FORECASTING MORTALITY RATES AND MODELLING LONGEVITY RISK OF SSNIT PENSIONERS

UNIVERSITY OF GHANA

BY

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DECLARATION

Candidate's Declaration

I, Emmanuel Kwabena Ofori-Amanfo hereby declare that apart from references to other people's publications, which have been duly acknowledged, this thesis is a result of my independent ideas, thought, deliberations and has not been submitted for the award of any degree at this institution and other universities elsewhere.

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Supervisors' Declaration

We hereby certify that this thesis was prepared from the candidate's own work and supervised in accordance with guidelines on supervision of thesis laid down by the University of Ghana.

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DEDICATION

This thesis is dedicated to the love, sacrifices and prayers of my cherished mom, Madam Mary Nyardy. Thank you for the seeds you sowed in me. My special dedication to Elder Foster Doteeb who has always been the source of my strength, joy and desire to excel academically. I thank God for having you in my life daddy.

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CAP 30	-	1950 British Colonial Ordinance (Pension Ordinance No. 42)
CBD	-	Cairns, Blake and Dowd
DB	-	Defined Benefit
DC	-	Defined Contribution
IMF	-	International Monetary Fund
LC	-	Lee-Carter
MAE	-	Minimum Absolute Error
MAPE	-	Maximum Absolute Percentage Error
NPRA	-	National Pensions Regulatory Authority
RMSE	-	Root Mean Square Error
SSNIT	-	Social Security and National Insurance Trust
SVD	-	Singular Value Decomposition
WHO	-	World Health Organization

ABSTRACT

Predicting mortality trend and hedging of longevity risk in recent times has gained attention at a period when life expectancies are increasing unexpectedly. Life insurance companies may make more profit owing to the advancements in technology, hygiene, and medical procedures which leads to decreasing liabilities emanating from payment of reducing death benefit payments. However, longevity improvements may result in losses for annuity insurance schemes such as pensions. The study investigated the existence or otherwise of longevity risk in Ghana and the financial approaches/insurance SSNIT ought to adopt for pensioners who outlive the 15-year guaranteed period. The Lee-Carter and Cairns-Blake-Dowd models were used to forecast mortality patterns through secondary source information from SSNIT covering 1991 to 2017. Results of the study show that mortality rates are on the decline whiles life expectancy, on the other hand, is rising. The study recommended that, since there is an improvement in longevity, SSNIT can hedge individual longevity risk by obtaining a life annuity from life insurance companies. The study further called for prudent financial management and operational efficiency that will ensure long term solvency.

CHAPTER ONE

INTRODUCTION

1.1 Background of the study

Human mortality has been on a decelerating trajectory since the beginning of the twentieth century (Wrigley-field, 2019). People live longer today than in the past. This phenomenon has become a vital subject amongst demographers and actuaries worldwide. Research surrounding this area has gained momentum because of the immense importance of understanding human mortality trends. Government-led population studies and also insurance and pension industries have increased research in this area because the findings enable governments to plan effectively and allocate resources equitably and enable accurate pricing and risk assessments by insurances and pension plans respectively.

Despite the numerous researches in the area longevity risk, only developed countries have been well represented. This has led to governments and life insurance sectors continually using less reliable forecasts of mortality and life expectancy. This is attributed to the paucity of quality data that plays a key role in most models that estimate future rates. Ghana relies heavily on international organizations such as United Nations, World Health Organization and the World Bank to fill the gap in this area.

According to WHO (2015), the length of time people are expected to remain alive in Ghana has increased by 25 to 30 years during the last century (World Health Organization, 2015). The latest WHO data published in 2018 for Ghana's life expectancy was 63.5 years.

Significant medical progress, improved hygiene and living standards, generally healthier lifestyles and the absence of wars and pandemic crises are some of the key factors responsible for the rising life expectancy (Crawford et., 2008). These gains in life expectancy are good news. However, policy-makers, insurance companies and pension

firms worry about the impact that these gains may have on retirement finances. As long as gains in life expectancy are foreseeable and are taken into account when planning retirement, they would have a negligible effect on retirement finances. Unfortunately, improvements in mortality and life expectancy are uncertain. In this regard, longevity risk is associated with the risk that future mortality and life expectancy outcomes turn out differently from expected.

The historic rise in life expectancy shows little sign of slowing, and increased survival is a significant contributor to population ageing. In this context, forecasting mortality has gained prominence. The future of mortality is of interest not only in its own right, but also in the context of population forecasting, on which economic, social and health planning is based. The future provision of health and social security for ageing populations is now a central concern of countries throughout the developed world.

DB pension schemes, companies, governments, insurers, reinsurers, long-term healthcare providers and individuals are all subjected to the uncertainties associated with increased life expectancy. Although some of these uncertainties can be diversified by aggregating individual lives into large groups, they cannot be eliminated completely and can have significant economic consequences. These uncertainties are usually referred to as *longevity risk* or *mortality risk* depending on the context and have become a vital concern for DB pension schemes and the respective interested parties. Pension providers are faced with the risk that pensioners will live longer than expected and since they have to pay monthly pension to the pensioners until their death; longevity risk may affect the annuity provider's solvency.

Longevity risk refers to the risk that the actual survival rates and life expectancy will exceed expectations or pricing assumptions, resulting in greater-than-anticipated retirement cash

flow needs. Longevity risk is the risk to which pension schemes or annuity providers are exposed to paying out higher amounts of benefits than expected in future. Longevity basis risk arises because different populations have different survival rates and hence longevity and life expectancy. The risk exists due to increasing life expectancy trends among policyholders and pensioners.

The need to manage longevity risk has become very important as employers and employees become aware of their exposure to longevity risk and their need to mitigate it. For individuals, longevity risk is the risk of outliving one's income, resulting in a lower standard of living, reduced care, or a return to employment at old age. For those institutions providing covered individuals with guaranteed retirement income, longevity risk is the risk of undervaluing survival rates, resulting in increased liabilities to sufficiently cover promised payments.

According to the American Center for Insurance Policy and Research of the National Association of Insurance Commissioners (NAIC), key drivers of the growing need to address longevity risk include an ageing population, increasing life expectancy, a shift in who bears the responsibility of sufficient retirement income, uncertainty of government benefits and economic volatility (Blake and Turner, 2014).

Understanding and managing longevity risk is a key challenge for pension funds and life insurers Ageing populations have been recognized as one of the risks in pension schemes in the world. When Longevity risk is not catered for, it can cause insolvency and cause individuals to lose their hard-earned retirement income. Pension schemes and annuity providers need to effectively manage the longevity they are exposed to.

Defined benefit pension plans and annuities which guarantee lifetime benefits for pensioners are the main types of plans exposed to longevity risk. Annuitants receiving income till death may live longer than expected or accounted for in the actuarial calculations of defined benefit pension plans and thus longevity risk.

Over the past few years, most companies have established a defined contribution (DC) scheme to reduce the risks that are associated with defined benefit (DB) schemes. Both DB and DC plans are meant to provide members with sufficient financial means to be able to retire and maintain an adequate standard of living throughout retirement. In a DB scheme, the risk of increasing longevity is borne by the scheme sponsor. Therefore, companies with Defined Benefit pension plans have paid attention to rising longevity as the rising life expectancy affects the funding costs of DB pension plans. Some companies have specifically a reserve that is purely to take care of longevity risk (longevity reserve). Even though longevity has no direct effect on the funding costs of DC plans, longevity is much of an issue in DC plans. At retirement, the total fund credit in the DC scheme member's account is converted to a life annuity using an annuity factor. Therefore, if the life expectancy is underestimated, this will result to annuity providers paying out more than expected.

As a result of this uncertainty surrounding future developments in mortality and life expectancy, individuals run the risk of outliving their resources and being forced to reduce their standard of living at old ages. Pension funds and life annuity providers, on the other hand, run the risk that the net present value of their annuity payments will turn out higher than expected, as they will have to pay out a periodic sum of income that will last for an uncertain life span. In this context, individuals bear the full extent of the longevity risk when this risk is 'uncovered'. However, private pension funds and national governments

providing defined retirement benefits, as well as financial institutions providing lifetime annuity payments face this longevity risk.

In the fields of Demography and Actuarial Science, there have been many attempts to develop appropriate models that represent mortality (Danesi, Haberman, & Millossovich, 2015). Traditionally, a parametric curve, like the ones suggested by De Moivre (1729), Gompertz (1825) and Weibull (1951), was used to fit annual death rates (Andreozzi et al., 2011). Over the past years, a significant number of new approaches were developed to forecast mortality by using stochastic models. The Lee-Carter model became one of the most well-known models, and it is applied in different countries around the world to forecast age-specific death rates (Danesi et al., 2015). The method proposed by Lee and Carter (1992), has become the leading statistical model of mortality forecasting in the demographic literature. It was used as a benchmark for recent Census Bureau population forecasts of the United States (Girosi & King, 2007). Lee and Carter developed their approach specifically for U.S. mortality data,1933-1987. However, the method is now being applied to all-cause and cause-specific mortality data from many countries and time-periods, beyond the application for which it was designed (Girosi & King, 2007). Blake, Cairns and Dowd (2006), extended Lee and Carter to allow not only for a cohort effect but also for a quadratic age effect in their CBD model that are found to solve all the problems Lee and Carter's model had. The study of longevity and mortality forecast is even more crucial in the present context. Therefore, this study assesses how pension schemes, annuity providers, insurance companies and the regulatory authorities can assess future improvements in mortality and life expectancy.

1.2 Statement of Problem

According to IMF (2012), survival probability is still considerably underestimated particularly in the recent past decades. Longevity has not only increased; the trend has

become more uncertain. Since the 20th century, life expectancy has increased dramatically. Due to advances made in medical technology, people changing their lifestyles and other factors, life expectancy has increased continually(Steptoe et al., 2015). Blake et al. (2012), found that each additional year of life adds 3-4% to the value of pension liability. Pension providers are obliged to pay a fixed amount to a pensioner monthly for as long as the pensioner remains alive.

Life expectancy in Ghana increased by16.58 years from 1960 to 2015 and by 3.76 years ten years from the year 2005 to 2015 (World Bank, 2015). In addition to increasing life expectancy, contributions made to SSNIT has decreased from 18.5 per cent to 11.5 per cent. Also, the guarantee period for pensioners has risen from 12 years to 15 years (NPRA, 2010). As a result, SSNIT now receives less income but will pay out benefits for a more extended period. This situation exposes SSNIT to the likelihood that at a future date, it may not be able to meet its financial obligations to pensioners.

Prior actuarial models of forecasting the trends mortality disregarded the stochastic nature of mortality. A stochastic approach is a better approach to assess the uncertainty and associated longevity risks adequately by modelling life expectancy and mortality as it attaches probabilities in different forecasts (Antolín, 2007). Therefore, understanding how the future mortality trend using the stochastic models is likely to interest the actuary in pricing and reserving of annuities. Lee and Carter's model is the first stochastic model to consider increased life expectancy which has become widely used and recently CBD model with its several extensions and modifications have been proposed to arrest the main features of mortality intensity. But then in developing countries such as Ghana, there is a paucity of literature on the impact of longevity risk on pension fund and the need for work of this nature cannot be downplayed. In this study, the focus is forecast mortality rates and model longevity risk of SSNIT pensioners using Lee-Carter and CBD models.

1.3 Objectives of the Study

The main objective is to forecast mortality rates and model the longevity risk of SSNIT pensioners. The specific objectives are to:

- Estimate mortality rate over time.
- Forecast stochastically future life expectancies using Lee-Carter and CBD models.
- To compare the forecast error of the two models.

1.4. Significance of the Study

The project would be relevant to practitioners and academicians both in the private and public sector by contributing to the existing body of knowledge in the area of mortality forecasting and accessing and quantifying longevity risk. The findings will help insurance and pensions providers to use national population-based mortality indices to manage longevity risk in pension schemes and annuity portfolios.

The research will be relevant to practitioners who would like to come up with more reasonably priced products suited for the Ghanaian population or any other developing country to enable them to manage and transfer longevity risk as it guides longevity modelling. This will result to the application of the risk transfer options: buy-out, buy-in and longevity swap used in other countries. Additionally, securities such as longevity bonds and indexes may be priced to ensure that longevity risk is hedged.

1.5 Limitation of The Study

In this study, other risks, such as interest rate risk, adverse policy changes and unfavorable investment returns, were not taken into consideration. The assumption was that there would be no default.

1.6 Overview of the Study

The study has been organized into five chapters. Chapter One highlights the rationale of the study, inform and establishes the decreasing mortality trend in the world and Ghana in particular, provide objectives, justification and limitations of the study.

In Chapter Two, we reviewed existing literature on longevity risk, Pensions in Ghana, and age-specific mortality and the deviations are discussed by the Lee-Carter and Cairns, Blake and Dowd (CBD) models. Stochastic mortality models were contrasted with deterministic mortality models, and a case was established for Stochastic models to help arrest the systematic mortality risk. Stochastic mortality models which were reviewed in this chapter include Lee-Carter (1992) and CBD Models. The structural and parametric changes to the Pension Scheme after the pension reform in 2008 were also outlined.

Chapter Three outlined the methodologies used in the study, including source and nature of data, assumptions used in the forecasting and modelling of mortality risk and the analytical tools used in the study.

Chapter Four, a section on Data Analysis and Discussion encompassed the life expectancy forecast of both Lee-Carter and CBD, out of sample validation, comparison of the forecast errors using MAE, MAPE and RSME and discussions on the analyzed results.

In the concluding chapter, we made conclusions and recommendations based on our findings. It was concluded that the SSNIT is exposed to mortality risk and the CBD model fit the data best since it recorded the least errors. Recommendations were made to help arrest the ever-soaring longevity risk.

CHAPTER TWO

LITERATURE REVIEW

2.1 Introduction

Existing literature on longevity risk, the average age-specific mortality and the deviations are discussed by the Lee-Carter model and Cairns, Blake and Dowd (CBD) model and forecasting future mortality were reviewed in this chapter.

2.2 Understanding Risk of Longevity

Longevity risks are any potential risks linked to a rise in the life expectancy of annuity policy-holders and pensioners which can ultimately transform many pension firms and insurance companies into higher than expected payouts (Antolin, 2007). This risk is existent in any product where there is exposure to financial losses in the event that policyholders live longer than anticipated. This happens when the issuer pays policyholders on the condition of survivorship.

Conventionally, these products are meant to give protection to individuals for outliving their income with the issuers being insurance companies. In recent times, there has been a rise in financial instruments number and type which are exposed to longevity risk. Although the transfer of longevity risk may not be the main objective of the transaction, it may occur.

