

## Research Article

# Martingale Transforms between Martingale Hardy-amalgam Spaces

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We discuss martingale transforms between martingale Hardy-amalgam spaces  $H_{p,q}^s$ ,  $\mathcal{Q}_{p,q}$  and  $\mathcal{P}_{p,q}$ . Let  $0 < p < q < \infty$ ,  $p_1 < p$  and  $q_1 < q$  and let  $f = (f_n, n \in \mathbb{N})$  be a martingale in  $\mathcal{P}_{p_1,q_1}$ ; then, we show that its martingale transforms are the martingales in  $\mathcal{P}_{p,q}$  for some  $p, q$  and similarly for  $H_{p,q}^s$  and  $\mathcal{Q}_{p,q}$ .

## 1. Introduction

Martingale came to existence through Doob as part of his seminal work [1]. The concept of classical martingale Hardy spaces came to light when Burkholder and Gundy extended an inequality due to R. E. A. C. Paley in 1970 [2]. This was made possible due to the introduction of the operators  $s(f)$  and  $S(f)$ , by Burkholder, which will be made clear in the sequel. Afterwards, playmakers including Burkholder himself, Garcia, Cairoli, and Davis contributed to the growth of the classical martingale Hardy space theory [2–8].

However, in 1966, Burkholder had already introduced the notion of martingale transforms [9] which became an indispensable tool in the study of some relations between classical martingale Hardy spaces, mostly the predictive spaces  $\mathcal{P}_p$  in the classical settings [6, 10]. In the past years, various authors have generalized the classical martingale Hardy spaces of the classical Lebesgue spaces to Lorentz spaces, Orlicz spaces, and Orlicz-Musielak spaces [11–17].

The interest in this paper is to discuss the martingale transforms between the martingale Hardy-amalgam spaces  $H_{p,q}^s$ ,  $\mathcal{Q}_{p,q}$  and  $\mathcal{P}_{p,q}$  introduced in [18]. These spaces, as indicated in [18], are generalizations of the classical martingale Hardy spaces. More precisely if  $p_1 < p$  and  $q_1 < q$  and  $f$  is a martingale in  $\mathcal{P}_{p_1,q_1}$ , then its martingale transforms are the martingales in  $\mathcal{P}_{p,q}$  and similarly for  $H_{p,q}^s$  and  $\mathcal{Q}_{p,q}$ .

The motivation to look for the various martingale transforms in these spaces comes from the various applications of martingale transforms in general. Especially, with the use of martingale transforms, the upcrossing theorem of martingales was established; the convergence of martingales has also been proved using martingale transforms and  $L^1$ -characterization of martingales [9, 10].

The next section introduces the basic definitions and the notations needed in subsequent sections. This is followed by a presentation on martingale transforms and their convergence. Presentation of the main results in this paper is given in Section 4. In Section 5, we prove the results and finally we conclude.

## 2. Basic Definitions and Notations

We introduce in this section some function spaces that will be relevant in the subsequent sections of this paper.

**2.1. Wiener Amalgam Spaces.** Let  $\Omega$  be an arbitrary nonempty set and let  $\{\Omega_j\}_{j \in \mathbb{Z}}$  be a sequence of nonempty subsets of  $\Omega$  such that  $\Omega_j \cap \Omega_i = \emptyset$  for  $j \neq i$ , and

$$\bigcup_{j \in \mathbb{Z}} \Omega_j = \Omega. \quad (1)$$

For  $0 < p, q \leq \infty$ , the classical amalgam of  $L_p$  and  $l_q$ , denoted  $L_{p,q}$ , on  $\Omega$  consists of functions which are locally in  $L_p$  and have  $l_q$  behaviour globally [19]. More precisely,  $f \in L_{p,q}$  if for  $0 < p < \infty$  and  $0 < q < \infty$ ,

$$\begin{aligned} \|f\|_{L_{p,q}(\Omega)} &:= \left( \sum_{j \in \mathbb{Z}} \left( \int_{\Omega} |f|^p \mathbf{1}_{\Omega_j} d\mathbb{P} \right)^{q/p} \right)^{1/q} < \infty, \\ \|f\|_{L_{p,\infty}(\Omega)} &:= \sup_{j \in \mathbb{Z}} \left( \int_{\Omega} |f|^p \mathbf{1}_{\Omega_j} d\mathbb{P} \right)^{1/p} < \infty, \end{aligned} \tag{2}$$

if  $q = \infty$ .  $\mathbf{1}_A$  is the usual indicator function of the set  $A$ . For more on amalgam spaces, we refer the reader to [19–22].

**2.2. Martingale Hardy Spaces via Amalgams.** The same notations used in [18] is adopted in this paper. Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space and let  $(\mathcal{F}_n)_{n \geq 0} := (\mathcal{F}_n)_{n \in \mathbb{Z}_+}$  be a nondecreasing sequence of  $\sigma$ -algebra with respect to the complete ordering on  $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$  such that

$$\sigma \left( \bigcup_{n \in \mathbb{N}} \mathcal{F}_n \right) = \mathcal{F}. \tag{3}$$

For  $n \in \mathbb{Z}_+$ ,  $\mathbb{E}$  and  $\mathbb{E}_n$  are the expectation operator and the conditional expectation operator, respectively, relative to  $\mathcal{F}$  and  $\mathcal{F}_n$ , respectively. We denote by  $\mathcal{M}$  the set of all martingales  $f = (f_n)_{n \geq 0}$  relative to the filtration  $(\mathcal{F}_n)_{n \geq 0}$  such that  $f_0 = 0$ . We recall that for  $f \in \mathcal{M}$ , its martingale difference is denoted  $d_n f = f_n - f_{n-1}$ ,  $n \geq 0$  with the convention that  $d_0 f = 0$ .

