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Path integral in position-deformed Heisenberg algebra with maximal length uncertainty

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ABSTRACT

In this work, we study the path integral in a position-deformed Heisenberg algebra with quadratic deformation which implements both minimal momentum and maximal length uncertainties. We construct propagators of path integrals within this deformed algebra using the position space representation on the one hand and the Fourier transform and its inverse representations on the other. The result is remarkably similar to the one obtained by Pramanik (2022) from the Perivolaropoulos's deformed algebra (Perivolaropoulos, 2017). Then, the propagators and the corresponding actions of a free particle and a simple harmonic oscillator are discussed as examples. We also show that the actions which describe the classical trajectories of both systems are bounded by the ordinary ones of classical mechanics due to the existence of this maximal length. Consequently, particles of these systems travel faster from one point to another with low kinetic and mechanical energies.

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1. Introduction

It has long been argued that quantum gravity should lead to a minimal observable length. This minimal length should however be described quantum mechanically as a nonzero minimal uncertainty in position measurements. The past two decades has seen the development and exploration of the general framework for the implementation of the appearance of a nonzero minimal uncertainty [1–19]. It is known that, the existence of this minimal length uncertainty presents the issue of high energy requirements that are beyond the scope of any experimental feasibility. To circumvent this requirement, the first author has recently proposed a position-deformed Heisenberg algebra [20] in two dimensions (2D) that introduces a simultaneous existence of minimal and maximal length uncertainties. The emergence of this maximal length demonstrated quantum deformation effects in this space and predicted the detection of low-energy gravity particles [21,22]. These effects have been confirmed in [23] by the study of statistical properties of ideal gas in this deformed-Heisenberg algebra with maximal length uncertainty [21]. This maximal length induces logarithmic corrections to the thermodynamic quantities of this gas which are consequences of strong quantum deformation effects. Furthermore, the mathematical and statistical properties of Gazeau-Klauder coherent states for a free particle in a square well potential have also been investigated within this position-deformed Heisenberg algebra [24].

In the present work, we investigate the effects of this maximal length uncertainty on the trajectories of systems by studying the path integral in the position-deformed Heisenberg algebra [21]. To do so, we construct the position space representation describing this maximal length, as well as the corresponding Fourier transform and its inverse representations. We derive the propagators of path integrals and the classical action in these different representations. The results in the position representation are consistent with the recent one obtained by Pramanick in [25] from the Perivolaropoulos's position-deformed Heisenberg algebra [26]. Then, the Hamiltonian's principle of least action is used to generate the classical equations of motion. We compute the propagators and the actions of a free particle and a simple harmonic oscillator as applications. We show that these deformed actions which describe the classical trajectories of both systems are bounded by the standard ones of classical mechanics. This indicates that particles of these systems travel quickly from one point to another with low kinetic and mechanical energies. This result perfectly strengthens the claim that the recently proposed position-deformed algebra [21,22] induces strong deformation of the quantum levels allowing particles to jump from one state to another with low energies.

This paper is outlined as follows: in Section 2, we establish the Hilbert space representations of wave functions associated with this deformed algebra. In Section 3, we construct the path integrals in these wave function representations and deduce the corresponding quantum propagators and classical actions. As examples, we compute the propagators and the actions for some simple models such as the free particle and the harmonic oscillator. In the last section, we present our conclusion.

2. Position deformed Heisenberg algebra with maximal length

Let $\mathcal{H} = \mathcal{L}^2(\mathbb{R})$ be the Hilbert space of square integrable functions. The Hermitian operators $\hat{x}$ and $\hat{p}$ that act on this space satisfy the condition

$$[\hat{x}, \hat{p}] = i\hbar\mathbb{I}. \tag{1}$$

The corresponding Heisenberg uncertainty principle is given by

$$\Delta x \Delta p \geq \frac{\hbar}{2}. \tag{2}$$

Let $\{|x\rangle\} \in \mathcal{H}$ be the complete position basis vectors. The action operators in Eq. (1) on this basis vector reads as follows

$$\hat{x}|x\rangle = x|x\rangle \quad \text{and} \quad \hat{p}|x\rangle = -i\hbar\frac{d}{dx}|x\rangle, \quad x \in \mathbb{R}. \tag{3}$$

The completeness and orthogonality relations are given by [27]

$$\langle x'|x\rangle = \delta(x-x') \quad \text{and} \quad \int_{-\infty}^{+\infty} dx|x\rangle\langle x| = \mathbb{I}. \tag{4}$$

Another useful choice of basis vectors is the momentum vector $\{|p\rangle\} \in \mathcal{H}$ defined by taking Fourier transforms

$$|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} dx e^{\frac{i}{\hbar}px}|x\rangle \quad \text{with} \quad p \in \mathbb{R} \tag{5}$$

and its inverse is defined as follows

$$|x\rangle = \int_{-\infty}^{+\infty} \frac{dp}{\sqrt{2\pi\hbar}} e^{-\frac{i}{\hbar}px}|p\rangle. \tag{6}$$

The inner product and completeness relations are given by [27]

$$\langle p'|p\rangle = \delta(p-p') \quad \text{and} \quad \int_{-\infty}^{+\infty} dp|p\rangle\langle p| = \mathbb{I}. \tag{7}$$

The action of the operators in (1) on the vector $|p\rangle$ is given by

$$\hat{p}|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} dx \left(-i\hbar \frac{d}{dx} e^{\frac{i}{\hbar}px}\right)|x\rangle = p|p\rangle, \tag{8}$$

$$\hat{x}|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} dx i\hbar \frac{d}{dp}\left(e^{\frac{i}{\hbar}px}\right)|x\rangle = i\hbar\frac{d}{dp}|p\rangle. \tag{9}$$

We introduce new operators $\hat{X}$ and $\hat{P}$ acting on $\mathcal{H}$. They are defined by

$$\hat{X} = \hat{x}, \quad \hat{P} = (\mathbb{I} - \tau\hat{x} + \tau^2\hat{x}^2)\hat{p}. \tag{10}$$

They satisfy the following relation [25]

$$[\hat{X}, \hat{P}] = i\hbar(\mathbb{I} - \tau\hat{X} + \tau^2\hat{X}^2), \tag{11}$$

where $\tau \in (0,1)$ is the generalized uncertainty principle parameter related to quantum deformation effects in this space [4,20–22]. Obviously by taking $\tau \to 0$, we recover the algebra (1). This algebra (11) is consistent with the one proposed by Perivolaropoulos [26].

