1}{\lambda} \log \mathbb{P}\Big\{ (L_1^\lambda, L_2^\lambda) \in F_{3,\mathcal{X}}^c \Big\} \le - \inf_{(\omega,\varpi) \in F_{3,\mathcal{X}}^c} I(\omega,\varpi).$$

It suffices for us to show that I is strictly positive where $\varepsilon > 0$. Suppose there is a sequence $(\omega_\lambda, \pi_\lambda) \to (\omega, \pi)$ such that $I(\omega_\lambda, \varpi_\lambda) \downarrow I(\omega, \varpi) = 0$. This implies $\pi = \omega \otimes \mathcal{Q}$ and $\pi = h\omega \otimes \mathcal{Q} \times \omega \otimes \mathcal{Q}$ which contradicts $(\omega, \pi) \in F_3^c$. This ends the proof of the Lemma. $\square$

Now, the distribution of the marked PPP $P(x) = \mathbb{P}\Big\{X^\lambda = x\Big\}$ is given by

$$P_\lambda(x) = \prod_{i=1}^{I} |m \otimes \mathcal{Q}(x_i, \sigma_i) \prod_{(i,j)\in E} \frac{P^{x^\lambda}([x_i, \sigma_i], [y_j, \sigma_j])}{1 - P^{x^\lambda}([x_i, \sigma_i], [y_j, \sigma_j])} \times$$

$$\prod_{(i,j)\in \mathcal{E}} (1 - P^{x^\lambda}([x_i, \sigma_i], [y_j, \sigma_j])) \prod_{i=1}^{I} (1 - P^{x^\lambda}([x_i, \sigma_i], [y_j, \sigma_j]))$$

$$\begin{aligned} -\frac{1}{a_\lambda \lambda^2 \log \lambda} \log P_\lambda(x) = & \frac{1}{a_\lambda \lambda \log \lambda} \Big\langle - \log m \otimes Q \,, L_1^\lambda \Big\rangle + \frac{1}{\log \lambda} \Big\langle - \log \Big(\tfrac{P^{x^\lambda}}{1 - P^{x^\lambda}} \Big) \,, L_2^\lambda \Big\rangle \\ & + \frac{1}{a_\lambda \log \lambda} \Big\langle - \log(1 - P^{x^\lambda}) \,, L_1^\lambda \otimes L_1^\lambda \Big\rangle \\ & + \frac{1}{a_\lambda \lambda \log \lambda} \Big\langle - \log(1 - P^{x^\lambda}) \,, L_\Delta^\lambda \Big\rangle \end{aligned}$$

Notice,

$$\lim_{\lambda\to\infty} \frac{1}{a_\lambda \lambda \log \lambda} \Big\langle - \log m \otimes \mathcal{Q} \,, L_1^\lambda \Big\rangle = \lim_{\lambda\to\infty} \frac{1}{\lambda} \Big\langle - \log(1 - P^{x^\lambda} \,, L_\Delta^\lambda \Big\rangle =$$

$$\lim_{\lambda\to\infty} \frac{1}{a_\lambda \log \lambda} \Big\langle - \log(1 - P^{x^\lambda}) \,, L_1^\lambda \otimes L_1^\lambda \Big\rangle = 0.$$

Using, Lemma 4.4.1 we have

$$\lim_{\lambda\to\infty} \frac{1}{\log \lambda} \Big\langle - \log \Big(P^{x^\lambda} / (1 - P^{x^\lambda} \Big) \,, L_2^\lambda \Big\rangle = \Big\langle \mathbb{1} \,, hm \otimes \mathcal{Q} \times m \otimes \mathcal{Q} \Big\rangle$$

which concludes the proof of Theorem 4.2.3.

4.5 Proof of Theorem 4.2.4, Corollary 4.2.5, Corollary 4.2.6

For $\omega \in \mathcal{M}(\mathcal{X})$, we define the spectral potential of the marked SINR graph (X^{λ}) conditional on the event $\{L_1^{\lambda} = \omega\}$, $U_Q(g, \omega)$ as

$$U_Q(g, \omega) = \Big\langle -(1 - e^{g}), h\omega \otimes \omega \Big\rangle. \tag{4.12}$$

Note that remarkable properties of a spectral potential, see or Sakyi-Yeboah et al. (2020) holds for U_Q.

For $\pi \in \mathcal{M}(\mathcal{X} \times \mathcal{X})$, we observe that $I_{\omega}(\pi)$ is the Kullback action of the marked SINR graph X^{λ}.

Lemma 4.5.1 *The following hold for the Kullback action or divergence function $I_{\omega}(\pi)$:*

(i)

$$I_{\omega}(\pi) = \sup_{g \in \mathcal{C}} \left\{ \langle g, \pi \rangle - U_Q(g, \omega) \right\}$$

(ii) The function $I_{\omega}(\pi)$ is convex and lower semi-continuous on the space $\mathcal{M}(\mathcal{X} \times \mathcal{X})$.

(iii) For any real α, the set $\Big\{\pi \in \mathcal{M}(\mathcal{X} \times \mathcal{X}) : I_{\omega}(\pi) \leq \alpha\Big\}$ is weakly compact.

The proof of Lemma 4.5.1 is omitted from the chapter. Interested readers may refer to Doku-Amponsah (2017) for similar proof for empirical measures of the Typed Random Graph Processes or See, for example Doku-Amponsah (2016) for the multitype Galton-Watson processes and/or the reference therein, Bakhtin (2015), for proof of the lemma for empirical measures on measurable spaces.

Note from Lemma 4.5.1 that, for any $\varepsilon > 0$, there exists some function $g \in \mathcal{X} \times \mathcal{X}$ such that

$$I_{\omega}(\pi) - \tfrac{\varepsilon}{2} < \langle g \, , \, \pi \rangle - U_Q(g, \omega).$$

We define the probability distribution of the powered X by P_ω by

$$P_\omega(x) = \prod_{(i,j)\in E} e^{g(x_i,x_j)} \prod_{(i,j)\in\mathcal{E}} e^{h_\lambda(x_i,x_j)},$$

where

$$h_\lambda(x,y) = \frac{1}{a_\lambda}\log\Big[1 - P^{x^\lambda}(x,y) + P^{x^\lambda}(x,y)e^{g(x,y)}\Big]$$

Then, observe that

$$\begin{aligned}\frac{dP_\omega}{d\tilde{P}_\omega}(x) &= \prod_{(i,j)\in E} e^{-g(x_i,x_j)} \prod_{(i,j)\in\mathcal{E}} e^{-h_\lambda(x_i,x_j)a_\lambda} \\ &= e^{-\lambda^2 a_\lambda(\langle\frac{1}{2}g,L_2^\lambda\rangle - \lambda^2 a_\lambda\langle\frac{1}{2}h_\lambda,L_1^\lambda\otimes L_1^\lambda\rangle) + \langle\frac{1}{2}h_\lambda,L_\Delta^\lambda\rangle}\end{aligned}$$

Now define the neighbourhood of ν, B_ν by

$$B_\nu := \Big\{\pi\in\mathcal{M}(\mathcal{X}\times\mathcal{X}) : \langle g,\pi\rangle - U_Q(g,\omega) > \langle g,\pi\rangle - U_Q(g,\omega) - \varepsilon/2\Big\}$$

Note that under the condition $L_2^\lambda \in B_\nu$, we have

$$\frac{dP_\omega}{d\tilde{P}_\omega}(x) < e^{-\lambda^2 a_\lambda(\langle\frac{1}{2}g,L_2^\lambda\rangle - \lambda^2 a_\lambda\langle\frac{1}{2}t_\lambda,L_1^\lambda\otimes L_1^\lambda\rangle) + \langle\frac{1}{2}h_\lambda,L_\Delta^\lambda\rangle} < e^{-\lambda^2 a_\lambda I_\omega(\pi) + \lambda^2 a_\lambda \varepsilon}$$

Therefore, we obtain

$$\begin{aligned}P_\omega\Big\{X^\lambda\in\mathcal{G}_P\Big|L_2^\lambda\in B_\nu\Big\} &\le \int \mathbb{1}_{\{L_2^\lambda\in B_\nu\}} d\tilde{P}_\omega(x^\lambda)(x) \le \int e^{-\lambda^2 a_\lambda I_\omega(\pi) - \lambda\varepsilon} d\tilde{P}_\omega(x^\lambda) \\ &\le e^{-\lambda^2 a_\lambda I_\omega(\pi) - \lambda^2 a_\lambda \varepsilon}.\end{aligned}$$

.