A martingale  $f = (f_n)_{n \geq 0}$  is said to be  $L_p$  bounded ( $0 < p \leq \infty$ ) if  $f_n \in L_p$  for all  $n \in \mathbb{Z}_+$  and

$$\|f\|_p := \sup_{n \in \mathbb{N}} \|f_n\|_p < \infty. \tag{4}$$

We recall that

$$\|f\|_p = (\mathbb{E}(|f|^p))^{1/p}. \tag{5}$$

For a martingale  $f = (f_n)_{n \geq 0}$ , the quadratic variation,  $S(f)$ , and the conditional quadratic variation,  $s(f)$ , of  $f$  are defined by

$$\begin{aligned} S(f) &= \left( \sum_{n \in \mathbb{N}} |d_n f|^2 \right)^{1/2}, \\ s(f) &= \left( \sum_{n \in \mathbb{N}} \mathbb{E}_{n-1} |d_n f|^2 \right)^{1/2}, \end{aligned} \tag{6}$$

respectively. In this regard, we have

$$\begin{aligned} S_n(f) &= \left( \sum_{i=1}^n |d_i f|^2 \right)^{1/2}, \\ s_n(f) &= \left( \sum_{i=1}^n \mathbb{E}_{i-1} |d_i f|^2 \right)^{1/2}. \end{aligned} \tag{7}$$

The maximal function  $f^*$  or  $M(f)$  of the martingale  $f$  is defined by

$$M(f) = f^* = \sup_{n \in \mathbb{N}} |f_n|. \tag{8}$$

The martingale Hardy-amalgam spaces  $H_{p,q}^s$ ,  $\mathcal{Q}_{p,q}$ , and  $\mathcal{P}_{p,q}$  were introduced in [18] but we list them here for completeness purposes. Let  $0 < p < \infty$  and  $0 < q \leq \infty$ . The spaces are defined as follows:

$$H_{p,q}^s(\Omega) = \left\{ f \in \mathcal{M} : \|f\|_{H_{p,q}^s(\Omega)} := \|s(f)\|_{p,q} < \infty \right\} \tag{9}$$

Let  $\Pi$  be the set of all sequences  $\beta = (\beta_n)_{n \geq 0}$  of adapted, nondecreasing, nonnegative functions and define also

$$\beta_\infty := \lim_{n \rightarrow \infty} \beta_n. \tag{10}$$

(i) The space  $\mathcal{Q}_{p,q}(\Omega)$ , called the space of predictive quadratic variations, is the set

$$\{f \in \mathcal{M} : \text{for } \beta \in \Pi, S_n(f) \leq \beta_{n-1} \text{ and } \beta_\infty \in L_{p,q}(\Omega)\}, \tag{11}$$

endowed with the (quasi)-norm

$$\|f\|_{\mathcal{Q}_{p,q}(\Omega)} := \inf_{\beta \in \Pi} \|\beta_\infty\|_{p,q} \tag{12}$$

(ii) The space  $\mathcal{P}_{p,q}(\Omega)$ , called the space of predictive martingales, is the set

$$\{f \in \mathcal{M} : \text{for } \beta \in \Pi, |f_n| \leq \beta_{n-1} \text{ and } \beta_\infty \in L_{p,q}(\Omega)\}, \tag{13}$$

endowed with the (quasi)-norm

$$\|f\|_{\mathcal{P}_{p,q}(\Omega)} := \inf_{\beta \in \Pi} \|\beta_\infty\|_{p,q} \tag{14}$$

The sequence  $\beta = (\beta_n)_{n \geq 0}$  is called the predictable sequence. In the subsequent sections, we shall sometimes write  $H_{p,q}^s(\Omega)$ ,  $\mathcal{Q}_{p,q}(\Omega)$ ,  $\mathcal{P}_{p,q}(\Omega)$  and  $L_{p,q}(\Omega)$  simply as  $H_{p,q}^s$ ,  $\mathcal{Q}_{p,q}$ ,  $\mathcal{P}_{p,q}$  and  $L_{p,q}$ , respectively, and we shall do the same for their respective (quasi)-norms. The atomic decompositions and duality characterizations of these spaces are in [18].

We note that the “infimum” taken in  $\mathcal{P}_{p,q}$  and  $\mathcal{Q}_{p,q}$  norms is attained [23]. Henceforth, such an “infimum” sequence will be referred to as optimal.