The action of the operators (10) on the following unit basis vectors $\{|x\rangle\}, \{|p\rangle\}$ reads as follows

$$\hat{X}|x\rangle = x|x\rangle \quad \text{and} \quad \hat{P}|x\rangle = -i\hbar(1-\tau x+\tau^2x^2)\partial_x|x\rangle, \quad x \in \mathbb{R}. \tag{12}$$

$$\hat{X}|p\rangle = i\hbar\partial_p|p\rangle \quad \text{and} \quad \hat{P}|p\rangle = (1 - i\tau\hbar\partial_p - \tau^2\hbar^2\partial_p^2)p|p\rangle, \quad p \in \mathbb{R}. \tag{13}$$

Let us consider an arbitrary vector $|\phi\rangle \in \mathcal{H}$, the projection of this vector on the unit vectors $\{|x\rangle\}$ and $\{|p\rangle\}$ generates the functions $\phi(x) = \langle x|\phi\rangle$ and $\phi(p) = \langle p|\phi\rangle$. As a result, we can write the above equations as follows

$$\hat{X}\phi(x) = x\phi(x) \quad \text{and} \quad \hat{P}\phi(x) = -i\hbar(1-\tau x+\tau^2x^2)\partial_x\phi(x), \tag{14}$$

$$\hat{X}\phi(p) = i\hbar\partial_p\phi(p) \quad \text{and} \quad \hat{P}\phi(p) = (1 - i\tau\hbar\partial_p - \tau^2\hbar^2\partial_p^2)p\phi(p). \tag{15}$$

An interesting feature can be observed from the commutation relation (11) through the following uncertainty relation:

$$\Delta X\Delta P \geq \frac{\hbar}{2}\left(1 - \tau\langle\hat{X}\rangle + \tau^2\langle\hat{X}^2\rangle\right), \tag{16}$$

where $\langle\hat{X}\rangle$ and $\langle\hat{X}^2\rangle$ are the expectation values of the operators $\hat{X}$ and $\hat{X}^2$ respectively for any space representations. Using the relation $\langle\hat{X}^2\rangle = (\Delta X)^2 + \langle\hat{X}\rangle^2$, Eq. (16) can be rewritten as a second order equation for ΔX

$$\Delta X^2 - \frac{2}{\hbar\tau^2}\Delta P\Delta X + \langle\hat{X}\rangle^2 - \frac{1}{\tau}\langle\hat{X}\rangle + \frac{1}{\tau^2} \leq 0. \tag{17}$$

By setting Eq. (17) into

$$\Delta X^2 - \frac{2}{\hbar\tau^2}\Delta P\Delta X + \langle\hat{X}\rangle^2 - \frac{1}{\tau}\langle\hat{X}\rangle + \frac{1}{\tau^2} = 0, \tag{18}$$

the solutions ΔX are given by

$$\begin{aligned}\Delta X &= \frac{\Delta P}{\hbar\tau^2} \pm \sqrt{\left(\frac{\Delta P}{\hbar\tau^2}\right)^2 + \frac{1}{\tau}\langle\hat{X}\rangle - \langle\hat{X}\rangle^2 - \frac{1}{\tau^2}} \\ &= \frac{\Delta P}{\hbar\tau^2} \pm \sqrt{\left(\frac{\Delta P}{\hbar\tau^2}\right)^2 - \left[\left(\langle\hat{X}\rangle - \frac{1}{2\tau}\right)^2 + \frac{3}{4\tau^2}\right]}.\end{aligned} \tag{19}$$

The reality of solutions (19) gives the following minimum value for ΔP

$$\Delta P_{min} = \hbar\tau^2\sqrt{\left(\langle\hat{X}\rangle - \frac{1}{2\tau}\right)^2 + \frac{3}{4\tau^2}}. \tag{20}$$

In order to determine the absolute minimum measurable momentum of this deformed algebra, we take only the physical states into account which satisfy the condition $\langle\hat{X}\rangle = \frac{1}{2\tau}$. Then, the above Eq. (19) and (20) are reduced into the absolute minimum momentum ΔP_{min}^{abs} and maximal length ΔX_{max}^{abs} respectevely

$$\Delta P_{min}^{abs} = \frac{\hbar\tau\sqrt{3}}{2} \quad \text{and} \quad \Delta X_{max}^{abs} = \frac{\sqrt{3}}{2\tau} = l_{max}. \tag{21}$$

These provides the precise scale for the maximum length and minimum momentum which are significantly different from the physical condition imposed in [20–24]

It is well known in [4] that, the existence of minimal uncertainty raises the question of the loss of representation i.e., the space is inevitably bounded by minimal quantity beyond which any further localization of particles is not possible. In the present situation, the minimal momentum ΔP_{min}^{abs} leads to a loss of $\phi(p)$-representation and a maximal $\phi(x)$-representation. Thus, the corresponding representation of operators are given by

$$\hat{X}\phi(x) = x\phi(x) \quad \text{and} \quad \hat{P}\phi(x) = -i\hbar D_x\phi(x), \tag{22}$$

where $D_x = (1 - \tau x + \tau^2x^2)\partial_x$ is a deformed derivative. Using Eq. (22), one can recover the algebra (11). As one can see from the representation of operators in Eq. (10) or in Eq. (22), the position operator $\hat{X}$ is Hermitian while the momentum operator $\hat{P}$ is not

$$\hat{X}^\dagger = \hat{X} \quad \text{and} \quad \hat{P}^\dagger = \hat{P} + i\hbar\tau(\mathbb{I} - 2\tau\hat{X}) \implies \hat{P}^\dagger \neq \hat{P}. \tag{23}$$

In order to guarantee the Hermicity of this operator, we arbitrarily restrict the study from the infinite dimensional Hilbert space $\mathcal{H}$ into its bounded dense domain $\mathcal{D}_\tau = \mathcal{L}^2(-l_{max}, +l_{max})$ in such a away that, for $\tau \to 0$, one recovers the entire space $\mathcal{H}$. As will be shown in the forthcoming development, the restriction of the domain guarantees the physical meaning of the eigenstates of the momentum operator (see more detail information in the Appendix). Furthermore, this restriction perfectly fits with the work of Nozari and Etemadi did in momentum space [12]. In addition to this condition, we propose the following deformed completeness relation to get the Hermicity of the operator $\hat{P}$

$$\int_{-l_{max}}^{+l_{max}} \frac{dx}{1 - \tau x + \tau^2x^2}|x\rangle\langle x| = \mathbb{I}. \tag{24}$$

Consequently, the scalar product between two states $|\Psi\rangle, |\Phi\rangle \in \mathcal{D}_\tau$ and the orthogonality of eigenstates become [12]

$$\langle\Psi|\Phi\rangle = \int_{-l_{max}}^{+l_{max}} \frac{dx}{1 - \tau x + \tau^2x^2}\Psi^*(x)\Phi(x), \tag{25}$$

$$\langle x|x'\rangle = (1-\tau x+\tau^2x^2)\delta(x-x'). \tag{26}$$

Now let us consider the operator $\hat{P}$ in its closed interval

$$\mathcal{D}_\tau(\hat{P}) = \{\varphi, -i\hbar D_x\varphi \in \mathcal{L}^2(-l_{max}, +l_{max}), \quad \varphi(-l_{max}) = 0 = \varphi(+l_{max})\}, \tag{27}$$

and its adjoint domain defined by

$$\mathcal{D}_\tau(\hat{P}^\dagger) = \{\xi, -i\hbar D_x\xi \in \mathcal{L}^2(-l_{max}, +l_{max})\}. \tag{28}$$

