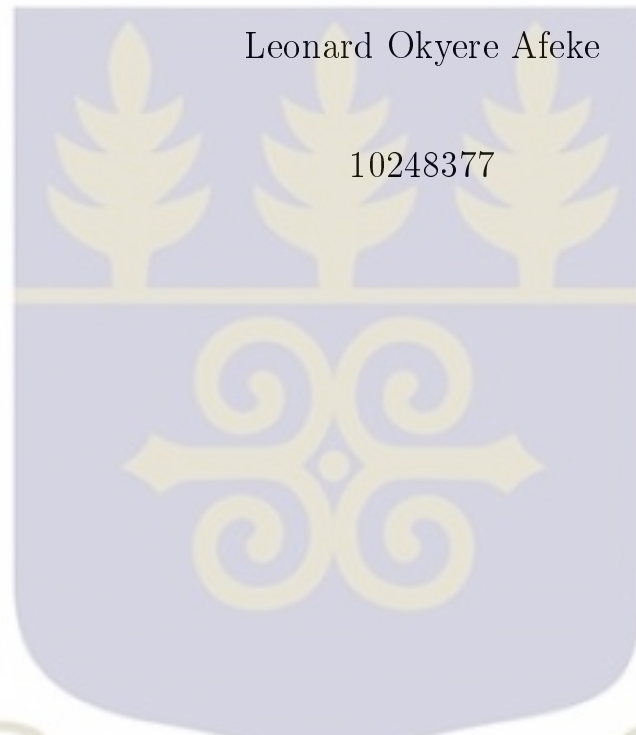


Khovanov Homology and Presheaves



THIS THESIS IS SUBMITTED TO THE UNIVERSITY OF GHANA,
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Declaration

This thesis was written in the Department of Mathematics, University of Ghana, Legon from September 2014 to July 2015 in partial fulfilment of the requirements for the award of Master of Philosophy degree in Mathematics under the supervision of Dr. Margaret McIntyre and Dr. Ralph Twum of the University of Ghana

I hereby declare that except where due acknowledgement is made, this work has never been presented wholly or in part for the award of a degree at the University of Ghana or any other University.

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Abstract

We show that the right derived functors of the limits of the Khovanov presheaf describes the Khovanov homology. We also look at the cellular cohomology of a poset \mathbf{P} with coefficients in a presheaf F and show by example that the Khovanov homology can be computed cellularly.



Dedication

I dedicate this work to the loving memory of my mother Juliana Abena Asantewaa, my uncle Mr. Samuel Asante, my siblings and all my loved ones.



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I first of all thank the Lord Jesus Christ who is the embodiment of all wisdom, knowledge and understanding for helping me complete this thesis successfully.

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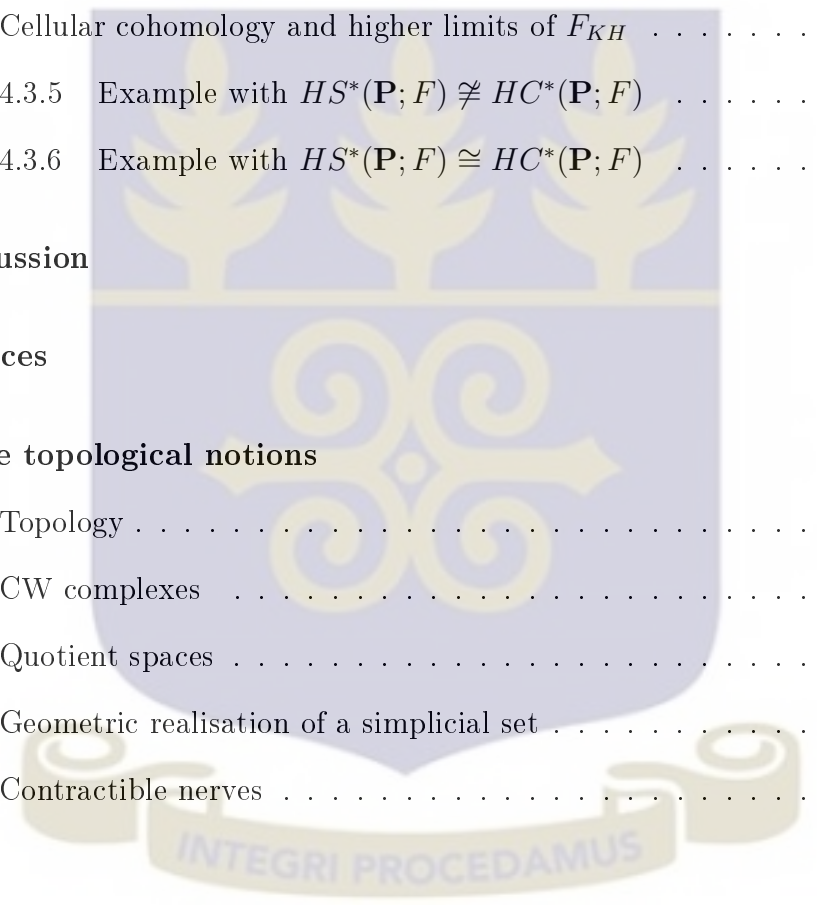
Finally, I thank my family and friends for their support.



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Chapter 1

Introduction and Motivation

A mathematical *knot* unlike a knot in everyday life is a knot with no loose ends. Formally, it is an embedding of S^1 , the 1-sphere in \mathbb{R}^3 . A *link* is a collection of knots that are all tangled up together. *Knot theory* is the study of these mathematical knots and links and one of the major areas of study in knot theory is *knot invariants*. Knots are 1-dimensional topological objects living in 3-dimensional space and so we may represent them using diagrams. A *knot diagram* is a projection of a knot into a 2-dimensional plane. A standard reference for the study of knot theory is the book by Collins C. Adams [1].

An important question in knot theory is that given two knots (or links), how can we tell them apart? That is, how can we tell if they are distinct or the same? Knot invariants are here to address this question. A *knot invariant* is a mathematical object assigned to each knot and which remains unchanged for equivalent knots¹. Some examples of knot invariants include tricolorability, integer-valued invariants like unknotting number and crossing number, polynomial invariants like the Jones polynomial and homological invariants like the Khovanov homology. If an invariant assigns a different object to two knots (or links), then the two knots must be distinct. On the other hand if an invariant assigns the same object to two knots (or links) then they are not necessarily the same and this depends on the strength of the invariant. If an invariant say I_1 tells apart more knots (or links) than the other say I_2 , then we say I_1 is stronger than I_2 .

One of the more interesting knot invariants is the Jones polynomial and it was developed by Vaughan Jones. It is constructed from the *Kauffman bracket* and we give a brief description of it below. The Kauffman bracket of a link D written $\langle D \rangle$ is defined recursively by the following three rules:

¹Two knots are equivalent if one can be deformed topologically into the other without breaking it.

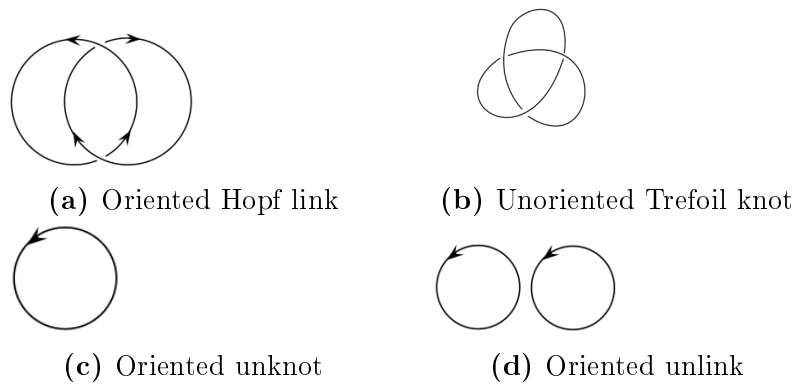
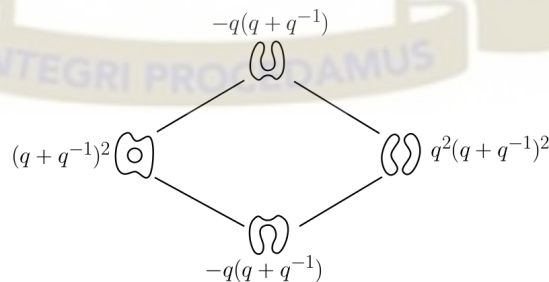


Figure 1.1: Examples of knots and links

(1) $\langle \emptyset \rangle = 1$, (2) $\langle \times \rangle := \langle \smile \rangle - q \langle \frown \rangle$ and (3) $\langle 0 \cup D \rangle := (q + q^{-1}) \langle D \rangle$ where \smile and \frown are the 0- and 1-resolution of the crossing \times respectively. Let D be a link diagram with n crossings. Let n_- be the number of negative crossings \times and n_+ the number of positive crossings \times . We have $n = n_- + n_+$. When the Kauffman bracket is computed, then we can define the unnormalised Jones polynomial to be $\hat{J}(D) = (-1)^{n_-} q^{n_+ - 2n_-} \langle D \rangle$ which when multiplied by $(q + q^{-1})^{-1}$ gives us the Jones polynomial: $J(D) = \hat{J}(D)(q + q^{-1})^{-1}$. An element α in the n -dimensional cube $\{0, 1\}^n$ corresponds to a **smoothing** (when all crossings of the link D are resolved to form a disjoint union of cycles) S_α of the link D . To calculate $\hat{J}(D)$, we replace each smoothing S_α of k_α cycles with a term $(-1)^{r_\alpha} q^{r_\alpha} (q + q^{-1})^{k_\alpha}$, where r_α is the number of 1's in α . Sum up all the terms over α to get $\langle D \rangle = \sum_{\alpha} (-1)^{r_\alpha} q^{r_\alpha} (q + q^{-1})^{k_\alpha}$, the Kauffman bracket of D , from which we compute $\hat{J}(D)$ and hence $J(D)$. For example if we consider the Hopf link (see Figure 1.1a) with $n_- = 2$ and $n_+ = 0$, we get the following diagram of smoothings with their corresponding terms $(-1)^{r_\alpha} q^{r_\alpha} (q + q^{-1})^{k_\alpha}$.



When we sum up all the terms we get $\langle D \rangle = q^4 + q^2 + q^{-2} + 1$ from which we get $\hat{J}(D) = q^{-4} \langle D \rangle = 1 + q^{-2} + q^{-6} + q^{-4}$ and hence the Jones polynomial $J(D) = \hat{J}(D)(q + q^{-1})^{-1}$. Even though the Jones polynomial has been very useful in telling knots(links) apart, it is



(a) A knot with 5 crossings. (b) A knot with 10 crossings

Figure 1.2: Two different knots with the same Jones polynomial

unable to tell some knots apart. For example the Jones polynomial cannot tell apart the knots in figure 1.2. They are two different knots with the same Jones polynomial. It is also still unknown whether or not it detects the unknot(see 1.1c) although S. Eliahou et al in their paper [3] have given an analogue for links. They show that there is an infinite family of distinct 2-component links with Jones polynomial equal to those of the corresponding unlinks(see 1.1d). That is if L is a 2-component link, then $J(L) = J(\bigcirc\bigcirc)$. Since the Jones polynomial is unable to tell all knots apart, the quest for a stronger knot invariant continues.

In search for a stronger knot invariant, M. Khovanov developed a homological knot invariant called the Khovanov homology. It is able to tell apart all distinct knots that the Jones polynomial can, and can tell apart some distinct knots that the Jones polynomial cannot distinguish. It is interesting to know that the Khovanov homology encodes more information about knots than the Jones polynomial. For example the Euler characteristic of the Khovanov complex (see Theorem 3.2.4) is the Jones Polynomial. That is, it determines the Jones polynomial, but not the other way around. In 2010, the Khovanov homology was shown to be the unknot detector by Kronheimer and Mrowka in their paper [6]. That is, it is able to tell if a given knot is the unknot. It is also important to know that the Khovanov homology is a *functor* where as the Jones polynomial is not. The above information about the Khovanov homology tells us that it is a stronger knot invariant than the Jones polynomial. It is therefore very interesting to study Khovanov homology and we do so using the language of category theory.

Category theory can be thought of as the study of the universal properties of mathematical structures. A *category* is a collection of objects and a collection of maps between objects called arrows. A *functor* is a map between two categories. Some examples of categories include posets, the category of abelian groups and the category of sets. Examples of functors are the identity functor, presheaves and right derived functors. A reference for category theory is [7].