### 3. Martingale Transforms and Their Convergence

Let  $\mathbf{Ps} := (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_n\}_{n \geq 0})$  be a probability space with the filtration  $\{\mathcal{F}_n\}_{n \geq 0}$ . Let  $v = (v_n)_{n \geq 0}$  be an adapted process such that for all  $n = 0, 1, 2, \dots$ ,  $v_n$  is  $\mathcal{F}_{n-1}$ -measurable, normally referred to as multiplier sequence. If  $f = (f_n, n \in \mathbb{N})$  is a martingale, then the following process:

$$g_n = \sum_{k=1}^n v_k d_k f, g_0 = 0, \tag{15}$$

is called a martingale transform where  $d_k f$  is the usual martingale difference sequence. The martingale transform need not be a martingale; however, it is a martingale if and only if  $g_n \in L^1$  (i.e.,  $\mathbb{E}g_n < \infty$ ) [9, 10]. As a trivial example, let  $\tau$  be a stopping time. Then, the following process  $f_n^\tau$ , referred to as stopped process, is an example of a martingale transform as  $\mathbf{1}_{\{k \leq \tau\}}$  is  $\mathcal{F}_{k-1}$ -measurable;

$$f_n^\tau = f_{n \wedge \tau} = \sum_{k=1}^n \mathbf{1}_{\{k \leq \tau\}} d_k f. \tag{16}$$

The usefulness of martingale transforms has aided various authors to study the relations of the predictive spaces in their various generalizations such as the martingale transforms between Hardy-Orlicz spaces among others [24–26]. Martingale transforms also share some properties with fractional integrals [27–29]. Burkholder has established the almost everywhere convergence of the martingale transform based on the condition that the maximal function of the multiplier sequence is finite. In fact, we can find the proof of the following convergence result in [2], [Theorem 1].

**Theorem 1.** *Let  $f = (f_n, n \in \mathbb{N}), f_0 = 0$  be an  $L^1$  bounded martingale and  $g$  be  $f$ 's martingale transform with multiplier sequence  $v = (v_k)_{k \geq 0}$  defined below:*

$$g_n = \sum_{k=1}^n v_k d_k f, n \geq 1, g_0 = 0. \tag{17}$$

*Then,  $g = (g_n)_{n \geq 0}$  converges almost everywhere on the set  $\{\sup_k |v_k| < \infty\}$ .*

### 4. Presentation of Main Results

In this section, we present the results that will be discussed in this paper. Their proofs are discussed in the next section. We start with the results on  $\mathcal{P}_{p,q}$  followed by  $H_{p,q}^s$  and finally  $\mathcal{Q}_{p,q}$ .

**Theorem 2** (Relation between  $\mathcal{P}_{p,q}$  and  $\mathcal{P}_{p_1,q_1}$ ). *Let  $0 < p < q < \infty, 0 < p_1 < p, 0 < q_1 = (p_1/p)q < q$ , and  $\alpha = 1 - (p_1/p)$ . Let  $f = (f_n, n \in \mathbb{N})$  be a martingale define on  $\mathbf{Ps}$  and suppose that  $f \in \mathcal{P}_{p_1,q_1}$ . Let  $v = (v_k)_{k \geq 0}$  be the optimal bounded positive increasing adapted process such that  $|f_n| \leq v_{n-1}$  and  $v_\infty \in L_{p_1,q_1}$ . Then, the process defined by*

$$g_n = \sum_{k=1}^n \frac{1}{v_{k-1}^\alpha} d_k f, g_0 = 0, \tag{18}$$

*is a martingale transform of  $f$  and converges almost everywhere.*

*Moreover,  $g = (g_n, n \in \mathbb{N}) \in \mathcal{P}_{p,q}$  and*

$$\|g\|_{\mathcal{P}_{p,q}}^q \leq \left(\frac{p}{p_1} + 1\right)^q \|f\|_{\mathcal{P}_{p_1,q_1}}^{q_1}. \tag{19}$$

*The next theorem states that if  $g \in \mathcal{P}_{p,q}$  is the martingale transform of  $f$ , then  $f \in \mathcal{P}_{p_1,q_1}$  which in turn is the martingale transform of  $g$ .*

**Theorem 3** (Relation between  $\mathcal{P}_{p,q}$  and  $\mathcal{P}_{p_1,q_1}$ ). *Let  $0 < p < q < \infty, 0 < p_1 < p, 0 < q_1 = (p_1/p)q < q$ , and  $\alpha = 1 - (p_1/p)$ . Let  $f = (f_n, n \in \mathbb{N})$  be a martingale define on  $\mathbf{Ps}$  and let  $v = (v_k)_{k \geq 0}$  be a bounded positive increasing adapted process such that  $v_\infty \in L_{p_1,q_1}$ . Let  $g = (g_n, n \in \mathbb{N}) \in \mathcal{P}_{p,q}$  be a martingale transform of  $f$  defined by*

$$g_n = \sum_{k=1}^n \frac{1}{v_{k-1}^\alpha} d_k f, g_0 = 0, \tag{20}$$

*such that  $\mathbb{E}(g_n) < \infty$ . Then,*

- (a)  $f_n = \sum_{k=1}^n v_{k-1}^\alpha d_k g$  and converges almost everywhere and
- (b)  $f \in \mathcal{P}_{p_1,q_1}$  and moreover

$$\|f\|_{\mathcal{P}_{p_1,q_1}} \leq x \|f\|_{\mathcal{P}_{p,q}} \|v_\infty\|_{L_{p_1,q_1}}^{1-(p_1/p)}. \tag{21}$$

*Let us now look at the relations between  $H_{p,q}^s$  and  $H_{p_1,q_1}^s$ .*

**Theorem 4** (Relation between  $H_{p,q}^s$  and  $H_{p_1,q_1}^s$ ). *Let  $0 < p < q < \infty, 0 < p_1 < p, 0 < q_1 = (p_1/p)q < q$ , and  $\alpha = 1 - (p_1/p)$ . Let  $f = (f_n, n \in \mathbb{N})$  be a martingale define on  $\mathbf{Ps}$ . Let  $s(f)$  be the conditional quadratic variation operator which we assume it is bounded and nonzero. Then, the process defined by*

$$g_n = \sum_{k=1}^n \frac{1}{s_k^\alpha(f)} d_k f, g_0 = 0, \tag{22}$$

*is a martingale transform of  $f$  and converges almost everywhere. Moreover, if  $f \in H_{p_1,q_1}^s$ , then  $g \in H_{p,q}^s$  and*

$$\|g\|_{H_{p,q}^s}^q \leq \left(\frac{p}{p_1}\right)^{q/2} \|f\|_{H_{p_1,q_1}^{q_1}}^{q_1}. \quad (23)$$

The converse of the above theorem is the following theorem that is given that  $g \in H_{p,q}^s$  is the martingale transform of  $f$ , then  $f$  is the martingale transform of  $g$  and  $f \in H_{p_1,q_1}^s$ .