Thus, we may write $\mathcal{D}_\tau(\hat{P}) \subset \mathcal{D}_\tau(\hat{P}^\dagger)$, which means that the domain of $\hat{P}$ is a proper subset of the domain of its adjoint $\hat{P}^\dagger$. To show the Hermicity of the operator $\hat{P}$, we consider a functional $F(\phi, \psi)$ defined by

$$F(\varphi, \xi) := \langle \xi|\hat{P}\varphi\rangle - \langle \hat{P}^\dagger\xi|\varphi\rangle. \tag{29}$$

Using the relation (24) and by a straightforward computation of this functional, we have

$$\begin{aligned} F(\varphi,\xi) &= \int_{-l_{max}}^{+l_{max}} \frac{dx}{1-\tau x+\tau^2x^2}\left[\xi^*(x)\left(-i\hbar D_x\varphi(x)\right) - \left(-i\hbar D_x\xi(x)\right)^*\varphi(x)\right] \\ &= -i\hbar\int_{-l_{max}}^{+l_{max}} d\left(\xi^*(x)\varphi(x)\right) = -i\hbar\left[\xi^*(x)\varphi(x)\right]_{-l_{max}}^{+l_{max}}. \end{aligned} \tag{30}$$

Since $\varphi(\pm l_{max}) = 0$, and $\xi(x)$ can reach any arbitrary value at the boundaries. This lead to the vanishing of $F(\phi, \psi)$ i.e., $F(\phi, \psi) = 0$. Consequently, the operator $\hat{P}$ is symmetric in $\mathcal{D}(\hat{P})$ such that

$$\langle \xi|\hat{P}\varphi\rangle = \langle \hat{P}^\dagger\xi|\varphi\rangle \implies \hat{P} = \hat{P}^\dagger. \tag{31}$$

Despite the fact that the momentum is Hermitian, it is not always a self-adjoint operator because its domain includes the domain of $\hat{P}^\dagger$. It could have none or have an infinite number of self-adjoint extensions. Note that, as a rule in quantum mechanics, the operators that act on square integrable functions are essentially self-adjoint. However, there are exceptions to this rule. This is because the basic quantization requirement that operators whose expectation values are real do not strictly require these operators be self-adjoint. Indeed, the Hermicity result (31) is a sufficient condition to ensure that all expectation values of the momentum operator are real.

To construct a Hilbert space representation that describes the maximal length and the minimal momentum uncertainties, one has to solve the eigenvalue problem

$$-i\hbar D_x\phi_\rho(x) = \rho\phi_\rho(x) \quad \text{with} \quad \rho \in \mathbb{R}. \tag{32}$$

The solution of this equation is given by

$$\phi_\rho(x) = A\exp\left(i\frac{2\rho}{\tau\hbar\sqrt{3}}\left[\arctan\left(\frac{2\tau x-1}{\sqrt{3}}\right) + \frac{\pi}{6}\right]\right), \tag{33}$$

where A is an abritrary constant. Then by normalization, $\langle\phi_\rho|\phi_\rho\rangle = 1$, we have

$$A = \left(\int_{-l_{max}}^{l_{max}} \frac{dx}{1-\tau x+\tau^2x^2}\right)^{-\frac{1}{2}} = \sqrt{\frac{\tau\sqrt{3}}{2\arctan(6)}} \tag{34}$$

Substituting Eq. (34) into Eq. (33) gives

$$\phi_\rho(x) = \sqrt{\frac{\tau\sqrt{3}}{2\arctan(6)}}\exp\left(i\frac{2\rho}{\tau\hbar\sqrt{3}}\left[\arctan\left(\frac{2\tau x-1}{\sqrt{3}}\right) + \frac{\pi}{6}\right]\right). \tag{35}$$

This wave function describes simultaneously the maximal length and the minimal momentum uncertainties. As one can see, the eigenvectors $|\phi_\rho\rangle$ are physical states. This is because the expectation values of position energy operator $\hat{X}^n$ $(n \in \mathbb{N})$ is finite:

$$\langle\phi_\rho|\hat{X}^n|\phi_\rho\rangle = \frac{\tau\sqrt{3}}{2\arctan(6)}\int_{-l_{max}}^{+l_{max}} \frac{x^n dx}{1-\tau x+\tau^2x^2} < \infty. \tag{36}$$

In comparison with Kempf et al. formalism [4], the expectation value of the operator $\hat{X}^2$ in this framework does not diverge as the one obtained in the momentum space [28]. According to this formalism, any state that has a well-defined minimal uncertainty measurement which is inside of a forbidden gap cannot have a finite energy, so cannot be accepted as physical states. Conversely, here the energy operator $\hat{X}^2$ is well defined therefore the states $|\phi_\rho\rangle$ are physically relevant.

As a consequence of this fact, we can define a new identity operator from this position wave function (35) which will play the role of the completeness relation of the momentum eigenstates in the derivation of the path-integral. It reads as follows

$$\int_{-\infty}^{+\infty} \frac{\arctan(6)}{\pi\hbar\tau\sqrt{3}} d\rho|\rho\rangle\langle\rho| = \mathbb{I}. \tag{37}$$

To prove this, we refer to the work of Bernardo and Esguerra [29,30] and computing the orthogonality of the states $\langle x|x'\rangle$, we have

$$\begin{aligned}\langle x|x'\rangle &= \langle x| \int_{-\infty}^{+\infty} \frac{\arctan(6)}{\pi\hbar\tau\sqrt{3}} d\rho|\rho\rangle\langle\rho|x'\rangle \\ &= \int_{-\infty}^{+\infty} \frac{\arctan(6)}{\pi\hbar\tau\sqrt{3}} d\rho\langle x|\rho\rangle\langle\rho|x'\rangle = \int_{-\infty}^{+\infty} \frac{\arctan(6)}{\pi\hbar\tau\sqrt{3}} d\rho\phi_\rho(x)\phi_\rho^*(x'). \end{aligned}\tag{38}$$

By using Eq. (35), we recover Eq. (26) as follows

$$\begin{aligned}\langle x|x'\rangle &= \frac{1}{2\pi\hbar} \int_{-\infty}^{+\infty} d\rho \exp\left(i\frac{2\rho}{\tau\hbar\sqrt{3}}\left[\arctan\left(\frac{2\tau x-1}{\sqrt{3}}\right) - \arctan\left(\frac{2\tau x'-1}{\sqrt{3}}\right)\right]\right) \\ &= \frac{\tau\sqrt{3}}{2}\delta\left(\arctan\left(\frac{2\tau x-1}{\sqrt{3}}\right) - \arctan\left(\frac{2\tau x'-1}{\sqrt{3}}\right)\right) \\ &= (1-\tau x+\tau^2x^2)\delta(x-x'). \end{aligned}\tag{39}$$

This confirm the claim that Eq. (37) is a correct expression for the identity.