One major challenge that life annuity providers and pension schemes face is longevity risk. The pensioner and annuity holder are paid by life annuity and pension providers respectively for life. In the case where assumptions made with respect to mortality are undervalued the financial stability of the paying institution can be endangered. There is evidence that mortality improves over time, as a result of technological advances in medical sciences and enhanced knowledge about healthy lifestyles.

In some years to come, if there is a decrease in mortality, there will be a reduction in the liabilities of the life insurer as payments would delay beyond the first time of estimation. With regards to pension scheme, a decrease in mortality implies benefits would have to be paid for extended periods than anticipated and there would be a loss on the annuity portfolio relative to initial expectations of annuity providers (Cox and Lin, 2004).

The life insurer makes more losses while more profits are made by the annuity providers than anticipated. A small number of researchers looked into the issue of natural hedging. Research done digs into the effect of mortality changes on life insurance and annuities distinctively, or inquiries into a simple combination of life and pure endowment life contracts (D. Blake et al., 2012). Research on the impact of mortality changes on life insurance centres on adverse shocks while those on annuities centre on good shocks. Wang, Hsu and Chen (2009), analyses the impact of the changes of underlined with basis on the mortality experience in Taiwan, factors that guide the process of the mortality hazard rates proposes an immunization model by which the optimal level of product mix between an annuity and life insurance is calculated in hedging longevity risks. Nevertheless, separate mortality tables are not used in exploring life insurance and annuity mortality experience. Practically, life insurance and annuity mortality experience can differ, so there is basis risk involved in the use of annuities to hedge life insurance mortality risk. The model cannot pick up this basis risk.

2.2.1 Consequences of Underestimating Longevity

Although the risk of longevity develops and reveals itself slowly over time, if left unhandled, its effect on financial stability is the build-up of important vulnerabilities in public and private balance sheets (IMF, 2012).

When a pension scheme is exposed to longevity risk, there would be an exposure of the pension scheme to other linked risks including:

- Interest rate risk: The longer people live, the expectation is that pension funds must be invested for longer periods to meet its liabilities and hence the exposure to interest rate volatilities.
- Increased inflation risk: The pension fund may also become exposed to adverse inflation rates in the long term.

2.3 Quantification and Management of Longevity

According to Crawford et al. (2008), in ensuring that pension funds are effectively managed against being exposed to longevity risk, there should be a measure of longevity risk as well as its impact by actuaries. Only when the nature of the risk is fully understood can we then design risk management tools effective at mitigating it.

Several efforts, including Blake et al. (2006) and Cox and Lin (2013), have been made towards managing longevity risk. There is a proposal of an extension of the normal retirement age by some researchers. Belgium, for instance, is in the process of reviewing the retirement age in line with life expectancy with the current retirement age of 65 being raised to 67 by 2030. In Germany, the retirement age currently 65 years, three months would be gradually increased to 67 years in 2029 (Leng & Peng, 2016). In Ghana, the retirement age was increased from 50 to 60 in 1965.

Blake et al. (2006), have proposed channelling longevity risk to the capital market or to a third party through different types of financial instruments. These financial instruments come in the form of financial derivatives with some longevity index as the fundamental asset. They are traded as special agreements between the parties since they are not

standardized. In a bid to quantify longevity risk, a model is required in the prediction of future mortality pattern, which can be compared to current mortality pattern.

2.4 Initial Actuarial Models

Mortality trends and their effects on pension annuities were detected at the start of the 20th century. For example, Nordenmark (1906) pointed out that, mortality improvements must be regarded with care when life annuities are priced and, especially, cohort mortality should be dealt with to avoid undervaluation of mortality linked liabilities.

Prior models for mortality rates were deterministic, and include the Gompertz and exponential model. Baingana and Bos, (2006), provides an acceptable fit to adult mortality, but overrates death rates at ages higher than 80. The Perks (1932) logistic model (a generalization of the Gompertz curve) provides a relatively good fit to mortality rates over the whole adult range.

The Heligman and Pollard (1980), curve gives a relatively good fit to mortality rates over all ages. Advances made in computational algorithms aid dealing with complex models, and the number of parameters is no longer an issue. Leng and Peng (2016), reviews early contributions to mortality forecast and current models. Some current studies have exhibited that the mortality rates estimated from the classic parametric formulas were changeable (Halicioglu, 2011). Stochastic models seem more attractive, as they link a condense error to each estimate. In 1992, Lee and Carter showed a stochastic model, based on a factor analytic approach, for fitting and predicting mortality rates for the United States. Since then, because of its simplicity and relatively good performance, the Lee-Carter (LC) model has been extensively used for demographic and actuarial applications in various countries.

The earliest formula was by a French Mathematician De Moivre (1725) who wrote the survival function as

$$s(x) = 1 - \frac{x}{\omega} \qquad (2.1)$$

where ω is the limiting age and with the assumption that deaths are uniformly distributed. Later on, British actuary Zaks (2018), suggested that a law of geometric progression penetrates mortality after a certain age. Even though death at ages greater than 80 is overestimated in the model, he observed that for the age grouping of between 20 and 60 years, the force of mortality increased almost exponentially with age hence he suggested the following model:

$$\mu(x) = \alpha e^{\beta x} \qquad (2.2)$$

Where α and β are positive parameters.

Makeham (1860) extended the Gompertz model by adding a constant to give:

$$\mu(x) = \gamma + \alpha e^{\beta x} \qquad (2.3)$$

where all the parameters used are positive real numbers. The right-hand side has two terms, the mortality γ which is independent of age and the mortality $\alpha e^{\beta x}$ which depends on age.

To correct the weakness of the Gompertz model, several models were proposed. Thorvald Thiele in 1867 focused on the following model that represents the whole lifespan:

$$\mu(x) = \alpha_1 \exp(-\beta_1 x) + \alpha_2 \exp(-\beta_2 (x - \eta^2) + \alpha_3 \exp(\beta_3 x) \qquad (2.4)$$

The parameter used is positive real numbers. The first part of the right-hand side denotes the decreasing mortality at very young ages after the young ones have survived the risks at birth. The second part denotes the mortality hump at young-adult ages as a result of accidents or drug abuse that is, it is at the young-adult age that mortality increases due to lifestyle and accidental effects such as excessive drinking, careless driving or drug abuse.

The third part represents mortality at adult and old ages. Note that if $\alpha_1 = \beta_1 = \beta_2 = 0$, we obtain a special case of Thiele model known as Makeham law.

Perks (1932) logistic model which is a linear generalization of the Gompertz curve gives a relatively good fit to mortality rates over the entire adult range. The model is represented by:

$$\mu(x) = \frac{\alpha \exp(\beta x) + \gamma}{\varepsilon \exp(-\beta x) + \delta \exp(\beta x) + 1} \qquad (2.5)$$

All the parameters used are positive real numbers. If we let $\varepsilon = \delta = 0$ we get Makehams law. Heligman and Pollad (1980) curve also give a relatively good fit to mortality rates over all ages and the number of parameters is no longer an issue. He proposed the following model:

$$\frac{q_x}{p_x} = A^{(x+b)^c} + Dexp[-E(\ln x - \ln F)^2] + GH^x \qquad (2.6)$$

However, studies were conducted to prove the practicability of the early actuarial models, for instance, Leng and Peng (2016), revealed many errors in the forecasts using the deterministic models. Also, they noted that the decline in the old age mortality was also underestimated and increases in life expectancy under projected. Therefore, the use of deterministic actuarial models in fitting and forecasting yields wrong forecasts and hence leads to erroneous conclusions. Further reviews of earlier contributions to mortality forecasts were provided by Danesi et al. (2015) and Neves, Fernandes, and Hoeltgebaum (2017).

Recent advances in the actuarial practices, especially in pensions and life mathematics, have resulted in proposal of more models for describing and projecting mortality. Pitacco, Denuit, Haberman and Oliviera (2009), carried out a convenient survey and exposition of the

models. One of the most essential features of the recent models is that they are stochastic as opposed to being deterministic. Stochastic models seem more appealing because they associate a confidence error to each estimate. Also, the value of annuity or any similar pension product is a nonlinear function of future mortality, and thus calculations of annuity values should be based upon the entire distribution rather than the expected future mortality.

2.5. Stochastic Mortality Models

Fitting mortality rates and hence longevity risk quantification dynamically continues to be a challenge, especially in the developing countries. Earlier development relied on one-factor model proposed by Lee and Carter (1992). However, the Lee and Carter model is widely applied since it has been found to provide reasonably accurate estimations and population projections for both the academicians and practitioners. Renshaw and Heberma (2003) analysed the Lee-Carter model and proposed a new model. Recently two-factor models were proposed and the cohort effect was considered in longevity modelling which Lee and Carter's model lacked. For instance, Renshaw and Haberman (2003) applied a cohort effect and later Currie (2006) introduces an age-period-cohort (APC) model. In the most recent proposals Cairns, Blake and Dowd (2006b) allow not only for a cohort effect but also for a quadratic age effect in their CBD model that are found to solve all the problems Lee and Carter's model had.

2.5.1 Age-Specific Death Rates

An age-specific death rate, $m_{x;t}$, is the ratio of the number of deaths within a specified age group in a specific geographic area during a certain period to the corresponding population at risk of the same group, in the same geographic area during the specified time of the study (Cairns et al. 2011).

$$m_{x;t} = \frac{D_{x;t}}{N_{x;t}} \tag{2.7}$$

where

$m_{x;t}$ -is the age-specific death rate at age group x at year t

$D_{x;t}$-is the number of deaths at age group x at year t

$N_{x;t}$-is the population at age group x at year t

2.5.2. The Lee-Carter Model (1992)

The Lee-Carter (LC) model (Lee and Carter 1992) has become the leading statistical model of mortality in the demographic literature (Andreozzi et al., 2011). It has served as a foundation for numerous related models and forecasting methods (Cairns, Blake, Dowd, Coughlan, & Epstein, 2011; Hyndman, Booth, & Yasmeen, 2011; Mircea, Covrig, & Serban, 2014). Lee–Carter and related models are used by demographers, actuaries, statisticians, economists and others to produce forecasts that inform government decision–making, that enable the effective operation of insurers, pension funds and financial markets, and that enhance understanding of mortality risk (Jong, Tickle, & Xu, 2014).

Lee and Carter (1992) came up with a stochastic model where the log of a time series of age-specific death rates is the sum of age-specific component and a component that is a product of a time-varying parameter. However, deterministic projections are possible, and it works in discrete age or time framework.

The Lee-Carter model (1992) has become widely used and there have been various extensions and modifications proposed to attain a broader interpretation and to capture the main features of the dynamics of the mortality intensity. The Lee-Carter methodology is a milestone in the mortality projections actuarial literature. The model describes the logarithm of the observed mortality rate for age x and year t, $m_{x,t}$ as the sum of an age-specific component, α_x , that is independent of time and another component that is the product of

time-varying parameter, κ_t reflecting the general level of mortality and an age-specific component, β_x, that represents how mortality at each age varies when the general level of mortality changes (Dowd et al., 2012).

Generally, Lee-Carter express the log of a time series of age-specific death rates $m_{x,t}$ as the sum of an age-specific component α_x that is independent on time and a component that is the product of a time-varying parameters κ_t reflecting the general level of mortality, and an age-specific component β_x that represents the rate of mortality changes at each age.

$$\mu_{x,t} = \ln(m_{x,t}) = \alpha_x + \beta_x \kappa_t + \varepsilon_{x,t} \qquad x = 0, \ldots, p\,, \quad t = 1, \ldots, N\,. \qquad (2.8)$$

Here there are $p + 1$ ages x and N time periods t of observation. In applications, the age range may start from, for example, 1 or some other age.

With the following parameter interpretations:

κ_t is the time index representing the level of mortality at time t;

α_x represents the average trend of mortality on the time horizon at age x;

β_x represents a measure of the sensitivity in movement from the parameter κ_t. In particular, β_x describes the relative speed of mortality changes, at each age, when κ_t changes; and

$\varepsilon_{x,t}$ is the homoskedastic error term, which incorporates historical trends not considered by the model. It is assumed to be $\varepsilon_{x,t} \sim N(0, \sigma^2)$.

The equations must be augmented with constraints on the α_x ,β_x and κ_t to ensure β_x and κ_t are identified. Since the parameters in the model are not fully identified, Lee and Carter (1992) enforced the constraints κ_t should satisfy the constraints:

$$\sum_{x=1}^{\omega} \beta_x = 1$$

and

$$\sum_{t=1}^{n} \kappa_t = 0$$

These two constraints together with $\sum_t \varepsilon_{x,t} = 0$ imply the estimated α_x is the average log mortality at age x across time.

Lists of approaches have been proposed on how the parameters in the Lee-Carter model can be estimated. For instance, in the original paper of Lee and Carter (1992), he used the method of single value decomposition where he assumed that errors in the observed rates compared to those fitted by the model are independent and identically distributed standard variables. The values of κ_t were then adjusted slightly to ensure that the total number of deaths that are predicted by the model across all ages are similar to the observed number of deaths across all ages. The Singular Value Decomposition approach has however been replaced by other formal statistical models proposed for example Li and Hare (2017), proposed a fitting procedure that takes advantage of the assumption that the death count can be assumed to be a Poisson variable and that the natural logarithms is the canonical link function for the Poisson distribution to use the maximum likelihood estimation in estimating the parameters. This means that in the maximum likelihood estimation method, the errors obtained between the fitted and the observed rates are allowed with varying age unlike that one proposed by Lee and Carter (1992) method.

Dowd, Cairns, Blake, Coughlan, and Epstein (2010), proposed a different method where he expressed that the goodness of fit in the final year in the data set should be looked into keenly. For Lee and Miller, they observed that they observed that modelling objective is to project the mortality rates. Nevertheless, in general, the usual statistical procedures aim to fit the historical data so well overall past years.

2.5.2.1 Fitting Lee Carter model

Lee Carter original model, Lee and Carter (1992), focuses, as main forecasting methodologies, on the central mortality rates $m_{x;t}$ for age x in year t defined as the ratio between the number of deaths $D_{x;t}$, recorded during the calendar year t for people aged x, and the exposure to risk $\varepsilon_{x;t}$ obtained as the average number of people living during the calendar year t.