In Section 3.3, we give a description of how to compute the Khovanov homology of a knot from a poset associated to the knot. Let D be a knot diagram with an associated poset \mathbf{Q} . Since \mathbf{Q} is a category, we may define a presheaf $F_{KH} : \mathbf{Q}^{op} \rightarrow \mathbf{Ab}$ which assigns to each element x in \mathbf{Q} an abelian group $F_{KH}(x)$ and to each map $x \leq y$ in \mathbf{Q} a homomorphism $F_{KH}(y) \rightarrow F_{KH}(x)$. We call F_{KH} the **Khovanov presheaf**. It is interesting to know that the unnormalised Khovanov homology can be identified with the right derived functors of the limits of the Khovanov presheaf and this is our following theorem stated and proved in Section 3.5.

Theorem: *Let D be a link diagram and let $F_{KH} : \mathbf{Q}^{op} \rightarrow \mathbf{Ab}$ be the Khovanov presheaf. Then $\overline{KH}^i(D) \cong \varprojlim_{\mathbf{Q}^{op}}^i F_{KH}$.*

Given a category \mathcal{C} and a presheaf F on \mathcal{C} we define a cochain complex $S^*(\mathcal{C}; F)$. The cohomology $HS^*(\mathcal{C}; F)$ of this complex is called the *cohomology of \mathcal{C} with coefficients in F* . The cohomology groups of a poset \mathbf{P} , with coefficients in a presheaf F are the right derived functors of the limits of F (see Proposition 4.1.3 and Remark 4.1.4). Since a knot is a topological object, we would like to compute its Khovanov homology cellularly. Everitt and Turner in [4] proposed a definition for **cellular cohomology** $HC^*(\mathbf{P}; F)$ of a poset \mathbf{P} with coefficients in a presheaf F on \mathbf{P} . It turns out that the cohomology $HS^*(\mathbf{P}; F)$ under certain conditions is isomorphic to the cellular cohomology $HC^*(\mathbf{P}; F)$ (see Theorem 4.3.4). This isomorphism is a generalisation of our Theorem 3.5.4, which is also stated above.

The thesis is organised as follows. **Chapter 1** provides an introduction and motivation to this thesis. In **Chapter 2**, we review some concepts in homology theory and category theory. **Chapter 3** looks at the Khovanov homology and the limits of the Khovanov presheaf. It first provides a review on how the Khovanov homology of a knot is computed and goes ahead to look at the comparison between the Khovanov homology and the Jones polynomial. We look at how Khovanov homology is constructed from a poset which is based on the papers [5] and [9]. We discuss the limit and its derived functors and finally establish the relation between the Khovanov homology and the limits of the Khovanov presheaf. In **Chapter 4**, we look at Khovanov homology and presheaves. We first discuss Khovanov homology and the classifying space of a category. A relation between Khovanov homology and the cohomology of a poset \mathbf{Q} with coefficients in a presheaf is then established using the classifying space of the poset \mathbf{Q} . We then look at the cellular cohomology of a poset with coefficients in a presheaf proposed by Everitt and Turner in [4] and we then establish a relation between the cellular cohomology and the limits of the Khovanov presheaf based on a certain condition. We conclude by discussing two examples. **Chapter 5** gives us a discussion of the thesis and some future directions.

Chapter 2

Preliminaries

This chapter looks at some concepts in homology theory and category theory. A reference for these concepts is [7].

2.1 Homology theory

Let R be a ring. We assume R to have an identity element 1.

Definition 2.1.1. A *left R -module* is an additive abelian group M together with an operation of R on M , that is $R \times M \rightarrow M$ defined by $(a, x) \mapsto ax$ such that, for all $a, b \in R$ and $x, y \in M$, the following axioms are satisfied:

- | | |
|---------------------------|----------------------|
| (a) $(a + b)x = ax + bx.$ | (c) $(ab)x = a(bx).$ |
| (b) $a(x + y) = ax + ay.$ | (d) $1x = x$ |

Definition 2.1.2. Let M and N be R -modules. A map $f : M \rightarrow N$ is called an *R -module homomorphism* if for all $a, b \in M$ and $r \in R$, we have $f(a + rb) = f(a) + rf(b)$.

A *right R -module* is an abelian group M together with an operation of R on M such that, for all $a, b \in R$ and $x, y \in M$, the axioms of Definition 2.1.1 are satisfied but with the places of the ring and the abelian group switched.

Examples of modules include the ring R itself and abelian groups. Every abelian group is a left \mathbb{Z} -module.

Definition 2.1.3. Let R be a ring, A a right R -module, B a left R -module and G an (additive) abelian group. A function $f : A \times B \rightarrow G$ is called *R -biadditive* if for all $a, a_1 \in A, b, b_1 \in B, r \in R$, we have

- (a) $f(a + a_1, b) = f(a, b) + f(a_1, b)$. (c) $f(ar, b) = f(a, rb)$.
 (b) $f(a, b + b_1) = f(a, b) + f(a, b_1)$.

The *tensor product* of A and B is an abelian group $A \otimes_R B$ and an R -biadditive function $h : A \times B \rightarrow A \otimes_R B$ such that for every R -biadditive function $f : A \times B \rightarrow G$, there exists a unique \mathbb{Z} -homomorphism $\gamma : A \otimes_R B \rightarrow G$ such that $f = \gamma \circ h$.

Definition 2.1.4. A sequence $M \xrightarrow{f} M' \xrightarrow{g} M''$ of R -modules is called an *exact sequence* if $\text{im} f = \ker g$ ² where f and g are module homomorphisms.

Example 2.1.5. Examples of exact sequences:

- The sequence $0 \rightarrow M \xrightarrow{\gamma} M'$ is exact if and only if γ is an injective homomorphism.
- The sequence $M \xrightarrow{\psi} N \rightarrow 0$ is exact if and only if ψ is a surjective homomorphism.
- The sequence $0 \rightarrow M \xrightarrow{f} M' \xrightarrow{g} M'' \rightarrow 0$ is an exact sequence if f is injective, g is surjective and $\text{im} f = \ker g$, and it is called a *short exact sequence*. For example if $\gamma : M \rightarrow N$ is a homomorphism, then the sequence $0 \rightarrow \ker \gamma \xrightarrow{i} M \xrightarrow{\gamma} \text{im} \gamma \rightarrow 0$ is a short exact sequence.

Definition 2.1.6. A *chain complex* is a sequence of modules and homomorphisms (M_n, f_n) ,

$$\dots \xrightarrow{f_{n+2}} M_{n+1} \xrightarrow{f_{n+1}} M_n \xrightarrow{f_n} M_{n-1} \xrightarrow{f_{n-1}} \dots$$

where n ranges over all integers and f_n maps M_n to M_{n-1} , such that $f_n \circ f_{n+1} = 0$ for all n .

Definition 2.1.7. A *cochain complex* is a sequence of modules and homomorphisms (M^n, d^n) ,

$$\dots \xrightarrow{d^{n-2}} M^{n-1} \xrightarrow{d^{n-1}} M^n \xrightarrow{d^n} M^{n+1} \xrightarrow{d^{n+1}} \dots$$

where n ranges over all integers and d^n maps M^n to M^{n+1} , such that $d^{n+1} \circ d^n = 0$ for all n .

We write (M_*, f) for a chain complex and (M^*, d) for a cochain complex.

² $\text{im} f$ is the image of a map f and $\ker f$ is the kernel of f .

Definition 2.1.8. Let (M_*, f) be a chain complex. Let $Z_n = \ker f_{n-1}$ and $B_n = \operatorname{im} f_n$. Then the n th **homology group** of the complex is the quotient $H^n(M_*) = Z_n/B_n$.

Definition 2.1.9. Let (M^*, d) be a cochain complex. Let $Z^n = \ker d^n$ and $B^n = \operatorname{im} d^{n-1}$. Then the n th **cohomology group** of the complex is the quotient $H^n(M^*) = Z^n/B^n$.

Definition 2.1.10. A **morphism of cochain complexes** or **cochain map** $h : (A^*, f) \rightarrow (B^*, g)$ is a sequence of homomorphisms $h^n : A^n \rightarrow B^n$ for which the following diagram commutes.

$$\begin{array}{ccc} A^n & \xrightarrow{f^n} & A^{n+1} \\ h^n \downarrow & & \downarrow h^{n+1} \\ B^n & \xrightarrow{g^n} & B^{n+1}. \end{array}$$

Definition 2.1.11. A left R -module P is called **projective** if, whenever $s : A \rightarrow A'$ is surjective, and $h : P \rightarrow A'$, there exists a map $g : P \rightarrow A$ making the following diagram commute. The map $s : A \rightarrow A'$ is surjective if and only if $A \xrightarrow{s} A' \rightarrow 0$ is exact.

$$\begin{array}{ccccc} & & P & & \\ & & \downarrow h & & \\ A & \xrightarrow{s} & A' & \longrightarrow & 0. \\ & \swarrow g & & & \end{array}$$

A **projective resolution** of a module M is an exact sequence

$$\dots \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} M \rightarrow 0,$$

where each P_i is projective. The map ε is a surjection by exactness.

Definition 2.1.12. A left R -module I is called **injective** if, whenever $i : A' \rightarrow A$ is injective, and $h : A' \rightarrow I$, there exists a map $g : A \rightarrow I$ making the following diagram commute. The map $i : A' \rightarrow A$ is injective if and only if $0 \rightarrow A' \xrightarrow{i} A$ is exact.

$$\begin{array}{ccccc} & & I & & \\ & & \uparrow h & & \\ 0 & \longrightarrow & A' & \xrightarrow{i} & A. \\ & & & \swarrow g & \end{array}$$

An **injective resolution** of a module M is an exact sequence

$$0 \rightarrow M \xrightarrow{\eta} I_0 \xrightarrow{d_0} I_1 \xrightarrow{d_1} I_2 \xrightarrow{d_2} \dots,$$

where each I_i is injective. The map η is an injection by exactness.

Definition 2.1.13. (*Short exact sequence of cochain complexes*) Let $f : A^* \rightarrow B^*$ and $g : B^* \rightarrow C^*$ be cochain maps. The sequence $0 \rightarrow A^* \xrightarrow{f} B^* \xrightarrow{g} C^* \rightarrow 0$ is a short exact sequence if $\ker f = 0$, $\text{im} f = \ker g$ and $\text{im} g = C^*$.

In Definition 2.1.13, all square commute and each horizontal sequence is exact. The following theorem will be useful in proving some lemmas in Section 4.2.

Theorem 2.1.14. ([10], Theorem 1.3.1) *If $0 \rightarrow A^* \xrightarrow{f} B^* \xrightarrow{g} C^* \rightarrow 0$ is a short exact sequence of cochain complexes, there are natural maps (connecting homomorphisms) $\delta : H^n C^* \rightarrow H^{n+1} A^*$ and a long exact sequence*

$$\dots \xrightarrow{g} H^{n-1} C^* \xrightarrow{\delta} H^n A^* \xrightarrow{f} H^n B^* \xrightarrow{g} H^n C^* \xrightarrow{\delta} H^{n+1} A^* \xrightarrow{f} \dots$$

Remark 2.1.15. Consider the long exact sequence in Theorem 2.1.14. Let $c \in C^n$ be a cocycle. Then the connecting homomorphism δ can be defined to be the map sending the homology class $[c] \in H^n C^*$ to the homology class $[m] \in H^{n+1} A^*$, where m is a cocycle in A^{n+1} .

2.2 Category theory

Definition 2.2.1. A *category* \mathcal{C} consists of the following data:

- a collection $\text{obj}(\mathcal{C})$ of objects and a collection $\text{arr}(\mathcal{C})$ of arrows also called morphisms.
- for each arrow $f \in \text{arr}(\mathcal{C})$ there are given objects $A, B \in \text{obj}(\mathcal{C})$ such that $f : A \rightarrow B$. A is called the domain of f and B is called the codomain of f . The collection of morphisms with domain A and codomain B forms a set which we denote $\text{Hom}(A, B)$.
- given arrows $f : A \rightarrow B$ and $g : B \rightarrow C$, there is an arrow $g \circ f : A \rightarrow C$.

These data satisfy the following two axioms.

- (i) for each object A , there is an arrow $1_A : A \rightarrow A$ called the identity arrow such that $f \circ 1_A = f$ and $1_B \circ f = f$ for every $f : A \rightarrow B$,
- (ii) If $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$, then $h \circ (g \circ f) = (h \circ g) \circ f$.

The collection $\text{Hom}(A, B)$ is called a hom-set and if $f \in \text{Hom}(A, B)$, then $f : A \rightarrow B$ is an arrow.

Example 2.2.2. Examples of categories are:

- (i) A *partially ordered set* or *poset* is a set \mathbf{P} equipped with a binary relation $x \leq y$ such that the following conditions hold for all $x, y, z \in \mathbf{P}$:
- reflexivity: $x \leq x$,
 - transitivity: if $x \leq y$ and $y \leq z$, then $x \leq z$,
 - antisymmetry: if $x \leq y$ and $y \leq x$, then $x = y$.