Since the states $\phi_\rho(x)$ are physically meaningful and are well localized, one can obtain the quasi-momentum representation by projecting an arbitrary state $|\psi\rangle \in \mathcal{H}$ onto these localized states $|\phi_\rho\rangle$ and one can obtain the quasi-momentum representation, that is

$$\psi(\rho) = \langle\phi_\rho|\psi\rangle = \sqrt{\frac{\tau\sqrt{3}}{2\arctan(6)}} \int_{-l_{max}}^{+l_{max}} \frac{dx\psi(x)}{1-\tau x+\tau^2x^2} e^{-i\frac{2\rho}{\tau\hbar\sqrt{3}}\left[\arctan\left(\frac{2\tau x-1}{\sqrt{3}}\right)+\frac{\pi}{6}\right]}. \tag{40}$$

This mapping defined the generalized Fourier transform of the representation in Eq. (35). Its inverse representation is given by

$$\psi(x) = \frac{\sqrt{\arctan(6)}}{\pi\hbar\sqrt{2\tau\sqrt{3}}} \int_{-\infty}^{+\infty} d\rho\psi(\rho)e^{i\frac{2\rho}{\tau\hbar\sqrt{3}}\left[\arctan\left(\frac{2\tau x-1}{\sqrt{3}}\right)+\frac{\pi}{6}\right]}. \tag{41}$$

Similar to the action of $\hat{X}$ on $\psi(p)$ in Eq. (15), here the action of the operator $\hat{X}$ on the quasi-momentum wavefunction (40) reads as follows

$$\begin{aligned}\hat{X}\psi(\rho) = i\hbar\frac{d}{d\rho}\psi(\rho) &= i\hbar\sqrt{\frac{\tau\sqrt{3}}{2\arctan(6)}} \int_{-l_{max}}^{+l_{max}} \frac{dx\psi(x)}{1-\tau x+\tau^2x^2} \\ &\quad\times\frac{d}{d\rho}e^{-i\frac{2\rho}{\tau\hbar\sqrt{3}}\left[\arctan\left(\frac{2\tau x-1}{\sqrt{3}}\right)+\frac{\pi}{6}\right]} \end{aligned}\tag{42}$$

$$\frac{d}{d\rho}\psi(\rho) = -i\frac{2}{\tau\hbar\sqrt{3}}\left[\arctan\left(\frac{2\tau x-1}{\sqrt{3}}\right)+\frac{\pi}{6}\right]\psi(\rho). \tag{43}$$

This equation is equivalent to

$$i\frac{\tau\hbar\sqrt{3}}{2}\frac{d}{d\rho} = \left[\arctan\left(\frac{2\tau x-1}{\sqrt{3}}\right)+\frac{\pi}{6}\right] = \left[\arctan\left(\frac{2\tau x-1}{\sqrt{3}}\right)+\arctan\left(\frac{1}{\sqrt{3}}\right)\right]. \tag{44}$$

From the following relation [31]

$$\arctan\alpha + \arctan\beta = \arctan\left(\frac{\alpha+\beta}{1-\alpha\beta}\right), \quad \text{with} \quad \alpha\beta < 1, \tag{45}$$

we deduce that

$$\tan\left[\arctan\left(\frac{2\tau x - 1}{\sqrt{3}}\right) + \arctan\left(\frac{1}{\sqrt{3}}\right)\right] = \frac{\tau x\sqrt{3}}{2-\tau x}. \tag{46}$$

In Eq. (44), we can see that the position operator is represented as

$$\hat{X} = \frac{2}{\tau}\frac{\tan\left(i\frac{\tau\hbar\sqrt{3}}{2}\frac{d}{d\rho}\right)}{\sqrt{3}+\tan\left(i\frac{\tau\hbar\sqrt{3}}{2}\frac{d}{d\rho}\right)}\mathbb{I}, \tag{47}$$

$$\hat{X}\psi(\rho) = \frac{2}{\tau}\frac{\tan\left(i\frac{\tau\hbar\sqrt{3}}{2}\frac{d}{d\rho}\right)}{\sqrt{3}+\tan\left(i\frac{\tau\hbar\sqrt{3}}{2}\frac{d}{d\rho}\right)}\psi(\rho). \tag{48}$$

From the action of $\hat{P}$ on the quasi-representation (41) and using Eq. (14), we have

$$\hat{P}\psi(\rho) = \rho\psi(\rho). \tag{49}$$

Note that in the limit $\tau \to 0$, we recover the corresponding ordinary quantum mechanics results in momentum space (8)

$$\lim_{\tau\to 0}\hat{X}\psi(\rho) = i\hbar\frac{d}{d\rho}\psi(\rho) \quad \text{and} \quad \lim_{\tau\to 0}\hat{P}\psi(\rho) = \rho\psi(\rho). \tag{50}$$

3. Path integral and propagator in position-deformed algebra

From the path integrals within this position-deformed Heisenberg algebra, we construct the propagator depending on the position-representation and on the Fourier transform and its inverse representations. We compute propagators and deduce the actions of a free particle and a harmonic oscillator as applications.

3.1. Path integral and propagator in position-space representation

The Hamiltonian operator for a particle with mass m living in one spatial dimension is given by

$$\hat{H} = \frac{\hat{P}^2}{2m} + V(\hat{X}), \tag{51}$$

where V is the potential energy of the system. The time-dependent deformed Schrödinger equation in the position representation is given by

$$\hat{H}|\phi_\rho(t)\rangle = -\frac{\hbar^2}{2m}D_x^2|\phi_\rho(t)\rangle + V(x)|\phi_\rho(t)\rangle = i\hbar\partial_t|\phi_\rho(t)\rangle. \tag{52}$$

The time-evolution process is described by

$$|\phi_\rho(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}(t-t')}|\phi_\rho(t')\rangle, \tag{53}$$

Multiplication of $\langle x|$ from the left of Eq. (53) gives

$$\phi_\rho(x,t) = \int_{-l_{max}}^{+l_{max}}\frac{dx'}{1-\tau x'+\tau^2 x'^2}K(x,t,x',t')\phi_\rho(x',t') \tag{54}$$

where K is the kernel in the Hamiltonian or the amplitude for a particle to propagate from the state with position x' to the state with position $x(x > x')$ in a time interval $\Delta t = t - t'(t > t')$ [32,33] and it is defined as

$$K(x, t, x', t') = \langle x|e^{-\frac{i}{\hbar}\hat{H}(t-t')}|x'\rangle. \tag{55}$$