**Theorem 5** (Relation between  $H_{p,q}^s$  and  $H_{p_1,q_1}^s$ ). Let  $0 < p < q < \infty$ ,  $0 < p_1 < p$ ,  $0 < q_1 = (p_1/p)q < q$ , and  $\alpha = 1 - (p_1/p)$ . Let  $f = (f_n, n \in \mathbb{N})$  be a martingale define on  $\mathbf{Ps}$  and  $s(f)$  be the conditional quadratic variation operator which we assume it is bounded and nonzero. Let  $g = (g_n, n \in \mathbb{N}) \in H_{p,q}^s$  be a martingale transform of  $f$  defined by

$$g_n = \sum_{k=1}^n \frac{1}{s_k^\alpha(f)} d_k f, g_0 = 0, \quad (24)$$

such that  $\mathbb{E}(g_n) < \infty$ . Then

- (a)  $f_n = \sum_{k=1}^n s_k^\alpha(f) d_k g$  and converges almost everywhere and  
 (b)  $f \in H_{p_1,q_1}^s$  and moreover

$$\|f\|_{H_{p_1,q_1}^{q_1/4}} \leq \|g\|_{H_{p,q}^s}. \quad (25)$$

We now come to the predictive quadratic variation spaces.

**Theorem 6** (Relation between  $\mathcal{P}_{p,q}$  and  $\mathcal{P}_{p_1,q_1}$ ). Let  $0 < p < q < \infty$ ,  $0 < p_1 < p$ ,  $0 < q_1 = (p_1/p)q < q$ , and  $\alpha = 1 - (p_1/p)$ . Let  $f = (f_n, n \in \mathbb{N})$  be a martingale define on  $\mathbf{Ps}$  and suppose that  $f \in \mathcal{Q}_{p_1,q_1}$ . Let  $\nu = (\nu_k)_{k \geq 0}$  be the optimal bounded positive increasing adapted process such that  $S_n(f) \leq \nu_{n-1}$  and  $\nu_\infty \in L_{p_1,q_1}$ . Then, the process defined by

$$g_n = \sum_{k=1}^n \frac{1}{\nu_{k-1}^\alpha} d_k f, g_0 = 0, \quad (26)$$

is a martingale transform of  $f$  and converges almost everywhere.

Moreover,  $g = (g_n, n \in \mathbb{N}) \in \mathcal{Q}_{p,q}$  and

$$\|g\|_{\mathcal{Q}_{p,q}}^q \leq \left(\frac{p}{p_1} + 1\right)^{q/2} \|f\|_{\mathcal{Q}_{p_1,q_1}}^{q_1}. \quad (27)$$

Similarly, if  $g \in \mathcal{Q}_{p,q}$  is the martingale transform of  $f$ , then  $f \in \mathcal{Q}_{p_1,q_1}$ , and moreover,  $f$  is the martingale transform of  $g$ . This is the statement below.

**Theorem 7** (Relation between  $\mathcal{P}_{p,q}$  and  $\mathcal{P}_{p_1,q_1}$ ). Let  $0 < p < q < \infty$ ,  $0 < p_1 < p$ ,  $0 < q_1 = (p_1/p)q < q$ , and  $\alpha = 1 - (p_1/p)$ . Let  $f = (f_n, n \in \mathbb{N})$  be a martingale define on  $\mathbf{Ps}$  and let  $\nu = (\nu_k)_{k \geq 0}$  be a bounded positive increasing adapted process such

that  $\nu_\infty \in L_{p_1,q_1}$ . Let  $g = (g_n, n \in \mathbb{N}) \in \mathcal{Q}_{p,q}$  be a martingale transform of  $f$  defined by

$$g_n = \sum_{k=1}^n \frac{1}{\nu_{k-1}^\alpha} d_k f, g_0 = 0, \quad (28)$$

such that  $\mathbb{E}(g_n) < \infty$ . Then,

- (a)  $f_n = \sum_{k=1}^n \nu_{k-1}^\alpha d_k g$  and converges almost everywhere and  
 (b)  $f \in \mathcal{Q}_{p_1,q_1}$  and moreover

$$\|f\|_{\mathcal{Q}_{p_1,q_1}} \leq \|g\|_{\mathcal{Q}_{p,q}} \|\nu_\infty\|_{L_{p_1,q_1}}^{1-(p_1/p)} \quad (29)$$

## 5. Proof of Results

*Proof of Theorem 2.* Let  $f \in \mathcal{P}_{p_1,q_1}$ . By the hypothesis,  $|f_n| \leq \nu_{n-1}$  and  $\nu_\infty \in L_{p_1,q_1}$  ( $\nu = (\nu_n)_{n \geq 0}$  optimal). From equation (18)

$$g_n = \frac{f_n}{\nu_{n-1}^\alpha} + \sum_{k=1}^{n-1} f_k \left( \frac{1}{\nu_{k-1}^\alpha} - \frac{1}{\nu_k^\alpha} \right) \quad (30)$$