A poset \mathbf{P} can be regarded as the category whose objects are the elements of \mathbf{P} , whose hom sets are either empty or have only one element:

$$\text{Hom}_{\mathbf{P}}(x, y) = \begin{cases} \{i_y^x\} & \text{if } x \leq y \\ \emptyset & \text{if } x \not\leq y. \end{cases}$$

- (ii) **Sets**: Objects are sets, morphisms are functions, and composition is the usual composition of functions.
- (iii) $R\mathbf{Mod}$: Objects are left R -modules, morphisms are R -homomorphisms, and composition is the usual composition.
- (iv) **Ab**: Objects are abelian groups, morphisms are group homomorphisms, and composition is the usual composition.
- (v) The *simplex category*, **SCat**: Objects of **SCat** are non-empty ordered sets of the form $[n] = \{0, 1, \dots, n\}$ with $n \geq 0$. The morphisms in **SCat** are order-preserving functions between sets (that is functions $f : [n] \rightarrow [m]$ such that $i \leq j \implies f(i) \leq f(j)$, for $0 \leq i, j \leq n$).
- (vi) Let \mathcal{C} be a category and fix an object $x \in \mathcal{C}$. Then the *coslice category* x/\mathcal{C} has as objects for any $b \in \mathcal{C}$ all morphisms $f : x \rightarrow b$; that is all morphisms f with domain equal to x . The morphisms are maps $h : (f : x \rightarrow b) \rightarrow (f' : x \rightarrow b')$ such that the following diagram commutes:

$$\begin{array}{ccc} & x & \\ f \swarrow & & \searrow f' \\ b & \xrightarrow{h} & b' \end{array}$$

that is $f' = h \circ f$.

Definition 2.2.3. Let \mathcal{C} be a category. Then the *opposite category* \mathcal{C}^{op} is the category with objects same as \mathcal{C} and morphisms same as \mathcal{C} but in reverse direction.

Definition 2.2.4. A *small category* \mathcal{C} is a category such that $\text{obj}(\mathcal{C})$ is a set, and not just a collection.

Definition 2.2.5. Let \mathcal{C} and \mathcal{D} be categories. A (*covariant*) *functor* $F : \mathcal{C} \rightarrow \mathcal{D}$ is a map that assigns to each object $A \in \mathcal{C}$ an object $F(A) \in \mathcal{D}$ and to each morphism $f : A \rightarrow B$ a morphism $F(f) : F(A) \rightarrow F(B)$ in \mathcal{D} such that

$$(i) F(g \circ f) = F(g) \circ F(f) \text{ for } A \xrightarrow{f} B \xrightarrow{g} C \text{ and } (ii) F(1_A) = 1_{F(A)}.$$

Example 2.2.6. Examples of covariant functors include:

- (i) If \mathcal{C} is a category, then the *identity functor* $1_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C}$ is defined by $1_{\mathcal{C}}(A) = A$ for all objects A and $1_{\mathcal{C}}(f) = f$ for all morphisms f .
- (ii) If \mathcal{C} is a category and $A \in \text{obj}(\mathcal{C})$, then the *hom functor* $\text{Hom}_{\mathcal{C}}(A, -) : \mathcal{C} \rightarrow \mathbf{Sets}$ is defined by

$$\text{Hom}_{\mathcal{C}}(A, -)(B) = \text{Hom}_{\mathcal{C}}(A, B) \text{ for all } B \in \text{obj}(\mathcal{C}),$$

and for $f : B \rightarrow D$ in \mathcal{C} we have $\text{Hom}_{\mathcal{C}}(A, f) : \text{Hom}_{\mathcal{C}}(A, B) \rightarrow \text{Hom}_{\mathcal{C}}(A, D)$. The map $\text{Hom}_{\mathcal{C}}(A, f)$ is called the *induced map* and it is denoted by f_* . For $h \in \text{Hom}_{\mathcal{C}}(A, B)$, define $f_* : \text{Hom}_{\mathcal{C}}(A, B) \rightarrow \text{Hom}_{\mathcal{C}}(A, D)$ by

$$f_*(h) = fh \in \text{Hom}_{\mathcal{C}}(A, D), \text{ (} fh \text{ is the composition of } f \text{ and } h\text{)}.$$

Definition 2.2.7. A *contravariant functor* $F : \mathcal{C} \rightarrow \mathcal{D}$ is a map that assigns to each object $A \in \mathcal{C}$ an object $F(A) \in \mathcal{D}$ and to each morphism $f : A \rightarrow B$ a morphism $F(f) : F(B) \rightarrow F(A)$ in \mathcal{D} such that

$$(i) F(g \circ f) = F(f) \circ F(g) \text{ for } A \xrightarrow{f} B \xrightarrow{g} C \text{ and } (ii) F(1_A) = 1_{F(A)}.$$

Example 2.2.8. Examples of contravariant functors include:

- (i) Presheaves (see Definition 2.2.9) are contravariant functors
- (ii) If \mathcal{C} is a category and $B \in \text{obj}(\mathcal{C})$, then the *hom functor* $\text{Hom}_{\mathcal{C}}(-, B) : \mathcal{C} \rightarrow \mathbf{Sets}$ is defined by

$$\text{Hom}_{\mathcal{C}}(-, B)(C) = \text{Hom}_{\mathcal{C}}(C, B) \text{ for all } C \in \text{obj}(\mathcal{C}),$$

and for $f : D \rightarrow D'$ in \mathcal{C} we have $\text{Hom}_{\mathcal{C}}(f, B) : \text{Hom}_{\mathcal{C}}(D', B) \rightarrow \text{Hom}_{\mathcal{C}}(D, B)$. The map $\text{Hom}_{\mathcal{C}}(f, B)$ is called the **induced map** and it is denoted by f^* . For $g \in \text{Hom}_{\mathcal{C}}(D', B)$, define $f^* : \text{Hom}_{\mathcal{C}}(D', B) \rightarrow \text{Hom}_{\mathcal{C}}(D, B)$ by

$$f^*(g) = gf \in \text{Hom}_{\mathcal{C}}(D, B), \quad (gf \text{ is the composition of } g \text{ and } f).$$

Definition 2.2.9. A presheaf on a small category \mathcal{C} is a covariant functor $\mathcal{F} : \mathcal{C}^{op} \rightarrow \mathbf{V}$ where \mathbf{V} is a category.

Note that one can also define it as a contravariant functor $\mathcal{F} : \mathcal{C} \rightarrow \mathbf{V}$.

We will consider the case where \mathbf{V} is the category of abelian groups \mathbf{Ab} , otherwise the category will be stated. Below are some examples of presheaves.

Example 2.2.10. Examples of presheaves include:

- (i) The **constant presheaf**: if $A \in \mathbf{Ab}$ define $\Delta A : \mathcal{C}^{op} \rightarrow \mathbf{Ab}$ by $\Delta A(x) = A$ for all $x \in \mathcal{C}$ and for all morphisms $x \rightarrow y$ in \mathcal{C} let $\Delta A(x \rightarrow y) = 1 : \Delta A(y) \rightarrow \Delta A(x)$.
- (ii) A **simplicial set** X is a covariant functor $X : \mathbf{SCat}^{op} \rightarrow \mathbf{Sets}$ where \mathbf{SCat} is the simplex category (see 2.2.2(v)) and \mathbf{Sets} is the category of sets (see 2.2.2(ii)). This is also an example of a presheaf.
- (iii) Let (\mathbf{P}, \leq) be a poset. Let $A \in \mathbf{Ab}$ and $x \in \mathbf{P}$. Then the **Yoneda presheaf** $\Upsilon_x A$ is defined by

$$\Upsilon_x A(y) = \begin{cases} A, & \text{if } y \leq x \\ 0, & \text{otherwise.} \end{cases}$$

For $\Upsilon_x A(y \leq z) : \Upsilon_x A(z) \rightarrow \Upsilon_x A(y)$ we have $\Upsilon_x A(y \leq z) = 1$ when $y \leq z \leq x$. When $y \leq x$ and $z \not\leq x$, we have $\Upsilon_x A(z) = 0$ so that $\Upsilon_x A(y \leq z)(0) = A$. $\Upsilon_x A(y \leq z)$ is the zero map otherwise. This shows that $\Upsilon_x A = \Delta A$ on $\mathbf{P}_{\leq x} = \{y \in \mathbf{P} : y \leq x\}$ and it is the zero presheaf on the rest of \mathbf{P} .

Definition 2.2.11. Let $F, S : \mathcal{A} \rightarrow \mathcal{B}$ be covariant functors. A **natural transformation** $\eta : F \rightarrow S$ is a one parameter family of morphisms in \mathcal{B} , $\eta = (\eta_x : F(x) \rightarrow S(x))_{x \in \text{obj}(\mathcal{A})}$, making the following diagram commute for all $f : x \rightarrow y$ in \mathcal{A} :

$$\begin{array}{ccc} F(x) & \xrightarrow{\eta_x} & S(x) \\ F(f) \downarrow & & \downarrow S(f) \\ F(y) & \xrightarrow{\eta_y} & S(y). \end{array}$$

Natural transformations between contravariant functors are defined in a similar way.

Natural transformations can be composed, for if $\tau : F \rightarrow S$ and $\beta : S \rightarrow T$ are natural transformations, where $F, S, T : \mathcal{A} \rightarrow \mathcal{B}$ are functors with the same variance, then define $(\beta\tau) : F \rightarrow T$ by $(\beta\tau)_x = \beta_x\tau_x$ for all objects x in \mathcal{A} . One can check that $\beta\tau$ is a natural transformation.

If \mathcal{A} and \mathcal{B} are categories, then there is a category $\mathcal{B}^{\mathcal{A}}$ called a **functor category** which has as its objects all covariant functors $F : \mathcal{A} \rightarrow \mathcal{B}$ and as its morphisms all natural transformations between these functors.

Example 2.2.12. An example of this functor category is $\mathbf{pSh}(\mathcal{C}) = \mathbf{Ab}^{\mathcal{C}^{op}}$ which has as its objects all presheaves $\mathcal{P}, \mathcal{F} : \mathcal{C}^{op} \rightarrow \mathbf{Ab}$ and as its morphisms the natural transformations $\tau : \mathcal{P} \rightarrow \mathcal{F}$.

2.2.13 Limits

Let I be a poset and \mathcal{C} be a small category. An **inverse system** in \mathcal{C} is an ordered pair $((M_i)_{i \in I}, (\psi_i^j)_{i \leq j})$ denoted by $\{M_i, \psi_i^j\}$, where (M_i) is an indexed family of objects in \mathcal{C} and $(\psi_i^j : M_j \rightarrow M_i)$ is an indexed family of morphisms for which $\psi_i^i = 1_{M_i}$ for all $i \in I$, such that the following diagram commutes whenever $i \leq j \leq k$.

$$\begin{array}{ccc}
 M_k & \xrightarrow{\psi_i^k} & M_i \\
 & \searrow \psi_j^k & \nearrow \psi_i^j \\
 & M_j &
 \end{array}$$

Definition 2.2.14. Let $\{M_i, \psi_i^j\}$ be an inverse system in a category \mathcal{C} over a poset I . The **limit** of $\{M_i, \psi_i^j\}$ is an object $\varprojlim M_i$ and a family of projections $(\pi_i : \varprojlim M_i \rightarrow M_i)_{i \in I}$ such that

- i $\psi_i^j \pi_j = \pi_i$ whenever $i \leq j$,
- ii for every object X in \mathcal{C} and all morphisms $f_i : X \rightarrow M_i$ satisfying $\psi_i^j f_j = f_i$ for all $i \leq j$, there exists a unique morphism $\theta : X \rightarrow \varprojlim M_i$ making the following diagram

commute.

$$\begin{array}{ccc}
 \varprojlim M_i & \xleftarrow{\theta} & X \\
 \searrow \pi_i & & \swarrow f_i \\
 & M_i & \\
 \swarrow \pi_j & \psi_i^j \uparrow & \searrow f_j \\
 & M_j &
 \end{array}$$

Remark 2.2.15. One can check that an inverse system in \mathcal{C} over I is a contravariant functor $M : I \rightarrow \mathcal{C}$ and so by the universality of $\varprojlim M_i$ we can have a covariant functor $\varprojlim : \mathcal{C}^{I^{op}} \rightarrow \mathcal{C}$.