Starting by this sample notation, Lee and Carter (1992) proposed to describe the logarithm of central mortality rates as a linear combination of parameters as expressed by Equation (2.8):

$$\ln(m_{x,t}) = \alpha_x + \beta_x \kappa_t + \varepsilon_{x,t}$$

where α_x describes the general shape of mortality according to different ages and it represents the logarithm of the geometric mean of empirical mortality rates, averaged over historical years. e^{α_x} measure indeed the general shape across the age of the mortality schedule. Furthermore, κ_t reproduces the underlying time trend, while a term β_x is considered in order to take into account the different effect of time t at each age. β_x is assumed to be invariant over time and it explains how rates decline rapidly or slowly in response to change in κ_t. Finally, $\varepsilon_{x,t}$ are independent and identical distributed random variables $N(0;\ \sigma^2)$ taking into account the age and time-specific trends not fully captured by the model.

In the original version, parameters have been estimated by a two-stage process where Singular Value Decomposition (SVD) of the matrix of centred age profiles $\ln(m_{x,t}) - \alpha_x$, allows the first estimation of parameters $\beta_x$and κ_t.

To assure a unique solution for the system of the equation of the model, Lee and Carter proposed the following constraints: $\sum_t \kappa_t = 0$ and $\sum_x \beta_x = 1$

A second step, based on a refitting of $\hat{\kappa}_t$ on the number of deaths, is usually suggested in order to assure a better convergence between estimated and observed deaths. The aim is to find the $\hat{\kappa}_t$ such that $D_{(x;\,t)} = E_{(x;\,t)} exp(\hat{\alpha}_x + \hat{\beta}_x \hat{\kappa}_t)$.

2.5.2.2 Advantages of Lee and Carter (1992)

According to Andreozzi et al. (2011), the advantage of the lee- carter model includes the fact that it provides a good fit to historical data. Even though the shape of the mortality tables can be complex at the early ages, the α_x age function in the lee-carter allows the model to be used across all ages. In addition, κ_t term captures the dominant trend in the evolution of mortality.

Again, it very simple in fitting and projecting mortality rates as compared to other models. Currie and Currie (2016) recounted that, the use of the model provides an easy way of fitting and projecting since the parameters in the model are relatively few in comparison to the other models. The singular value decomposition and Poisson likelihood methods are likewise simple to put into practice.

Lastly, it is easy to project since the linear trend in the $\kappa_t's$ is common in most of the data used. The random walk with drift time series time structure is widely used to give estimates of future central mortality rates (Plat, 2009).

However, the model has several drawbacks (Girosi and King, 2007). The model is a one-factor model which means that the mortality improvements at all ages in the datasets are perfectly correlated. Search results are unrealistic and pause a problem when looking into how risky the liabilities are based on central mortality.

According to Dowd et al. (2012), the β_x age effect in the model is measured as the average improvement at all age x . However, β_x is also used in obtaining the level of uncertainty in future mortality rates age x , which therefore results, therefore;

$$Var\left[\log m_{x,n}/m_t\right] = \beta_x Var[\kappa_t^2] \qquad (2.9)$$

Historically, the rates of improvement have been lower at the very old ages; meaning that the projected future death rates uncertainty will be lower at old ages Chiang (1984).

Mircea et al. (2014), posited that the Lee-Carter model does not contain any allowance for cohort effects depending on an individual's year of birth. In recent years, models based on the Lee-Carter model incorporating cohort effects have since been introduced for instance. Alternative frameworks have been proposed over the years to improve some drawbacks of the original Lee-Carter model (Kang, Yang, Pan, Yu, & Zhou, 2014).

2.5.3 Lee-Carter model under a Poisson setting

have implemented the Lee-Carter model assuming a Poisson distribution of the number of deaths and using the log *link function* with respect to the force of mortality $\mu_{x,t}$. The predictor structure proposed by Lee and Carter (1992) assumes that there is a static age function, α_x, a unique non-parametric age-period term ($N = 1$), and no cohort effect.

Thus, the predictor is given by:

$$\log m_{x,t} = \alpha_x + \beta_x^{(1)}\kappa_t^{(1)}$$

To project mortality, the time index $\kappa_t^{(1)}$ is modelled and forecasted using ARIMA processes. Typically, a random walk with drift has been shown to provide a reasonable fit, that is,

$$\kappa_t^{(1)} = \delta + \kappa_{t-1}^{(1)} + \xi_t \qquad \xi_t \sim N(0, \sigma_k^2) \qquad i.i.d.,$$

where δ is the drift parameter and ξ_t is a Gaussian white noise process with variance σ_k^2.

2.5.4 Cairns, Blake and Dowd (2006)

Given the problems with the preceding models, a range of alternatives have been explored to find a model that incorporates a parsimonious, multi-factor age-period structure with a cohort effect that lacked in the previous models.

Cairns *et al.* (2006) propose a predictor structure with two age-period terms ($N = 2$) with pre-specified age-modulating parameters $\beta_x^{(1)} = 1 \; and \; \beta_x^{(2)} = x - \bar{x}$, no static age function and no cohort effect. Thus, the predictor of the CBD model is given by:

The model was expressed as:

$$\text{logit } q(t,x) = \kappa_t^{(1)} + (x - \bar{x})\kappa_t^{(2)}, \tag{2.10}$$

where $\bar{x}$ is the average age in the data. Cairns et al. (2006) obtain mortality forecasts by projecting the period effects $\kappa_t^{(1)}$) and$\kappa_t^{(2)}$ using a bivariate random walk with drift.

The CBD model does not have identifiability issues and hence the set of parameter constraints is empty. To estimate the parameter of the CBD model we can follow Haberman and Renshaw (2011) and assume a Binomial distribution of deaths using a logit link function targeting the one-year death probabilities $q_{x,t}$.

2.5.5 The Cairns-Blake-Dowd model with a cohort effect

Dowd et al. (2010), extend the original CBD model by adding a cohort effect and a quadratic age effect to obtain the predictor:

$$\text{logit } q(t,x) = \kappa_t^{(1)} + (x - \bar{x})\kappa_t^{(2)} + ((x - \bar{x})^2 - \hat{\sigma}_x^2)\kappa_t^{(3)} + \gamma_{t-x}, \tag{2.11}$$

where $\hat{\sigma}_x^2$ is the average value of $(x - \bar{x})^2$.

Compared with the original model of Cairns et al. (2011), there are two additional components.

CBD with a cohort effect adds a quadratic term into the age effect and still maintains the cohort effects.

Cairns et al. (2011), also consider the simpler predictor structures

$$\text{logit}\, q(t,x) = \kappa_t^{(1)} + (x - \bar{x})\kappa_t^{(2)} + \gamma_{t-x}, \tag{2.12}$$

$$\text{logit}\, q(t,x) = \kappa_t^{(1)} + (x - \bar{x})\kappa_t^{(2)} + (x_c - x)\gamma_{t-x}, \tag{2.13}$$

where x_c is a constant parameter to be estimated.

2.5.5.1 Advantages of the CBD model

1. The mortality prediction of CBD is very accurate when the age is above 40 years.

2. The CBD uses two-time length parameter impact to take advantage of mortality rates and the difference in rates at high age.

3. The model is robust and easy adjusted to parameter uncertainty.

2.5.5.2 Disadvantages of the CBD model

There are few disadvantages of the CBD namely;

1. The model is not appropriate for small set data

2. It is not suitable for predicting the mortalities aged below 40 years.

2.5.5.3 Implication of the CBD Mortality Indexes

As older ages are been approached, pay-outs by annuity insurers and sponsors of pensions are positively related to mortality productivity. The total productivity of mortality is higher than expected when financial duties are huge. (k1t is lower than expected).

Mortality productivity at old age is higher than at younger age and as such annuity providers and pension sponsors problem worsens in a fixed total mortality productivity. (k2t is lower than expected).

Pay-outs are negatively connected to productivity of mortality at younger ages for life insurers selling term insurance products therefore total mortality productivity is lower than expected. k_2^t is higher than expected)

Mortality productivity at younger age is less at higher age and as such life insurers' problem worsens in a fixed total mortality productivity. (k_2^t is lower than expected).

2.6 Mortality Index

Mortality indexes provide an objective method of measuring longevity risk. They broadly indicate the pace at which the mortality of a population is changing, enabling the measurement of longevity risk by comparing the difference between the expected and actual paths of the index. There have been a number of attempts by the industry to create indexes (Li & Hare, 2017).

- In 2006, Credit Suisse started a longevity index with the life expectancy at birth of the US population as its basis.
- In 2007, Goldman Sachs launched the QX index, which is based on the number of survivors in the reference population.
- JP Morgan introduced the Life Metrics in 2007, which renders death rates and period life expectancy figures.
- In 2008, Deutsche Borse released the Expect Cohort Index, and it is linked to the number of survivors of a certain birth cohort.

2.6.1 Properties of the mortality indexes

Apart from being a good representative of varying age pattern of mortality improvement and being readily interpretable, a CBD and Lee- Carter mortality indexes have other desirable properties that mortality indexes in general, should satisfy. Here we explain additional properties of the model.

Unambiguous: The population on which the mortality indexes are based must be defined in detail. In this study, the population used was males and females under the pension scheme who retired at the normal retirement age of 60.

Transparent: The method used to calculate the index value must be clear. While there exist multiple methods for estimating the Lee-carter model, the index provider can use one method. In this study, a computer program for fitting the model was used.

Objectivity: The method used to calculate the index should have as little as subjective input as possible. The Lee-carter model used in this study meets this requirement because given our data and age range, the estimation of the parameters requires no subjective input.

Appropriateness: The indexes should react the compositions of the populations requiring the hedging. If the Lee-carter mortality indexes are based on national populations, then this criterion may not be met as the mortality experience requiring hedging may be different from the mortality experience of the entire population. In this study, the reference populations are males under the SSNIT pension plan who retired at the normal retirement age of 60 and this is the population that requires the hedge hence the indexes are appropriate.

2.7 Model Selection Criteria

Comparison of the stochastic model is advised in order to know if the model is a good one or not. Cairns et al. (2011), proposed a list of qualities to check and evaluate the models and compare them with the other proposed models.

Consistency with historical data

Cairns et al. (2011), describes a good mortality model as one which consistent with historical patterns of mortality. If this is not the case, much greater doubt is placed on the validity of any forecasts produced by the model. Once a model is consistent with historical patterns, it gives confidence in the use of the forecasted values as opposed to inconsistent one.

Therefore, Cairns et al. (2011), compared different models using the maximum likelihood and using the method that penalize over-parametrized models. In their results, they suggested that improvements in the Lee and Carter (1992) and Cairns et al. (2011), models can be obtained by incorporating period and cohort effects.

Biological reasonableness

Cairns et al. (2011), introduce the concept of biological reasonableness, drawing on the concept of economic reasonableness from interest-rate modelling. Different modellers might have their own idea about what constitutes a biologically reasonable model, but Sleeman, Chung and Counsell (2016), offered the following examples of what might be considered biologically unreasonable:

Period mortality tables have historically exhibited increasing rates of mortality with age at higher ages. A forecasting model that gives rise to the possibility of period mortality tables that have mortality rates falling with age *might* be considered biologically unreasonable.

Short-term mean reversion might be considered to be biologically reasonable due to annual environmental variation. Long-run mean reversion around a deterministic trend *might*, on the other hand, be considered biologically un-reasonable. In the long term, mortality improvements will, amongst other reasons, be the result of medical advances, such as a cure for cancer. It is very difficult to predict when such advances will happen or what the impact of a new treatment might be.

Ease of Implementation

A good model according to Dowd et al. (2012), should be the one that requires less computing time in that the model can easily be programmed using the available software. All the stochastic models discussed requires some programming in that codes that run should be devised, a good model therefore is the one that is easy to program. If a model will require excessive amounts of computing time, then it should only be used if the model yields an acceptable goodness of fit.

Dowd et al. (2010), further posited that, there is little point in having a great model if it requires excessive amounts of computing time to calculate important quantities of interest. If this happens, then a compromise needs to be reached, ideally without sacrificing too much in terms of statistical goodness of fit.

Parsimony

Models that are excessively parameterized should be avoided. This is done by the use of the Bayes Information Criterion (BIC) in order to ensure that parameters in the model are only included if the improvement in the fit is significant. Therefore, the less the number of parameters the better the model so long as the model has an acceptable goodness of fit. Each of the models described has a large number of parameters, therefore all are non-parsimonious. All the same, some models are parsimonious in that they have fewer effective parameters to estimate.

Transparency

Except for the P-splines, all the model's results are straight forward to analyse and thus deemed to be transparent. The P-splines model is less transparent because its output is smooth surface fitted to historical data and then projected.

Plausibility of forecasts

The plausibility of forecasts is again a rather subjective issue, discussed by (Cairns et al., 2011). In general, one cannot normally make a definitive statement that a set of forecasts look reasonable. However, Cairns et al. (2011), do provide examples of models that provide a statistically good fit to the historical data, but then produce entirely implausible projections of mortality. Examples, of implausible forecasts include a sudden and dramatic change in the central trend in mortality rates at certain ages; and prediction intervals for mortality that are either extremely wide or extremely narrow.

Sample paths and prediction intervals

According to Dowd et al. (2010), except for P-splines models, most of the models generate sample paths. This means that an assessment of the uncertainty in future mortality-linked cash flows and pricing of the cash flows is allowed.

Parameter uncertainty

The parameters fitted and projected will often be subject to estimation errors because normally, we will have limited data to estimate the parameters. With this in mind, it is wise to include parameter uncertainty into the programming so that we can be in a position to know the impact of the estimation errors. Therefore, in their study Cairns et al. (2011), CMI working paper 15(2005) demonstrated that parameter uncertainty forms a significant element of the uncertainty in the fitting and forecasting of the future mortality. Any model that does not allow for parameter uncertainty is in danger of significantly underestimating uncertainty in its forecasts.

An additional criterion is that the model is that it should be applicable for a full age range. The annuities providers and pension funds would want to model the mortality rates and their dependencies for the whole portfolio consistently; therefore, the model should be applicable

for the entire age range. Some authors have recently sought to identify the similarities amongst stochastic mortality models. For instance, Steptoe et al. (2015) describe an Age-Period-Cohort model structure which encompasses the vast majority of stochastic mortality models.

Dowd et al. (2010), shows that many conventional mortality models can be expressed in the standard terminology of generalized linear or non-linear models. In the previous research, the models are used to fit historical data. The resulting estimates of the time-varying parameter is then modelled and forecast as stochastic time series using standard Box-Jenkins methods. From the forecast of the general level of mortality, the age-specific rates are derived using the estimated age-specific rates are derived using the estimated age effects. Li and Hare (2017), resorted to Poisson log-bilinear regression model to build projected life tables.