Note that $I_{Sc}(\pi)=\infty$ implies Theorem 4.2.3 (ii), hence it is sufficient for us to deduce that the result is true for a probability distribution of the form $\pi = e^g\omega\otimes\omega$ and for $I_\omega(\pi) = \frac{1}{2}\mathcal{H}(\pi\|h\omega\otimes\omega)$. Fix any number $\varepsilon>0$ and any neigbourhood

$B_\nu \subset \mathcal{M}(\mathcal{X} \times \mathcal{X})$. Now define the sequence of sets

$$\mathcal{G}_p^\lambda = \Big\{x \in \mathcal{G}_p : L_2^\lambda(x) \in B_\nu \Big| \langle g, L_2^\lambda \rangle - U_Q(g,\omega) \Big| \leq \tfrac{\varepsilon}{2} \Big\}.$$

Note that for all $x \in \mathcal{G}_p^\lambda$ we have

$$\frac{dP_\omega}{d\tilde{P}_\omega} > e^{-\lambda^2 a_\lambda \langle \frac{1}{2} g, \pi \rangle + \lambda^2 a_\lambda U_Q(g,\omega) + \lambda^2 a_\lambda \frac{\varepsilon}{2}}.$$

This yields

$$P_\omega(\mathcal{G}_P^\lambda) = \int_{\mathcal{G}_P^\lambda} dP_\omega(x) \geq \int e^{-\lambda^2 a_\lambda \langle \frac{1}{2} g, \pi \rangle + \lambda^2 a_\lambda U_Q(g,\omega) + \lambda^2 a_\lambda \frac{\varepsilon}{2}} d\tilde{P}_\omega(x)$$

$$\geq e^{-\lambda^2 a_\lambda \frac{1}{2} \mathcal{H}(\pi \| h\omega \otimes \omega) + \lambda^2 a_\lambda \varepsilon} \tilde{P}_\omega(\mathcal{G}_P^\lambda).$$

Applying the law of large numbers, we have that $\lim_{\lambda \to \infty} \tilde{P}_\omega(\mathcal{G}_P^\lambda) = 1$. This completes of the Theorem.

Proof of Corollary 4.2.5

The proof of Corollary 4.2.5 follows from the definition of the Kullback action and Theorem 4.2.4 if we set $\omega = m \otimes Q$ and $\lambda\omega \otimes \omega(x,y) = \|\lambda\omega \otimes \omega\|$, for all $(x,y) \in \mathcal{X} \times \mathcal{X}$.

Proof of Corollary 4.2.6

We observe that, by Lemma 4.3.3 the law of empirical connectivity measure is exponentially tight. Henceforth, without loss of generality we can assume that the set F in Corollary 4.2.6(ii) above is relatively compact. If we choose any $\varepsilon > 0$; then for each functional $\nu \in F$ we can find a weak neigbourhood such that the estimate of Theorem 4.2.4(i) above holds. From all these neigbhourhoods, we choose a finite cover of $\mathcal{G}_P$ and sum up over the estimate in Corollary 4.2.6(i) above to obtain

$$\limsup_{\lambda\to\infty}\frac{1}{\lambda}\log\mathbb{P}_{\omega}\Big\{X^{\lambda}\in\mathcal{G}_P\,\Big|\,L_2^{\lambda}\in F\Big\}\le-\inf_{\pi\in F}I_{\omega}(\pi)+\varepsilon.$$

As ε was arbitrarily chosen and the lower bound in Corollary 4.2.6(ii) implies the lower bound in Theorem 2.2(i) we get the desired results which completes the proof.

4.6 Conclusion

In this chapter, we have presented a joint large deviation principle for the empirical power measure and the empirical connectivity measure of telecommunication networks in the $\tau-$ topology. From this large deviation principle, we deduce an asymptotic equipartition property for the telecommunication network modelled as the SINR network model.

We have also presented a Local large deviation principle for the empirical connectivity measure given the empirical power measure and from this result we deduce the classical MacMillian theorem and an asymptotic bound for the set all possible SINR network process. Finally, we also presented a large deviation principle for the SINR networks. This chapter may be regarded as a first step in the proof of a Lossy asymptotic equipartition property for the SINR networks. See, Doku-Amponsah (2017) and Doku-Amponsah (2016) for similar results for the networked data structures modelled as colored random graph process and for the hierarchical data structure modelled as Galton-Watson tree process.

4.7 Summary

In this chapter, we obtain large deviation asymptotics for supercritical communication networks modelled as Signal-Interference-Noise Ratio networks. To do this, we define the empirical power measure and the empirical connectivity measure, and prove joint large deviation principles(LDPs) for the two empirical measures on two different scales i.e. λ and $\lambda^2 a_\lambda$, where λ is the intensity measure of the Poisson Point Process (PPP) which defines the SINR random network. Using this joint LDPs we prove an asymptotic equipartition property for the stochastic telecommunication Networks modelled as the SINR networks. Further, we prove a Local Large Deviation Principle (LLDP) for the SINR Network. From the LLDP we prove the a large deviation principle, and a classical McMillian Theorem for the stochastic SINR network processes. Note, for tupical empirical connectivity measure, $h\omega \otimes \omega$, we can deduce from the LLDP a bound on the cardinality of the space of SINR networks to be approximately equal to $e^{\lambda^2 a_\lambda \|h\omega\otimes\omega\| H\left(h\omega\otimes\omega/\|h\omega\otimes\omega\|\right)}$, where the connectivity probability of the network, P^{x^λ}, satisfies $a_\lambda^{-1} P^{x^\lambda} \to h$. Observe, the LDP for the empirical measures of the stochastic SINR network were obtained on spaces of measures equipped with the $\tau-$ topology, and the LLDPs were obtained in the space of SINR network process without any topological restrictions.

Chapter 5

Large Deviations and Information theory for Sub-Critical for the Signal -to- Interference -Plus- Noise Ratio Random Network Models.

This chapter has already appeared in the co-authored Sakyi-Yeboah et al. (2021a).

5.1 Introduction and Background

5.1.1 Introduction

In telecommunication, Wireless networks are usually modelled by the SINR random networks. In the SINR random network model two nodes are deemed to communicate if SINR is bigger than a certain threshold as specified by some technical constant. In the process of addressing the additional requirement imposed on wireless communication networks, in particular, a higher availability of a highly accurate modeling of the SINR is required. Example, each transmission may be equipped with some battery power which may be called the mark of the node and the quantity SINR defined by the inclusion of the marks in the definition. Further study of the SINR network model has shown that an SINR model of interference is a more realistic model of interference than the protocol model of interference: a receiver node receives a packet so long as the signal to interference plus noise ratio is above a certain threshold. See, (Bakshi et al., 2017). Furthermore, Manesh and Kaabouch (2017) stated that SINR is successful if the desired receiver surpasses the threshold. This enables the transmitted signal to decoded with satisfaction root error probability.

There are many applications of large deviation techniques to the SINR networks, which are used as models for telecommunication networks. Some of these applications include, the analysis of bi-stability in networks, such as notorious bi-stability in multiple access protocols the Aloha, and the stochastic behaviour of ATM admission control, sizing of internal buffers, and the simulation of ATM models, see,(Weiss, 1995). and prevention of cyber-attacks on wireless telecommunication networks, see example (Paschalidis and Chen, 2008).

Cybersecurity of the devices in a telecommunication system is a major issue when the devices become increasingly dependent on computer and other local networks. And an anomaly detection in the devices networks is key to avoiding disruption in the telecommunication systems. Cybersecurity of the intelligent electronic devices in telecommunication substations has been recognized as a critical issue for smooth running of the system. One main approach to dealing with these issues is to develop new technologies to detect and disrupt any malicious activities over the networks. An Anomaly detection may be regarded as an early warning mechanism to extract relevant cybersecurity events from devices locations and correlate these events. Large deviation principles have played a key role in the formulation of efficient anomaly inference algorithm for systems such as power grid, Wireless Sensor Network systems and Telecommunication systems.

In this chapter, we prove joint large deviation principles on the scales λ and $\lambda^2 a_\lambda$, where λ is the intensity measure of the underlining PPP of the subcritical SINR model. See, Sakyi-Yeboah et al. (2020) or Sakyi-Yeboah et al. (2021c) or Sakyi-Yeboah et al. (2021b) for similar results and also for the dense SINR random network models. From these LDPs, we prove an asymptotic equipartition property; see example Sakyi-Yeboah et al. (2020), for the SINR models.

Further, the study shows a LLDP for the SINR models. See example, Sakyi-Yeboah et al. (2020) and references therein. From the LLDP, we deduce asymptotic bounds on the cardinality of the set of SINR models for a given typical empirical marked measure. In addition, the study deduce the prove of the LLDP an LDP for the SINR modelled processes.

5.1.2 Background

This study set a dimension $d \in \mathbb{N}$ and some measureable set $\mathcal{D} \subset \mathbb{R}^d$ with reference to the Borel-σ algebra $\mathcal{B}(\mathbb{R}^d)$. Given $\lambda m : \mathcal{D} \to [0,1]$, an intensity measure and probability kernel density function from $\mathcal{D}$ to $\mathbb{R}^+$, $\mathcal{Q}$ and a path loss model, $\ell(r) = r^{-\alpha}$, where $\alpha \in \mathbb{R}^+$, and some technical constraint; $\gamma^{(\lambda)}, \tau^{(\lambda)} : \mathbb{R}^+ \to \mathbb{R}^+$. The study defined the SINR network model as follows:

(i) We select $X = (X_i)_{i \in I}$, a Poisson Point Process (PPP) with rate measure $\lambda m : \mathcal{D} \to [0,1]$.

(ii) Given the process X, the locations, each X_i is assigned a mark or power $\sigma(X_i) = \sigma_i$, independently according to the kernel density function $\mathcal{Q}(\cdot\,, X_i)$.