□

We note that when  $\nu_{k-1} \leq t \leq \nu_k$  then  $1/\nu_k^\alpha \leq 1/t^\alpha$  which implies that

$$\int_{\nu_{k-1}}^{\nu_k} \frac{1}{\nu_k^\alpha} dt \leq \int_{\nu_{k-1}}^{\nu_k} \frac{1}{t^\alpha} dt \Rightarrow \frac{\nu_k - \nu_{k-1}}{\nu_k^\alpha} \leq \int_{\nu_{k-1}}^{\nu_k} \frac{1}{t^\alpha} dt. \quad (31)$$

Therefore, from equation (30), we get that

$$|g_n| \leq \int_0^{\nu_{n-1}} \frac{1}{t^\alpha} dt = \left( \frac{1}{1-\alpha} + 1 \right) \nu_{n-1}^{1-\alpha}. \quad (32)$$

Indeed since  $(1/\nu_{k-1}^\alpha) - (1/\nu_k^\alpha) \geq 0$ , we get

$$\begin{aligned} |g_n| &\leq \frac{|f_n|}{\nu_{n-1}^\alpha} + \sum_{k=1}^{n-1} |f_k| \left( \frac{1}{\nu_{k-1}^\alpha} - \frac{1}{\nu_k^\alpha} \right) \\ &\leq \nu_{n-1}^{1-\alpha} + \sum_{k=1}^{n-1} \nu_{k-1} \left( \frac{1}{\nu_{k-1}^\alpha} - \frac{1}{\nu_k^\alpha} \right) \\ &= \nu_0^{1-\alpha} + \sum_{k=1}^{n-1} (\nu_k^{1-\alpha} - \nu_{k-1}^{1-\alpha}) + \sum_{k=1}^{n-1} \nu_{k-1}^{1-\alpha} - \frac{\nu_{k-1}}{\nu_k^\alpha} \\ &= \nu_0^{1-\alpha} + \sum_{k=1}^{n-1} \nu_k^{1-\alpha} + \sum_{k=1}^{n-1} - \frac{\nu_{k-1}}{\nu_k^\alpha} \\ &= \nu_0^{1-\alpha} + \sum_{k=1}^{n-1} \frac{\nu_k}{\nu_k^\alpha} - \frac{\nu_{k-1}}{\nu_k^\alpha} \leq \nu_0^{1-\alpha} + \sum_{k=1}^{n-1} \int_{\nu_{k-1}}^{\nu_k} \frac{1}{t^\alpha} dt \\ &\leq \nu_0^{1-\alpha} + \int_0^{\nu_{n-1}} \frac{1}{t^\alpha} dt \leq \left( \frac{1}{1-\alpha} + 1 \right) \nu_{n-1}^{1-\alpha} \end{aligned} \quad (33)$$

Hence,  $\sup_n |g_n| < \infty$  since  $\{v_n\}_{n \geq 0}$  is a bounded sequence. We note that  $v_k$  is  $\mathcal{F}_{k-1}$ -measurable as it is adapted and so is  $v_{k-1}^{-\alpha}$ . Now, the sequence  $w_k = v_{k-1}^{-\alpha}$  is a positive decreasing sequence which is bounded above by  $v_0^{-\alpha}$  since  $v_0 > 0$ . Therefore,  $\sup_k |v_k^{-\alpha}| < \infty$  since  $v_k^\alpha \neq 0$  for all  $k$ . Thus, equation (18) is a martingale transform and hence by Theorem 1,  $g_n$  converges almost everywhere.

We observe that as  $(1 - \alpha)p = p_1$ , we also have that  $\|v_\infty^{1-\alpha} \mathbf{1}_{\Omega_j}\|_p = \|v_\infty \mathbf{1}_{\Omega_j}\|_{p_1}^{p_1/p}$ . Therefore,

$$\|v_\infty^{1-\alpha}\|_{p,q} = \|v_\infty\|_{p_1,q_1}^{q_1/q} := \|f\|_{\mathcal{F}_{p_1,q_1}}^{q_1/q}. \quad (34)$$

Let  $\beta_{n-1} = ((1/(1 - \alpha) - 1) + 1)v_{n-1}^{1-\alpha}$ . Then, the sequence  $\beta = (\beta_n)_{n \geq 0}$  is also an increasing positive adapted process. Hence, by definition,

$$\|g\|_{\mathcal{F}_{p,q}} \leq \left\| \left( \frac{1}{1 - \alpha} + 1 \right) v_\infty^{1-\alpha} \right\|_{p,q} = \left( \frac{1}{1 - \alpha} + 1 \right) \|v_\infty^{1-\alpha}\|_{p,q}. \quad (35)$$

Equation (34) then gives us

$$\|g\|_{\mathcal{F}_{p,q}} \leq \left( \frac{p}{p_1} + 1 \right) \|f\|_{\mathcal{F}_{p_1,q_1}}^{q_1/q}, \quad (36)$$

establishing the result.

*Proof of Theorem 3.* Let  $g \in \mathcal{F}_{p,q}$ . Then, there exists an optimal positive increasing adapted sequence  $u = (u_n)_{n \geq 0}$ , such that  $|g_n| \leq u_{n-1}$  and  $u_\infty \in L_{p,q}$ .