Definition 2.2.16. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{C}$ be covariant functors. The ordered pair (F, G) is an **adjoint pair** if for each $C \in \mathcal{C}$ and $D \in \mathcal{D}$, there are bijections

$$\tau_{C,D} : \text{Hom}_{\mathcal{D}}(FC, D) \rightarrow \text{Hom}_{\mathcal{C}}(C, GD)$$

that are natural transformations in \mathcal{C} and in \mathcal{D} .

If (F, G) is an adjoint pair, then we say F is **left adjoint** to G and G is **right adjoint** to F .

Proposition 2.2.17. Let \mathcal{C} be a category. A sequence of presheaves on \mathcal{C} $F \xrightarrow{\tau} G \xrightarrow{\psi} H$ is exact if and only if for all $x \in \mathcal{C}$, the local sequence $F(x) \xrightarrow{\tau_x} G(x) \xrightarrow{\psi_x} H(x)$ is exact.

Proof. $F \xrightarrow{\tau} G \xrightarrow{\psi} H$ is exact

$$\iff \ker(\psi : G \rightarrow H) = \text{im}(\tau : F \rightarrow G)$$

$$\iff \forall x \in \mathcal{C}, \ker(\psi_x : G(x) \rightarrow H(x)) = \text{im}(\tau_x : F(x) \rightarrow G(x))$$

$$\iff \forall x \in \mathcal{C}, F(x) \xrightarrow{\tau_x} G(x) \xrightarrow{\psi_x} H(x) \text{ is exact.} \quad \square$$

Definition 2.2.18. A covariant functor F is called **left exact** if exactness of the sequence $0 \rightarrow A \rightarrow B \rightarrow C$ implies exactness of the sequence $0 \rightarrow F(A) \rightarrow F(B) \rightarrow F(C)$.

Proposition 2.2.19. Let $\mathcal{C} = {}_R\mathbf{Mod}$, the category of left R -modules.

Then $\text{Hom}_{\mathcal{C}}(M, -) : \mathcal{C} \rightarrow \mathbf{Ab}$ is a left exact functor.

Proof. Suppose $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C$ is an exact sequence in $\mathcal{C} = {}_R\mathbf{Mod}$. We show that the sequence $0 \rightarrow \text{Hom}_{\mathcal{C}}(M, A) \xrightarrow{f_*} \text{Hom}_{\mathcal{C}}(M, B) \xrightarrow{g_*} \text{Hom}_{\mathcal{C}}(M, C)$ is exact. That is:

(i) Exactness at $\text{Hom}_{\mathcal{C}}(M, A)$:

Let $\alpha \in \ker f_*$, then $\alpha : M \longrightarrow A$ and $f_*(\alpha) = f\alpha = 0$. This implies that $f(\alpha(x)) = 0$ for all $x \in M$. Since f is injective, $\alpha(x) = 0$ for all $x \in M$ which implies $\alpha = 0$. Thus $\ker f_* = \text{im}0 = \{0\}$ ³.

(ii) Exactness at $\text{Hom}_{\mathcal{C}}(M, B)$:

Let $\beta \in \text{im}f_*$. Then $\beta : M \longrightarrow B$. There is some $\psi \in \text{Hom}_{\mathcal{C}}(M, A)$ with $f_*(\psi) = f\psi = \beta$. Now $g_*(\beta) = g\beta = g(f\psi) = 0$ by assumption. Thus $\beta \in \ker g_*$ and so $\text{im}f_* \subseteq \ker g_*$.

Again let $\theta \in \ker g_*$. Then $\theta : M \longrightarrow B$ and $g_*(\theta) = g\theta = 0$ which implies $g\theta(x) = 0$ for all $x \in M$. Thus $\theta(x) \in \ker g = \text{im}f$. There is some $y \in A$ with $f(y) = \theta(x)$, $x \in M$, and y is unique since f is injective. Thus we have a well-defined function $\alpha \in \text{Hom}_{\mathcal{C}}(M, A)$ given by $\alpha(x) = y$ if $\theta(x) = f(y)$. Now $f_*(\alpha) = f\alpha$ and $f\alpha(x) = f(y) = \theta(x)$ for all $x \in M$. Thus $f_*(\alpha) = \theta$ and so $\theta \in \text{im}f_*$ which implies $\ker g_* \subseteq \text{im}f_*$.

Thus we have $\ker g_* = \text{im}f_*$.

The sequence is therefore exact and so $\text{Hom}_{\mathcal{C}}(M, -)$ is left exact. \square

Proposition 2.2.20. *A sequence $A \xrightarrow{f} B \xrightarrow{g} C$ in a category \mathcal{A} is exact if for each $M \in \text{obj}(\mathcal{A})$, the sequence*

$$\text{Hom}_{\mathcal{A}}(M, A) \xrightarrow{f_*} \text{Hom}_{\mathcal{A}}(M, B) \xrightarrow{g_*} \text{Hom}_{\mathcal{A}}(M, C) \quad (2.1)$$

is exact.

Proof. Suppose (2.1) is exact for each $M \in \text{obj}(\mathcal{A})$. Taking $M = A$, we have $\text{id} \in \text{Hom}_{\mathcal{A}}(A, A)$ and so $g_*f_*(\text{id}) = 0$ since $g_*f_* = 0$ by exactness. But $g_*f_*(\text{id}) = g_*(f \circ \text{id}) = g \circ f = gf$ and so $gf = 0$ which implies $\text{im}f \subseteq \ker g$.

Now take $M = \ker g$ and consider the inclusion map $i : M \longrightarrow B$. Then $g_*i = gi = 0$ by the definition of M . By the exactness of the sequence 2.1, there exists some $\beta \in \text{Hom}_{\mathcal{A}}(M, A)$ with $f_*(\beta) = i = f\beta$. Thus if $m \in M$ then $m = i(m) = f\beta(m)$. This shows that $m \in \text{im}f$ which implies $\ker g \subseteq \text{im}f$.

Thus $\ker g = \text{im}f$ and so the sequence $A \xrightarrow{f} B \xrightarrow{g} C$ is exact. \square

Proposition 2.2.21. *Suppose that $L : \mathcal{A} \longrightarrow \mathcal{B}$ and $F : \mathcal{B} \longrightarrow \mathcal{A}$ are an adjoint pair of functors with L left adjoint to F . Then F is left exact.*

³ $\text{im}0$ is the image of the zero map and the 0 in $\{0\}$ is the zero element

Proof. Suppose $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ is an exact sequence in \mathcal{B} . Let $D \in \mathcal{A}$. Applying the left exact functor $\text{Hom}_{\mathcal{B}}(L(D), -)$ (see Proposition 2.2.19) to the exact sequence, we get an exact sequence $0 \rightarrow \text{Hom}_{\mathcal{B}}(L(D), A) \xrightarrow{f^*} \text{Hom}_{\mathcal{B}}(L(D), B) \xrightarrow{g^*} \text{Hom}_{\mathcal{B}}(L(D), C)$. By definition of adjoint pair, we have the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Hom}_{\mathcal{B}}(L(D), A) & \xrightarrow{f^*} & \text{Hom}_{\mathcal{B}}(L(D), B) & \xrightarrow{g^*} & \text{Hom}_{\mathcal{B}}(L(D), C) \\ & & \downarrow \tau_{D,A} & & \downarrow \tau_{D,B} & & \downarrow \tau_{D,C} \\ 0 & \longrightarrow & \text{Hom}_{\mathcal{A}}(D, F(A)) & \xrightarrow{Ff^*} & \text{Hom}_{\mathcal{A}}(D, F(B)) & \xrightarrow{Fg^*} & \text{Hom}_{\mathcal{A}}(D, F(C)). \end{array}$$

The top row is exact and since $\tau_{D,A}, \tau_{D,B}, \tau_{D,C}$ are isomorphisms, we see that the bottom row is also exact. By Proposition 2.2.20, (stitching together the sequences $0 \rightarrow A \rightarrow B$ and $0 \rightarrow A \rightarrow B \rightarrow C$) we see that the sequence

$$0 \rightarrow F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C)$$

(stitching together the sequences $0 \rightarrow F(A) \rightarrow F(B)$ and $0 \rightarrow F(A) \rightarrow F(B) \rightarrow F(C)$) is exact which implies that F is left exact. \square

2.2.22 Right derived functors

Let $T : \mathcal{C} \rightarrow \mathbf{Ab}$ be a covariant left exact functor (for example $\text{Hom}_{\mathcal{C}}(M, -)$). Let

$$\mathbf{I}^M = 0 \rightarrow M \xrightarrow{\eta} I_0 \xrightarrow{d_0} I_1 \xrightarrow{d_1} I_2 \xrightarrow{d_2} \dots$$

be an injective resolution (see Definition 2.1.12) of M in \mathcal{C} . Let

$$\mathbf{I} = 0 \rightarrow I_0 \xrightarrow{d_0} I_1 \xrightarrow{d_1} I_2 \xrightarrow{d_2} \dots$$

and form the complex $T\mathbf{I} = 0 \rightarrow T(I_0) \xrightarrow{T(d_0)} T(I_1) \xrightarrow{T(d_1)} T(I_2) \xrightarrow{T(d_2)} \dots$. In general $T\mathbf{I}$ is **not** an exact sequence. Take homology of the complex $T\mathbf{I}$:

$$R^i T(M) = H^i(T\mathbf{I}) = \ker T(d_i) / \text{im} T(d_{i-1}).$$

$R^i T$ are called the **right derived functors** of the functor T for all $i \geq 0$. It can be shown that the right derived functors do not depend on the choice of injective resolutions since they yield naturally isomorphic functors.

The sequence $T\mathbf{I}^M$ (applying T to \mathbf{I}^M) is exact since the functor T is left exact. Since η is an injective map by exactness, it implies $T(\eta)$ is also an injective map by exactness.

Proposition 2.2.23. *Let T be a covariant left exact functor. Then $R^0T \cong T$.*

Proof. Exactness of $T\mathbf{I}^M = 0 \rightarrow T(M) \xrightarrow{T(\eta)} T(I_0) \xrightarrow{T(d_0)} T(I_1) \xrightarrow{T(d_1)} T(I_2) \xrightarrow{T(d_2)} \dots$ tells us that $\ker T(d_0) = \text{im} T(\eta) \cong T(M)$. The isomorphism is as a result of $T(\eta)$ being an injection. We have that $R^0T(M) = \ker T(d_0) / \{0\} \cong \ker T(d_0) \cong T(M)$. Thus $R^0T(M) \cong T(M)$ and so $R^0T \cong T$. \square



Chapter 3

Khovanov homology and limits

3.1 Khovanov homology

In this section, we describe how the Khovanov complex is constructed and then define the Khovanov homology. This will serve as a motivating factor when we look at how the Khovanov homology is constructed from a *poset* (see Example 2.2.2(i)) in Section 3.3. We follow the descriptions given by Turner in his lecture notes (see [8]).

The idea here is to assign a cochain complex $C^*(D)$ to an oriented link represented by a diagram D .

Definition 3.1.1. An *I-graded vector space* V is a decomposition of V into a direct sum of the form $V = \bigoplus_{m \in I} V_m$ where the V_m 's are called homogeneous components of degree m and I is an indexing set.

Let $V = \bigoplus_m V_m$ be a graded vector space over the field \mathbb{Q} where the V_m 's are vector spaces over \mathbb{Q} with degree m . Let $-\{l\}$ be the degree shift operator on the graded vector space. Then, we set $V\{l\}_m = V_{m-l}$. If $V = \bigoplus_j V_j$ and $U = \bigoplus_k U_k$ are graded vector spaces, then $V \oplus U$ and $V \otimes U$ are graded vector spaces with gradation $(V \oplus U)_i = V_i \oplus U_i$ and $(V \otimes U)_i = \bigoplus_{j+k=i} V_j \otimes U_k$ respectively.

For example, let $V = \mathbb{Q}[1, u]$ be the two-dimensional graded vector space with two basis elements 1 and u . Grade the two basis elements by $\deg(1) = 1$ and $\deg(u) = -1$ ⁴. Then

$$V = \mathbb{Q}[1, u] = \bigoplus_{m \in \{-1, 1\}} V_m \cong \mathbb{Q} \oplus \mathbb{Q}, \text{ where } V_m = \begin{cases} \mathbb{Q}, & m = -1, 1 \\ 0, & \text{otherwise} \end{cases}.$$

⁴ $\deg(x)$ denotes the degree of x

- The direct sum of $V \oplus V$ is:

$$V \oplus V = \{(a1, bu) : a, b \in \mathbb{Q}\} \oplus \{(c1, du) : c, d \in \mathbb{Q}\} = \{(a, c)(1), (b, d)(u)\} \cong \mathbb{Q}^2 \oplus \mathbb{Q}^2$$

and

- The tensor product $V \otimes V$ is:

$$\begin{aligned} V \otimes V &= \langle u \otimes u \rangle \oplus (\langle 1 \otimes u \rangle \oplus \langle u \otimes 1 \rangle) \oplus \langle 1 \otimes 1 \rangle \\ &\cong (\mathbb{Q} \otimes \mathbb{Q}) \oplus ((\mathbb{Q} \otimes \mathbb{Q}) \oplus (\mathbb{Q} \otimes \mathbb{Q})) \oplus (\mathbb{Q} \otimes \mathbb{Q}). \end{aligned}$$

Let D represent a link diagram and let n be the number of crossings in D . A crossing in D can be resolved in two ways, that is a 0-resolution which looks like this: \frown and a 1-resolution which looks like this: \smile . A **smoothing** of a link D is a picture of planar circles in which all the crossings of the link have been resolved. If there are n crossings in D , then we will have 2^n smoothings of the link D .