Among the discussed models, the Lee and Carter model has been widely discussed and used to model the mortality rates and thus quantify the longevity risks involved. For instance, Andreozzi et al. (2011), reviewed the Lee-Carter model and provided recommendations for forecasters. Lee Carter has been found suitable for actuarial applications for several reasons including, the fact that the model has relatively few parameters that are easy to interpret.

In addition, future mortality trends can easily be generated using the stochastic components of the model; hence the actuaries are in a position of quantifying the unanticipated mortality improvements using the relevant risks measures. Therefore, in this paper, we have used the Lee-Carter model to forecast the mortality rates and show that indeed the life expectancy has been increasing with time and it is expected to increase in time hence longevity risk.

2.8 Pension

Pensions, in a broad sense, are regular payments given to retired workers. At retirement, salaries are no more paid hence a decline or a complete cut-off of income. To sustain a living after retirement for employees, most employers, including government-run a pension scheme. This pension scheme is meant to support employees who go on retirement for various reasons. Employees and employers make regular contributions to the scheme during their years of service and these contributions are invested. It is obvious that pensions are necessary as in many cases it becomes the only source of livelihood for elderly people.

2.8.1 Retirement Benefit approaches

The different types of benefit structures are normally classified as follows:

- Defined Benefit (DB) approach
- Defined contribution (DC) approach

2.8.1.1 Defined Benefit (DB) approach

Under a defined benefit scheme, a retiree has a guaranteed benefit set by a formula specified on the onset. The retirement benefit is independent of market performance, employees are protected from the ups and downs of the market, the benefit is paid for as long as the retiree lives and the contributions to a defined benefit pension scheme are made by both the employer and the employee. According to Davies (1993), a traditional pension plan that defines a benefit for an employee upon that employee's retirement is a DB approach.

2.8.1.2 Defined contribution (DC) approach

Under the DC approach, contributions made by the worker and/or the employer are paid into an individual account for each member. The contributions are invested and the returns on the investment (which may be positive or negative) are credited to the individual's account.

According to Halley and Yanes (1990), on retirement, the balance in the member's account is used to provide retirement benefits. In particular, the balance in the account is paid out in full (i.e. a lump sum) and used to purchase an annuity or converted to an annual pension. So, for this arrangement, the contribution is known (defined) but the benefit is unknown. The benefits one receives at retirement is based on contributions made over the years and the investment income generated on those contributions (Bodie and Merton,1988).

There is no guaranteed retirement benefit under a defined contribution plan and the benefit is paid for as long as the money lasts. The employee bears the ultimate risk of investment performance and therefore makes pension amount unpredictable (Halley and Yanes (1990).

2.9 Ghanaian Pension Industry

The pensions industry in Ghana is regulated by the National Pensions Regulation Authority (NPRA) through the National Pensions Act. There are a few pension providers of which the Social Security and Insurance Trust (SSNIT) is the largest. For many years, Ghana operated a pension scheme known as CAP 30 which was created in 1950 for all public servants. The name CAP 30 was coined from chapter 30 of the pension ordinance of 1946. CAP 30 is a defined benefit scheme which gives members the option to choose between a lump sum payment on retirement or a monthly pension until death.

To qualify for a pension under the CAP 30 scheme, one must serve continuously for 10 years in the public service. Upon retirement, a member gets 80% of his final salary as pension. The CAP 30 was a non -contributory scheme, so members made no contributions to the scheme. It was entirely funded by the government.

In 1961, a compulsory savings Act (Act 70) was instituted to encourage National savings in Ghana and provide social security on a national scale. This was later replaced by the Social Security Act of 1965 (Act 279) which covered all establishment with five or more

employees except those already covered by the CAP 30 scheme. The benefits under the scheme varied from 50% to 80%.

In 2008, Act 766 was passed and was implemented in 2010. The Act establishes a three-tier Pension scheme. Tiers one and two are mandatory for formal sector workers and tier three was optional. However, the CAP 30 scheme is still in force but limited to a few sections of public servants such as the security agencies. Employers and employees contribute to the scheme. Employees make a mandatory contribution of 5.5% of salary whiles the Employers contribute 13% of the employee's salary.

The tier one is a defined benefit scheme which is mandatory for all formal sector workers both in the public and private sectors. The benefits depend on the average of the best three years' salary of the member. The tier one scheme is managed by the Social Security and National Insurance Trust (SSNIT) and the ranges from 37.5% to 50% of the member's pensionable salary.

Tier two is a mandatory defined contribution plan being managed privately by a chosen pension trustee. Upon retirement, a member is paid a lump sum which is the contributions made and investment returns for the entire period.

Tier three is a voluntary occupational fund. The funds in tier three can also be accessed after ten years of contribution and can also be used as collateral for mortgages. Tier two is also for workers in the informal sector. Contributions by both employers and employees are exempted from tax according to the National Pension Regulatory Authority (NPRA).

2.9.1 Pensionable Age

According to the Antolin (2007), Pensionable age is defined here as the age at which people can first qualify for full pension benefit without actuarial deduction for early retirement. Normal pensionable age in most countries are clearly set out in legislation. However, it may

be possible to retire before the normal age without an actuarial reduction in pension benefits (to react to the longer duration of benefit payment).

2.9.2 Benefits under SSNIT Pension Scheme

According to SSNIT, benefits are paid to members of the scheme when they qualify. There are three main contingencies under which benefits can be paid.

In Act 766, these categories are listed below:

Old Age Pension: This is a monthly payment made to retired members of the scheme. Members who retire at the normal pensionable age (age 60) and have made contributions of at least 180 months qualify for a full pension. Members who retire earlier than the normal pensionable age but have made contributions to the scheme for at least 180 months qualify for a reduced pension.

Invalidity Pension: Members who for one reason or the other are incapable of working for a living and have contributed 12 months within the last 36 months before the unfortunate incidence. The member must provide a medical certificate to prove he or she is unable to be gainfully employed due to a disability (physical or mental).

Survivor's Lump Sum Benefit- This is a lump sum paid to the beneficiary of a member of the scheme if the member dies in service or dies after retirement but before the age of 75. If a pensioner dies after the age of 75, nothing is paid to the beneficiary.

Other Benefits: With the three-tier scheme, members would have access to multiple retirement income for members.

2.9.3 Old Age or Retirement Pension

Qualifying Conditions

Full Pension: A member of the SSNIT pension scheme qualifies for a full pension if he or she is at least 60 years old and have made contributions to the scheme for at least 15 years.

Reduced Pension: Members who are between 55 and 59 years old and have contributed to the scheme for at least 15 years can apply for a reduced pension.

Earned Pension Right - Earned pension right is determined by the number of months of contribution and it ranges between 37.5% and 60%. A 15 years' service period guarantees a 50% pension rights and each additional month of contribution earns an additional 0.125%.

2.10 Conclusion

From the literature, it can be concluded that the studies conducted in Ghana on pensions mostly focus on the pension scheme reforms, governance and policy issues without considering the ever-soaring longevity risk that the funds are exposed to. The extent of underestimation of lifespans in individual retirement planning and policy settings in Ghana has not been quantified.

Moreover, existing studies on longevity risk forecasting and modelling of pension schemes and insurance products focus on developed countries. Very few studies have been made on the longevity risk of the Ghanaian pension industry. Despite indications that longevity risk exists, in Ghana and more generally, and some analysis within annuity or insurance settings, the extent of longevity risk in policy and by individuals remains unexplored.

Thus, while evidence accumulates to suggest an existent of longevity risk in both policy and individual settings, existing literature leaves a research gap on the significance of longevity risk in pension fund management.

CHAPTER THREE

METHODOLOGY

3.0 Introduction

In this chapter, we consider the data used for the study and also discuss the models used. It provides a description of the various methods and procedures that were used in addressing the main objective of the study: Forecasting mortality rates and modelling longevity risk of SSNIT pensioners.

3.1 Data Description

One of the sources of data used in modelling longevity risk is from the mortality of individual or aggregate pension plans and annuity providers such as insurance companies. Secondary data was obtained from Social Security and National Insurance Trust (SSNIT). SSNIT is the leading pension provider in Ghana with investment across various sectors of the economy. The data obtained was representative of Ghanaian Pensioners. It contains 26 years' times series data on SSNIT pensioners from 1991 to 2017. The data included the date of births, date of entry(pension), date of retirement, date of death for only males.

For this study, early retirements were ignored hence analysis was carried out for pensioners who retired at the mandatory retirement age of 60. Disability retirement and ill health retirement were also ignored. The data was organised such that the mortality pattern for each cohort, being a group of people retiring at age 60 in a particular year was studied separately.

3.2 Assumption

To model longevity risk, plethora of techniques have been suggested, including securitization of longevity risk and valuation methodology entailing the building of mortality indices. According to Lee and Carter (1992), modelling longevity risk involves a mortality index, heterogeneity and inter-age dependence and smoothing and closing tables.

Cairns et al., (2008b) summarized using specific criterion of the various models that were proposed. Prospective life tables provide a view of the future evolution of the mortality rates. In the past decades, longevity improvement has taken place and therefore using the standard life tables will lead to restrictions and underestimation of the real scenario of future mortality(Steptoe, Deaton, & Stone, 2015). This has the tendency of distorting annuity pricing and reserving.

This section describes the mortality assumptions, heterogeneity and inter-age dependence and smoothing and closing tables.

3.2.1 Mortality Assumption

Mortality Assumptions are projections of the expected death rates used to estimate pension obligations and price annuities. Mortality assumptions are based on the mortality tables. In most countries, the insurance and retirement benefits regulator provide a guideline on the mortality rates and assumptions to be used since the assumptions are crucial when it comes to pricing and reserving of annuities.

In the estimation of life expectancy at birth or retirement age, one of the critical factors considered is the mortality assumption. The life expectancy calculated will then be used to determine the long-term obligations of the pension fund and the annuity providers.

In the event the mortality assumptions are low, the long-term liability of the pension fund and the insurance company will be overestimated. On the other hand, if the assumptions are too high, the life expectancy of the pension plan will be underestimated and consequently underestimate the obligations of the pension plan and annuity providers.

3.2.2 Heterogeneity and inter-age dependence

For a given population, the level of heterogeneity differs from any other population. Heterogeneity is as a result of several observable factors, for example, gender, age, occupation and physiological factors or due to features of the living environment such as climate, population and nutritional standards. Pensioners or policyholders that are of higher socioeconomic status (assessed by occupation, income or education) have higher life expectancy or tend to experience lower rates of mortality. However, significant difference also exists within the same socio-economic status since generally females experience lower mortality rates compared to males. Longevity patterns and improvements therefore differ from one company to another.

3.2.3 Smoothing and closing tables

Age profiles of empirical annual mortality rates are inconsistent at high ages. Therefore, actuaries mostly close the mortality tables, i.e. extrapolate the shape of the survival functions at high ages from some exogenous assumptions. In the past, mortality after age 100 was not emphasized since it had a minimal impact on residual life expectations (and so annuities) for pensioners. With the recent longevity improvements, this is no longer the case, and it becomes crucial to have a better view on mortality and longevity risk for high ages since mortality is now improving for those ages.

3.3 Basic Building Blocks

3.3.1 Initial Rate of Mortality

The initial rate of mortality q measures the probability of death over the next year of age or, more generally, over the next rate interval. So the q -type rate applies to the age at the start of the interval.

The rate of mortality q_x is the probability of death over the next year of age for a person aged x last birthday.

Where;

$$q_x = \frac{d_x}{l_x} \tag{3.1}$$

d_x is the number of deaths over the next year.

l_x is the number of people alive at the start of the year.

3.3.2 Central Rate of Mortality

m_x is the probability of dying between exact ages x and $x+1$ per person-year lived between exact ages x and $x+1$. We define, T_x the total number of people lived beyond age x and L_x as the average number of lives in the interval $(x, x+1)$.

$$\mathrm{L_x} = \int_0^1 l_{x+t}\, dt \tag{3.2}$$

$$\mathrm{L_x} = \int_x^{x+1} l_y\, dy$$

$$\mathrm{L_x} = \int_x^{\infty} l_y\, dy - \int_{x+1}^{\infty} l_y\, dy$$

$$= T_x - T_{x+1}$$

The central death rate at age x;

$$\mathrm{m_x} = \frac{\mathrm{d_x}}{\mathrm{L_x}} \tag{3.3}$$

$$= \frac{\int_0^1 l_{x+t}\, \mu_{x+t} dt}{\int_0^1 l_{x+t}\, dt}$$

$$= \int_0^1 \mu_{x+t}\, dt$$

$$\cong \mu_{x+1/2} \tag{3.4}$$

3.3.3 Instantaneous Force of Mortality

This is the instantaneous death rate at exact time t for individuals aged $x + t$ at time t. μ_x is the instantaneous rate of mortality. This is the continuous equivalent of the discrete quantity q_x .

$$\mu_x = \lim_{h \to 0^+} \frac{1}{h} P[T_x \leq x + h \,|T_x > x] \tag{3.5}$$

The probability $P[T_x \leq x + h \,|T_x > x]$ is $F_X(h) = {}_hq_x$

$$\mu_x = \lim_{h \to 0^+} F_X(h)$$

$$\mu_x = \frac{f_X(x)}{1 - F_X(x)} \tag{3.6}$$

where $f_X(x)$ represent the probability density function (pdf) and $F_X(x)$ is the cumulative distribution function (CDF) of number of deaths.

The small h, we can ignore the limit and write:

${}_hq_x = h.\mu_x$ for small h

3.3.4 Expected Future Lifetime

The expected value of $T(x)$, denoted by $\dot{\boldsymbol{e}}_{\boldsymbol{x}}$, is called *the* ***complete-expectation-of-life.*** By definition and integration by parts, we have

$$\dot{\boldsymbol{e}}_{\boldsymbol{x}} = E\,[Tx] = \int_0^\infty t\ {}_tp_x\, \mu_{x+t}\, dt \tag{3.7}$$

where tp_x , the probability that an individual alive at age $x$will survive to age $x+t$, x is the age current age, t is number of years and μ_{x+t} is the instantaneous rate of mortality.

$$\dot{e}_x = \int_0^\infty t \; {}_t p_x \, \mu_{x+t} \, dt$$

$$= \int_0^\infty t(-\frac{d}{dt} \; {}_t p_x) \, dt$$

$$= -[\, t \; {}_t p_x \,]_0^\infty + \int_0^\infty tP_x . dt$$

$$= \int_0^\infty tP_x . dt \qquad (3.8)$$

Curtate Expectation of Life, e_x

The curtate future lifetime random variable is defined as the integer part of future lifetime, and is denoted by K_x for a life aged x. If we let $\lfloor \; \rfloor$ denote the floor function, we have

$$K_x = \lfloor T_x \rfloor \qquad (3.9)$$

where the $\lfloor \; \rfloor$ denote the integer part

The curtate future lifetime random variable K_x is defined as the integer part of future lifetime for a life aged x.