Part (a) follows since  $d_n f = v_{n-1}^\alpha d_n g$ . Since  $g$  is a martingale and  $v$  is a bounded positive increasing and adapted, and for all  $n$ ,  $v_n$  is  $\mathcal{F}_{n-1}$ -measurable, and the convergence of  $f_n$  follows from Theorem 1.

For part (b), we observe that since  $(u_n)_{n \geq 0}$  is increasing,

$$\begin{aligned} |f_n| &\leq |g_n| v_{n-1}^\alpha + \sum_{k=1}^{n-1} |g_k| d_k v^\alpha \\ &\leq u_{n-1} v_{n-1}^\alpha + \sum_{k=1}^{n-1} u_{k-1} d_k v^\alpha \\ &\leq u_{n-1} v_{n-1}^\alpha + u_{n-1} \sum_{k=1}^{n-1} d_k v^\alpha \\ &\leq 2u_{n-1} v_{n-1}^\alpha \leq 2u_\infty v_{n-1}^\alpha. \end{aligned} \quad (37)$$

Let  $\gamma_{n-1} = 2u_\infty v_{n-1}^\alpha$ . Then, the sequence  $\gamma = (\gamma_n)_{n \geq 0}$  is also positive increasing and bounded adapted process. Hence, by definition,

$$\|f\|_{\mathcal{F}_{p_1,q_1}} \leq \|\gamma_\infty\|_{p_1,q_1} = 2\|u_\infty v_\infty^\alpha\|_{p_1,q_1}. \quad (38)$$

Hence,

$$\|f\|_{\mathcal{F}_{p_1,q_1}} \leq 2\|u_\infty v_\infty^\alpha\|_{p_1,q_1}. \quad (39)$$

By definition,

$$\|u_\infty v_\infty^\alpha\|_{p_1,q_1} = \left( \sum_{j \in \mathbb{Z}} \left\| u_\infty v_\infty^\alpha \mathbf{1}_{\Omega_j} \right\|_{p_1}^{q_1} \right)^{1/q_1}. \quad (40)$$

With the choice of  $\alpha$ , and also observing that  $q_1 = (1 - \alpha)q$ , we apply Hölder's inequality to obtain

$$\|u_\infty v_\infty^\alpha\|_{p_1,q_1} \leq \|u_\infty\|_{p,q} \|v_\infty\|_{p_1,q_1}^{1-(p_1/p)}. \quad (41)$$

Inequality (6) then becomes

$$\|f\|_{\mathcal{F}_{p_1,q_1}} \leq 2\|u_\infty\|_{p,q} \|v_\infty\|_{p_1,q_1}^{1-(p_1/p)} = 2\|g\|_{\mathcal{F}_{p,q}} \|v_\infty\|_{p_1,q_1}^{1-(p_1/p)}, \quad (42)$$

and the theorem is proved.  $\square$

*Proof of Theorem 4.* Let  $s(f)$  be the conditional quadratic variation operator. Then, by definition,  $s_k(f)$  is  $\mathcal{F}_{k-1}$ -measurable. Thus,  $s_k^\alpha(f)$  is  $\mathcal{F}_{k-1}$ -measurable and positive and bounded adapted decreasing process. Hence,  $g_n$  is a martingale transform. Since  $s_k(f)$  is positive and bounded,  $s_k^\alpha(f)$  is also positive and bounded. That is,  $\sup_k |s_k^\alpha(f)| < \infty$ . Thus, by Theorem 1,  $g_n$  converges almost everywhere. Suppose that  $f \in H_{p_1,q_1}^s$ . Then,  $\|s(f)\|_{p_1,q_1} < \infty$ . From equation (2), we get that

$$d_k^2 g = \frac{d_k^2 f}{s_k^{2\alpha}(f)} \Rightarrow \mathbb{E}_{k-1} d_k^2 g = \frac{\mathbb{E}_{k-1} d_k^2 f}{s_k^{2\alpha}(f)}. \quad (43)$$

Since  $\mathbb{E}_{k-1} d_k^2 f = s_k^2(f) - s_{k-1}^2(f)$ , we sum both sides of the last equality to obtain

$$s_n^2(g) = \sum_{k=1}^n \frac{s_k^2(f) - s_{k-1}^2(f)}{s_k^{2\alpha}(f)}. \quad (44)$$

Since  $s_k(f)$  is increasing, we observe that for  $t > 0$ ,  $s_{k-1}^2(f) \leq t \leq s_k^2(f)$ , and thus,

$$\frac{1}{s_k^{2\alpha}(f)} \leq \frac{1}{t^\alpha}. \quad (45)$$

It follows that

$$\int_{s_{k-1}^2(f)}^{s_k^2(f)} \frac{1}{s_k^{2\alpha}(f)} dt \leq \int_{s_{k-1}^2(f)}^{s_k^2(f)} \frac{1}{t^\alpha} dt. \quad (46)$$

That is,

$$\frac{s_k^2(f) - s_{k-1}^2(f)}{s_k^{2\alpha}(f)} \leq \int_{s_{k-1}^2(f)}^{s_k^2(f)} \frac{1}{t^\alpha} dt. \tag{47}$$