Splitting the interval $t-t'$ into N intervals of length $\epsilon = (t_k-t_{k-1})/N$ and inserting the completeness relations in (25) and (37), the propagator (90) becomes

$$K(x, t, x', t') = \int_{-l_{max}}^{+l_{max}} \left(\prod_{k=1}^{N-1} \frac{dx_k}{1-\tau x_k+\tau^2 x_k^2}\right) \int_{-\infty}^{+\infty} \left(\prod_{k=1}^{N} \frac{\arctan(6)}{\pi\hbar\tau\sqrt{3}} d\rho_k\right) \times \langle x_k|\rho_k\rangle\langle\rho_k|e^{-\frac{i}{\hbar}\epsilon\hat{H}}|x_{k-1}\rangle. \tag{56}$$

Recall that

$$\langle x_k|\rho_k\rangle = \phi_{\rho_k}(x_k) = \sqrt{\frac{\tau\sqrt{3}}{2\arctan(6)}} e^{\left(i\frac{2\rho_k}{\tau\hbar\sqrt{3}}\left[\arctan\left(\frac{2\tau x_k-1}{\sqrt{3}}\right)+\frac{\pi}{6}\right]\right)}, \tag{57}$$

$$\langle\rho_k|e^{-\frac{i}{\hbar}\epsilon\hat{H}}|x_{k-1}\rangle \simeq e^{-\frac{i}{\hbar}\epsilon H(\rho_k, x_{k-1})}\langle\rho_k|x_{k-1}\rangle + \mathcal{O}(\epsilon^2) \simeq e^{-\frac{i}{\hbar}\epsilon H(\rho_k, x_{k-1})}\phi^*_{\rho_k}(x_{k-1}) + \mathcal{O}(\epsilon^2). \tag{58}$$

Substituting these expressions into Eq. (56) gives

$$K_{disc}(x, t, x', t') = \left[\int_{-l_{max}}^{+l_{max}} \left(\prod_{k=1}^{N-1} \frac{dx_k}{1-\tau x_k+\tau^2 x_k^2}\right)\right]\left[\int_{-\infty}^{+\infty} \left(\prod_{k=1}^{N} \frac{d\rho_k}{2\pi\hbar}\right)\right] e^{\frac{i}{\hbar}\epsilon S_{disc}}, \tag{59}$$

where the discrete action S_{disc} is given by

$$S_{disc} = \sum_{k=1}^{N-1} \frac{2\rho_k}{\tau\sqrt{3}} \left[\frac{\arctan\left(\frac{2\tau x_k-1}{\sqrt{3}}\right) - \arctan\left(\frac{2\tau x_{k-1}-1}{\sqrt{3}}\right)}{\epsilon}\right] - \sum_{k=1}^{N-1} H(\rho_k, x_{k-1}) \tag{60}$$

Finally, we take the limit $N \to \infty$, so that $\epsilon \to 0$. We obtain the final expression for the propagator as follows

$$K(x, t, x', t') = \int \mathcal{D}x\mathcal{D}\rho e^{\frac{i}{\hbar}S}, \tag{61}$$

where the integration measures $\mathcal{D}x$ and $\mathcal{D}\rho$ are defined as

$$\mathcal{D}x = \lim_{N\to\infty} \prod_{k=1}^{N-1} \frac{dx_k}{1-\tau x_k+\tau^2 x_k^2} \quad \text{and} \quad \mathcal{D}\rho = \lim_{N\to\infty} \prod_{k=1}^{N} \left(\frac{d\rho_k}{2\pi\hbar}\right). \tag{62}$$

and the continuous action S is given by

$$S\left[x(t), x(t')\right] = \int_{t'}^{t} d\nu \left[\frac{\dot{x}(\nu)}{1-\tau x(\nu)+\tau^2 x^2(\nu)}\rho(\nu) - H(\rho(\nu), x(\nu))\right], \tag{63}$$

where $\dot{x}(\nu) = dx/d\nu$. These results (61), (62) and (63) are remarkably similar to the one obtained by Pramanick [25] from the Perivolaropoulos space [26].

Now, taking the limit $\tau \to 0$, the deformed propagator (61) is reduced to

$$K^0(x, t, x', t') = \int \lim_{N\to\infty} \prod_{k=1}^{N-1} dx_k \prod_{k=1}^{N} \left(\frac{d\rho_k}{2\pi\hbar}\right) e^{\frac{i}{\hbar}S^0}, \tag{64}$$

where the undeformed action S^0 is given by

$$S^0\left[x(t), x(t')\right] = \int_{t'}^{t} d\nu \left[\dot{x}(\nu)\rho(\nu) - H(\rho(\nu), x(\nu))\right]. \tag{65}$$

It is straightforward to show that the following relations

$$K(x,x',t,t') \le K^0(x,x',t,t') \implies S \le S^0. \tag{66}$$

The stationary path (63) is obtained by using the variational principle

$$\delta S = \delta \int_{t'}^{t} d\nu L\left[\dot{x}(\nu), x(\nu)\right] = \int_{t'}^{t} d\nu \left(\frac{\partial L}{\partial x(\nu)}\delta x(\nu) + \frac{\partial L}{\partial \dot{x}(\nu)}\delta \dot{x}(\nu) \right) = 0, \tag{67}$$

where the Lagrangian L of the system is given by

$$L\left[\dot{x}(\nu), x(\nu)\right] = \frac{\dot{x}(\nu)}{1-\tau x(\nu)+\tau^2 x^2(\nu)}\rho(\nu) - H(\rho(\nu), x(\nu)). \tag{68}$$

The solutions of Eq. (67) generates the following differential equations

$$\dot{x} = (1-\tau x+\tau^2x^2)\frac{\partial H}{\partial \rho} = \{x,\rho\}_\tau \frac{\partial H}{\partial \rho}, \tag{69}$$

$$\dot{\rho} = -(1-\tau x+\tau^2x^2)\frac{\partial H}{\partial x} = -\{x,\rho\}_\tau \frac{\partial H}{\partial x}, \tag{70}$$

where $\{x,\rho\}_\tau = (1-\tau x+\tau^2x^2)$ is the position-deformed Poisson bracket. By taking the limit $\tau \to 0$, we recover the ordinary Hamilton's equations of motion.