Associate to each smoothing of D an element α in the set $\{0, 1\}^n$, the vertices of the n -dimensional cube. The set $\{0, 1\}^n$ with 2^n elements is the vertex set of a hyper-cube with edges between words ⁵ which differ in exactly one place. An example of the cube from the Hopf link is in figure 3.1.

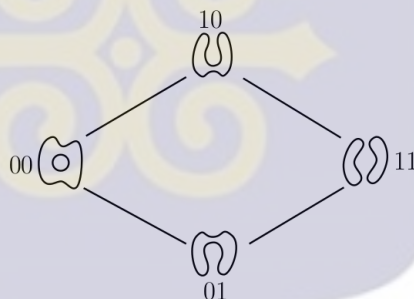


Figure 3.1: Hyper-cube from the Hopf link.

We can now construct the Khovanov complex $C^*(D)$ using the above information. Let D represent an oriented link diagram. Write $n = n_+ + n_-$ where n_- is the number of negative crossings \times and n_+ is the number of positive crossings \times . For $\alpha \in \{0, 1\}^n$, let $V_\alpha = V^{\otimes k_\alpha} \{r_\alpha + n_+ - 2n_-\}$ (where $\{r_\alpha + n_+ - 2n_-\}$ is the degree shift operation) and define

$$C^i(D) = \bigoplus_{\alpha \in \{0,1\}^n, r_\alpha = i + n_-} V_\alpha$$

⁵A word is a string of zeros and ones. The zero's and one's correspond to the 0 and 1-smoothings respectively.

where $i = -n_-, \dots, n_+$, r_α is the number of 1's in α and k_α is the number of planar circles in a smoothing corresponding to α . Each tensor factor corresponds to a planar circle.

We now need to define a differential map to turn $C^*(D)$ into a complex. Let $\alpha, \beta \in \{0, 1\}^n$ be vertices of an edge of the hyper-cube. Put a star (*) at the position where digits of the vertices differ. Label the edge d_s where s is the resulting vertex with a star (*). For example $00 \text{ --- } 01$ is an edge with vertices 00 and 01 . The digits 0 and 1 in the second position of 00 and 01 respectively differ and so we label the edge d_{0*} . Make edges into arrows by letting $* = 0$ be the tail and $* = 1$ be the head, for instance $00 \xrightarrow{d_{0*}} 01$.

Let $V = \mathbb{Q}[1, u]$ and define $d_s : V_\alpha \rightarrow V_\beta$ where $\text{tail}(s) = \alpha$ and $\text{head}(s) = \beta$ by using the linear maps $m : V \otimes V \rightarrow V$ and $\Delta : V \rightarrow V \otimes V$ defined on the basis vectors as follows.

$$m : 1 \otimes 1 \mapsto 1, 1 \otimes u \text{ and } u \otimes 1 \mapsto u, u \otimes u \mapsto 0$$

and

$$\Delta : 1 \mapsto 1 \otimes u + u \otimes 1, u \mapsto u \otimes u.$$

Note that m and Δ are extended to $V \otimes V$ and V by *linearity*. The map m corresponds to two planar circles fusing into one and the map Δ corresponds to one circle splitting into two.

Now for $v \in V_\alpha \subset C^i(D)$ define $d^i : C^i(D) \rightarrow C^{i+1}(D)$ by

$$d^i(v) = \sum_{s : \text{tail}(s)=\alpha} \text{sgn}(s) d_s(v)$$

where $\text{sgn}(s) = (-1)^{\text{number of 1's to the left of } * \text{ in } s}$. The presence of $\text{sgn}(s)$ ensures that the square faces of the cube anti-commute and hence the d^i 's are differentials.

Definition 3.1.2. The *Khovanov homology* of the oriented link diagram D is defined as the cohomology of the Khovanov complex (C^*, d) .

The Khovanov homology of an oriented link is invariant. The proof of this statement can be found in the paper [2] by D. Bar-Natan.

Example 3.1.3. We give an example to show how to get the Khovanov homology of the oriented Hopf link (see Figure 3.2a). It has two crossings ($n = 2$); two negative crossings ($n_- = 2$) and no positive crossings ($n_+ = 0$).

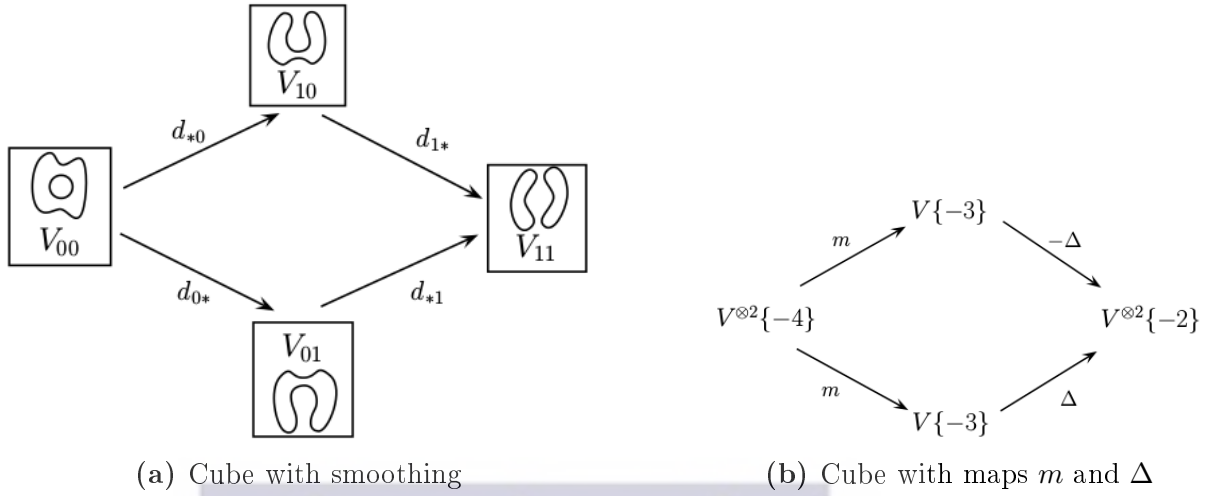


Figure 3.2: Defining differentials from the cube of the Hopf link

From figure 3.2b we get the following complex with only three non-trivial terms:

$$0 \longrightarrow C^{-2}(D) \xrightarrow{d^{-2}} C^{-1}(D) \xrightarrow{d^{-1}} C^0(D) \longrightarrow 0$$

where $C^{-2}(D) = V^{\otimes 2}\{-4\}$, $C^{-1}(D) = V\{-3\} \oplus V\{-3\}$ and $C^0(D) = V^{\otimes 2}\{-2\}$

Calculating the homology.

1. The homology at $C^{-2}(D)$:

Elements of $C^{-2}(D)$ are spanned by $1 \otimes 1, 1 \otimes u, u \otimes 1$ and $u \otimes u$. Let $v \in C^{-2}(D)$, then $d^{-2}(v) = (m(v), m(v))$, $m(v) = (a1, bu)$, where $a, b \in \mathbb{Q}$. The image of d^{-2} is therefore spanned by $(1, 0), (0, 1), (u, 0)$ and $(0, u)$.

The first map in the complex is the zero differential and so its image is $\{0\}$. For the kernel of d^{-2} observe that the only elements of $C^{-2}(D)$ that are sent to $(0, 0)$ by d^{-2} are the elements spanned by $u \otimes u, u \otimes 1 - 1 \otimes u$ since $m(u \otimes u) = 0$ and $m(u \otimes 1 - 1 \otimes u) = u - u = 0$. Thus $\ker d^{-2} = \langle u \otimes u, u \otimes 1 - 1 \otimes u \rangle$. The homology at $C^{-2}(D)$ is then

$$H(C^{-2}(D)) = \langle u \otimes u, u \otimes 1 - 1 \otimes u \rangle / \{0\} = \langle u \otimes u, u \otimes 1 - 1 \otimes u \rangle \cong \mathbb{Q} \oplus \mathbb{Q}.$$

2. The homology at $C^{-1}(D)$:

Elements of $C^{-1}(D)$ are spanned by $(1, 0), (0, 1), (u, 0)$ and $(0, u)$. Let $(x, y) \in C^{-1}$, then $d^{-1}(x, y) = -\Delta(x) + \Delta(y)$. From 1, we have $\text{im}d^{-2} = \langle (1, 0), (0, 1), (u, 0), (0, u) \rangle$.

By the definition of Δ , we see that

$$d^{-1}((1, 0)) + d^{-1}((0, 1)) = d^{-1}((1, 1)) = -\Delta(1) + \Delta(1) = 0$$

and

$$d^{-1}((u, 0)) + d^{-1}((0, u)) = d^{-1}(u, u) = -\Delta(u) + \Delta(u) = 0.$$

Notice that d^{-1} maps $C^{-1}(D) = \langle (1, 0), (0, 1), (u, 0), (0, u) \rangle$ to

$$\langle -1 \otimes u - u \otimes 1, 1 \otimes u + u \otimes 1, -u \otimes u, u \otimes u \rangle = \langle 1 \otimes u + u \otimes 1, u \otimes u \rangle.$$

Thus image of d^{-1} is $\langle 1 \otimes u + u \otimes 1, u \otimes u \rangle$. Now we have that $\dim(C^{-1}(D)) = 4$ and $\dim(\text{im}(d^{-1})) = 2$. By the Rank-Nullity Theorem in linear algebra, we see that

$$\dim(\ker d^{-1}) = \dim(C^{-1}(D)) - \dim(\text{im}(d^{-1})) = 2.$$

Thus $\ker d^{-1} = \langle (1, 1), (u, u) \rangle$. The homology at $C^{-1}(D)$ is therefore

$$H(C^{-1}(D)) = \ker d^{-1} / \text{im} d^{-2} = 0.$$

3. The homology at $C^0(D)$:

From 2, we have the image of d^{-1} to be $\langle 1 \otimes u + u \otimes 1, u \otimes u \rangle$. The kernel of the last map is $C^0(D) = \langle 1 \otimes 1, 1 \otimes u, u \otimes 1, u \otimes u \rangle$ since it is the zero differential. An element in $\ker(0)$ is of the form $a(1 \otimes 1) + b(1 \otimes u) + c(u \otimes 1) + d(u \otimes u)$ and so in the quotient $\ker(0)/\text{im}(d^{-1})$, an element will be of the form

$$\begin{aligned} \beta &= a(1 \otimes 1) + b(1 \otimes u) + c(u \otimes 1) + d(u \otimes u) + [\lambda(1 \otimes u + u \otimes 1) + \theta(u \otimes u)] \\ &= a(1 \otimes 1) + (c - b)(u \otimes 1) + [\lambda(1 \otimes u + u \otimes 1) + \theta(u \otimes u)], \end{aligned}$$

where the coefficients a, b, c, d, λ and θ are in \mathbb{Q} . Thus $\ker(0)/\text{im}(d^{-1}) \cong \langle 1 \otimes 1, u \otimes 1 \rangle$ and so the homology at $C^0(D)$ is $H(C^0(D)) = \langle 1 \otimes 1, u \otimes 1 \rangle \cong \mathbb{Q} \oplus \mathbb{Q}$

An element $v \in C^i(D)$ is said to have homological-grading i and q -grading $j = \deg(v) + i + n_+ - n_-$. For example if we consider the Hopf link and $H(C^{-2}(D)) = \langle u \otimes u, u \otimes 1 - 1 \otimes u \rangle \cong \mathbb{Q} \oplus \mathbb{Q}$, then the homological-grading for $u \otimes u$ and $u \otimes 1 - 1 \otimes u$ is $i = -2$. The q -grading for $u \otimes u$ is $j = \deg(u \otimes u) + -2 + 0 - 2 = -6$, where $\deg(u \otimes u) = \deg(u) + \deg(u) = -1 - 1 = -2$. Also the q -grading for $u \otimes 1 - 1 \otimes u$ is $j = -4$.