We can find the probability function of K_x by noting that for $k = 0, 1, 2, \ldots, K_x = k$ if and only if (x) dies between the ages of $x + k$ and $x + k + 1$. Thus for $k = 0, 1, 2, \ldots$

$$Pr[Kx = k] = Pr[k \leq Tx < k + 1]$$

$$= k \,|qx$$

$$= {}_k p_x - {}_{k+1} p_x$$

$$= {}_k p_x - {}_k p_x \times p_{x+k}$$

$$= {}_kp_x \, q_{x+k}. \tag{3.10}$$

The expected value of K_x is denoted by e_x, so that $e_x = E[K_x]$, and is referred to as the curtate expectation of life. So

$$E[Kx] = e_x$$

$$= \sum_{k=0}^{\infty} kPr[K_x = k]$$

$$= \sum_{k=0}^{\infty} k({}_kP_x - {}_{k+1}P_x)$$

$$= ({}_1p_x - {}_2p_x) + 2({}_2p_x - {}_3px) + 3({}_3p_x - {}_4p_x) + \cdots$$

$$e_x = E\lfloor T_x \rfloor$$

$$= \sum_{k=1}^{\infty} {}_kP_x \tag{3.11}$$

3.3.5 Age-Specific Death Rates

Age-specific death rate is the total number of deaths to residents of a specified age or age group in a specified geographical area divided by the population of the same age or age group in the same geographical area (for a specified time period, usually a calendar year) and multiplied by 100, 000.

$$= \frac{Total\ Deaths\ in\ specified\ age\ group}{Total\ number\ of\ persons\ \ in\ that\ age\ group} \times 100{,}000$$

3.3.6 Stochastic Modelling for Mortality

From equation 3.2, we present definitions and notations regarding mortality models. According to Waegenaere, Melenberg and Stevens (2010), the one-year death probability, is defined as $q_{x,t}^{(a)}$.This represents the probability that an individual belonging to the year

group a, aged x in year t will not survive to age $x+1$ and the probability that the individual survives another year to age $x + 1$ is given by

$$p_{x,t}^{(a)} + q_{x,t}^{(a)} = 1 \qquad (3.12)$$

The total number of deaths occurring in each year for each cohort was obtained from the data and the proportion of deaths in the cohort was obtained using the relation. From (3.12),

$$q_{x,t} = \frac{d_{x,t}}{l_{x,t}} \qquad (3.13)$$

where $q_{x,t}$ = probability that a life aged x in year t dies before attaining age $x+1$

$d_{x,t}$ = the number of people aged x and dies in year t,

$l_{x,t}$= the number of people aged x in year t.

3.4 The Lee and Carter Model (1992)

Lee and Carter initially developed their approach specifically for U.S.A mortality data. However, the method has become the leading statistical model of mortality (forecasting) in the demographic literature.

Lee and Carter (1992), suggested a log-bilinear form of the force of mortality $\mu_{x,t}$, as follows:

$$\mu_{x,t} = \ln(m_{x,t}) = \alpha_x + \beta_x \kappa_t + \varepsilon_{x,t} \qquad (3.14)$$

$$x = 1, \ldots\ldots, \omega$$

$$t = 1, \ldots\ldots, n$$

In the Lee-Carter model, α_x is an age-dependent term independent of time, κ_t is the mortality index factor which depends on the calendar year, and β_x is the age-dependent term

which measures response speed and age in the mortality rate of change in the mortality index factor.

Interpretation of parameters

$\mu_{x,t}$ Denotes the central mortality rate for age x at time t

$m_{x,t}$ Describes the logarithmically transformed age-specific central rate of death

α_x The average of $m_{x,t}$ over time t, describing the general pattern of mortality at different ages

κ_t Represents a time-trend index of general mortality model levels, describing the general level of mortality at different times. It captures the most important trend in death rates at all ages. Since mortality is a decreasing function, we can expect this trend to decrease.

β_x Deviations from the age when $\kappa_t$varies. β_x is an age-specific constant which describes the relative speed of mortality changes at each age, when κ_t is changing. The model allows for both positive and negative values of β_x . A negative value of β_x shows us that the mortality rate for a specific age is rising with increasing time.

However, in practice, this *usually* does not matter in the long run (Lee and Carter, 1992). When the model is adjusted over a period that is long enough, β_x *mostly* has the same characteristics (Lee and Miller, 2001), with some exceptions, for instance for some European and Central countries (Scherp, 2007)

$\varepsilon_{x,t}$ Is a zero-mean Gaussian error $N(0, \sigma^2)$. The error term, including systematic as well as purely random deviations.

The coefficient α_x is age-specific constants that describe the general shape of the age mortality profile while the index κ_t serves to capture the main temporal level of mortality. Since the parameterization in equation 3.14 is invariant with respect to the transformations:

$$\{\alpha_x, \beta_x, \kappa_t\} \rightarrow \{\alpha_x, \beta_x / c\,, c\kappa_t\}$$

$$\{\alpha_x, \beta_x, \kappa_t\} \rightarrow \{\alpha_x - c\beta_x, \beta_x, \kappa_t + c\} \qquad (3.15)$$

for any constant c , then to ensure identifiability of equation 3.15, that is obtained, unique solutions to the model the parameters β_x and κ_t should satisfy the constraints:

$$\sum_{x=1}^{\omega} \beta_x = 1$$

and

$$\sum_{t=1}^{n} \kappa_t = 0 \qquad (3.16)$$

The constraint $\sum_{t=1}^{n} \kappa_t = 0$ implies that by summing over the years t the estimates of parameters α_x are given by the averages of the force of mortality over the time period. That is

$$\hat{\alpha}_x = \frac{1}{n}\sum_{t=1}^{n} \mu_{x,t} \qquad (3.17)$$

where $\hat{\alpha}_x$is the average pattern of mortality at age x.

An estimate of κ_t is obtained by summing both sides of equation 3.16 over the ages and using $\sum_{x=1}^{\omega} \beta_x = 1$ to obtain

$$\kappa_t = \sum_x (\ln(m_{x,t}) - \hat{\alpha}_x). \qquad (3.18)$$

An estimate for $\beta_x$is obtained by differentiating both sides of equation 3.18 with respect to time t to obtain

$$\hat{\beta}_x = (\partial \ln(m_{x,t}/\partial t)/\partial \hat{\kappa}/\partial t \,. \qquad (3.19)$$

Then the parameter β_x captures the relative density of the logarithm of the central death rates to change in the mortality index κ_t. The function moderates the time-dependent element $\kappa_t$by age.

Lee and Carter (1992) proposed the method of singular value decomposition (SVD) in model fitting.

3.5. Estimation Approaches

3.5.1. The Singular Value Decomposition Approach

Lee and Carter used Single Value Decomposition to estimate the parameters of the equation $\ln(m_{x,t}) = \alpha_x + \beta_x\kappa_t + \varepsilon_{x,t}$ in their first paper. First the parameter vector α_x is computed as the average overtime of the logarithm of the central death. That is:

$$\hat{\alpha}_x = \frac{1}{n}\sum_{t=1}^{n} \mu_{x,t}$$

The Singular Value Decomposition is applied to matrix $y = \ln(m) - \hat{\alpha}_x$. To obtain β_x and κ_t , singular value decomposition is applied to the matrix $Y_{x,t} = \ln(m_{x,t}) - \hat{\alpha}_x$.

3.5.1.1 Theorem of Low-Rank Approximation

Low-rank approximation problem involves the approximation of a matrix D with another matrix $\hat{\mathrm{D}}$, said truncated which has a specific rank r. If now the approximation is by minimizing Frobenius norm of the difference between D and $\hat{\mathrm{D}}$ under the constraint rank $\hat{\mathrm{D}} \leq r$, that is $\boldsymbol{Minimize\ over}\ \hat{\mathrm{D}} \parallel \mathrm{D} - \hat{\mathrm{D}} \parallel_{\mathrm{F}}\ \boldsymbol{subject\ to\ rank}\ (\hat{\mathrm{D}}) \leq r$, we obtain the solution by Singular Value Decomposition of the data matrix to obtain the matrix approximation lemma or Eckart-Young Mirsky (1936).

$$A = U_n \Sigma_n \mathrm{V}_n^T \qquad (3.20)$$

Where $U_n$and V_n^T are orthogonal matrix, and Σ_n is a diagonal matrix with entries$(\sigma_1, \sigma_2, \ldots \sigma_n)$ Such that $(\sigma_n \leq \sigma_{n-1} \leq \cdots \leq \sigma_1)$

The best approximation for A is given by

$$A^k = \sum_{i=1}^{k} u_i \sigma_i v_i \qquad (3.21)$$

To prove that A^k is indeed the best approximation that is $\| A - A^k \|$ is minimum.

Proof by Contradiction:

Let us suppose $\exists B$ such that $\| A - B \|_2^2 < \| A - A^k \|_2^2 = \sigma_{k+1}^2$ rank $(B) \leq k$ (Assuming in Low-Rank Approximation through $dim(null(B) + rank(B) = N \rightarrow dim(null\,(B)) \geq n - k$.

The null space of the matrix(B),

$null(B) = \{x: Bx = 0\}$ is the set of vector that B maps to zero.

Let $\omega \in null(B)$

$then \| (A - B)\omega \|_2 = \| A\omega \|_2 < \sigma_{k+1}$

We know that $\exists (k+1)$dimension space $(v_1, v_2, \ldots . v_n)$ such that $V \in (v_1, v_2, \ldots . v_n)$ and

$\| AV \|_2 \geq \sigma_{k+1}$ since $n - k + k + 1 > n$.

Therefore, by contradiction, we get that A^k is the best approximation.

Let, $$D = U\Sigma V^T \qquad (3.22)$$

be the singular value decomposition of D and partition U,

where

U -is an $m \times m$ orthogonal matrix

Σ -is an $m \times n$ matrix with non-negative numbers on the diagonal.

V^T -is the conjugate transpose of the $n \times n$ orthogonal matrix V

Writing in matrix, we have

$$U = \begin{bmatrix} u_{1,1} & \cdots & u_{1,m} \\ \vdots & \ddots & \vdots \\ u_{m,1} & \cdots & u_{m,m} \end{bmatrix} \Sigma = \begin{bmatrix} \sigma_{1,1} & \cdots & \sigma_{1,n} \\ \vdots & \ddots & \vdots \\ \sigma_{m,1} & \cdots & \sigma_{m,n} \end{bmatrix} V^T = \begin{bmatrix} v_{1,1} & \cdots & v_{1,n} \\ \vdots & \ddots & \vdots \\ v_{n,1} & \cdots & v_{n,n} \end{bmatrix}$$

Such factorization is called the singular value decomposition of D (Skoufranis,2010). The diagonal entries $\sigma_{i,i}, i = 1,2, \ldots\ldots, m$ of Σ are the singular values of D. These singular values are the square roots of the eigenvalues used to obtain matrix U, and listed in descending order, and D uniquely determines the diagonal matrix Σ.

The singular value decomposition Method factorization results in real or complex orthogonal matrix U and V^T where $U \times U^T = V \times V^T = I$, where I is the identity matrix.

$$U = [U_1, U_2],\ \Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \text{ and } V = [V_1, V_2],$$

where Σ is an $r \times r$, U is $m \times r$ and V_1 is $n \times r$. Then the rank-r matrix obtained from the truncated singular value decomposition is:

$$\widehat{D}^* = U_1 \Sigma_1 V_1$$

Is such that

$$\| D - \widehat{D}^* \|_F = \min_{rank(\widehat{D}) \leq r} \| D - \widehat{D} \|_F = \sqrt{\sigma_{r+1}^2 + \ldots\ldots + \sigma_m^2} \qquad (3.23)$$

The minimized $\widehat{D}^*$ is unique if and only if $\sigma_{r+1} \neq \sigma_r$

3.5.2 Maximum Likelihood Estimation

Enchev, Kleinow and Cairns (2017), proposed using Maximum Likelihood Estimation to find the parameters in the Lee and Carter model (3.14). We use the Poisson approximation of deaths as follows. Based on the Poisson approximation of the number of deaths $D_{x,t}$:

$$D_{x,t} \approx Poisson\left(E_{x,t}m_{x,t}\right) \tag{3.24}$$

in which $m_{x,t} = \exp(\alpha_x + \beta_x\kappa_t)$ and $E_{x,t}$ represents the exposures to the risk of death (in other words the number of years from which $D_{x,t}$ occurs) and where the parameters are subjected constraints:

$$\sum_{x=1}^{\omega} \beta_x = 1$$

and

$$\sum_{t=1}^{n} \kappa_t = 0$$

The estimation of the parameters a_x, β_x and κ_t in equation 3.24 takes place using the maximum likelihood method, that is maximizing the log-likelihood of model given by:

$$L(\alpha, \beta, \kappa, D) = \log \prod_{x,t} f(\ D_{x,t}; \alpha, \beta, \kappa) \tag{3.25}$$

$$= \sum_{x,t} [\ D_{x,t}\left(\ln(E_{x,t}m_{x,t}\right) - E_{x,t}\exp(\alpha_x + \beta_x\kappa_t) - \ln(D_{x,t}!)\ ]$$

Where

$$\alpha = (\alpha_1, \alpha_2, \dots\dots, \alpha_m)$$

$$\beta = (\beta_1, \beta_2, \dots\dots, \beta_m)$$

$$\kappa = (\kappa_1, \kappa_2, \dots\dots, \kappa_n)$$

where m is the total number of data points for each calendar years, and n is the total number of calendar years.

The maximum likelihood estimation allows non-additive heteroscedastic Mircea et al. (2014) and avoids the assumption of errors with constant variance present in the SVD approach (Lee and Carter 1992). The MLE formulation of the LC model is often referred to as the Poisson log-bilinear model from the paper Danesi et al. (2015), which provides an algorithm to minimize the equation. The MLE was used to model death rates from Belgium, UK and Wales (Renshaw and Haberman 2003) and the Nordic countries (Koissi et al., 2006b).

3.5.3. Weighted Least Squares

Wilmoth (1993) proposed fitting the Lee-Carter model using weighted least squares. Basically, we want to estimate the parameters a_x, β_x and κ_t . The estimation a_x which minimizes the sum of least squares of errors $s = \sum_{x,t} \varepsilon_{x,t}^2$ is the average of $m_{x,t}$. That is,

$$a_x = \frac{1}{n}\sum_t m_{x,t}\,. \tag{3.26}$$

Where n in the total number of calendar years.