(iii) For any two set of marked points $((X_i, \sigma_i), (X_j, \sigma_j))$ we link an edge if and only if

$$SINR(X_i, X_j, X) \geq \tau^{(\lambda)}(\sigma_j) \text{ and } SINR(X_i, X_j, X) \geq \tau^{(\lambda)}(\sigma_i),$$

where

$$SINR(X_j, X_i, X) = \frac{\sigma_i \ell(\|X_i - X_j\|)}{N_0 + \gamma^{(\lambda)}(\sigma_j) \sum_{i \in I \setminus \{j\}} \sigma_i \ell(\|X_i - X_j\|)}$$

We let E denote the set of edges in the SINR random network and observe $X^\lambda := X^\lambda(m, \mathcal{Q}, \ell) = \Big\{[(X_i, \sigma_i), i \in I], E\Big\}$ under the joint law of the marked

PPP and the network. In this chapter, we call X^λ an SINR Network model and $(X_i, \sigma_i) := X_i^\lambda$ as the mark of site i. Recall from Sakyi-Yeboah et al. (2020) that if $N_0 = 0$, then the connectivity function of the SINR random network model, P^λ, is defined as $P^\lambda((x,\sigma_x),(y,\sigma_y)) = e^{-\lambda h_\lambda^{\mathcal{D}}((x,\sigma_x),(y,\sigma_y))}$, where

$$h_\lambda^{\mathcal{D}}((x,\sigma_x),(y,\sigma_y)) = \int_D \Big[\frac{\tau^{(\lambda)}(\sigma_x)\gamma^{(\lambda)}(\sigma_x)}{\gamma^{(\lambda)}(\sigma_x)\tau^{(\lambda)}(\sigma_x)+(\|z\|^\alpha/\|x-y\|^\alpha)} + \frac{\tau^{(\lambda)}(\sigma_y)\gamma^{(\lambda)}(\sigma_y)}{\tau^{(\lambda)}(\sigma_y)\gamma^{(\lambda)}(\sigma_y)+(\|z\|^\alpha/\|y-x\|^\alpha)}\Big] m(dz).$$

This chapter assumes that there exists a_λ and a function $h : \mathcal{D} \times \mathbb{R}_+ \to (0, \infty)$ such that $\lambda^2 a_\lambda \to 0$ and

$$\lim_{\lambda\uparrow\infty} a_\lambda^{-1} P^\lambda((x,\sigma_x),(y,\sigma_y)) = h((x,\sigma_x),(y,\sigma_y)).$$

Sakyi-Yeboah et al. (2021b) and Sakyi-Yeboah et al. (2021c) investigates the critical SINR network model (that is $\lambda a_\lambda \to 1$) and super-critical SINR network model (that is $\lambda a_\lambda \to \infty$) respectively . In this chapter, we shall focus this study on sub-critical SINR Networks(that is $\lim_{\lambda\to\infty} \lambda a_\lambda \to 0$).

For a given set $\mathcal{D}$ we define $\mathcal{S}(\mathcal{D})$ by

$$\mathcal{S}(\mathcal{D}) = \cup_{x\subset\mathcal{D}}\Big\{x : \ |x \cap A| < \infty \ , \text{for any bounded } A \subset \mathcal{D} \Big\}. \tag{5.1}$$

Let $\mathcal{X} = \mathcal{S}(\mathcal{D} \times \mathbb{R}_+)$ and $\mathcal{M}(\mathcal{X})$, represent the space of positive measures on the space $\mathcal{X}$ equipped with $\tau-$ topology. Note, $\mathcal{X}$ is a locally finite subset of the set $\mathcal{D}\times\mathbb{R}_+$. See, example, (Sakyi-Yeboah et al., 2021c). Without abuse of notation we shall refer to $\mathcal{M}(\mathcal{X} \times \mathcal{X})$ as the space of symmetric measure on $\mathcal{X} \times \mathcal{X}$ endowed with the $\tau-$ topology. For any SINR random network model X^λ we define a probability measure, the *empirical power measure*, $L_1^\lambda \in \mathcal{M}(\mathcal{X})$, by

$$L_1^\lambda\big((x,\sigma_x)\big) := \frac{1}{\lambda}\sum_{i\in I} \delta_{X_i^\lambda}\big((x,\sigma_x)\big)$$

and a finite measure, the *empirical connectivity measure* $L_2^\lambda \in \mathcal{M}(\mathcal{X} \times \mathcal{X})$, by

$$L_2^\lambda\big((x,\sigma_x),(y,\sigma_y)\big) := \frac{1}{\lambda^2 a_\lambda} \sum_{(i,j)\in E} [\delta_{(X_i^\lambda, X_j^\lambda)} + \delta_{(X_j^\lambda, X_i^\lambda)}]\big((x,\sigma_x),(y,\sigma_y)\big).$$

It should be noted that the total mass $\|L_1^\lambda\|$ of the empirical power measure is 1 and total mass of the empirical connectivity measure is $2|E|/\lambda^2 a_\lambda$.

Note that, we shall use L_1^λ and L_2^λ interchangeable as $L_1^{X^\lambda}$ and $L_2^{X^\lambda}$ in this chapter.

5.1.3 Motivation: Anomaly detection in spatial networks

Consider, SINR random network model as a model that account for the connectivity structure of the Wireless telecommunication networks (WTN). In particular, consider the subcritical SINR random networks as model for the WTNs since, in the implementation, the multihop network formed by the sensor nodes may adopt a network structure. The network will be formed randomly according to an arbitrary rule that is dependent on the distances between the device locations. Assume the device locations are marked according to their battery power, and the propagation of events is un-directed on the network. Our objective is to estimate network parameters and possibly identify possible deviations from the actual values.

For instance, given a long sequence of realization $X^{\lambda,k}$ of this sub-critical marked SINR random network, one would like to approximate parameter of the model, $m \otimes \mathcal{Q}$ and h, by taking the average frequencies of the corresponding samples. In particular, if $L_1^{X^{\lambda,k}}$ and $L_2^{X^{\lambda,k}}$; the empirical power measure and the empirical connectivity measure of X^λ, up to the k^{th} realization then

$$\lim_{k\to\infty} \tfrac{1}{k} \sum_{r=1}^{k} L_1^{X^{\lambda,r}}(x,\sigma_x) \to m \otimes \mathcal{Q}(x,\,\sigma_x)$$

and

$$\lim_{k\to\infty}\Big[\frac{1}{k}\sum_{r=1}^{k}L_2^{X^{\lambda,r}}\Big((x,\sigma_x),(y,\sigma_y)\Big)/\frac{1}{k}\sum_{r=1}^{k}L_2^{X^{\lambda,r}}(x,\sigma_x)\otimes\frac{1}{k}\sum_{r=1}^{k}L_1^{X^{\lambda,r}}(y,\sigma_y)\Big]\to h\Big((x,\sigma_x),(y,\sigma_y)\Big),$$

with probability 1.

Assuming that we have estimated $m\otimes\mathcal{Q}$ and h. We are interested in a test that determines whether a particular realization X^{λ} is typical or not. Thus, we want to differentiate between $m\times\mathcal{Q}$ and h (Hypothesis H_0) and any other unknown law (Hypothesis H_1). Theorem 5.2.1 will be the bases of providing generalized Neyman-Pearson criterion,See (Zeitouni and Dembo, 1998, pp.96-100),and hence an anomaly detection test for the sub-critical marked SINR random networks.

This chapter is structured as follows: Section 5.2 presents the main results; Theorem 5.2.1, Theorem 5.2.2, Theorem 5.2.3, Corollary 5.2.4 and Corollary 5.2.5. In Section 5.3 we prove the main results of the article, Theorem 5.2.1. Section 5.4 provides the proof of the AEP, see Theorem 5.2.2 and Section 5.5; Proof of Theorem 5.2.3, Corollary 5.2.4 and Corollary 5.2.5. Lastly, Section 5.6 presents the conclusion to the chapter.

5.2 Main Results

Theorem 5.2.1, be a joint large deviation principle for the empirical meassures of the SINR network models.With reference from Subsection 5.1.2, we recall the definition of $h_{\lambda}^{\mathcal{D}}$ as

$$h_{\lambda}^{\mathcal{D}}((x,\sigma_x),(y,\sigma_y))=\int_D\Big[\frac{\tau^{(\lambda)}(\sigma_x)\gamma^{(\lambda)}(\sigma_x)}{\gamma^{\lambda}(\sigma_x)\tau^{\lambda}(\sigma_x)+(\|z\|^{\alpha}/\|x-y\|^{\alpha})}+\frac{\tau^{(\lambda)}(\sigma_y)\gamma^{(\lambda)}(\sigma_y)}{\tau^{(\lambda)}(\sigma_y)\gamma^{(\lambda)}(\sigma_y)+(\|z\|^{\alpha}/\|y-x\|^{\alpha})}\Big]m(dz)$$

and note that

$$h\omega \otimes \omega((x,\sigma_x),(y,\sigma_y))) := h((x,\sigma_x),(y,\sigma_y))\omega((x,\sigma_x))\omega((y,\sigma_y)).$$

Theorem 5.2.1 *Let X^λ be a sub-critical marked SINR network model with rate measure $\lambda m : \mathcal{D} \to [0,1]$ and a power transition kernel function $\mathcal{Q}(\cdot,\sigma) = ce^{-c\sigma}, \sigma > 0$ and path loss function $\ell(r) = r^{-\alpha}$, for $\alpha > 0$. Thus, the link kernel function P^λ of X^λ satisfies $a_\lambda^{-1}P^\lambda \to h$ and $\lambda^2 a_\lambda \to 0$. Then, as $\lambda \to \infty$, the pair of measures $(L_1^{X^\lambda}, L_2^{X^\lambda})$ satisfies a large deviation principle in the space $\mathcal{M}(\mathcal{X}) \times \mathcal{M}(\mathcal{X}\times\mathcal{X})$*