We summarise the computation of the Khovanov homology of the Hopf link in the following table with the homological-grading i and the q -grading j in horizontal and vertical respectively.

$j \backslash i$	-2	-1	0
0			\mathbb{Q}
-1			
-2			\mathbb{Q}
-3			
-4	\mathbb{Q}		
-5			
-6	\mathbb{Q}		

Table 3.1: Summarised Khovanov homology of the Hopf link

3.2 Khovanov homology versus the Jones polynomial

Let I_1 and I_2 be two knot invariants. We say I_1 is a stronger invariant than I_2 if I_1 tells more distinct knots(links) apart than I_2 . The Jones polynomial and the Khovanov homology are two strong knot invariants that are used in telling knots apart but the Khovanov homology is the stronger of the two. It tells all the distinct knots(links) that the Jones polynomial can and cannot distinguish apart. We illustrate this with an example below.

Let us consider examples of knots(links) that both the Jones polynomial and the Kkovanov homology tell apart. Let D_1 be the Hopf link and D_2 be the knot in Figure 1.2a. The unnormalised Jones polynomial for D_1 is $\widehat{J}(D_1) = 1 + q^{-2} + q^{-6} + q^{-4}$ and that for D_2 can be found to be $\widehat{J}(D_2) = q^{-3} + q^{-5} + q^{-7} - q^{-15}$. The summarised Khovanov homology for D_1 and D_2 are given in Tables 3.1 and 3.2 respectively.

The above example shows that $\widehat{J}(D_1) \neq \widehat{J}(D_2)$ and $H(C^*(D_1)) \neq H(C^*(D_2))$. Thus both the Jones polynomial and the Khovanov homology tell D_1 and D_2 apart.

Now let D_3 be the knot in Figure 1.2b. Then the unnormalised Jones polynomial for D_3 can be found to be $\widehat{J}(D_3) = q^{-3} + q^{-5} + q^{-7} - q^{-15}$. Thus $\widehat{J}(D_2) = \widehat{J}(D_3)$. We therefore see that the Jones polynomial cannot distinguish between D_2 and D_3 . The summarised Khovanov homology for D_3 is displayed in the following table. Observe from Tables 3.2 and 3.3 that $H(C^*(D_2)) \neq H(C^*(D_3))$. Thus the Khovanov homology tells D_2 and D_3 apart.

$j \backslash i$	-5	-4	-3	-2	-1	0
-3						\mathbb{Q}
-5						\mathbb{Q}
-7				\mathbb{Q}		
-9						
-11		\mathbb{Q}	\mathbb{Q}			
-13						
-15	\mathbb{Q}					

Table 3.2: Summarised Khovanov homology $H(C^i(D_2))$ of the link D_2 .

$j \backslash i$	-7	-6	-5	-4	-3	-2	-1	0
-1							\mathbb{Q}	\mathbb{Q}
-3								\mathbb{Q}
-5					\mathbb{Q}	$\mathbb{Q} \oplus \mathbb{Q}$		
-7				\mathbb{Q}				
-9				\mathbb{Q}	\mathbb{Q}			
-11		\mathbb{Q}	\mathbb{Q}					
-13								
-15	\mathbb{Q}							

Table 3.3: Summarised Khovanov homology $H(C^i(D_3))$ of the link D_3 .

The above example demonstrates that the Khovanov homology is a stronger invariant than the Jones polynomial.

Definition 3.2.1. Let $C := \dots \rightarrow C^{r-1} \xrightarrow{d^{r-1}} C^r \xrightarrow{d^r} C^{r+1} \xrightarrow{d^{r+1}} \dots$ be a cochain complex where the cochain groups are graded vector spaces, that is $C^r = \bigoplus_m C_m^r$ for all r . We say the differential d^r has degree k if $d^r(C_m^r) \subset C_{m+k}^{r+1}$ for all m .

Definition 3.2.2. Let $V = \bigoplus_m V_m$ be a graded vector space. Then the graded dimension is defined to be the power series $q \dim V := \sum_m q^m \dim V_m$.

For example if $V = \mathbb{Q}[1, u] = \bigoplus_{m \in \{-1, 1\}} V_m$ with $V_{-1}, V_1 = \mathbb{Q}$, then $q \dim V = q + q^{-1}$.

From the definition we have $q \dim V \cdot \{k\} = q^k \dim V$. If W is another graded vector space then we can have

$$q \dim(V \otimes W) = (q \dim V)(q \dim W) \text{ and } q \dim(V \oplus W) = q \dim V + q \dim W.$$

Definition 3.2.3. The graded Euler characteristic of a chain complex C , $\chi_q(C)$ is given by $\chi_q(C) = \sum_r (-1)^r q \dim(H^r)$, where $\dim(H^r)$ is the dimension of the homology groups of the complex.

It can be shown that if the degree of the differential is zero and if all the cochain groups are finite dimensional, then $\chi_q(C) = \sum_r (-1)^r q \dim(H^r) = \sum_r (-1)^r q \dim(C^r)$. Apart from being a stronger invariant than the Jones polynomial, the Khovanov homology also determines the unnormalised Jones polynomial. This is stated as a theorem:

Theorem 3.2.4. ([2], Theorem 1) *The graded Euler characteristic of the Khovanov complex $C^*(D)$ is the unnormalised Jones polynomial $\widehat{J}(D)$: $\chi_q(C^*(D)) = \widehat{J}(D)$.*

We illustrate Theorem 3.2.4 with an example. Consider the Khovanov complex of the Hopf link with cochain groups $C^{-2}(D) = V^{\otimes 2}\{-4\}$, $C^{-1}(D) = V\{-3\} \oplus V\{-3\}$ and $C^0(D) = V^{\otimes 2}\{-2\}$ in Example 3.1.3. The Euler characteristic of the complex is computed as follows:

$$\begin{aligned} \chi_q(C^*(D)) &= q \dim(C^{-2}(D)) + q \dim(C^{-1}(D)) + q \dim(C^0(D)) \\ &= q^{-4}(q + q^{-1})^2 - 2q^{-3}(q + q^{-1}) + q^{-2}(q + q^{-1})^2 \\ &= 1 + q^{-2} + q^{-4} + q^{-6} = \widehat{J}(D). \end{aligned}$$

The Khovanov homology is therefore very interesting to study and we do so via category theory.

3.3 Khovanov homology from a poset

In this section, we will look at how the Khovanov homology can be constructed from a poset and we follow the ideas used in [5] and [9].

Let D represent a link diagram with crossings labelled $1, 2, 3, \dots, n$. Let $\chi = \{1, 2, \dots, n\}$ be the set of crossings of D . Let $\mathbf{B} = \mathbf{B}_\chi$ be the poset (see Example 2.2.2) of subsets of χ ordered by reverse inclusion. We write $x \leq y$ when $y \subseteq x$ for the partial order and $x \prec y$ when x is obtained from y by adding a single element for $x, y \in \mathbf{B}$.

\mathbf{B} is a category and so we can construct a covariant functor $F_{KH} : \mathbf{B}^{op} \rightarrow \mathbf{Ab}$. Thus F_{KH} assigns abelian groups to the elements of \mathbf{B} and homomorphisms between the groups associated to comparable elements.

Definition 3.3.1. The functor F_{KH} is called the *Khovanov presheaf*.

The Hasse diagram formed from \mathbf{B} is the $|\chi|$ -dimensional cube with edges given by the relations $x \prec y$ for $x, y \in \mathbf{B}$. This cube is sent to the Khovanov cube by F_{KH} where each square face of the cube \mathbf{B} is sent to a commutative diagram of abelian groups.

Let $x \in \mathbf{B}$. If $c \in x$ then do a 1-resolution and if $c \notin x$ do a 0-resolution (see Figure 3.3). The result of these resolutions yield a collection $D(x)$ of closed planar circles. $D(x)$ is called the complete resolution associated to the set $x \in \mathbf{B}$. For example see Figure 3.4 for the complete resolution of the Hopf link where $\mathbf{B} = \text{Powerset}(\{1, 2\})$ in this case.



Figure 3.3: arrow to the left is a 0-resolution, arrow to the right is a 1-resolution.

Let $V = \mathbb{Z}[1, u]$ be the free abelian⁶ group generated by the set $\{1, u\}$ and let $F_{KH}(x) = V^{\otimes k}$ for each $x \in \mathbf{B}$ where k is the number of tensor factors corresponding to the planar circles in $D(x)$.

If $x \prec y$ is a covering relation in \mathbf{B} then $D(x)$ results from 1-resolving a crossing that was 0-resolved in $D(y)$. The outcome of this is that two of the circles in $D(y)$ fuse into one in $D(x)$ or one of the circles in $D(y)$ splits into two in $D(x)$. In the first case we have the map $F_{KH}(x \prec y) : F_{KH}(y) = V^{\otimes k} \rightarrow V^{\otimes k-1} = F_{KH}(x)$ defined by using the map m (see Section 3.1) on the tensor factors which corresponds to the fused circles and the identity on the rest. In the second case we have the map $F_{KH}(x \prec y) : F_{KH}(y) = V^{\otimes k} \rightarrow V^{\otimes k+1} = F_{KH}(x)$ defined by using Δ (see Section 3.1) on the tensor factors corresponding to the splitting of one circle into two and the identity on the rest.

Using the above descriptions, the Khovanov cochain complex can be constructed. We will now let K^* represent the Khovanov complex. Let $K^n = \bigoplus_{|x|=n} F_{KH}(x)$ be the n -cochain in K^* and this is the direct sum over $x \in \mathbf{B}$ of size $|x| = n$. To get the differentials, it is enough to add \pm signs to the edges of the cube of abelian groups so that the square faces anti-commute. Each square face has an odd number of $-$ sign on its edges. Let $[x, y]$ be the sign associated to the edge $x \prec y$ in \mathbf{B} . Define $[x, y] = (-1)^{\sum_{j < i} 1}$ where the sum is over $j \in x$ and i is the number corresponding to the crossing that was 0-resolved in $D(y)$ but 1-resolved in $D(x)$. Thus the differential $d : K^{n-1} \rightarrow K^n$ is given by

$$d = \sum_{x \prec y: |x|=n-1} [x, y] F_{KH}(x \prec y).$$

⁶A group G is free abelian if it is isomorphic to a direct sum of copies of the group of integers, that is $G \cong \bigoplus_{i \in I} \mathbb{Z}$, where I is some set.

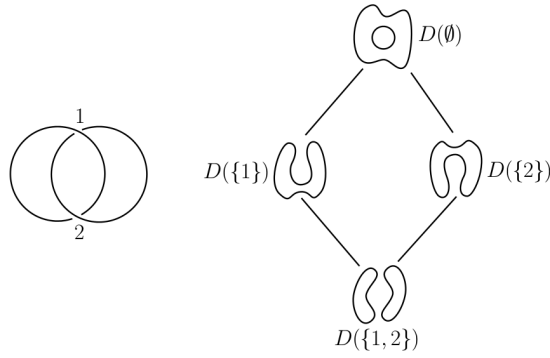


Figure 3.4: A projection of the Hopf link(left) and the corresponding diagram of complete resolutions of each set.

Definition 3.3.2. The *unnormalised Khovanov homology* of a link diagram D is defined as the cohomology of the Khovanov cochain complex: $\overline{KH}^*(D) = H(K^*, d)$.

The *normalised Khovanov homology* of an oriented link diagram D with m negative crossings is defined as $KH^*(D) = \overline{KH}^{*+m}(D)$.