The difference in matrix is formed as;

$Z_{x,t} = m_{x,t} - \alpha_x$ and it satisfies $\sum_t Z_{x,t} = 0$ for all x.

To achieve a unique solution, the following restrictions are used;

$\sum_t \kappa_t = 0$ and $\sum_x (\beta_x)^2 = 1$

$$Q = \sum_{x,t} (\kappa_t \beta_x - Z_{x,t})^2$$

To find the values that minimize Q we introduce the Lagrangian multiplier a and b that minimizes:

$$R = Q - a\sum \kappa_t - b\sum \beta_x^2$$

Differentiation on R with respect of x and t give us

$\frac{dR}{d\kappa_t} = 2\sum_x \beta_x(\kappa_t\beta_x - Z_{x,t}) - a$ for every t

$\frac{dR}{d\beta_x} = 2\sum_t \kappa_t(\beta_x\kappa_t - Z_{x,t}) - 2b$ for every x

If the derivatives, $\frac{dR}{d\kappa_t}, \frac{dR}{d\beta_x}$ are set equal to zero, we obtain

$$\frac{a}{2} = \kappa_t \sum_x \beta_x^2 - \sum_x \beta_x Z_{x,t}$$

If we add the sums with respect to t we get that $a = 0$. We then solve for $\beta_x \; and \; \kappa_t$ from the systems of the equation to get;

$$\kappa_t = \sum_x \beta_x Z_{x,t} \qquad (3.27)$$

and variance of κ_t is given as

$$\beta_x = \frac{\sum_t \kappa_t Z_{x,t}}{\sqrt{\sum_x (\sum_t \kappa_t Z_{x,t})^2}} \qquad (3.28)$$

The equations (3.27) and (3.28) cannot be solved explicitly and we cannot use ordinary regression. On the other hand, it is easy to iterate to reach a solution. We start with $\beta_x = 1/\sqrt{m}$ is independent of x, where m is the number of ages that are observed. We use our values of β_x and input these into (3.27), which gives us a value on, and then we use these to

update in β_x (3.28). Then we repeat this cycle. The iteration converges surprisingly rapidly. After less than ten iterations, we have a solution with high precision (Kang et al., 2014).

3.6 Forecasting

After estimating the parameters, the second stage involves finding a modified $\kappa_t^{(1)}$which adjusts the total number of deaths $\sum_x d_{x,t}$ to the estimated number of deaths as follows:

$$\sum_x d_{x,t} = \sum_x E_{x,t} \exp(\hat{\alpha}_x + \sum_i \hat{\beta}_x^{(i)} \hat{\kappa}_t^{(i)}), \qquad \forall t \tag{3.29}$$

Where $E_{x,t}$ and $d_{x,t}$ are exposure to risk and actual numbers of death at age x and time t. This step is necessary because the LC model fits the logarithm of the death rates instead of the current death rates.

Predicting mortality with LC is reduced to forecasting the index κ_t using time series approaches (Andreozzi et al., 2011). To forecast future mortality rates, Lee and Carter assumes that α_x and β_x remains constant over time and the time trend κ_t is intrinsically viewed as a stochastic process. Lee and Carter (1992) suggested the following random walk with drift to model k_t.

$$\hat{\kappa}_t = \hat{\kappa}_{t-1} + \varphi + C\varepsilon_t \tag{3.30}$$

In which φ is a constant drift term, C is a constant volatility and ε_t is a one-dimensional $i.i.d\ N(0,1)$ error.

The dynamics of κ_t are modelled by autoregressive integrated moving average (ARIMA) models.

An appropriate ARIMA (p, d, q) model for the mortality index κ_t is found by carrying out the standard Box and Jenkins methodology (identification-estimation-diagnosis). In general

an ARIMA (0,1,0) with drift $\hat{\kappa}_t = \hat{\kappa}_{t-1} + \mathrm{C} + \xi_t$ is found suitable, though other ARIMA forms provided a better fit to some data (Andreozzi et al., 2011).

After having found an appropriate ARIMA model, the variables, the mortality index can be forecasted. Let $\hat{\kappa}_{t_n+s}$ denote the s-period ahead forecast of the mortality index, then in case of the Poisson Lee-Carter model, the expected value of future death count is given by:

$$E\left[D_{x,t_n+s}\right] = E_{x,t_n+s}\,\hat{m}_{x,t_n+s} \qquad (3.31)$$

Where E_{x,t_n+s} is the future exposure and $\hat{m}_{x,t_n+s}$ forecasts of future death rates with:

$$\hat{m}_{x,t_n+s} = \exp\left(\hat{\alpha}_x + \hat{\beta}_x\,\hat{\kappa}_{x,t_n+s}\right) \qquad (3.32)$$

Using $\hat{m}_{x,t_n+s}$, we calculate life expectancies.

3.7 Cairns-Blake-Dowd (2006)

The Cairns-Blake-Dowd (2006) model is a stochastic mortality model. The CBD model is suitable for higher age mortality modelling. Cairns et al. (2006)' two-factor stochastic mortality model is formulated as

$$\log\left(\frac{q_{x,t}}{1-q_{x,t}}\right) = k_t^1 + k_t^2\left(x-\bar{x}\right) \qquad (3.33)$$

$$\log it\left(q_{x,t}\right) = k_t^1 + k_t^2\left(x-\bar{x}\right) \qquad (3.34)$$

Where

$q_{x,t}$: The probability that a person aged x will die at time t.

$\bar{x}$: Average of the age range under consideration

k_t^1, k_t^2 : Are the period mortality indexes

3.8 Holt Linear Trend Method

Holt linear trend is used for forecasting data with either or both seasonality and trend. The method is made up of a forecast equation and two smoothing equations for the level and the trend as described below:

Forecast equation: $\hat{F}_{t+h} = \hat{y}_{t+h/t} = s_t + h\mu_t$ (3.35)

Level equation: $s_t = \alpha y_t + (1-\alpha)(s_{t-1} + \mu_{t-1})$ (3.36)

Trend equation: $\mu_t = \beta(s_t - s_{t-1}) + (1-\beta)\mu_{t-1}$ (3.37)

Where estimate of the level of series at time t is denoted s_t an estimate of trend also known as slope of series at time t is μ_t. The smoothing parameter α for level where $\leq \alpha \leq 1$ and β is the smoothing parameter for trend where $0 \leq \beta \leq 1$. The time to be forecasted is represented as h. The $h - step$ forecast is equal to the last estimate of level added to h multiplied by the last trend value estimated.

3.9 Estimation of Life Expectancies

Life expectancy is a key characteristic of human longevity and development. Chaing (1970) developed a methodology for estimating life expectancies using mortalities observed at each age x; this computed the life expectancies using mortalities predicted using Lee Carter and Cairns-Blake Dowd model.

$$\hat{e}_x = \frac{T_x}{d_x \mu_x} \quad (3.38)$$

where, T_x is the total number of people lived beyond age x, d_x is the number of deaths at age x and μ_x is the estimated mortality rate from Lee Carter and Cairns-Blake Dowd model at age x

and the standard error is estimated as

$$s(e) = \sqrt{\sum_{i=0}^{\omega}(1-\mu)^2\left[e_{i+1} + \left(1 - \frac{1}{2}/n_i\right)\right]^2 s(\mu)^2} \quad (3.39)$$

where, n_i is the age interval , e_{i+1} is the life expectancy at age $x+1$ and μ_x is the estimated mortality rate from Lee-Carter and Cairns-Blake Dowd model at age x, $s(\mu)^2$ is the variance of the estimated mortality.

3.10 Models for testing Out of Sampling Validation

The out of sampling validation was tested using Root Mean Square Error (RMSE), The minimum absolute error (MAE) and Maximum Absolute Percentage Error (MAPE).

3.10.1 Root Mean Square Error (RMSE)

Root Mean Square Error (RMSE) is defined as the standard deviation of the residuals (prediction errors). The Root Mean Square Error was described in Brooks (2008) as

$$RMSE = \sqrt{\frac{1}{T-(T_1-1)}\sum_{i=T_1}^{T}\left(y_{t+s} - f_{t,s}\right)^2} \quad (3.40)$$

Where;

T is the total sample size (in-sample + out-of-sample),

T_1 is the first out-of-sample forecast observation. Thus in-sample model estimation initially runs from observation 1 to $(T_1 - 1)$, and observations T_1 to T are available for out-of-sample estimation, i.e. a total holdout sample of $T - (T_1 - 1)$.

y_{t+s} is the actual value at time t,

$f_{t,s}$ forecast value at time t.

.

3.10.2 Mean Absolute Error (MAE)

Mean absolute error (MAE) measures the average absolute forecast error, and is given in Brooks (2008) as

$$MAE = \frac{1}{T-(T_1-1)}\sum_{i=T_1}^{T}\left|y_{t+s}-f_{t,s}\right| \tag{3.41}$$

3.10.3 Maximum Absolute Percentage Error (MAPE)

The MAPE (Mean Absolute Percent Error) measures the size of the error in percentage terms.

$$\text{MAPE} = \frac{1}{T-(T_1-1)}\sum_{i=T_1}^{T}\left|\frac{y_{t+s}-f_{t,s}}{y_{t+s}}\right| \tag{3.42}$$

CHAPTER FOUR

DATA ANALYSIS AND DISCUSSION

4.1 Introduction

This part of the study presents a discussion and detailed results of the collected data that was analysed. The purpose of this chapter is to present and discuss the empirical findings of the study.

4.2 Descriptive Statistics

The descriptive statistics of pensioners of the SSNIT pension scheme was done to determine information on the number of people who retire in a year and the proportion of those people who survive within the first years. The results can be seen in Table 4.1 below

Table 4.1: Descriptive Statistics

Year	Number enrolled	number survives	Proportion
1991	1032	1019	0.9874031
1992	1065	1051	0.9868545
1993	1421	1404	0.9880366
1994	1198	1187	0.9908180
1995	1858	1846	0.9935414
1996	1516	1504	0.9920844
1997	1728	1711	0.9901620
1998	1613	1602	0.9931804
1999	2264	2243	0.9907244
2000	2331	2308	0.9901330
2001	1469	1452	0.9884275
2002	2352	2328	0.9897959
2003	1932	1917	0.9922360
2004	2383	2360	0.9903483
2005	2947	2924	0.9921955
2006	2818	2790	0.9900639
2007	3389	3360	0.9914429
2008	3845	3814	0.9919376
2009	4098	4066	0.9921913
2010	4626	4589	0.9920017
2011	6972	6952	0.9971314
2012	10274	10249	0.9975667
2013	9782	9760	0.9977510
2014	12771	12744	0.9978858
2015	10304	10274	0.9970885
2016	11719	11688	0.9973547
2017	11958	11929	0.9975748

From Table 4.1, it can be seen that the proportion of those people who survive within the first years' increases when the number of people who survive is high and vice versa. The proportion of those people who survive within the first year of retirement rises as the years' increases and 2017 recorded the highest proportion of 0.9975748. It was found that the number of people that retired on the average also increase over the years. This stands to reason that the number of people who register under SSNIT increases over the years hence leading to increments in the number of people who retired (Kwabla-King, 2017).

4.3 Mortality Rate from 1991 to 2017

Figure 4.1, displayed the mortality rate of pensioners from 1991 to 2017. The mortality rate was estimated using stochastic modelling for mortality in equation 3.9.

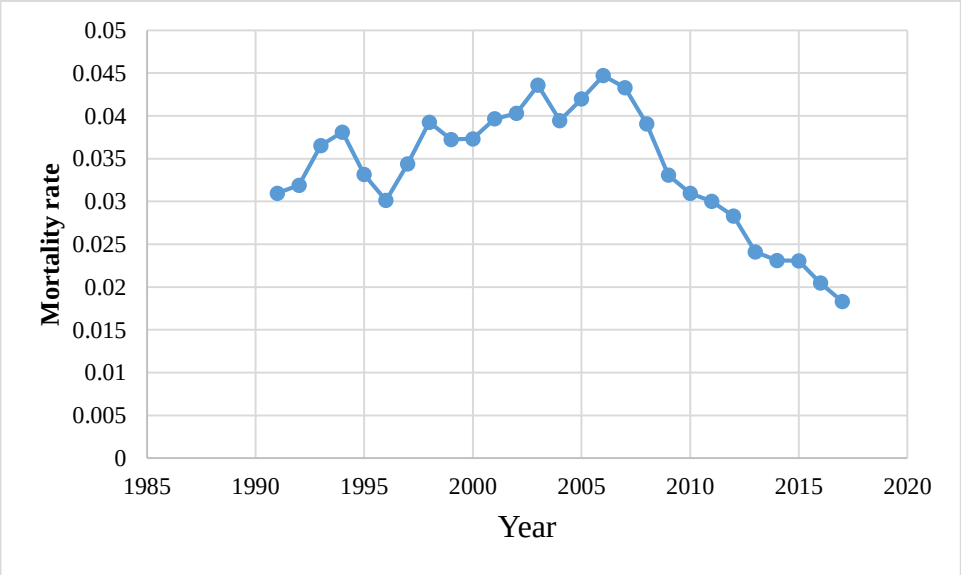

Figure 4.1: Mortality Rate from 1991 to 2017

The mortality increases when the number of deaths within the quarter is high and vice versa. There is a rise in mortality from 1991 to 2006, but from 2009 to 2017 there was a fall in mortality and 2017 recorded the least mortality rate. These results confirm Gitau (2017), who found out that mortality rate of recent years has been decreasing over time due to improvement in health facilities and healthy lifestyles of the populace

4.4 Life Expectancy CBD model

The life expectancies of the pensioners were computed also using CBD model. Table 4.2 represents the results of the life expectancies of ages 70, 75, 80, 85 and 90 over the years (1991 to 2017).