3.2. Path integral and propagator in Fourier transform and its inverse representions

Using the generalized Fourier transform and its inverse representations (40), (41) and taking into account Eq. (54), we have

$$\begin{aligned}\psi(\rho,t) = &\sqrt{\frac{\tau\sqrt{3}}{2\arctan(6)}}\int_{-l_{max}}^{+l_{max}} \frac{dx}{1-\tau x+\tau^2x^2} e^{-i\frac{2\rho}{\tau\hbar\sqrt{3}}\left[\arctan\left(\frac{2\tau x-1}{\sqrt{3}}\right)+\frac{\pi}{6}\right]} \\ &\times \int_{-l_{max}}^{+l_{max}} \frac{K(x,t,x',t')}{1-\tau x'+\tau^2x'^2}dx' \frac{\sqrt{\arctan(6)}}{\pi\hbar\sqrt{2\tau\sqrt{3}}} \\ &\times \int_{-\infty}^{+\infty} d\rho' e^{i\frac{2\rho'}{\tau\hbar\sqrt{3}}\left[\arctan\left(\frac{2\tau x'-1}{\sqrt{3}}\right)+\frac{\pi}{6}\right]}\psi(\rho',t'). \end{aligned} \tag{71}$$

This path integral can be rewritten as follows

$$\psi(\rho,t) = \int_{-\infty}^{+\infty} d\rho' \mathcal{K}(\rho,t,\rho',t')\psi(\rho',t'), \tag{72}$$

where $\mathcal{K}$ is the propagator in Fourier transform and its inverse representions for a particle to go from a state $\psi(\rho')$ to a state $\psi(\rho)$ in a time interval $t-t'$ is

$$\begin{aligned}\mathcal{K}(\rho,t,\rho',t') &= \frac{1}{2\pi\hbar}\int_{-l_{max}}^{+l_{max}} \frac{dx}{1-\tau x+\tau^2x^2}\frac{dx'}{1-\tau x'+\tau^2x'^2} \\ &\quad\times e^{-i\frac{2}{\tau\hbar\sqrt{3}}\left[\rho\arctan\left(\frac{2\tau x-1}{\sqrt{3}}\right)-\rho'\arctan\left(\frac{2\tau x'-1}{\sqrt{3}}\right)\right]}K(x,t,x',t'), \\ &= \frac{1}{2\pi\hbar}\int \mathcal{D}x\mathcal{D}\rho \frac{dx}{1-\tau x+\tau^2x^2}\frac{dx'}{1-\tau x'+\tau^2x'^2}e^{\frac{i}{\hbar}\mathcal{S}}, \end{aligned} \tag{73}$$

with $\mathcal{S}$ the action given by

$$\mathcal{S}(\rho,t,\rho',t') = S - \frac{2}{\tau\sqrt{3}}\left[\rho\arctan\left(\frac{2\tau x-1}{\sqrt{3}}\right) - \rho'\arctan\left(\frac{2\tau x'-1}{\sqrt{3}}\right)\right]. \tag{74}$$

3.3. Propagators for a free particle and for a harmonic oscillator

In this section, we compute the propagator in position-space (55) and the one in Fourier transform and its inverse representations (73) for the Hamiltonians of a free particle and a simple harmonic oscillator. From these propagators, we deduce the actions of both systems.

The generalized form of the Hamiltonian we chose is

$$\hat{H} = \frac{\hat{P}^2}{2m} + V(\hat{X}). \tag{75}$$

The action of this Hamiltonian of the functions $\phi(x)$ and $\psi(\rho)$ reads as follows

$$\hat{H}\phi(x) = \left(-\frac{\hbar^2}{2m}D_x^2 + V(x)\right)\phi(x), \tag{76}$$

$$\hat{H}\psi(\rho) = \left(\frac{\rho^2}{2m} + V\left(\frac{2}{\tau}\frac{\tan\left(i\frac{\tau\hbar\sqrt{3}}{2}\frac{d}{d\rho}\right)}{\sqrt{3}+\tan\left(i\frac{\tau\hbar\sqrt{3}}{2}\frac{d}{d\rho}\right)}\right)\right)\psi(\rho). \tag{77}$$

3.3.1. Propagator of a free particle

The free particle problem defined by the Hamiltonian is given by

$$\hat{H}_{fp} = \frac{\hat{P}^2}{2m}. \tag{78}$$

The propagator in position-representation in the time interval $\Delta t = t - t'$ is given by

$$\begin{aligned} K_{fp}(x, x', \Delta t) &= \langle x|e^{-\frac{i}{\hbar}\frac{\hat{P}^2}{2m}\Delta t}|x'\rangle \\ &= \frac{\arctan(6)}{\pi\hbar\tau\sqrt{3}}\langle x|\int_{-\infty}^{+\infty} d\rho e^{-\frac{i}{\hbar}\frac{\hat{P}^2}{2m}\Delta t}|\rho\rangle\langle\rho|x'\rangle \\ &= \frac{\arctan(6)}{\pi\hbar\tau\sqrt{3}}\int_{-\infty}^{+\infty} d\rho e^{-\frac{i}{\hbar}\frac{\rho^2}{2m}\Delta t}\langle x|\rho\rangle\langle\rho|x'\rangle \\ &= \int_{-\infty}^{+\infty}\frac{d\rho}{2\pi\hbar}e^{\left(i\frac{2\rho}{\tau\hbar\sqrt{3}}\left[\arctan\left(\frac{2\tau x-1}{\sqrt{3}}\right)-\arctan\left(\frac{2\tau x'-1}{\sqrt{3}}\right)\right]-\frac{i}{\hbar}\frac{\rho^2}{2m}\Delta t\right)}. \end{aligned} \tag{79}$$

Completing this Gaussian integral (79), we have

$$K_{fp}(x, x', \Delta t) = \sqrt{\frac{m}{2\pi\hbar i\Delta t}}e^{i\frac{2m}{\hbar 3\tau^2\Delta t}\left[\arctan\left(\frac{2\tau x-1}{\sqrt{3}}\right)-\arctan\left(\frac{2\tau x'-1}{\sqrt{3}}\right)\right]^2}. \tag{80}$$

Thus the deformed-classical action is given by

$$S_{fp} = \frac{2m}{3\tau^2\Delta t}\left[\arctan\left(\frac{2\tau x-1}{\sqrt{3}}\right) - \arctan\left(\frac{2\tau x'-1}{\sqrt{3}}\right)\right]^2. \tag{81}$$

The limit $\tau \to 0$, the latter propagator properly reduces to the well-known result in ordinary quantum mechanics for a free particle [32,33] that is

$$\lim_{\tau\to 0} K_{fp}(x, x', \Delta t) = K_{fp}^0(x, x', \Delta t) = \sqrt{\frac{m}{2\pi\hbar i\Delta t}}e^{\frac{i}{\hbar}\frac{m(x-x')^2}{2\Delta t}}, \tag{82}$$

and the corresponding classical action is given by

$$\lim_{\tau\to 0} S_{fp} = S_{fp}^0 = \frac{m}{2}\frac{(x-x')^2}{\Delta t}, \tag{83}$$

$$\frac{S_{fp}^0}{\Delta t} = \frac{m}{2}\frac{(x-x')^2}{(\Delta t)^2} = T^0, \tag{84}$$

where T^0 is the standard kinetic energy of the particle. Also, it is straightforward to show the following relations

$$K_{fp}(x, x', \Delta t) \leq K^0_{fp}(x, x', \Delta t) \implies S_{fp} \leq S^0_{fp} \implies T \leq T^0, \tag{85}$$

where T is the deformed kinetic energy of the particle

$$T = \frac{2m}{3\tau^2(\Delta t)^2}\left[\arctan\left(\frac{2\tau x - 1}{\sqrt{3}}\right) - \arctan\left(\frac{2\tau x' - 1}{\sqrt{3}}\right)\right]^2. \tag{86}$$