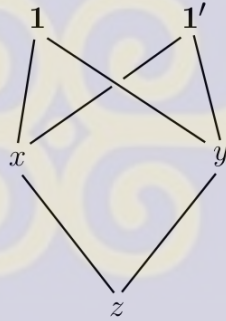


Figure 3.5: Hasse diagram of the poset $\mathbf{Q} = \{\mathbf{1}, x, y, z\} \cup \{\mathbf{1}'\}$ obtained from the Hopf link

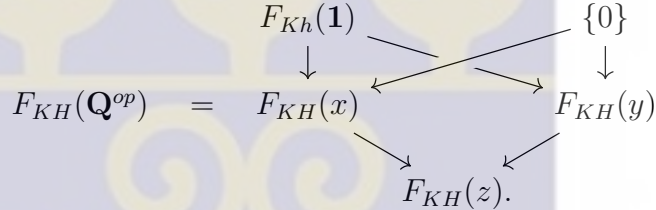
The poset \mathbf{B} has a unique maximal element $\mathbf{1} \in \mathbf{B}$ which represents the empty subset of χ and we have $x \leq \mathbf{1}$ for all $x \in \mathbf{B}$. Now add another maximal element $\mathbf{1}'$ to \mathbf{B} such that $x \leq \mathbf{1}'$ with $x \neq \mathbf{1}$ to form another poset. The poset formed will be denoted by $\mathbf{Q} = \mathbf{Q}_x$. Let $F_{KH}(\mathbf{1}') = 0$ and let $F_{KH}(x \leq \mathbf{1}') : F_{KH}(\mathbf{1}') \longrightarrow F_{KH}(x)$ be the only possible homomorphism. Then we can extend F_{KH} to the functor $F_{KH} : \mathbf{Q}^{op} \longrightarrow \mathbf{Ab}$. We construct the cochain complex K^* over \mathbf{Q}^{op} in the same way we constructed the one over \mathbf{B}^{op} with some little inclusions. Let $[x, \mathbf{1}'] = -1$ for all $x \prec \mathbf{1}'$ and let $[x, y]$ be defined as before for all edges $x \leq y$. $K^0 = F_{KH}(\mathbf{1}) \oplus \{0\}$. The homology resulting from this slight modification is nothing other than the unnormalised Khovanov homology.

3.4 The limit and its derived functors

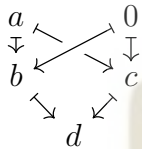
Let \mathcal{C} be a small category and $F : \mathcal{C}^{op} \rightarrow \mathbf{Ab}$ be a (covariant) functor. The limit (see Definition 2.2.14) $\varprojlim_{\mathcal{C}^{op}} F$ is an abelian group constructed in the following way. Take a subgroup of the product $\prod_{x \in \text{obj}(\mathcal{C})} F(x)$ consisting of the tuples $(a_x)_{x \in \text{obj}(\mathcal{C})}$ so that for all morphisms $x \rightarrow y$, the induced map $F_x^y : F(y) \rightarrow F(x)$ sends a_y to a_x . That is

$$\varprojlim_{\mathcal{C}^{op}} F = \left\{ (a_x)_{x \in \text{obj}(\mathcal{C})} \in \prod_{x \in \text{obj}(\mathcal{C})} F(x) \mid F_x^y(a_y) = a_x \text{ for all } x \rightarrow y \text{ in } \mathcal{C} \right\}.$$

Let $F = F_{KH}$. Then we will show that $\varprojlim_{\mathbf{Q}^{op}} F_{KH} = \ker d^0$ where d^0 is the differential of the cochain complex K^* with degree zero. Let us consider the poset \mathbf{Q} (see Figure 3.5) obtained from the Hopf link. F_{KH} on the opposite category \mathbf{Q}^{op} gives

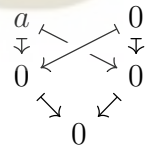


Now the elements of $\varprojlim F_{KH}(\mathbf{Q}^{op}) = \varprojlim_{\mathbf{Q}^{op}} F_{KH}$ are all 5-tuples formed from the set $S = \{0, a, b, c, d : 0 \in \{0\}, a \in F_{KH}(\mathbf{1}), b \in F_{KH}(x), c \in F_{KH}(y), d \in F_{KH}(z)\}$. Each element of S appears only once in each possible tuple. All the 5-tuples must be of the form:



. Now since $0 \mapsto 0$ under the homomorphism, we must have $b = c = 0$ which

implies $d = 0$ and so we have the tuples being of the form



. This is equivalent

to all $a \in F_{KH}(\mathbf{1})$ such that $a \mapsto 0$ under both maps $F_{KH}(\mathbf{1}) \rightarrow F_{KH}(x)$ and $F_{KH}(\mathbf{1}) \rightarrow F_{KH}(y)$. This is by definition the kernel of the differential d^0 of the Khovanov complex K^* . Thus $\varprojlim_{\mathbf{Q}^{op}} F_{KH} = \ker d^0$. The homology at K^0 is nothing but $\overline{KH}^0 = \ker d^0$ and so we have

$$\varprojlim_{\mathbf{Q}^{op}} F_{KH} \cong \overline{KH}^0(D). \tag{3.1}$$

The above isomorphism (3.1) is also true for the general case because of the presence of $F_{KH}(\mathbf{1}') = \{0\}$. This tells us that the limit of the presheaf $F_{KH} : \mathbf{Q}^{op} \rightarrow \mathbf{Ab}$ captures the unnormalised Khovanov homology in degree zero.

Let us consider the constant presheaf $\Delta A : \mathcal{C}^{op} \rightarrow \mathbf{Ab}$ in example 2.2.10. If $f : A \rightarrow B$ is a map of abelian groups in \mathbf{Ab} then there is a natural transformation (see Definition 2.2.11) $\eta : \Delta A \rightarrow \Delta B$ with $\eta_x = f : \Delta A(x) \rightarrow \Delta B(x)$. From this, we can have the constant presheaf functor $\Delta : \mathbf{Ab} \rightarrow \mathbf{pSh}(\mathcal{C})$. Recall from Remark 2.2.15 that $\varprojlim : \mathbf{pSh}(\mathcal{C}) \rightarrow \mathbf{Ab}$ is a covariant functor.

Proposition 3.4.1. *Let $A \in \mathbf{Ab}$ and $F \in \mathbf{pSh}(\mathcal{C})$. Then there are natural bijections*

$$\mathrm{Hom}_{\mathbf{pSh}(\mathcal{C})}(\Delta A, F) \cong \mathrm{Hom}_{\mathbb{Z}}(A, \varprojlim F). \quad (3.2)$$

Proof. Let $\tau_{A,F} : \mathrm{Hom}_{\mathbf{pSh}(\mathcal{C})}(\Delta A, F) \rightarrow \mathrm{Hom}_{\mathbb{Z}}(A, \varprojlim F)$ be defined by

$$\tau_{A,F}(\alpha) = \alpha_x : \Delta A(x) \rightarrow F(x)$$

where $x \in \mathcal{C}$ and $\alpha : \Delta A \rightarrow F$ is a natural transformation in $\mathbf{pSh}(\mathcal{C})$. Now α_x is an element of $\mathrm{Hom}_{\mathbb{Z}}(A, \varprojlim F)$ since for all $a \in A$, $\alpha_x(a) \in F(x)$ is a component of $\varprojlim F$.

For naturality, let $f : A \rightarrow B$ be in \mathbf{Ab} and $g : F \rightarrow G$ be a natural transformation in $\mathbf{pSh}(\mathcal{C})$. Then we show that the following diagrams commute, that is

(i) $f^* \tau_{B,F} = \tau_{A,F} (\Delta f)^*$.

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbf{pSh}(\mathcal{C})}(\Delta B, F) & \xrightarrow{\tau_{B,F}} & \mathrm{Hom}_{\mathbb{Z}}(B, \varprojlim F) \\ \downarrow (\Delta f)^* & & \downarrow f^* \\ \mathrm{Hom}_{\mathbf{pSh}(\mathcal{C})}(\Delta A, F) & \xrightarrow{\tau_{A,F}} & \mathrm{Hom}_{\mathbb{Z}}(A, \varprojlim F). \end{array}$$

Let $\alpha \in \mathrm{Hom}_{\mathbf{pSh}(\mathcal{C})}(\Delta B, F)$, then

$$f^* \tau_{B,F}(\alpha) = f^*(\alpha_x) = \alpha_x f \text{ and } \tau_{A,F}(\Delta f)^*(\alpha) = \tau_{A,F}(\alpha \Delta f) = (\alpha \Delta f)_x = \alpha_x f.$$

And

(ii) $(\varprojlim g)_* \tau_{A,F} = \tau_{A,G} g_*$.

$$\begin{array}{ccc}
 \mathrm{Hom}_{\mathbf{pSh}(\mathcal{C})}(\Delta A, F) & \xrightarrow{\tau_{A,F}} & \mathrm{Hom}_{\mathbb{Z}}(A, \varprojlim F) \\
 \downarrow g_* & & \downarrow (\varprojlim g)_* \\
 \mathrm{Hom}_{\mathbf{pSh}(\mathcal{C})}(\Delta A, G) & \xrightarrow{\tau_{A,G}} & \mathrm{Hom}_{\mathbb{Z}}(A, \varprojlim G).
 \end{array}$$

For $\alpha \in \mathrm{Hom}_{\mathbf{pSh}(\mathcal{C})}(\Delta A, F)$ we have

$$(\varprojlim g)_* \tau_{A,F}(\alpha) = (\varprojlim g)_* \alpha_x = \varprojlim g \alpha_x \text{ and } (\tau_{A,G})_* g_*(\alpha) = \tau_{A,G}(g\alpha) = (g\alpha)_x = g_x \alpha_x,$$

where $g_x = \varprojlim g$.

For bijectivity, observe that for all $\alpha, \eta : \Delta A \rightarrow F$, if $\tau_{A,F}(\alpha) = \tau_{A,F}(\eta)$ then $\alpha_x = \eta_x$ implies $\alpha = \eta$ and by universality of $\varprojlim F$, we see that for each $x \in \mathcal{C}$, and $\alpha_x \in \mathrm{Hom}_{\mathbb{Z}}(A, \varprojlim F)$, there is $\alpha \in \mathrm{Hom}_{\mathbf{pSh}(\mathcal{C})}(\Delta A, F)$ such that $\tau_{A,F}(\alpha) = \alpha_x$. Thus (3.2) holds. \square

Remark 3.4.2. Proposition 3.4.1 tells us that \varprojlim is right adjoint (see Definition 2.2.16) to Δ and Proposition 2.2.13 tells us that \varprojlim is left exact. We can therefore have right derived functors (see Subsection 2.2.22) $R^i \varprojlim : \mathbf{pSh}(\mathcal{C}) \rightarrow \mathbf{Ab}$ of \varprojlim . Let $\varprojlim^i := R^i \varprojlim$. By proposition 2.2.23, we have $\varprojlim^0 \cong \varprojlim$. We call the right derived functors of the limit functor **higher limits** and we will see in Section 4.1 that these higher limits are the cohomology of a small category with coefficients in a presheaf.

We give a Corollary to Proposition 3.4.1.

Corollary 3.4.3. *Let $F \in \mathbf{pSh}(\mathcal{C})$. Then there is an isomorphism*

$$\mathrm{Hom}_{\mathbf{pSh}(\mathcal{C})}(\Delta \mathbb{Z}, F) \cong \varprojlim F. \quad (3.3)$$

Proof. Define $h : \mathrm{Hom}_{\mathbf{pSh}(\mathcal{C})}(\Delta \mathbb{Z}, F) \rightarrow \varprojlim F$ by $h(\eta) = (\eta_x(1))_{x \in \mathcal{C}}$.

Let $\tau, \sigma \in \mathrm{Hom}_{\mathbf{pSh}(\mathcal{C})}(\Delta \mathbb{Z}, F)$ and let $x \in \mathcal{C}$. Then $h(\tau\sigma) = ((\tau\sigma)_x(1)) = ((\tau_x\sigma_x)(1)) = h(\tau)h(\sigma)$.

Now for h being a bijection, observe that if $h(\tau) = h(\sigma)$, then $(\tau_x(1)) = (\sigma_x(1))$ implies $\tau_x(1) = \sigma_x(1)$ which implies $\tau = \sigma$. Also by universality of $\varprojlim F$, we see that h is a surjection. Hence h is an isomorphism. \square

Remark 3.4.4. Corollary 3.4.3, tells us that we have a natural isomorphism of functors $\varprojlim \cong \text{Hom}_{\mathbf{pSh}(\mathcal{C})}(\Delta\mathbb{Z}, -)$ and hence

$$\varprojlim^i \cong R^i \text{Hom}_{\mathbf{pSh}(\mathcal{C})}(\Delta\mathbb{Z}, -) \text{ for all } i \geq 0. \quad (3.4)$$

The following theorem tells us that the derived functors of the covariant hom functor $\text{Hom}_{\mathbf{pSh}(\mathcal{C})}(\Delta\mathbb{Z}, -)$ can be replaced by the derived functors of the contravariant hom functor.

Theorem 3.4.5. ([10], Theorem 2.7.6) *For every pair of S -modules A and B , and for all n , $R^n \text{Hom}_S(A, -)(B) \cong R^n \text{Hom}_S(-, B)(A)$.*

It can be shown that Theorem 3.4.5 holds if the S -modules are replaced by presheaves and we state it as a proposition below.

Proposition 3.4.6. *Let F, G be presheaves over the small category \mathcal{C} . Then*

$$R^i \text{Hom}_{\mathbf{pSh}(\mathcal{C})}(F, -)(G) \cong R^i \text{Hom}_{\mathbf{pSh}(\mathcal{C})}(-, G)(F).$$

for all $i \geq 0$.