Table 4.2: CBD model Life Expectancy

Year	e70	se(70)	e75	se(e75)	e80	se(e80)	e85	se(e85)	e90	se(e90)
1991	6.126	0.145	4.890	0.182	3.075	0.200	2.454	0.182	0.023	0.165
1992	6.376	0.196	5.090	0.245	3.325	0.269	2.654	0.245	0.274	0.223
1993	7.429	0.326	5.931	0.408	4.378	0.448	3.495	0.408	1.326	0.371
1994	7.590	0.340	6.059	0.425	4.539	0.467	3.624	0.425	1.488	0.387
1995	7.924	0.485	6.326	0.606	4.873	0.665	3.890	0.606	1.822	0.551
1996	8.060	0.538	6.435	0.671	5.009	0.738	3.999	0.671	1.958	0.611
1997	8.602	0.662	6.867	0.826	5.550	0.908	4.431	0.826	2.499	0.752
1998	8.619	0.628	6.881	0.785	5.568	0.862	4.445	0.785	2.516	0.714
1999	9.012	0.838	7.194	1.046	5.961	1.150	4.759	1.046	2.909	0.952
2000	9.528	1.004	7.607	1.254	6.477	1.378	5.171	1.254	3.426	1.141
2001	9.219	0.254	7.360	0.317	6.168	0.348	4.924	0.317	3.117	0.289
2002	9.579	0.434	7.647	0.542	6.527	0.596	5.211	0.542	3.476	0.493
2003	9.625	0.463	7.684	0.578	6.573	0.635	5.248	0.578	3.522	0.526
2004	9.908	0.688	7.910	0.859	6.856	0.944	5.474	0.859	3.805	0.782
2005	10.000	0.684	7.983	0.854	6.949	0.939	5.547	0.854	3.897	0.777
2006	10.202	0.840	8.145	1.049	7.151	1.153	5.709	1.049	4.099	0.955
2007	10.573	0.398	8.441	0.497	7.522	0.547	6.005	0.497	4.470	0.453
2008	10.853	0.619	8.664	0.773	7.802	0.850	6.228	0.773	4.750	0.704
2009	10.529	0.349	8.405	0.436	7.477	0.479	5.969	0.436	4.426	0.397
2010	10.921	0.453	8.718	0.566	7.869	0.622	6.282	0.566	4.818	0.515
2011	10.942	0.935	8.736	1.167	7.891	1.283	6.300	1.167	4.840	1.062
2012	11.413	1.110	9.111	1.386	8.361	1.524	6.675	1.386	5.310	1.262
2013	11.400	1.177	9.101	1.470	8.348	1.615	6.665	1.470	5.297	1.338
2014	11.634	1.428	9.288	1.784	8.583	1.960	6.852	1.784	5.531	1.623
2015	11.800	1.524	9.420	1.903	8.749	2.091	6.984	1.903	5.697	1.732
2016	11.773	1.664	9.399	2.078	8.721	2.283	6.963	2.078	5.670	1.891
2017	12.002	1.698	9.582	2.120	8.951	2.330	7.146	2.120	5.900	1.929

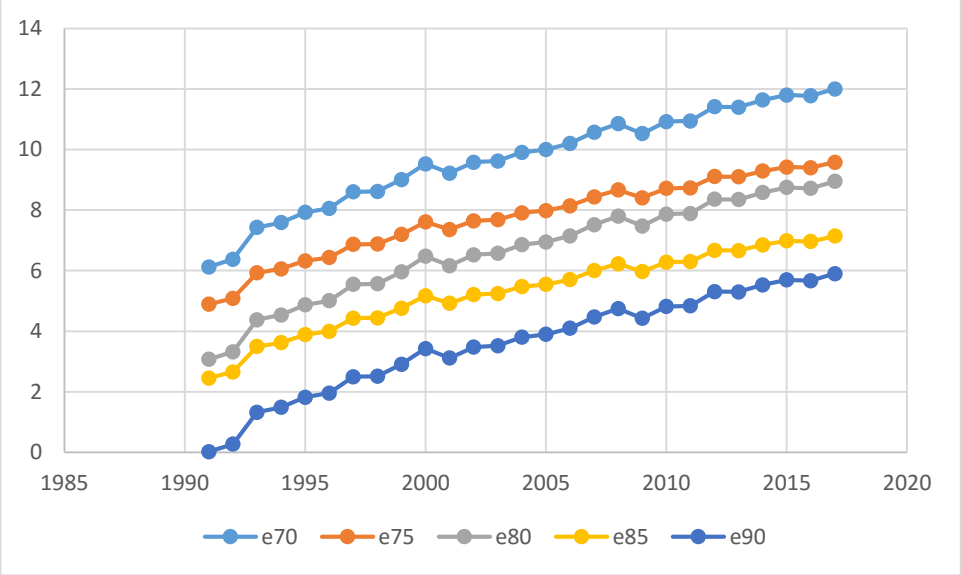

Figure 4.2: CBD model Life Expectancy

From Table 4.2, it was revealed that life expectancy at all the ages increases over the years. For example, in 1991, the life expectancy at age 70 was 6.1258 whiles in 2017, the life expectancy was 12. years. Figure 4.3 presents the two mortality index of Cairns-Blake-Dowd (CBD) model, it was found that kt(1) the mortality curve moves down over time, which means that overall mortality decreases over the years. This results also confirmed Chan and Li (2017) research in which they used mortality experiences in the UK. The other mortality curve kt(2), becomes steeper, which means mortality declines faster at younger ages than at older ages.

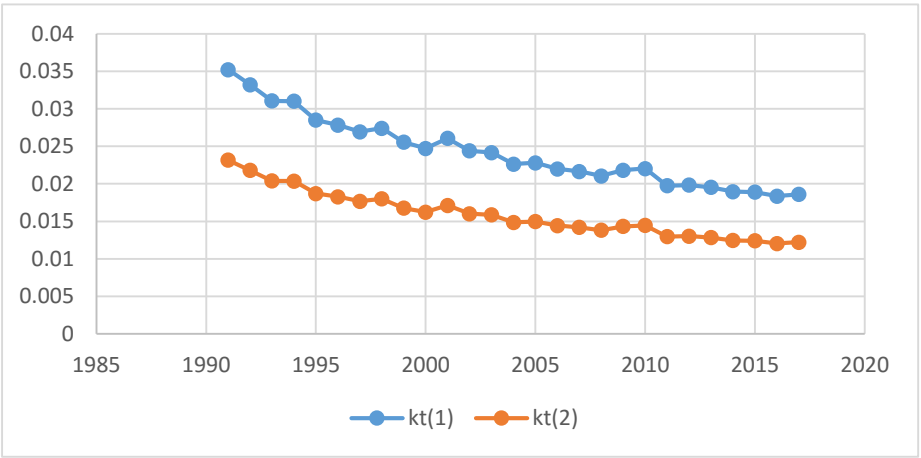

Figure 4.3: Mortality index of kt(1), kt(2)

4.5 Out of Sample Validation

A good way to test the accuracy of a model and to realistically compare its forecasting performance against actual values is to perform out-of-sample validation (Austin & Steyerberg, 2017). This is the process of withholding some of the sample data from the model identification and estimation process, then use the model to make predictions for the hold-out data to check the accuracy (Austin & Steyerberg, 2017). Based on this assertion, the study constructed an out of sample validation for Lee–Carter and CBD model using data obtained from 1991 to 2015 and forecasted for 2016.

Table 4.3: Out of Sample Validation

Year	Actual	Lee–Carter			CBD model		
		Forecast	Error	% error	Forecast	Error	% error
70	12.0040	12.1404	-0.1364	-1.1363	12.0022	0.0018	0.0150
75	9.3052	9.6195	-0.3143	-3.3772	9.5818	-0.2766	-2.9724
80	9.1017	9.2927	-0.1910	-2.0985	8.9509	0.1508	1.6568
85	6.2104	5.6029	0.6076	9.7834	6.1459	0.0646	1.0399
90	5.9321	6.1414	-0.2093	-3.5283	5.9900	-0.0579	-0.9755
MAE			0.2917			0.1103	
RMSE			0.5401			0.3321	
MAPE				3.9845%			1.3318%

The mean absolute percentage error (MAPE), Mean Absolute Error (MAE) and Root Mean Square Errors (RMSE) were computed from Lee–Carter and CBD model and results obtained were presented in Table 4.3. The mean or average of the absolute percentage errors of forecasts for the Lee-Carter and CBD model were 3.9845% and 1.3318%. The study results show that the MAPE are 3.9845% and 1.3318% this implies that, on average, the forecasting model is giving errors of 3.9845% and 1.3318%. According to Hyndnam and Koehler (2006), the smaller the MAPE, the better the forecast hence its CBD model is more accurate in forecasting than Lee–Carter.

4.6 Life Expectancy Using Lee–Carter

The life expectancies of the pensioners at age 70, 75, 80, 85 and 90 were computed using the Lee-Carter model; Table 4.4 represents the results. The estimates of parameters ax and bx for 70, 75, 80, 85, 90 for 1991 to 2007 obtained are presented in Table 4.4 whiles Figure 4.6 presents the k_t mortality index.

Table 4.4: Lee–Carter model Life Expectancy

Year	Age 70				Age 75				Age 80			
	ax	bx	e70	se(e70)	ax	bx	e75	se(e75)	ax	bx	e80	se(e80)
1991	0.5	0.023	6.412	0.535	0.5	0.029	5.119	0.241	0.5	3.075	3.360	0.241
1992	0.5	0.022	6.801	0.475	0.5	0.027	5.429	0.325	0.5	3.325	3.749	0.325
1993	0.5	0.020	7.270	0.416	0.5	0.026	5.804	0.541	0.5	4.378	4.218	0.541
1994	0.5	0.020	7.282	0.415	0.5	0.026	5.813	0.564	0.5	4.539	4.231	0.564
1995	0.5	0.019	7.924	0.350	0.5	0.023	6.326	0.804	0.5	4.873	4.873	0.804
1996	0.5	0.018	8.112	0.334	0.5	0.023	6.476	0.891	0.5	5.009	5.061	0.891
1997	0.5	0.018	8.384	0.313	0.5	0.022	6.694	1.096	0.5	5.550	5.333	1.096
1998	0.5	0.018	8.241	0.324	0.5	0.023	6.579	1.041	0.5	5.568	5.189	1.041
1999	0.5	0.017	8.843	0.281	0.5	0.021	7.059	1.388	0.5	5.961	5.791	1.388
2000	0.5	0.016	9.134	0.264	0.5	0.020	7.292	1.664	0.5	6.477	6.083	1.664
2001	0.5	0.017	8.663	0.293	0.5	0.021	6.916	0.421	0.5	6.168	5.612	0.421
2002	0.5	0.016	9.257	0.257	0.5	0.020	7.391	0.719	0.5	6.527	6.206	0.719
2003	0.5	0.016	9.343	0.252	0.5	0.020	7.459	0.767	0.5	6.573	6.292	0.767
2004	0.5	0.015	9.981	0.221	0.5	0.019	7.968	1.140	0.5	6.856	6.930	1.140
2005	0.5	0.015	9.906	0.224	0.5	0.019	7.908	1.134	0.5	6.949	6.854	1.134
2006	0.5	0.014	10.268	0.209	0.5	0.018	8.197	1.393	0.5	7.151	7.217	1.393
2007	0.5	0.014	10.437	0.202	0.5	0.018	8.332	0.660	0.5	7.522	7.385	0.660
2008	0.5	0.014	10.735	0.191	0.5	0.017	8.570	1.026	0.5	7.802	7.684	1.026
2009	0.5	0.014	10.350	0.205	0.5	0.018	8.262	0.578	0.5	7.477	7.298	0.578
2010	0.5	0.014	10.249	0.209	0.5	0.018	8.182	0.751	0.5	7.869	7.198	0.751
2011	0.5	0.013	11.443	0.168	0.5	0.016	9.135	1.549	0.5	7.891	8.392	1.549
2012	0.5	0.013	11.388	0.170	0.5	0.016	9.092	1.840	0.5	8.361	8.337	1.840
2013	0.5	0.013	11.561	0.165	0.5	0.016	9.229	1.951	0.5	8.348	8.509	1.951
2014	0.5	0.012	11.910	0.155	0.5	0.016	9.509	2.367	0.5	8.583	8.859	2.367
2015	0.5	0.012	11.960	0.154	0.5	0.016	9.548	2.525	0.5	8.749	8.908	2.525
2016	0.5	0.012	12.307	0.145	0.5	0.015	9.825	2.757	0.5	8.721	9.256	2.757
2017	0.5	0.012	12.144	0.149	0.5	0.015	9.695	2.813	0.5	8.951	9.093	2.813

Table 4.4: Lee–Carter model Life Expectancy

Year	Age 85				Age 90			
	ax	bx	e85	se(e85)	ax	bx	e90	se(e90)
1991	0.5	3.523	2.246	0.060	0.5	3.523	0.309	0.105
1992	0.5	3.912	2.494	0.169	0.5	3.912	0.698	0.214
1993	0.5	4.381	2.793	0.274	0.5	4.381	1.167	0.320
1994	0.5	4.393	2.801	0.277	0.5	4.393	1.179	0.322
1995	0.5	5.035	3.211	0.389	0.5	5.035	1.822	0.434
1996	0.5	5.223	3.330	0.416	0.5	5.223	2.009	0.462
1997	0.5	5.495	3.504	0.453	0.5	5.495	2.282	0.498
1998	0.5	5.351	3.412	0.434	0.5	5.351	2.138	0.479
1999	0.5	5.954	3.796	0.507	0.5	5.954	2.740	0.552
2000	0.5	6.245	3.982	0.537	0.5	6.245	3.032	0.583
2001	0.5	5.774	3.682	0.696	0.5	5.774	2.561	0.765
2002	0.5	6.368	4.060	0.720	0.5	6.368	3.155	0.765
2003	0.5	6.454	4.115	0.720	0.5	6.454	3.241	0.765
2004	0.5	7.092	4.522	0.720	0.5	7.092	3.879	0.765
2005	0.5	7.017	4.474	0.720	0.5	7.017	3.803	0.765
2006	0.5	7.379	4.705	0.730	0.5	7.379	4.165	0.765
2007	0.5	7.548	4.812	0.714	0.5	7.548	4.334	0.765
2008	0.5	7.846	5.003	0.725	0.5	7.846	4.633	0.765
2009	0.5	7.461	4.757	0.724	0.5	7.461	4.247	0.765
2010	0.5	7.360	4.693	0.720	0.5	7.360	4.146	0.765
2011	0.5	8.554	5.454	0.819	0.5	8.554	5.340	0.854
2012	0.5	8.499	5.419	0.809	0.5	8.499	5.286	0.854
2013	0.5	8.672	5.529	0.829	0.5	8.672	5.458	0.754
2014	0.5	9.021	5.752	0.809	0.5	9.021	5.808	0.854
2015	0.5	9.071	5.783	0.809	0.5	9.071	5.857	0.854
2016	0.5	9.418	6.005	0.799	0.5	9.418	6.204	0.754
2017	0.5	9.255	5.901	0.789	0.5	9.255	6.041	0.854