We deduce from this and (7) that

$$s_n^2(g) \leq \sum_{k=1}^n \int_{s_{k-1}^2(f)}^{s_k^2(f)} \frac{1}{t^\alpha} dt, \tag{48}$$

and hence, we obtain that

$$s_n^2(g) \leq \int_0^{s_n^2(f)} \frac{1}{t^\alpha} dt \leq \frac{1}{1-\alpha} (s(f))^{2-2\alpha}, \tag{49}$$

which gives us

$$s(g) \leq \left(\frac{1}{1-\alpha}\right)^{1/2} s^{1-\alpha}(f). \tag{50}$$

Now, since

$$\|s^{1-\alpha}(f)\|_{p,q} = \|s(f)\|_{p_1,q_1}^{q_1/q}, \tag{51}$$

we have that

$$\|s(g)\|_{p,q} \leq \left(\frac{p}{p_1}\right)^{1/2} \|s(f)\|_{p_1,q_1}^{q_1/q}. \tag{52}$$

Thus, by definition,

$$\|g\|_{H_{p,q}^s} \leq \left(\frac{p}{p_1}\right)^{1/2} \|f\|_{H_{p_1,q_1}^s}^{q_1/q}. \tag{53}$$

and the theorem is proved.  $\square$

*Proof of Theorem 5.* The proof of part (a) follows obviously. For part (b), we have from equation (22) that  $d_k f = d_k g s_k^\alpha(f)$  and by measurability and the increasing property of  $s(\cdot)$ , we get that

$$\mathbb{E}_{k-1} d_k^2 f \leq s^{2\alpha}(f) \mathbb{E}_{k-1} d_k^2 g. \tag{54}$$

Summing both sides, we shall obtain

$$s^2(f) \leq s^{2\alpha}(f) s^2(g), \tag{55}$$

and thus,

$$s^{1-\alpha}(f) \leq s(g). \tag{56}$$

But

$$\|s^{1-\alpha}(f)\|_{p,q} = \|s(f)\|_{p_1,q_1}^{q_1/q}. \tag{57}$$

Therefore,

$$\|s(f)\|_{p_1,q_1}^{q_1/q} \leq \|s(g)\|_{p,q}. \tag{58}$$

The proof is complete.  $\square$

*Proof of Theorem 6.* Let  $f \in \mathcal{Q}_{p_1,q_1}$ . By hypothesis,  $S_n(f) \leq v_{n-1}$  and  $v_\infty \in L_{p_1,q_1}$  ( $v = (v_n)_{n \geq 0}$  optimal). Considering equation (26), we have that

$$|d_n g|^2 = v_{n-1}^{-2\alpha} |d_n f|^2. \tag{59}$$

Since  $v = (v_k)_{k \geq 0}$  is increasing and  $S_k(f) \leq v_{k-1}$ , we observe from equation (59) that (as  $S_0(f) = 0$ ),

$$\begin{aligned} S_n^2(g) &= \sum_{k=1}^n \frac{S_k^2(f) - S_{k-1}^2(f)}{v_{k-1}^{2\alpha}} \\ &= \frac{S_n^2(f)}{v_{n-1}^{2\alpha}} - \frac{S_{n-1}^2(f)}{v_{n-1}^{2\alpha}} + \sum_{k=1}^{n-1} \frac{S_k^2(f) - S_{k-1}^2(f)}{v_{k-1}^{2\alpha}} \\ &= \frac{S_n^2(f)}{v_{n-1}^{2\alpha}} - \left( \sum_{k=1}^{n-1} \frac{S_k^2(f)}{v_k^{2\alpha}} - \frac{S_{k-1}^2(f)}{v_{k-1}^{2\alpha}} \right) + \sum_{k=1}^{n-1} \frac{S_k^2(f) - S_{k-1}^2(f)}{v_{k-1}^{2\alpha}} \\ &= \frac{S_n^2(f)}{v_{n-1}^{2\alpha}} + \sum_{k=1}^{n-1} \frac{S_k^2(f)}{v_{k-1}^{2\alpha}} - \frac{S_k^2(f)}{v_k^{2\alpha}} \\ &\leq v_{n-1}^{2-2\alpha} + \sum_{k=1}^{n-1} S_k^2(f) \left( \frac{1}{v_{k-1}^{2\alpha}} - \frac{1}{v_k^{2\alpha}} \right) \\ &\leq v_{n-1}^{2-2\alpha} + \sum_{k=1}^{n-1} v_{k-1}^2 \left( \frac{1}{v_{k-1}^{2\alpha}} - \frac{1}{v_k^{2\alpha}} \right) \\ &= v_0^{2-2\alpha} + \sum_{k=1}^{n-1} (v_k^{2-2\alpha} - v_{k-1}^{2-2\alpha}) + \sum_{k=1}^{n-1} v_{k-1}^{2-2\alpha} - \frac{v_{k-1}^2}{v_k^{2\alpha}} \\ &= v_0^{2-2\alpha} + \sum_{k=1}^{n-1} v_k^{2-2\alpha} + \sum_{k=1}^{n-1} -\frac{v_{k-1}^2}{v_k^{2\alpha}} \\ &= v_0^{2-2\alpha} + \sum_{k=1}^{n-1} \frac{v_k^2}{v_k^{2\alpha}} - \frac{v_{k-1}^2}{v_k^{2\alpha}} \leq v_0^{2-2\alpha} + \sum_{k=1}^{n-1} \int_{v_{k-1}^2}^{v_k^2} \frac{1}{t^\alpha} dt \\ &\leq v_0^{2-2\alpha} + \int_0^{v_{n-1}^2} \frac{1}{t^\alpha} dt \leq \left( \frac{1}{1-\alpha} + 1 \right) v_{n-1}^{2-2\alpha}. \end{aligned} \tag{60}$$

Hence, we get that

$$S_n(g) \leq \left( \frac{1}{1-\alpha} + 1 \right)^{1/2} v_{n-1}^{1-\alpha}. \tag{61}$$

This implies that  $S(g) \leq ((p/p_1) + 1)^{1/2} v_\infty^{p_1/p} < \infty$ . Also,  $v_{k-1}^{-\alpha}$  is adapted. Thus,  $g_n$  is a martingale transform and by Theorem 1,  $g_n$  converges almost everywhere since  $\sup_k |v_k^{-\alpha}| < \infty$  (as  $v_0 > 0$ ) and  $f$  is a martingale.