This indicates that the deformed propagator and actions of the free particle are dominated by the standard ones without quantum deformation. These results indicate that the quantum deformation effects in this space shortens the paths of particles, allowing them to move from one point to another in a short time. In one way or another, as one can see from Eq. (85), these results can be understood as free particles use low kinetic energies to travel faster in this deformed space. This confirms our recent results [21,22] and strengthens the claim that the position deformed-algebra (11) induces strong deformation of the quantum levels allowing particles to jump from state to another with low energy transitions [21,22].

The propagator for the Fourier transform and its inverse represensions is given by

$$\begin{aligned}\mathcal{K}(\rho, \rho', \Delta t) = \frac{1}{2\pi\hbar}\sqrt{\frac{m}{2\hbar\pi i\Delta t}}\int_{-l_{max}}^{+l_{max}}\frac{dx}{1-\tau x+\tau^2x^2}\int_{-l_{max}}^{+l_{max}}\frac{dx'}{1-\tau x'+\tau^2x'^2}\\ \times e^{-i\frac{2}{\tau\hbar\sqrt{3}}\left[\rho\arctan\left(\frac{2\tau x-1}{\sqrt{3}}\right)-\rho'\arctan\left(\frac{2\tau x'-1}{\sqrt{3}}\right)\right]}\\ \times e^{i\frac{2m}{\hbar 3\tau^2\Delta t}\left[\arctan\left(\frac{2\tau x-1}{\sqrt{3}}\right)-\arctan\left(\frac{2\tau x'-1}{\sqrt{3}}\right)\right]^2}.\end{aligned} \tag{87}$$

The corresponding action is given by

$$\mathcal{S}_{fp} = S_{fp} - \frac{2}{\tau\sqrt{3}}\left[\rho\arctan\left(\frac{2\tau x - 1}{\sqrt{3}}\right) - \rho'\arctan\left(\frac{2\tau x' - 1}{\sqrt{3}}\right)\right]. \tag{88}$$

3.3.2. *Reduced propagator of a simple harmonic oscillator*

The simple harmonic oscillator problem is defined by the Hamiltonian

$$\hat{H} = \frac{\hat{P}^2}{2m} + \frac{1}{2}m\omega^2\hat{X}^2. \tag{89}$$

The propagator in position representation is given by

$$K_{ho}(x, x', \Delta t) = \langle x|e^{-\frac{i}{\hbar}\left(\frac{\hat{P}^2}{2m}+\frac{1}{2}m\omega^2\hat{X}^2\right)\Delta t}|x'\rangle \tag{90}$$

For a sufficiently small time $\Delta t = \epsilon$, the time evolution operator is factorizable as a consequence of the Baker-Campbell-Hausdorff formula [32]. The propagator (90) is rewritten as follows

$$\begin{aligned}K_{ho}(x, x', \epsilon) &= \langle x|e^{-i\frac{\hat{P}^2}{2m\hbar}\epsilon}e^{-\frac{i}{2\hbar}m\omega^2\hat{X}^2\epsilon}|x'\rangle + \mathcal{O}(\epsilon^2)\\ &= \frac{\arctan(6)}{\pi\hbar\tau\sqrt{3}}\langle x|e^{-\frac{i}{2\hbar}m\omega^2x'^2\epsilon}\int_{-\infty}^{+\infty}d\rho e^{-\frac{i}{\hbar}\frac{\hat{P}^2}{2m}\epsilon}|\rho\rangle\langle\rho|x'\rangle + \mathcal{O}(\epsilon^2)\\ &= \int_{-\infty}^{+\infty}\frac{d\rho}{2\pi\hbar}\\ &\quad\times e^{\left(i\frac{2\rho}{\tau\hbar\sqrt{3}}\left[\arctan\left(\frac{2\tau x-1}{\sqrt{3}}\right)-\arctan\left(\frac{2\tau x'-1}{\sqrt{3}}\right)\right]-\frac{i}{\hbar}\left(\frac{\rho^2}{2m}+\frac{1}{2}m\omega^2x'^2\right)\epsilon\right)}.\end{aligned} \tag{91}$$

Computing the Gaussian integral (91), we have

$$K_{ho}(x, x', \epsilon) = \sqrt{\frac{m}{2\pi\hbar i\epsilon}}e^{i\frac{2m}{\hbar 3\tau^2\epsilon}\left[\arctan\left(\frac{2\tau x-1}{\sqrt{3}}\right)-\arctan\left(\frac{2\tau x'-1}{\sqrt{3}}\right)\right]^2-\frac{i}{2\hbar}m\omega^2x'^2\epsilon}, \tag{92}$$

and the corresponding deformed classical action is given by

$$S_{ho} = \frac{2m}{3\tau^2\epsilon}\left[\arctan\left(\frac{2\tau x-1}{\sqrt{3}}\right) - \arctan\left(\frac{2\tau x'-1}{\sqrt{3}}\right)\right]^2 - \frac{1}{2}m\omega^2 x'^2\epsilon. \tag{93}$$

At the limit $\tau \to 0$, we recover the ordinary propagator and the classical action of the simple harmonic oscillator [32,33]

$$\lim_{\tau\to 0} K_{ho}(x,x',\epsilon) = K^0_{ho}(x,x',\epsilon) = \sqrt{\frac{m}{2\pi\hbar i\epsilon}}e^{\frac{i}{\hbar}\left(\frac{m(x-x')^2}{2\epsilon} - \frac{1}{2}m\omega^2 x'^2\epsilon\right)},$$

$$\lim_{\tau\to 0} S_{ho} = S^0_{ho} = \frac{m(x-x')^2}{2\epsilon} - \frac{1}{2}m\omega^2 x'^2\epsilon, \tag{94}$$

$$\frac{S^0_{ho}}{\epsilon} = \frac{m(x-x')^2}{2(\epsilon)^2} - \frac{1}{2}m\omega^2 x'^2 = E^0_m, \tag{95}$$

where E^0_m is the standard mechanical energy of a simple harmonic mechanics. Like in the prior instance (85) it is simple to demonstrate that

$$K_{ho}(x,x',\epsilon) \le K^0_{ho}(x,x',\epsilon) \implies S_{ho} \le S^0_{ho} \implies E_m \le E^0_m, \tag{96}$$

where E_m is the deformed mechanical energy of harmonic oscillator

$$E_m = \frac{2m}{3\tau^2\epsilon^2}\left[\arctan\left(\frac{2\tau x-1}{\sqrt{3}}\right) - \arctan\left(\frac{2\tau x'-1}{\sqrt{3}}\right)\right]^2 - \frac{1}{2}m\omega^2 x'^2. \tag{97}$$