(i) with speed λ and a good rate function

$$I^1\big(\omega,\pi\big) = \begin{cases} H\Big(\omega\Big|m\otimes\mathcal{Q}\Big) & \text{if } \pi = h\omega\otimes\omega \\ \infty & \text{elsewhere.}\end{cases} \tag{5.2}$$

(ii) with speed $\lambda^2 a_\lambda$ and good rate function

$$I^2\big(\omega,\pi\big) = \begin{cases} \mathcal{H}(\pi\|h\omega\otimes\omega), & \text{if } \omega = m\otimes\mathcal{Q} \\ \infty & \text{elsewhere.}\end{cases} \tag{5.3}$$

where

$$\mathcal{H}(\pi\|h\omega\otimes\omega) := \begin{cases} H\Big(\pi\ \Big|h\omega\otimes\omega\Big) + \Big(\|h\omega\otimes\omega\| - \|\pi\|\Big), & \text{if } \|\pi\| > 0. \\ \infty & \text{elsewhere.}\end{cases} \tag{5.4}$$

Theorem 5.2.2 *Suppose X^λ is a sub-critical marked SINR network model with rate measure $\lambda m : D \to [0,1]$ and a power probability function $\mathcal{Q}(\cdot,\sigma) = ce^{-c\sigma}, \sigma > 0$ and path loss function $\ell(r) = r^{-\alpha}$, for $\alpha > 0$. Thus, the connectivity probability P^λ of X^λ satisfies $\lim_{\lambda\to\infty} a_\lambda^{-1}P^\lambda \to h$ and $\lambda^2 a_\lambda \to 0$. Suppose the* *sequence a_λ of X^λ is such that* $\lim_{\lambda\to\infty} \lambda a_\lambda \log\lambda \to 0$ *and* $\lim_{\lambda\to\infty} a_\lambda/\log\lambda \to -1$. *Then, we have*

$$\lim_{\lambda\to\infty} \mathbb{P}\Big\{\Big| -\frac{1}{a_\lambda \lambda^2 \log\lambda} \log P(X^\lambda) - \mathbb{E}_f\Big[h((\cdot,\cdot),(\cdot,\cdot))\Big]\Big| \geq \varepsilon\Big\} = 0,$$

where the expectation was taken with respect to the distribution function

$$f((x,\sigma_x),(y,\sigma_y)) = c^2 e^{-c(x+y)} m(d\sigma_x) m(d\sigma_y) dxdy, \quad x>0, y>0, \sigma_x>0, \sigma_y>0.$$

Note that the $H(f) := \mathbb{E}_f\Big[h((\cdot,\cdot),(\cdot,\cdot))\Big]$ is an entropy.

Interpretation: To transmit information contained in a large SINR random network models, one require, with a large probability

$$-\lambda^2 a_\lambda \log\lambda \Big[H(f)\Big]/\log 2 \textbf{ bits.}$$

Let $\mathcal{G}$ be the set of all SINR networks with rate measure $\lambda m : \mathcal{D} \to [0,1]$ and state the Local Large deviation principle as follows:

Theorem 5.2.3 *Suppose X^λ is a sub-critical marked SINR network model with rate measure $\lambda m : \mathcal{D} \to [0,1]$ and a mark transition kernel $\mathcal{Q}(\sigma) = ce^{-c\sigma}, \sigma > 0$ and path loss function $\ell(r) = r^{-\alpha}$, for $r > 0$ and $\alpha > 0$. Thus, the link probability P^λ of X^λ satisfies $\lim_{\lambda\to\infty} a_\lambda^{-1} P^\lambda \to h$ and $\lambda^2 a_\lambda \to 0$. Then,*

(i) for any functional $\omega \in \mathcal{M}_\omega$ and a number $\varepsilon > 0$, there exists a weak neighbourhood B_ν such that

$$\mathbb{P}_\omega\Big\{X^\lambda \in \mathcal{G} \,\Big|\, L_2^{X^\lambda} \in B_\nu\Big\} \leq e^{-\frac{1}{2}\lambda^2 a_\lambda \mathcal{H}(\pi \| h\omega\otimes\omega) - \lambda a_\lambda \varepsilon}, \text{ where } \omega = m \otimes \mathcal{Q}.$$

(ii) for any $\omega \in \mathcal{M}_\omega$, a number $\varepsilon > 0$ and a fine neighbourhood B_ν, we have the

compute:

$$\mathbb{P}_\omega\Big\{X^\lambda \in \mathcal{G}\,\Big|\, L_2^{X^\lambda} \in B_\nu\Big\} \geq e^{-\frac{1}{2}\lambda^2 a_\lambda \mathcal{H}(\pi\|h\omega\otimes\omega)+\lambda_\lambda a_\lambda \varepsilon}, \text{ where } \omega = m\otimes\mathcal{Q}.$$

For the given telecommunication network model, we define an entropy as $h : \mathcal{M}(\mathcal{X}\times\mathcal{X}) \to [0.\infty]$ by

$$h(\nu) := \Big(\|\pi\| - \|\lambda\omega\otimes\omega\| - \Big\langle \pi\,,\,\log\frac{\pi}{\|h\omega\otimes\omega\|}\Big\rangle\Big)/2, \quad \text{where } \omega = m\otimes\mathcal{Q}. \tag{5.5}$$

Corollary 5.2.4 (McMillian Theorem) *Let* X^λ *be a sub-critical marked SINR network model with rate measure* $\lambda m : \mathcal{D} \to [0,1]$ *and a mark transition kernel* $\mathcal{Q}(\cdot,\sigma) = ce^{-c\sigma}, \sigma > 0$ *and path loss function* $\ell(r) = r^{-\alpha}$*, for* $r > 0$ *and where* $\omega = m\otimes\mathcal{Q}$ *and* $\alpha > 0$*. Thus, the link probability* P^λ *of every* $X^\lambda \in \mathcal{G}$ *satisfies* $\lim_{\lambda\to\infty} a_\lambda^{-1} P^\lambda \to h$ *and* $\lambda a_\lambda \to 0$.

(i) For any empirical link measure π *on* $\mathcal{X}\times\mathcal{X}$ *and* $\varepsilon > 0$*, there exists a neighborhood* B_ν *such that*

$$Card\Big(\{X^\lambda \in \mathcal{G}\,|\, L_2^{X^\lambda} \in B_\nu\}\Big) \geq e^{\lambda^2 a_\lambda (h(\nu)-\varepsilon)}.$$

(ii) for any neighborhood B_ν *and* $\varepsilon > 0$*, we have*

$$Card\Big(\{X^\lambda \in \mathcal{G}\,|\, L_2^{X^\lambda} \in B_\nu\}\Big) \leq e^{\lambda^2 a_\lambda (h(\nu)+\varepsilon)},$$

where $Card(A)$ means the cardinality of A.

Remark 9 *Given* $\pi = h\omega\otimes\omega$*, we have* $Card\Big(\Big\{x \in \mathcal{G}\Big\}\Big) \approx e^{\lambda^2 a_\lambda \|h\omega\otimes\omega\| \mathcal{H}\left(h\omega\otimes\omega/\|h\omega\otimes\omega\|\right)}$*, where* $\omega = m\otimes\mathcal{Q}$.

Interpretation: Note from Corollary 5.2.4 that, for the typical empirical connectivity measure, $hm^2\otimes\mathcal{Q}^2$, the cardinality of the space of SINR models is nearly equal to $e^{\lambda^2 a_\lambda \|hm^2\otimes\mathcal{Q}^2\| H\left(hm^2\otimes\mathcal{Q}^2/\|hm^2\otimes\mathcal{Q}^2\|\right)}$.

The next theorem is the LDP for the SINR random network processes.

Corollary 5.2.5 *Let X^{λ} be a sub-critical marked SINR random network model with rate measure $\lambda m : \mathcal{D} \to [0,1]$ and a mark kernel function $\mathcal{Q}(\cdot, \sigma) = ce^{-c\sigma}, \sigma > 0$ and path loss function $\ell(r) = r^{-\alpha}$, for $\alpha > 0$. Thus, the link probability P^{λ} of X^{λ} satisfies $\lim_{\lambda\to\infty} a_{\lambda}^{-1} P^{\lambda} \to h$ and $\lambda^2 a_{\lambda} \to 0$.*

(i) Let F be closed subset $\mathcal{M}_{\omega}$. Then we have

$$\limsup_{\lambda\to\infty} \frac{1}{\lambda^2 a_{\lambda}} \log \mathbb{P}_{\omega}\Big\{X^{\lambda} \in \mathcal{G} \,\Big|\, L_2^{X^{\lambda}} \in F\Big\} \leq -\tfrac{1}{2} \inf_{\pi\in F} \Big\{\mathcal{H}(\pi \| hm\otimes\mathcal{Q}\times m\otimes\mathcal{Q})\Big\}$$

(ii) Let O be open subset $\mathcal{M}_{\omega}$. Then we have

$$\liminf_{\lambda\to\infty} \frac{1}{\lambda^2 a_{\lambda}} \log \mathbb{P}_{\omega}\Big\{X^{\lambda} \in \mathcal{G} \,\Big|\, L_2^{X^{\lambda}} \in O\Big\} \geq -\tfrac{1}{2} \inf_{\pi\in O} \Big\{\mathcal{H}(\pi \| hm\otimes\mathcal{Q}\times m\otimes\mathcal{Q})\Big\}.$$

5.3 Proof of Main Results

5.3.1 Proof of Theorem 5.2.1(i)

Suppose $A_1, ..., A_n$ is a decomposition of the space $\mathcal{D} \times \mathbb{R}_+$. Note that, for every $(x, y) \in A_i \times A_j$, $i, j = 1, 2, 3, ..., n$, $\lambda L_2^{X^{\lambda}}(x, y)$ given $\lambda L_1^{X^{\lambda}}(x) = \lambda\omega(x)$ denotes a number of bernoulli trial with parameters $\lambda^2\omega(x)\omega(y)/2$ and $P^{\lambda}(x, y)$. Consider $\mathcal{Q}$ to represent as the gamma distribution with mean $1/c$. With reference to the function $h_{\lambda}^{\mathcal{D}}$ from the preceding sections, we observe that Lemma 2.4.2 is fundamental in the application of the Gartner-Ellis Theorem. See, (Zeitouni and Dembo, 1998).