Let  $\beta_{n-1} = \sqrt{(1/(1-\alpha)) + 1}v_{n-1}^{1-\alpha}$ . Then, the sequence  $\beta = (\beta_n)_{n \geq 0}$  is also an increasing positive and bounded adapted process. Hence, by definition,

$$\begin{aligned} \|g\|_{\mathcal{Q}_{p,q}} &\leq \|\beta_\infty\|_{p,q} = \left\| \sqrt{\frac{1}{1-\alpha} + 1}v_{n-1}^{1-\alpha} \right\|_{p,q} \\ &= \sqrt{\frac{1}{1-\alpha} + 1} \|v_\infty^{1-\alpha}\|_{p,q}. \end{aligned} \tag{62}$$

In the same manner as we obtained equation (34), we also have that

$$\|v_\infty^{1-\alpha}\|_{p,q} = \|v_\infty\|_{p_1,q_1}^{q_1/q} := \|f\|_{\mathcal{Q}_{p_1,q_1}}^{q_1/q}. \tag{63}$$

Therefore,

In other words,  $\|g\|_{\mathcal{Q}_{p,q}}^q \leq ((p/p_1) + 1)^{q/2} \|f\|_{\mathcal{Q}_{p_1,q_1}}^{q_1}$ , (66) and the theorem is proved.  $\square$

*Proof of Theorem 7.* Similar to the proof of Theorem 3, part (a) is established.

For part (b), let  $g \in \mathcal{Q}_{p,q}$ . Then, there exist an optimal increasing positive adapted process  $u = (u_k)_{k \geq 0}$  such that  $S_n(g) \leq u_{n-1}$  and  $u_\infty \in L_{p,q}$ . From equation (59), we have that

$$|d_n f|^2 = v_{n-1}^{2\alpha} |d_n g|^2 \Rightarrow S_n^2(f) = \sum_{k=1}^n v_{k-1}^{2\alpha} [S_k^2(g) - S_{k-1}^2(g)]. \tag{64}$$

We also observe that

$$S_n^2(f) = S_n^2(g)v_{n-1}^{2\alpha} - \sum_{k=1}^{n-1} S_k^2(g)d_k v^{2\alpha}, \tag{65}$$

and therefore,

$$S_n(f) \leq \sqrt{2}u_\infty v_{n-1}^\alpha. \tag{66}$$

Let  $\gamma_{n-1} = \sqrt{2}u_\infty v_{n-1}^\alpha$ . Then, the sequence  $\gamma = (\gamma_n)_{n \geq 0}$  is also positive increasing and bounded adapted process. Hence, by definition,

$$\|f\|_{\mathcal{Q}_{p_1,q_1}} \leq \|\gamma_\infty\|_{p_1,q_1} = \sqrt{2}\|u_\infty v_\infty^\alpha\|_{p_1,q_1}. \tag{67}$$

Hence,

$$\|f\|_{\mathcal{Q}_{p_1,q_1}} \leq \sqrt{2}\|u_\infty v_\infty^\alpha\|_{p_1,q_1}. \tag{68}$$

By definition,

$$\|u_\infty v_\infty^\alpha\|_{p_1,q_1} = \left( \sum_{j \in \mathbb{Z}} \|u_\infty v_\infty^\alpha \mathbf{1}_{\Omega_j}\|_{p_1}^{q_1} \right)^{1/q_1}. \tag{69}$$

With the choice of  $\alpha$ , and noting that  $q_1 = (1-\alpha)q$ , we apply Hölder’s inequality to get

$$\|u_\infty v_\infty^\alpha\|_{p_1,q_1} \leq \|u_\infty\|_{p,q} \|v_\infty\|_{p_1,q_1}^{1-(p_1/p)}. \tag{70}$$

Inequality (9) then becomes

$$\|f\|_{\mathcal{Q}_{p_1,q_1}} \leq \sqrt{2}\|u_\infty\|_{p,q} \|v_\infty\|_{p_1,q_1}^{1-(p_1/p)} = \sqrt{2}\|g\|_{\mathcal{Q}_{p,q}} \|v_\infty\|_{p_1,q_1}^{1-(p_1/p)}, \tag{71}$$

and the theorem is proved.  $\square$

### 6. Conclusion

We are thus able to locate where the martingale transform of a particular martingale is, and also given a martingale, we are able to find its martingale transform in the spaces  $\mathcal{P}_{p,q}$ ,  $\mathcal{Q}_{p,q}$  and  $H_{p,q}^s$ . These are extended results corresponding to Garsia [6] where he considered the  $L_p$  space while this paper considers the  $L_{p,q}$ , the amalgam space of  $\ell_q$ , and  $L_p$  spaces, defined in previous pages.

### Data Availability

No data were used in support of this study.

### Conflicts of Interest

The author declares that there is no conflict of interests regarding the publication of this paper.

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