This also strengthens our obtained result in (85). In more general case, we can see that the harmonic oscillator potential does not affect the motion of the deformed free particle such that

$$K_{ho} \approx K_{fp} \le K^0_{fp} \approx K^0_{ho} \implies S_{ho} \approx S_{fp} \le S^0_{fp} \approx S^0_{ho}. \tag{98}$$

The propagator in Fourier transform and its inverse representions is given by

$$\begin{aligned}\mathcal{K}(\rho,\rho',\epsilon) &= \frac{1}{2\pi\hbar}\sqrt{\frac{m}{2\pi\hbar i\epsilon}}\int_{-l_{max}}^{+l_{max}}\frac{dx}{1-\tau x+\tau^2x^2}\int_{-l_{max}}^{+l_{max}}\frac{dx'}{1-\tau x'+\tau^2x'^2}\\ &\times e^{-i\frac{2}{\tau\hbar\sqrt{3}}\left[\rho\arctan\left(\frac{2\tau x-1}{\sqrt{3}}\right)-\rho'\arctan\left(\frac{2\tau x'-1}{\sqrt{3}}\right)\right]}\\ &\times e^{i\frac{2m}{\hbar 3\tau^2\epsilon}\left[\arctan\left(\frac{2\tau x-1}{\sqrt{3}}\right)-\arctan\left(\frac{2\tau x'-1}{\sqrt{3}}\right)\right]^2-\frac{i}{2\hbar}m\omega^2x'^2\epsilon},\end{aligned} \tag{99}$$

and its action is given by

$$\mathcal{S}_{ho} = S_{ho} - \frac{2}{\tau\sqrt{3}}\left[\rho\arctan\left(\frac{2\tau x-1}{\sqrt{3}}\right) - \rho'\arctan\left(\frac{2\tau x'-1}{\sqrt{3}}\right)\right]. \tag{100}$$

4. Conclusion

We have constructed path integrals in Euclidean position representation and in Fourier transform and its inverse representations within a position-deformed Heisenberg algebra (11). We have derived from these path integrals the propagators and the corresponding classical actions. These results are remarkably similar to the one obtained by Pramanick [25] from the Perivolaropoulos's algebra [26]. Then, the classical equations of motion are obtained by the principle of least action. The Hamiltonians of a free particle and a simple harmonic oscillator are used as examples to compute the propagators and the actions in position representation and in Fourier transform and its inverse representations. We have shown through these results that, the propagators and the actions of these systems in position space representation are properly bounded by the well-known results in the $\tau \to 0$ limit. This has indicated that the deformation induced by the maximal length shortens particle pathways, allowing them to travel faster from one point to the next with low kinetic and mechanical energies. This confirms our recent results [21,22] and strengthens the claim that the position deformed-algebra (11) induces deformation of the quantum levels allowing particles to jump from state to another with low transition energies. Finally, the propagators for Fourier transform and its inverse representations for both systems are given as integral expressions and we have deduced the corresponding actions.

CRediT authorship contribution statement

Latévi M. Lawson: Initiated the project, Performed the computations, Wrote the main manuscript, Discussion, Results analysis and Interpretation. **Prince K. Osei:** Initiated the project, Supervised and finalized the writing of the manuscript, Discussion, Results analysis and Interpretation. **Komi Sodoga:** Supervised and finalized the writing of the manuscript, Discussion, Results analysis and Interpretation. **Fred Soglohu:** Performed the computations, Wrote the main manuscript, Discussion, Results analysis and Interpretation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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Appendix

In this appendix we provide a detail information on the physically relevance of the states $|\phi_\rho\rangle$ (35) by restricting the Hilbert $\mathcal{H} = \mathcal{L}^2(\mathbb{R})$ into its dense bounded domain $\mathcal{D}_\tau = \mathcal{L}^2(-l_{max}, +l_{max})$.

To construct a Hilbert space representation that describes the maximal length and the minimal momentum uncertainties, one has to solve the eigenvalue problem

$$-i\hbar D_x\phi_\rho(x) = \rho\phi_\rho(x), \qquad \rho \in \mathbb{R}, \quad \phi_\rho(x) \in \mathcal{H}. \tag{101}$$

The solution of this equation is given by

$$\phi_\rho(x) = \sqrt{\frac{\tau\sqrt{3}}{2\pi}} \exp\left(i\frac{2\rho}{\tau\hbar\sqrt{3}}\left[\arctan\left(\frac{2\tau x - 1}{\sqrt{3}}\right) + \frac{\pi}{6}\right]\right).$$

As we can see, the expectation value of operator (energy) $\hat{X}^n$ $(n \geq 2)$ in this infinite dimensional Hilbert space diverges

$$\langle\phi_\rho|\hat{X}^n|\phi_\rho\rangle = \frac{\tau\sqrt{3}}{\pi}\int_{-\infty}^{+\infty}\frac{x^n dx}{1 - \tau x + \tau^2x^2} > \infty.$$

In comparaison to Kempf et al. formalism [4], the momentum eigenvectors $|\phi_\rho\rangle$ are not physical states. To circumvent this problem, we restrict the study from the infinite dimensional Hilbert space $\mathcal{H}$ into its bounded dense domaine $\mathcal{D}_\tau = \mathcal{L}^2(-l_{max}, +l_{max})$ in such away that, for $\tau \to 0$, one recovers the entire space $\mathcal{H} = \mathcal{L}^2(\mathbb{R})$. In this domaine, repeating the resolution of Eq. (101), we obtain the solution (35) of the paper given by

$$\phi_\rho(x) = \sqrt{\frac{\tau\sqrt{3}}{2\arctan(6)}}\exp\left(i\frac{2\rho}{\tau\hbar\sqrt{3}}\left[\arctan\left(\frac{2\tau x-1}{\sqrt{3}}\right)+\frac{\pi}{6}\right]\right).$$

With this solution in hand, we show that expectation values of position energy operator $\hat{X}^n$ $(n \in \mathbb{N})$ is finite:

$$\langle\phi_\rho|\hat{X}^n|\phi_\rho\rangle = \frac{\tau\sqrt{3}}{2\arctan(6)}\int_{-l_{max}}^{+l_{max}}\frac{x^n dx}{1-\tau x+\tau^2x^2} < \infty. \tag{102}$$

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