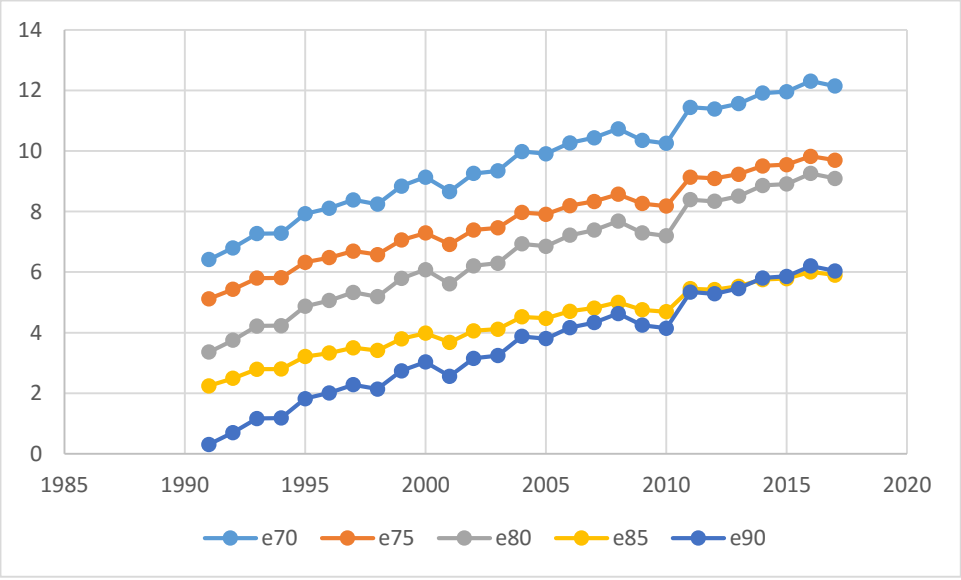

Figure 4.4: Lee–Carter model Life Expectancy

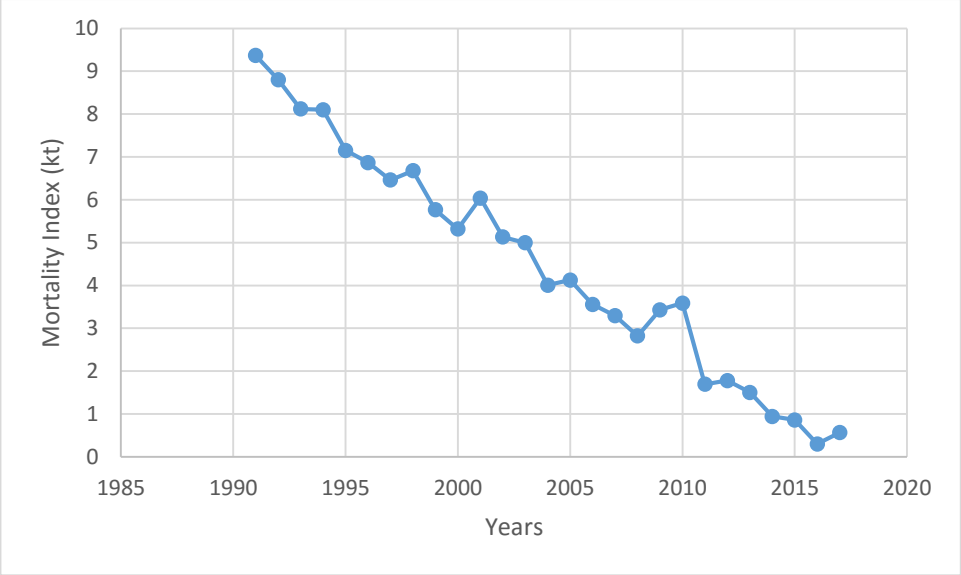

Figure 4.5: Mortality index of k_t

From Figure 4.5, it was observed that the average life expectancy increases over the years as the age increases, which is an indication of the reverse pattern in k_t. The $b_x$describes the tendency of mortality at age x to change as the general level of mortality k_t changes. This indicates that when b_x is large for some age, the death rate at a specific age varies a lot than the general level of mortality and vice versa.

4.7 Forecasted Life Expectancy

The life expectancies of ages were forecasted to determine the future behaviour of the various life expectancies using the Lee-Carter and CBD model. Table 4.5, reported the forecasted life expectancy.

Table 4.5: Forecasted Life Expectancy

Year	Lee–Carter			CBD model		
	70	80	90	70	80	90
2018	13.0313	9.9801	6.9288	12.8984	9.8471	6.7958
2019	12.9597	9.9084	6.8571	12.8309	9.7797	6.7284
2020	13.2328	10.1815	7.1302	12.6288	9.5775	6.5262
2021	13.3343	10.2831	7.2318	13.0952	10.0439	6.9927
2022	13.1049	10.0536	7.0023	14.1543	11.1030	8.0518
2023	13.9388	10.8875	7.8363	13.8328	10.7815	7.7302
2024	14.2672	11.2159	8.1646	14.3507	11.2994	8.2482
2025	14.2607	11.2094	8.1582	14.4870	11.4357	8.3845
2026	14.4541	11.4028	8.3515	14.1627	11.1114	8.0601
2027	14.7850	11.7337	8.6824	14.3663	11.3150	8.2637
2028	14.4629	11.4116	8.3603	14.4974	11.4461	8.3949
2029	14.5212	11.4700	8.4187	14.3946	11.3434	8.2921
2030	15.5866	12.5353	9.4840	14.8254	11.7741	8.7228

From Table 4.6, it was revealed that life expectancies would be increasing over the coming years. This implies that the cost of pension annuities and life insurance is expected to be higher in future as a result of people living longer than expected. Hence this will lead to longevity risk on the in-insurance companies.

4.8 Comparison of forecast errors in life expectancies by various models

The Minimum Absolute Error (MAE), Maximum Absolute Percentage Error (MAPE), Root Mean Square Error (RMSE). These model adequacies were computed to determine the best forecast model. Table 4.6 displayed the various forecast error of the Lee-Carter and CBD models.

Table 4.6 Comparison of forecast errors in life expectancies by various models

Criteria	CBD models	Lee–Carter
MAE	0.1103	0.2425
MAPE	1.3318	3.9845
RMSE	0.3321	0.5401

From 4.6, it was revealed that the MAE, MAPE and RMSE of the CBD models were less than the Lee-Carter model. Therefore, the forecast of the CBD model yields better results than the Lee-Carter models.

CHAPTER FIVE

SUMMARY OF FINDINGS, CONCLUSIONS AND RECOMMENDATIONS

5.1 Introduction

This chapter provides a summary of the findings, conclusions and recommendations of mortality forecast and longevity risk modelling of SSNIT pensioners in Ghana.

5.2 Summary

Earlier actuarial models of forecasting the trends disregarded the stochastic nature of mortality. Therefore, understanding the future mortality trends using the stochastic models is likely to be of interest to the actuary in pricing and reserving of annuities. Subsequently, demographers and actuaries explored distinct ranges of stochastic models using several stochastic approaches in the mortality prediction. Lee and Carter (1992) model was the first stochastic model to take into account increased life expectancy which has become widely used and recently CBD model several extensions and modifications have been proposed to arrest the main features of mortality intensity.

The research aimed at forecasting mortality rates and modelling the longevity risk of SSNIT pensioners. Secondary data on pensioners from 1991 to 2017 was obtained from the Social Security and National Insurance Trust (SSNIT). The largest pension provider in Ghana is SSNIT which has been mandated by the state (Act 766) to manage the first tier of the contributory three-tier scheme. SSNIT has investment across various sectors of the economy and hence the data obtained representative of Ghanaian Pensioners. For this study, early retirements were ignored; therefore, all analyses were carried out for pensioners who retire at the mandatory retirement age of 60.

There was a decrease in mortality from 2009 to 2017 whiles the life expectancy increased over the years at each age (70, 75, 80, 85, 90) years observed an increase. The mortality index of Cairns-Blake-Dowd (CBD) model, kt(1) was downwards sloping portraying that overall mortality decreases over the years and kt(2), the curve becomes steeper, signifying a faster decline in younger ages than at older ages. The study also revealed that the CBD model forecast better than Lee-carter. From the forecasted models, it was also revealed that life expectancies would be increasing over the coming years.

5.3 Conclusions

The following conclusions were drawn from the study:

There has been a decrease in mortality rate over the years and an increase in life expectancies. Stochastic forecasting of future life expectancies using Lee-Carter and CBD model yields good estimate with few errors and CBD model producing better results than the Lee-Carter. The life expectancies were forecasted to be high over the years.

5.4 Recommendations

On the basis of the aforementioned findings and conclusions, the study makes the following recommendations.

It is imperative for pension firms to enact appropriate policy interventions to be in step with the ever-soaring longevity risk they are confronted with. This entails inter alia, investing in longer-term less risky instruments that will beget the necessary returns to meet their liabilities to pensioners.

The study further recommended that the normal retirement age be adjusted to the decreasing mortality trend. This requires that the Ghanaian retirement age is harmonized with the increase in life expectancy.

Ghanaian Pension Providers such as SSNIT should use longevity swaps or other financial instruments to hedge against the risk of longevity.

Finally, pension firms need to rely on adequately predicted mortality data in their design of germane pensions and other financial products for profitability

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APPENDIX: R-CODES

```
### Lee carter Model #####
data1 <-read.csv(file.choose(),header=TRUE)

data <- mutate(data1 , age0 = as.numeric(str_extract(us$Age, "[0-9]+")),
age = ifelse(age0 >= 5, age0 + 2.5, ifelse(age0 > 0, 3, 0.5)))

le85 <- mutate(us, mx = ifelse(Age == "85-89", lx/Tx, mx))
damx <- select(us85, Year, age, mx) %>% filter(age < 90)
rates <- filter(usmx, Year <= 1987)$mx
M <- matrix(log(rates), 55, 19, byrow = TRUE)

### Plotting
damx2 <- filter(damx, Year==1933 | Year == 1987) %>%
mutate(year = factor(Year),
fit = c(exp(a + b * k[1]), exp(a + b * k[55])))
ggplot(usmx2, aes(age, mx, color=year)) + geom_point() +
geom_line(aes(age,fit,color=year)) + scale_y_log10() +
ggtitle("Lee-Carter Fits for 1933 and 1987")

ggsave("lcfig1r.png", width = 500/72, height = 400/72, dpi = 72)
rend <- data.frame(year = 1933:1987, k = k)

ggplot(trend, aes(year, k)) + geom_line() +
ggtitle("Lee-Carter k for 1933-1987")

ggsave("lcfig2r.png", width = 500/72, height = 400/72, dpi = 72)

### Age-Specific Forecasts for Lee carter model###
lc <- read.table("LeeCarter.dat", header = TRUE)
```

```
k2050 <- 33.3
z <- qnorm(0.975)
se <- sqrt(2050 - 1989) * 0.652
k2050 + c(-1, 1) * z * se

forecast <- data.frame(age = lc$age,
fit = exp(lc$a + lc$b * k2050),
low = exp(lc$a + lc$b * (k2050 - z * se)),
hi  = exp(lc$a + lc$b * (k2050 + z * se)))

ggplot(forecast, aes(age, fit)) + scale_y_log10() +
geom_ribbon(aes(ymin = low, ymax = hi), fill="#d0d0d0") +
geom_line() + ggtitle("Figure 4: Forecast for 2050 with 95% CI")
ggsave("lcfig4r.png", width = 500/72, height = 400/72, dpi = 72)

#Cross Validation
#Predicting with the model for example Total Vat
# Suppose "mod" is the fitted time series model
library(readr)
library(dplyr)
library(ggplot2)
library(scales)
library(quantmod)

L80=read.csv(file.choose(),header=TRUE)
L70
attach(L70)
names(L70)
MAE=function(Actual,Lee.Carter){mean(abs(Actual-Lee.Carter))}
```

```
MAPE=function(actual,predicted){mean(abs((actual-predicted)/actual)*100)}
MAPE(Actual,Lee.Carter)

(MSE1=mean((p1-TotalVat)^2))

RMSE<-function(actual,predicted){
  abb<-abs(actual-predicted)
  ss<-sum(abb)/length(predicted)
  sq<-sqrt(ss)
  return(sq)
}


RMSE(Actual,Lee.Carter)

aex1<- diffseries(ex1,0.4990817)
fitt2 <- arima(aex1, order=c(p=0, d=0, q=2))
summary(fitt2)

# set up the plot
plot(Actual,Lee.Carter, type="n", xlab="Years",
     ylab="Life expectancy at 70" )
colors <- rainbow(ntrees)
linetype <- c(1:ntrees)
plotchar <- seq(18,18+ntrees,1)
names(L80)
# add lines
for (i in 1:ntrees) {
  tree <- subset(Orange, Tree==i)
  lines(tree$age, tree$circumference, type="b", lwd=1.5,
```

```
      lty=linetype[i], col=colors[i], pch=plotchar[i])
}
MAE(L80$Actual,L80$Lee.Carter)
MAPE(L80$Actual,L80$Lee.Carter)
RMSE(L80$Actual,L80$Lee.Carter)

MAE(L80$Actual,L80$CBD.model)
MAPE(L80$Actual,L80$CBD.model)
RMSE(L80$Actual,L80$CBD.model)

# add a title and subtitle
title("Tree Growth", "example of line plot")

# add a legend
legend(xrange[1], yrange[2], 1:ntrees, cex=0.8, col=colors,
      pch=plotchar, lty=linetype, title="Tree")

CBD <- cbd()
CBDfit <- fit(CBD, data = central2initial(EWMaleData), ages.fit = 55:89)
plot(CBDfit, parametricbx = FALSE)

lct=read.csv(file.choose(),header=TRUE)
library(readr)
library(dplyr)
library(ggplot2)
library(scales)
library(quantmod)
# Read car and truck values from tab-delimited autos.dat
attach(lct)
names(lct)
```

```
ktt<- c(lct$kt, lct$kt.1., lct$kt.2.)
## colors <- c("black", "blue", "darkgreen")
colors <- c("black", "blue")
labels <- c("kt1", "kt2")
plot(lct$Year, lct$kt, type="l", lty=2, xlab="Year",
    ylab="Mortality index")

plot(lct$Year, lct$kt.1., type="l", lty=2, xlab="Year",
    ylab="Mortality index")

lines(lct$Year, lct$kt.1., lwd=2, col= "black")
lines(lct$Year, lct$kt.2., lwd=2, col= "blue")
## lines(L80$Year, L80$CBD.model, lwd=2, col= "darkgreen")

legend("topright", inset=.02,
      labels, lwd=2, lty=c(2, 1), col=colors)
```