Lemma 5.3.1 *Suppose X^{λ} is a sub-critical marked SINR random model with rate measure $\lambda m : \mathcal{D} \to [0,1]$ and a power probability function $\mathcal{Q}(\cdot, \sigma) = ce^{-c\sigma}, \sigma > 0$ and path loss function $\ell(r) = r^{-\alpha}$, for $r > 0$ and $\alpha > 0$. Thus, the link probability P^{λ} of X^{λ} satisfies $\lim_{\lambda\to\infty} a_{\lambda}^{-1} P^{\lambda} \to h$ and $\lambda^2 a_{\lambda} \to 0$. Suppose X^{λ} be a sub-critical*

SINR network model, conditional on the event $L_1^{X^\lambda} = \omega$. Let $g : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ be bounded function. Then,

$$\lim_{\lambda\to\infty} \frac{1}{\lambda} \log \mathbb{E}\Big\{ e^{\lambda\langle g,\, L_2^{X^\lambda}\rangle} \Big| L_1^{X^\lambda} = \omega \Big\} = \frac{1}{2} \lim_{n\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{n} \Big\langle g,\, h\omega \otimes \omega \Big\rangle_{A_i\times A_j}$$
$$= \frac{1}{2}\Big\langle g,\, h\omega \otimes \omega \Big\rangle_{\mathcal{X}\times\mathcal{X}}.$$

Proof: Now we observe that

$$\mathbb{E}\Big\{ e^{\int\int \lambda g(x,y) L_2^{X^\lambda}(dx,dy)/2} \Big| L_1^{X^\lambda} = \omega \Big\} = \mathbb{E}\Big\{ \prod_{x\in\mathcal{X}} \prod_{y\in\mathcal{X}} e^{\lambda g(x,y) L_2^{\lambda}(dx,dy)/2} \Big\}$$

$$\mathbb{E}\Big\{ \prod_{x\in\mathcal{X}} \prod_{y\in\mathcal{X}} e^{g(x,y)\lambda L_2^{X^\lambda}(dx,dy/2)} \Big\} = \prod_{i=1}^{n} \prod_{j=1}^{n} \prod_{x\in A_i} \prod_{y\in A_j} \mathbb{E}\Big\{ e^{g(x,y)\lambda L_2^{X^\lambda}(dx,dy)/2} \Big\}$$

$$\log \Big\{ e^{\lambda\langle g, L_2^{X^\lambda}\rangle/2} \Big| L_1^{X^\lambda} = \omega \Big\} =$$
$$\sum_{j=1}^{n} \sum_{i=1}^{n} \int_{A_j} \int_{A_i} \log \Big[1 - P^\lambda(x,y) + P^\lambda(x,y) e^{g(x,y)/\lambda a_\lambda} \Big]^{\lambda^2 \omega\otimes\omega(dx,dy)/2} + o(n)$$

Using the dominated convergence theorem

$$\frac{1}{\lambda} \log E\{ e^{\lambda\langle g, L_2^{X^\lambda}\rangle/2} \mid L_1^{X^\lambda} = \omega \} =$$

$$\frac{1}{\lambda} \sum_{j=1}^{n} \sum_{i=1}^{n} \int_{A_i} \int_{A_j} \log \Big[1 - (1 - e^{g(x,y)/\lambda a_\lambda}) P^\lambda(x,y) \Big]^{\lambda^2 \omega\otimes\omega(dx,dy)/2} + o(n)/\lambda$$

$$\frac{1}{\lambda} \log \mathbb{E}\{ e^{\lambda\langle g, L_2^{X^\lambda}\rangle/2} | L_1^{X^\lambda} = \omega \} =$$

$$\lim_{\lambda\to\infty} \sum_{j=1}^{n} \sum_{i=1}^{n} \int_{A_i} \int_{A_j} \log \Big[1 + g(x,y)h(x,y)/\lambda + o(\lambda)/\lambda \Big]^{\lambda\omega\otimes\omega(dx,dy)/2} + o(n)/\lambda$$

$$\lim_{\lambda\to\infty} \frac{1}{\lambda} \log \mathbb{E}\{ e^{\lambda\langle g, L_2^{X^\lambda}\rangle/2} \mid L_1^{X^\lambda} = \omega \} = \frac{1}{2} \sum_{j=1}^{n} \sum_{i=1}^{n} \Big\langle g,\, h\omega \otimes \omega \Big\rangle_{A_i\times A_j}$$

$$\lim_{\lambda\to\infty}\frac{1}{\lambda}\log\mathbb{E}\{e^{\lambda\langle g,L_2^{X^\lambda}\rangle/2}\Big|L_1^{X^\lambda}=\omega\}=\frac{1}{2}\lim_{n\to\infty}\sum_{j=1}^{n}\sum_{i=1}^{n}\Big\langle g,\,h\omega\otimes\omega\Big\rangle_{A_i\times A_j}$$
$$=\frac{1}{2}\Big\langle g,\,h\omega\otimes\omega\Big\rangle_{\mathcal{X}\times\mathcal{X}}.$$

Hence,by the Gartner-Ellis theorem, conditional on the event $\Big\{L_1^{X^\lambda}=\omega\Big\}$, $L_2^{X^\lambda}$ obey a large deviation principle with speed λ and variational formulation of the rate function

$$I_\omega(\pi)=\frac{1}{2}\sup_g\Big\{\Big\langle g,\,\pi\Big\rangle_{\mathcal{X}\times\mathcal{X}}-\Big\langle g,\,h\omega\otimes\omega\Big\rangle_{\mathcal{X}\times\mathcal{X}}\Big\}$$

the solution can be found, see example Doku-Amponsah (2012), would obviously reduces to the good rate function as such

$$I_\omega(\pi)=0. \tag{5.6}$$

□

5.3.2 Proof of Theorem 5.2.1(ii)

Analogously we consider $A_1,...,A_n$ as decomposition of the space $\mathcal{D}\times\mathbb{R}_+$. We refer to h_λ^D and observe that, Lemma 5.3.2 will play an important role in the application of the Gartner-Ellis Theorem. See, (Zeitouni and Dembo, 1998).

Lemma 5.3.2 *Let X^λ be a sub critical powered SINR network with rate measure $\lambda m:\mathcal{D}\to[0,1]$ and a power probability function $\mathcal{Q}(\sigma)=ce^{-c\sigma},\sigma>0$, and path loss function $\ell(r)=r^{-\alpha}$, for $r>0$, and $\alpha>0$. Thus, the link probability P^λ of X^λ satisfies $\lim_{\lambda\to\infty}a_\lambda^{-1}P^\lambda\to h$ and $\lambda^2a_\lambda\to\infty$. Let X^λ be a sub-critical SINR network, conditional on the event $L_1^{X^\lambda}=\omega$. Let $g:\mathcal{X}\times\mathcal{X}\to\mathbb{R}$ be bounded function. Then,*

$$\lim_{\lambda\to\infty}\frac{1}{\lambda^2 a_\lambda}\log\mathbb{E}\Big\{e^{\lambda^2 a_\lambda\langle g,\,L_2^{X^\lambda}\rangle}\Big|L_1^{X^\lambda}=\omega\Big\}=-\frac{1}{2}\lim_{n\to\infty}\sum_{j=1}^{n}\sum_{i=1}^{n}\Big\langle 1-e^g,\,h\omega\otimes\omega\Big\rangle_{A_i\times A_j}$$
$$=-\frac{1}{2}\Big\langle 1-e^g,\,h\omega\otimes\omega\Big\rangle_{\mathcal{X}\times\mathcal{X}}.$$

Proof: Now we note that

$$\mathbb{E}\Big\{e^{\int\int\lambda^2 a_\lambda g(x,y)L_2^{X^\lambda}(dx,dy)/2}\Big|L_1^{X^\lambda}=\omega\Big\}=\mathbb{E}\Big\{\prod_{i\in\mathcal{X}}\prod_{j\in\mathcal{X}}e^{\lambda^2 a_\lambda g(x,y)L_2^{X^\lambda}(dx,dy)/2}\Big\}$$

$$\mathbb{E}\Big\{\prod_{i\in\mathcal{X}}\prod_{j\in\mathcal{X}}e^{g(x,y)\lambda L_2^{X^\lambda}(dx,dy/2)}=\prod_{i=1}\prod_{j=1}\prod_{x\in A_i}\prod_{y\in A_j}\mathbb{E}\Big\{e^{\lambda^2 a_\lambda g(x,y)L_2^{X^\lambda}(dx,dy)/2}\Big\}\times e^{o(n)}$$

$$\log\Big\{e^{\lambda^2 a_\lambda\langle g,L_2^{X^\lambda}\rangle/2}\Big|L_1^{X^\lambda}=\omega\Big\}=$$
$$\sum_{j=1}^{n}\sum_{i=1}^{n}\int_{A_j}\int_{A_i}\log\Big[1-P^\lambda(x,y))+P^\lambda(x,y)e^{g(x,y)}\Big]^{\lambda^2\omega\otimes\omega(dx,dy)/2}+o(n)$$

Using the dominated convergence theorem

$$\frac{1}{\lambda^2 a_\lambda}\log E\{e^{\lambda\langle g,L_2^{X^\lambda}\rangle/2}\mid L_1^{X^\lambda}=\omega\}=$$
$$\frac{1}{\lambda^2 a_\lambda}\sum_{j=1}\sum_{i=1}\int_{A_i}\int_{A_j}\log\Big[1-(1-e^{g(x,y)})P^\lambda(x,y)\Big]^{\lambda^2\omega\otimes\omega(dx,dy)/2}+o(n)/\lambda^2 a_\lambda$$

$$\frac{1}{\lambda^2 a_\lambda}\log\mathbb{E}\{e^{\lambda\langle g,L_2^{X^\lambda}\rangle/2}|L_1^{X^\lambda}=\omega\}=$$
$$\lim_{\lambda\to\infty}\sum_{j=1}\sum_{i=1}\int_{A_i}\int_{A_j}\log\Big[1-(1-e^{g(x,y)})P^\lambda(x,y)\Big]^{\lambda\omega\otimes\omega(dx,dy)/2}+o(n)/\lambda^2 a_\lambda$$

$$\lim_{\lambda\to\infty}\frac{1}{\lambda^2 a_\lambda}\log\mathbb{E}\Big\{e^{\lambda\langle g,L_2^{X^\lambda}\rangle/2}\big|L_1^{X^\lambda}=\omega\Big\}=-\frac{1}{2}\sum_{j=1}\sum_{i=1}\int_{A_i}\int_{A_j}\Big[(1-e^{g(x,y)})h(x,y)\omega\otimes\omega(dx,dy)\Big]$$

$$\lim_{\lambda\to\infty}\frac{1}{\lambda^2 a_\lambda}\log\mathbb{E}\{e^{\lambda\langle g,L_2^{X^\lambda}\rangle/2}\big|L_1^{X^\lambda}=\omega\}=-\frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n}\Big\langle 1-e^g,\, h\omega\otimes\omega\Big\rangle_{A_i\times A_j}$$

$$\begin{aligned}\lim_{\lambda\to\infty}\frac{1}{\lambda^2 a_\lambda}\log\mathbb{E}\{e^{\lambda\langle g,L_2^{X^\lambda}\rangle/2}\big|L_1^{X^\lambda}=\omega\}&=-\frac{1}{2}\lim_{n\to\infty}\sum_{j=1}^{n}\sum_{i=1}^{n}\Big\langle 1-e^g,\, h\omega\otimes\omega\Big\rangle_{A_i\times A_j}\\&=-\frac{1}{2}\Big\langle 1-e^g,\, h\omega\otimes\omega\Big\rangle_{\mathcal{X}\times\mathcal{X}}\end{aligned}$$

Hence,by the Gartner-Ellis theorem, conditional on the event $\Big\{L_1^{X^\lambda}=\omega\Big\}$, $L_2^{X^\lambda}$ obey a large deviation principle with speed λ and variational formulation of the rate function is given by

$$I_\omega(\pi)=\frac{1}{2}\sup_g\Big\{\Big\langle g,\,\pi\Big\rangle_{\mathcal{X}\times\mathcal{X}}+\Big\langle 1-e^g,\, h\omega\otimes\omega\Big\rangle_{\mathcal{X}\times\mathcal{X}}\Big\}$$

which when solved, see example Doku-Amponsah (2012), will clearly reduce to the good rate function given by

$$I_\omega(\pi)=\frac{1}{2}\mathcal{H}(\pi\|h\omega\otimes\omega). \tag{5.7}$$

□

5.3.3 Proof of Theorem 5.2.1(ii) by Method of Mixtures.

For any $\lambda\in(0,\infty)$, we define

$$\begin{aligned}\mathcal{M}_\lambda(\mathcal{X})&:=\Big\{\omega\in\mathcal{M}(\mathcal{X})\,:\,\lambda\omega(x)\in\mathbb{N}\text{ for all }x\in\mathcal{X}\Big\},\\\tilde{\mathcal{M}}_\lambda(\mathcal{X}\times\mathcal{X})&:=\Big\{\pi\in\mathcal{M}(\mathcal{X}\times\mathcal{X})\,:\,\lambda\pi(x,y)\in\mathbb{N},\ \text{ for all }\ x,y\in\mathcal{X}\times\mathcal{X}\Big\}.\end{aligned}$$

We denote by $\Theta_\lambda := \mathcal{M}_\lambda(\mathcal{X})$ and $\Theta := \mathcal{M}(\mathcal{X})$. We write

$$P^{(\lambda)}_{\omega_\lambda}(\eta_\lambda) := \mathbb{P}\big\{L_2^{X^\lambda} = \eta_\lambda \,\big|\, L_1^{X^\lambda} = \omega_\lambda\big\},$$
$$P^{(\lambda)}(\omega_\lambda) := \mathbb{P}\big\{L_1^{X^\lambda} = \omega_\lambda\big\}$$

Th joint distribution of $L_1^{X^\lambda}$ and $L_2^{X^\lambda}$ is the mixture of $P^{(\lambda)}_{\omega_\lambda}$ with $P^{(\lambda)}(\omega_\lambda)$, given as follows:

$$d\tilde{P}^\lambda(\omega_\lambda, \eta_\lambda) := dP^{(\lambda)}_{\omega_\lambda}(\eta_\lambda)\, dP^{(\lambda)}(\omega_\lambda). \tag{5.8}$$

(Biggins et al., 2004, Theorem 5 (b)) provides condition for the validity of large deviation principles for the mixtures and for the goodness of the rate function if individual large deviation principles are known. The following three lemmas ensure validity of these conditions.

Note that the family of measures $(P^{(\lambda)}\colon \lambda \in (0,\infty))$ is exponentially tight on Θ.

Lemma 5.3.3 *(i) The family of measures $(\tilde{P}^\lambda\colon \lambda \in (0,\infty))$ is exponentially tight on $\Theta \times \tilde{\mathcal{M}}(\mathcal{X} \times \mathcal{X})$.*

(ii) The family of measures $(P^\lambda\colon \lambda \in (0,\infty))$ is exponentially tight on $\Theta \times \mathcal{M}(\mathcal{X} \times \mathcal{X})$.

We refer to (Sakyi-Yeboah et al., 2020, Lemma 4.3) for similar proof for Large Deviation Principle on the scale λ^2

Define the function $I^2_{sc}, I^1_{sc}\colon \Theta \times \mathcal{M}(\mathcal{X} \times \mathcal{X}) \to [0,\infty]$, by

$$I^1\big(\omega,\pi\big) = \begin{cases} H\Big(\omega\Big|m \otimes \mathcal{Q}\Big) & \text{if } \pi = h\omega \otimes \omega \\ \infty & \text{otherwise.} \end{cases} \tag{5.9}$$

$$I^2\big(\omega,\pi\big) = \frac{1}{2}\mathcal{H}\Big(\pi\|h\omega \otimes \omega\Big). \tag{5.10}$$

Lemma 5.3.4 *(i) I^1 is lower semi-continuous.*

(ii) I^2 is lower semi-continuous.

By (Biggins et al., 2004, Theorem 5(b)),the two previous lemmas, the LDP for the empirical power measure, see, (Sakyi-Yeboah et al., 2020, Theorem 2.1) and the large deviation principles we have established Theorem 5.2.1 ensure that under $(\tilde{P}^{\lambda})$ and P^{λ} the random variables $(\omega_{\lambda},\sigma_{\lambda})$ satisfy a large deviation principle on $\mathcal{M}(\mathcal{X})\times\mathcal{M}(\mathcal{X}\times\mathcal{X})$ and $\Theta\times\mathcal{M}_{\lambda}(\mathcal{X}\times\mathcal{X})$ on the speeds λ and $\lambda^2 a_{\lambda}$ with good rate functions I^1 and I^2 respectively, which ends the proof of Theorem 5.2.1.

5.4 Proof of Theorem 5.2.2 by Large deviations

To prove the Shannon-Mcmillian Breiman (SMB) or the AEP, we first prove a weak law of large numbers (WLLN) for the empirical marked measure and the empirical connectivity measure of the SINR network model.

Lemma 5.4.1 *Let Y^{λ} be a sub-critical marked SINR model with rate measure $\lambda m:\mathcal{D}\to[0,1]$, and a marked transition function $\mathcal{Q}(\cdot,\sigma)=ce^{-c\sigma},\sigma>0$ and path loss function $\ell(r)=r^{-\alpha}$, for $\alpha>0$. Thus, the link probability P^{λ} of X^{λ} satisfies $\lim_{\lambda\to\infty}a_{\lambda}^{-1}P^{\lambda}\to h$ and $\lambda^2 a_{\lambda}\to 0$. Then,*

$$\lim_{\lambda\to\infty}\mathbb{P}\Big\{\sup_{(x,\sigma_x)\in\mathcal{X}}\Big|L_1^{X^{\lambda}}(x,\sigma_x)-m\otimes\mathcal{Q}(x,\sigma_x)\Big|>\varepsilon\Big\}=0$$

and

$$\lim_{\lambda\to\infty}\mathbb{P}\Big\{\sup_{([x,\sigma_x],[y,\sigma_y])\in\mathcal{X}\times\mathcal{X}}\Big|L_2^{X^{\lambda}}([x,\sigma_x],[y,\sigma_y])-hm\otimes\mathcal{Q}\times m\otimes\mathcal{Q}([x,\sigma_x],[y,\sigma_y])\Big|>\varepsilon\Big\}=0$$

Proof: Let

$$F_{1,\mathcal{X}}=\Big\{\omega:\sup_{(x,\sigma_x)\in\mathcal{X}}|\omega(x,\sigma_x)-m\otimes\mathcal{Q}(x,\sigma_x)|>\varepsilon\Big\},$$

$$F_{2,\mathcal{X}}=\Big\{\pi:\sup_{([x,\sigma_x],[y,\sigma_y])\in\mathcal{X}\times\mathcal{X}}|\pi([x,\sigma_x],[y,\sigma_y])-hm\otimes\mathcal{Q}\times m\otimes\mathcal{Q}([x,\sigma_x],[y,\sigma_y])|>\varepsilon\Big\}$$

and $F_{3,\mathcal{X}}=F_{1,\mathcal{X}}\cup F_{2,\mathcal{X}}$. Now, observe from Theorem 5.2.1 that

$$\lim_{\lambda\to\infty}\frac{1}{\lambda}\log\mathbb{P}\Big\{(L_1^{X^\lambda},L_2^{X^\lambda})\in F_{3,\mathcal{X}}^c\Big\}\le-\inf_{(\omega,\varpi)\in F_{3,\mathcal{X}}^c}I(\omega,\pi).$$

It meets the requirement for the study to prove that I is strictly positive. For instance,there is a sequence $(\omega_\lambda,\pi_\lambda)\to(\omega,\pi)$ such that $I(\omega_\lambda,\pi_\lambda)\downarrow I(\omega,\pi)=0$. This means $\omega=m\otimes\mathcal{Q}$ and $\pi=hm\otimes\mathcal{Q}\times m\otimes\mathcal{Q}$ which contradicts $(\omega,\pi)\in F_3^c$. This ends the proof of the Lemma. □

We write $L_\Delta^\lambda=\frac{1}{\lambda}\sum_{i\in I}\delta_{(X_i^\lambda,X_j^\lambda)}$ and observe that the distribution of the marked SINR random network $P(x)=\mathbb{P}\Big\{X^\lambda=x\Big\}$ is given by

$$P_\lambda(x)=\prod_{i=1}^{I}|m\otimes\mathcal{Q}(x_i,\sigma_i)\prod_{(i,j)\in E}\frac{P^{x^\lambda}([x_i,\sigma_i],[y_j,\sigma_j])}{1-P^\lambda([x_i,\sigma_i],[y_j,\sigma_j])}\times$$

$$\prod_{(i,j)\in\mathcal{E}}(1-P^{x^\lambda}([x_i,\sigma_i],[y_j,\sigma_j]))\prod_{x=1}^{I}(1-P^{x^\lambda}([x_i,\sigma_i],[y_j,\sigma_j]))$$

$$\begin{aligned}-\frac{1}{a_\lambda\lambda^2\log\lambda}\log P_\lambda(x)=&\frac{1}{a_\lambda\lambda\log\lambda}\Big\langle-\log m\otimes Q\,,L_1^{X^\lambda}\Big\rangle\\&+\frac{1}{\log\lambda}\Big\langle-\log\Big(\frac{P^{x^\lambda}}{1-P^{x^\lambda}}\Big)\,,L_2^{X^\lambda}\Big\rangle\\&+\frac{1}{a_\lambda\log\lambda}\Big\langle-\log(1-P^{x^\lambda})\,,L_1^{X^\lambda}\otimes L_1^{X^\lambda}\Big\rangle\\&+\frac{1}{a_\lambda\lambda\log\lambda}\Big\langle-\log(1-P^{x^\lambda})\,,L_\Delta^\lambda\Big\rangle\end{aligned}$$

Notice,

$$\lim_{\lambda\to\infty}\frac{1}{a_\lambda\lambda\log\lambda}\Big\langle-\log m\otimes\mathcal{Q}\,,L_1^{X^\lambda}\Big\rangle=\lim_{\lambda\to\infty}\frac{1}{\lambda}\Big\langle-\log(1-P^\lambda\,,L_\Delta^\lambda\Big\rangle=$$

$$\lim_{\lambda\to\infty}\frac{1}{a_\lambda\log\lambda}\Big\langle-\log(1-P^{x^\lambda})\,,L_1^{X^\lambda}\otimes L_1^{X^\lambda}\Big\rangle=0.$$

Using, Lemma 5.4.1 we have

$$\lim_{\lambda\to\infty}\frac{1}{\log\lambda}\Big\langle-\log\Big(P^{x^\lambda}/(1-P^{x^\lambda}\Big)\,,L_2^{X^\lambda}\Big\rangle=\Big\langle\mathbb{1}\,,hm\otimes\mathcal{Q}\times m\otimes\mathcal{Q}\Big\rangle$$

which concludes the proof of Theorem 5.2.2.

5.5 Proof of Theorem 5.2.3, Corollary 5.2.4, Corollary 5.2.5

For $\omega \in \mathcal{M}(\mathcal{X})$ we define the spectral potential of the marked SINR graph (X^{λ}) conditional on the event $\{L_1^{X^{\lambda}} = \omega\}$, $U_Q(g,\omega)$ as

$$U_Q(g,\omega) = \Big\langle -(1-e^{g}),\, h\omega \otimes \omega \Big\rangle. \tag{5.11}$$

Note that remarkable properties of a spectral potential, see Bakhtin (2015) or Sakyi-Yeboah et al. (2020) holds for U_Q.
For $\omega \in \mathcal{M}(\mathcal{X} \times \mathcal{X})$, we observe that $I_\omega(\pi)$ is the Kullback action of the marked SINR graph X^{λ}.

Lemma 5.5.1 *The following hold for the Kullback action or divergence function $I_\omega(\pi)$:*

(i)

$$I_\omega(\pi) = \sup_{g \in \mathcal{C}} \{\langle g,\, \pi\rangle - U_Q(g,\omega)\}$$

(ii) The function $I_\omega(\pi)$ is convex and lower semi-continuous on the space $\mathcal{M}(\mathcal{X} \times \mathcal{X})$.

(iii) For any real α, the set $\Big\{\pi \in \mathcal{M}(\mathcal{X} \times \mathcal{X}) : I_\omega(\pi) \le \alpha\Big\}$ is weakly compact.

The proof of Lemma 5.5.1 is excluded from the article. Readers interested in the proved may rnfer to Sakyi-Yeboah et al. (2021c) for empirical measures of ‘ the supercritical marked SINR random network processes and/or the references therein for proof of the lemma for empirical measures on measurable spaces.

Note from Lemma 5.5.1 that, for any $\varepsilon > 0$, there exists some function $g \in \mathcal{X} \times \mathcal{X}$ such that

$$I_\omega(\pi) - \tfrac{\varepsilon}{2} < \langle g\,,\, \pi\rangle - U_Q(g,\omega).$$

We define the probability distribution of the powered X by P_ω by

$$P_\omega(x) = \prod_{(i,j)\in E} e^{g(x,y)} \prod_{(i,j)\in \mathcal{E}} e^{h_\lambda(x,y)},$$

where

$$h_\lambda(x,y) = \frac{1}{a_\lambda} \log\left[1 - P^\lambda(x,y) + P^\lambda(x,y)e^{g(x,y)}\right]$$

Then, clearly that

$$\frac{dP_\omega}{d\tilde{P}_\omega}(x) = \prod_{(i,j)\in E} e^{-g(x,y)} \prod_{(i,j)\in \mathcal{E}} e^{-h_\lambda(x,y)a_\lambda}$$
$$= e^{-\lambda^2 a_\lambda(\langle \frac{1}{2}g, L_2^{X^\lambda}\rangle - \lambda^2 a_\lambda \langle \frac{1}{2}h_\lambda, L_1^{X^\lambda}\otimes L_1^{X^\lambda}\rangle) + \langle \frac{1}{2}h_\lambda, L_\Delta^\lambda\rangle}$$

Now define the neighbourhood of ν, B_ν by

$$B_\nu := \left\{\omega \in \mathcal{M}(\mathcal{X}\times\mathcal{X}) : \langle g,\pi\rangle - U_Q(g,\omega) > \langle g,\pi\rangle - U_Q(g,\nu) - \varepsilon/2\right\}$$

Note that under the condition $L_2^{X^\lambda} \in B_\nu$ we have

$$\frac{dP_\omega}{d\tilde{P}_\omega}(x) < e^{-\lambda^2 a_\lambda(\langle \frac{1}{2}g, L_2^\lambda\rangle - \lambda^2 a_\lambda \langle \frac{1}{2}t_\lambda, L_1^{X^\lambda}\otimes L_1^{X^\lambda}\rangle) + \langle \frac{1}{2}h_\lambda, L_\Delta^\lambda\rangle} < e^{-\lambda^2 a_\lambda I_{sc}(\nu) + \lambda^2 a_\lambda \varepsilon}$$

Thus, the study can deduce that

$$P_\omega\left\{X^\lambda \in \mathcal{G}\middle| L_2^{X^\lambda} \in B_\nu\right\} \leq \int \mathbb{1}_{\{L_2^{X^\lambda}\in B_\nu\}} d\tilde{P}_\omega(X^\lambda) \leq \int e^{-\lambda^2 a_\lambda I_{sc}(\nu) - \lambda\varepsilon} d\tilde{P}_\omega(X^\lambda)$$
$$\leq e^{-\lambda^2 a_\lambda I_{sc}(\nu) - \lambda^2 a_\lambda \varepsilon}.$$

.

Given that $I^2(\pi) = 0$ satisfies the proof of Theorem 5.2.2 (ii), hence it is enough for us to obtain that the result is true for a probability distribution of the form $\pi = e^g\omega\otimes\omega$ and for $I^2(\pi) = \frac{1}{2}\mathcal{H}(\pi\|h\omega\otimes\omega)$, where $\omega = m\otimes\mathcal{Q}$. Fix any number

$\varepsilon > 0$ and any neigbourhood $B_\nu \subset \mathcal{M}(\mathcal{X} \times \mathcal{X})$. Now define the sequence of sets

$$\mathcal{G}^\lambda = \Big\{ x^\lambda \in \mathcal{G} : L_2^{x^\lambda} \in B_\nu \Big| \langle g, L_2^{x^\lambda} \rangle - U_Q(g, \omega) \Big| \leq \tfrac{\varepsilon}{2} \Big\}.$$

Note that for all $g \in \mathcal{G}^\lambda$ we have

$$\frac{dP_\omega}{d\tilde{P}_\omega}(x) > e^{-\lambda^2 a_\lambda \langle \frac{1}{2} g, \pi \rangle + \lambda^2 a_\lambda U_Q(g,\omega) + \lambda^2 a_\lambda \frac{\varepsilon}{2}}.$$

This yields

$$P_\omega(\mathcal{G}^\lambda) = \int_{\mathcal{G}^\lambda} dP_\omega(x) \geq \int e^{-\lambda^2 a_\lambda \langle \frac{1}{2} g, \pi \rangle + \lambda^2 a_\lambda U_Q(g,\omega) + \lambda^2 a_\lambda \frac{\varepsilon}{2}} d\tilde{P}_\omega(x)$$

$$\geq e^{-\lambda^2 a_\lambda \frac{1}{2} \mathcal{H}(\pi \| h\omega \otimes \omega) + \lambda^2 a_\lambda \varepsilon} \tilde{P}_\omega(\mathcal{G}^\lambda).$$

Applying the law of large numbers, we have that $\lim_{\lambda \to \infty} \tilde{P}_\omega(\mathcal{G}^\lambda) = 1$. This completes the proof of the Theorem.

Proof of Corollary 5.2.4

The proof of Corollary 5.2.4 follows from the definition of the Kullback action and Theorem 5.2.3 if we set $\omega = m \otimes \mathcal{Q}$ and $\lambda\omega \otimes \omega(x, y) = \|\lambda\omega \otimes \omega\|$, for all $(x, y) \in \mathcal{X} \times \mathcal{X}$.

Proof of Corollary 5.2.5

In this scenario, the result was obtained by Lemma 5.3.3, the law of empirical link measure is exponentially tight. Moreover, without loss of generality, we can assume that the set F in Corollary 5.2.5(ii) above is relatively compact. If the study chooses any $\varepsilon > 0$; then for each functional $\pi \in F$ one can find a weak neighborhood such that the estimate of Theorem 5.2.3(i) above holds. From all these neighborhood, the study select a finite cover of $\mathcal{G}$ and sums up over the value in Corollary 5.2.5(i) above to obtain

$$\limsup_{\lambda\to\infty} \frac{1}{\lambda} \log \mathbb{P}_\omega\Big\{X^\lambda \in \mathcal{G} \,\Big|\, L_2^\lambda \in F\Big\} \le -\inf_{\pi\in F} I_\omega(\pi) + \varepsilon, \quad \text{where } \omega = m \otimes \mathcal{Q}.$$

As ε was arbitrarily chosen and the lower bound in Theorem 5.2.1(ii) means the lower bound in Theorem 5.2.5 holds, we obtains the desired results which completes the proof.

5.6 Conclusion

The study provided a joint large deviation principle for the empirical power measure and the empirical connectivity measure of telecommunication networks in the $\tau-$ topology. Adopting the concept of the large deviations, we have proved Shannon-McMillian Breiman Theorem for the telecommunication network modelled as the sub-critical SINR network model. In addition, we have proved a local large deviation principle for the empirical connectivity measure given the empirical power measure and from this result; we have obtained the classical McMillian theorem and for a given PPP. Finally, we have obtained an asymptotic bound on the set of all possible sub-critical SINR network processes. Conclusively, we have presented large deviation principles for the sub-critical SINR networks. Note that, our results may form the bases for designing an anomaly inference algorithms for subcritical wireless telecommunication network models.

5.7 Summary

The chapter obtains large deviation asymptotic for sub-critical communication networks modelled as signal-interference-noise-ratio (SINR) random networks. To achieve this, we define the empirical power measure and the empirical connectivity measure, as well as prove joint large deviation principles (LDPs) for the two empirical measures on two different scales. Using the joint LDPs, we prove an Asymptotic equipartition property(AEP) for wireless telecommunication Networks modelled as the subcritical SINR random networks. Further, we prove a Local Large deviation principle (LLDP) for the sub-critical SINR random network. From the LLDPs, we prove the large deviation principle, and a classical McMillan Theorem for the stochastic SINR model processes. Note that, the LDPs for the empirical measures of this stochastic SINR random network model were derived on spaces of measures equipped with the $\tau-$ topology, and the LLDPs were deduced in the space of SINR model process without any topological limitations. We motivate the study by describing a possible anomaly detection test for SINR random networks.

Chapter 6

CONCLUSION AND RECOMMENDATION

This section presents the conclusion and recommendation of the study.

6.1 Conclusion

For a given Poisson Point Process, we have defined SINR and the SINR network as Telecommunication Networks. We defined the empirical marked measure and empirical paired measure for a class of Telecommunication networks. For a class of telecommunication networks, we have proved a joint large deviation principle for the empirical measures, with speed λ in the τ-topology. From these results, we proved the asymptotic equipartition property for the Telecommunication networks. Further, we obtained the local large deviation principle for the empirical measures and derived the classical MacMillian theorem as well as asymptotic bound for the set of all possible SINR Network processes. Finally, we have presented a large deviation principle for the SINR Network.

6.2 Recommendation

This study best describes the possible anomaly detection test for SINR random network. The result may form the basis for designing an anomaly inference algorithm for telecommunication models. We provided the proof of a Lossy asymptotic equipartition property, which may be regarded as the first step for the Telecommunication Network.

We recommended that future studies model the SINR network's noise component as random and incorporate fading effect in the interference. Also, future studies can be explored on the other forms of the path loss function and compare the result to the path loss function adopted for this study.

INDEX OF NOTATION

Table 6.1: Notations of the study symbol

Meaning	**Symbol**
Borel subset of $\mathbb{R}^d$	$\mathcal{B}$
Cardinality of the set A	$\|A\|$
Closure of the set F	$cl(F)$
Element of locally finite set	$\mathcal{S}(D)$
Empirical marked measure	L_1^λ
Empirical paired measure	L_2^λ
Entropy of x	$\mathcal{H}(x)$
Expectation of the function g with respect to the measure π	$\langle g\ ,\ \pi\rangle$
Indicator function of set Γ	$\mathbb{1}_\Gamma(x)$
Laplace transform random variable of $\mathcal{X}$	$\mathcal{L}_\mathcal{X}(s)$
Path loss function	$\ell(r)$
Positive real numbers	$\mathbb{R}^+$ or $\mathbb{R}_+$
Probability measure	$\mathbb{P}$
Real numbers	$\mathbb{R}$
Relative entropy of the probability measure π with respect to ω	$H(\pi\|\|\omega)$
Scale	λ or $\lambda^2 a_\lambda$
Set of counting measures on $\mathcal{X}$ equipped with discrete topology	$\mathcal{N}(\mathcal{X})$
Space of positive measures on $\mathcal{X}$ equipped with the weak topology	$\mathcal{M}_\lambda(\mathcal{X})$
Space of symmetric measure on $\mathcal{X}\times\mathcal{X}$ equipped with the weak topology	$\mathcal{M}_\lambda(\mathcal{X}\times\mathcal{X})$
Set of finite symmetric measure on $\mathcal{X}\times\mathcal{X}$ equipped with the weak topology	$\tilde{\mathcal{M}}_\lambda(\mathcal{X}\times\mathcal{X})$
Technical constant	τ_λ or γ_λ
Transitional probability	$q(\cdot, X_i)$

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