Let $F \in \mathbf{pSh}(\mathcal{C})$. Then (3.4) tells us that

$$\varprojlim^i(F) \cong R^i \text{Hom}_{\mathbf{pSh}(\mathcal{C})}(\Delta\mathbb{Z}, -)(F) \cong R^i \text{Hom}_{\mathbf{pSh}(\mathcal{C})}(-, F)(\Delta\mathbb{Z}). \quad (3.5)$$

for all $i \geq 0$.

3.5 Khovanov homology and the higher limits of F_{KH}

It turns out that the derived functors of the limit of the Khovanov presheaf can be used to describe the Khovanov homology. Everitt and Turner in their paper (see [5]) gave a theorem to this description. We have already seen that the limit of the Khovanov presheaf describes the unnormalised Khovanov homology in degree zero. We look at the general description in this section.

3.5.1 A projective resolution of $\Delta\mathbb{Z}$

Projective resolutions are needed to compute the right derived functors of a contravariant functor (see Definition 2.2.7) and $\text{Hom}_{\mathbf{pSh}(C)}(-, F)$ is no exception.

Recall the poset $\mathbf{Q} = \mathbf{Q}_X$ described in Section 3.3. Let us identify \mathbf{Q} with the poset of cells of a *regular CW complex* (see Section A.2).

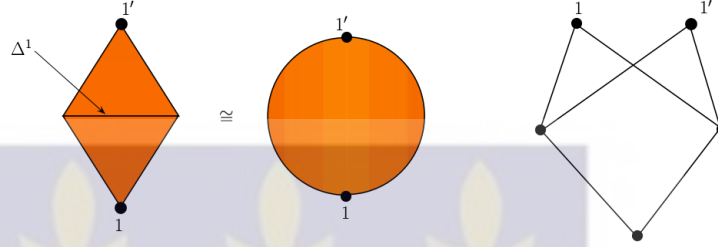


Figure 3.6: Regular CW complex $X = S\Delta^1$ (left) with cell poset \mathbf{Q}_X (right) for $|\chi| = 2$.

Suppose $|\chi| = n$ and let Δ^{n-1} be an $(n-1)$ -simplex. Let X be the suspension (see Section A.3) $S\Delta^{n-1}$ of Δ^{n-1} and take the CW decomposition of X with two 0-cells (the suspension points) 1 and $1'$ and all other cells. Suspension of cells preserves the inclusion $y \subseteq \bar{x}$ and so define a partial order on the cells of X by $x \leq y$ whenever $y \subseteq \bar{x}$. An element $x \in \mathbf{Q}$ corresponds to an $|x|$ -cell of X . The case when $|\chi| = n = 2$ is shown in Figure 3.6. The suspension points 1 and $1'$ are the elements corresponding to the maximal element in $x \in \mathbf{Q}$ and so we have that X has a cell poset \mathbf{Q} .

Let us construct a presheaf $P_n, n \in \mathbb{N}_{>0}$ in $\mathbf{pSh}(\mathbf{Q})$ where \mathbf{Q} is the cell poset of the regular CW complex X . Now for $x \in \mathbf{Q}$, set $P_n(x) := \mathbb{Z}[n\text{-cells of } X \text{ contained in the closure of the cell } x]$. Thus

$$P_n(x) = \begin{cases} 0 & \text{if } \dim x < n \\ \mathbb{Z}[x] & \text{if } \dim x = n \\ \mathbb{Z}[a \mid \dim(a) = n, a \in \bar{x}] & \text{if } \dim x > n \end{cases} .$$

If $x \leq y$ is a relation in \mathbf{Q} , then the map $P_n(x \leq y) : P_n(y) \longrightarrow P_n(x)$ will be the inclusion.

Proposition 3.5.2. For $F \in \mathbf{pSh}(\mathbf{Q})$ the map

$$f^n : \text{Hom}_{\mathbf{pSh}(\mathbf{Q})}(P_n, F) \longrightarrow \bigoplus_{\dim x=n} F(x)$$

defined by $f^n(\tau) = \sum_{\dim x=n} \tau_x(x)$, is an isomorphism of abelian groups.

Proof. Let $F \in \mathbf{pSh}(\mathbf{Q})$ and f^n be as in the proposition. We show first that f^n is a homomorphism.

Let $\tau, \gamma \in \text{Hom}_{\mathbf{pSh}(\mathbf{Q})}(P_n, F)$, then

$$f^n(\tau + \gamma) = \sum_{\dim x=n} (\tau + \gamma)_x(x) = \sum_{\dim x=n} \tau_x(x) + \sum_{\dim x=n} \gamma_x(x) = f^n(\tau) + f^n(\gamma),$$

since $(\tau + \gamma)_x = \tau_x + \gamma_x$. Thus f^n is a homomorphism.

We now show that f^n is a bijection.

(i) For injectivity we show that $\ker f^n = \{0\}$. We have that $f^n(\tau) = 0$ implies $\tau_x(x) = 0$ for all n -cells $x \in \mathbf{Q}$. To show that $\tau = 0$, we show that $\tau_a : P_n(a) \rightarrow F(a)$ is zero for all $a \in \mathbf{Q}$.

- If $\dim a < n$, then we have $P_n(a) = 0$ and so there is nothing to show.
- If $\dim a = n$, the $P_n(a) = \mathbb{Z}[a]$ and so $\tau_a(a) = 0$ since a is an n -cell.
- If $\dim a > 0$, then $P_n(a) = \mathbb{Z}[a_i \mid \dim(a_i) = n, a_i \in \bar{a}]$. We have $a \leq a_i$ and so consider the following diagram:

$$\begin{array}{ccc} P_n(a_i) & \xrightarrow{\tau_{a_i}} & F(a_i) \\ \downarrow P_n(a \leq a_i) & & \downarrow F(a \leq a_i) \\ P_n(a) & \xrightarrow{\tau_a} & F(a). \end{array}$$

From the diagram, we have

$$\begin{aligned} \tau_a(a_i) &= \tau_a(P_n(a \leq a_i)(a_i)) \\ &= F(a \leq a_i)(\tau_{a_i}(a_i)), \tau \text{ is a natural transformation} \\ &= F(a \leq a_i)(0), \text{ since } a_i \text{ is an } n\text{-cell} \\ &= 0, \text{ by inclusion.} \end{aligned}$$

We see from the above computation that $\tau_a(a) = 0$ for all $a \in \mathbf{Q}$ which implies $\tau_a = 0$. Thus $\tau = 0$ and so $\ker f^n = \{0\}$ which implies injectivity of f^n .

(ii) For each $x \in \mathbf{Q}$ with $\dim x = n$, let $\sum_{\dim x=n} \tau_x(x) \in \bigoplus_{\dim x=n} F(x)$. The component $\tau_x(x) \in F(x)$ specifies the map $\tau_x : P_n(x) \rightarrow F(x)$ which in turn specifies the natural

transformation $\tau : P_n \rightarrow F$. Thus for all $\sum_{\dim x=n} \tau_x(x) \in \bigoplus_{\dim x=n} F(x)$, there exists a $\tau \in \text{Hom}_{\mathbf{pSh}(\mathbf{Q})}(P_n, F)$ such that $f^n(\tau) = \sum_{\dim x=n} \tau_x(x)$. Hence f^n is surjective.

This completes the proof of the proposition. \square

The preceding proposition makes it possible for us to define a morphism $\tau : P_n \rightarrow F$ by specifying a tuple $\sum_{\dim x=n} \alpha_x \in \bigoplus F(x)$, $\alpha_x = \tau_x(x)$. Consider the following diagram of presheaves where the row is exact.

$$\begin{array}{ccc} & P_n & \\ & \downarrow \tau & \\ G & \xrightarrow{\sigma} & F \longrightarrow 0. \end{array}$$

Let $x \in \mathbf{Q}$. Then the local maps $\sigma_x : G(x) \rightarrow F(x)$ are surjections by exactness of the row. This implies that if $\sum_{\dim x=n} \alpha_x \in \bigoplus F(x)$ specifies the map $\tau : P_n \rightarrow F$ then for each x , there is a β_x such that $\sigma_x(\beta_x) = \alpha_x$. Thus there is a morphism $\tau' : P_n \rightarrow G$ specified by $\sum_{\dim x=n} \beta_x, \tau'_x(x)$. This makes the following diagram commute.

$$\begin{array}{ccc} & P_n & \\ & \swarrow \tau' & \downarrow \tau \\ G & \xrightarrow{\sigma} & F \longrightarrow 0. \end{array}$$

The above description says that the P_n 's are projective presheaves.

We are now ready to construct the projective resolution of $\Delta\mathbb{Z}$. Let $\delta_n : P_n \rightarrow P_{n-1}$ be a natural transformation. For $x \in \mathbf{Q}$ let $\delta_{n,x} : P_n(x) \rightarrow P_{n-1}(x)$ be the homomorphism defined by

$$\delta_{n,x}(y) = \sum_{y \prec z} [y, z]z$$

for any y an n -cell, $y \subset \bar{x}$ and the sum is over the $(n-1)$ -cells $z \subset \bar{y}$. We define $[y, z]$ in the same way as in Section 3.3. That is $[y, z] = (-1)^{\sum_{j < i} 1}$ where the $j \in y$, $|y| = n$ and i is the number corresponding to the crossing that was 0-resolved in $D(z)$ but 1-resolved in $D(y)$, $|z| = n-1$. These homomorphisms assemble into a morphism of presheaves $\delta_n : P_n \rightarrow P_{n-1}$ since for each $a \leq b$ in \mathbf{Q} the following diagram commutes:

$$\begin{array}{ccc}
 P_n(b) & \xrightarrow{\delta_{n,b}} & P_{n-1}(b) \\
 \downarrow P_n(a \leq b) & & \downarrow P_{n-1}(a \leq b) \\
 P_n(a) & \xrightarrow{\delta_{n,a}} & P_{n-1}(a).
 \end{array}$$

That is

$$j_* \delta_{n,b}(y) = j_* \left(\sum_{y \prec z} [y, z] z \right) = \sum_{y \prec z} [y, z] z, b \prec y \text{ and } \delta_{n,a} i_*(y) = \delta_{n,a}(y) = \sum_{y \prec z} [y, z] z, a \prec y$$

where $i_* = P_n(a \leq b)$ and $j_* = P_{n-1}(a \leq b)$ are the inclusion maps. Proposition 2.2.17 says that the sequence $P_* = \dots \xrightarrow{\delta_{n+1}} P_{n+1} \xrightarrow{\delta_n} P_n \xrightarrow{\delta_{n-1}} P_{n-1} \xrightarrow{\delta_{n-2}} \dots$ is exact at P_n if and only if each of the local sequences $P_*(x)$ is exact at $P_n(x)$. We will now see that the sequence P_* is exact at P_n for $n > 0$ using the cellular homology of the $\dim(x)$ -dimensional ball.

Cellular homology of the $\dim(x)$ -dimensional ball

In this subsection we look at the cellular homology of the $\dim(x)$ -dimensional ball which corresponds to the closure of the $\dim(x)$ -cell x in the cell poset \mathbf{Q} . The CW decomposition of the $\dim(x)$ -dimensional ball corresponding to $x \in \mathbf{Q}$ has two 0-cells and other cells in the decomposition with their dimensions less or equal to $\dim(x)$.

Let \mathfrak{B} be the $\dim(x)$ -dimensional ball. Then the cellular homology groups of the cellular chain complex $C_*(\mathfrak{B})$ are

$$H_k C_*(\mathfrak{B}) = \begin{cases} 0 & \text{if } k \neq 0 \\ \mathbb{Z} & \text{if } k = 0. \end{cases}$$

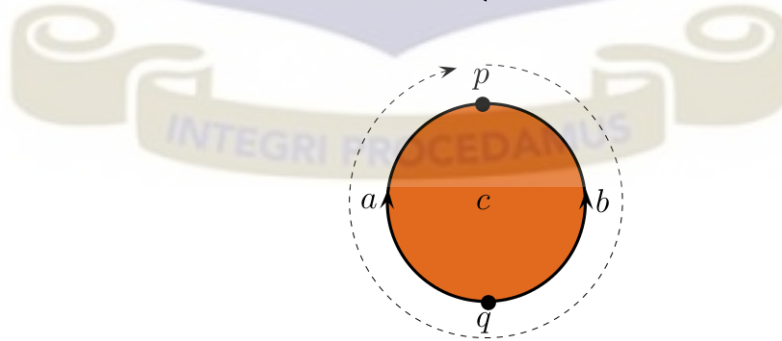


Figure 3.7: An oriented 2-dimensional ball

We illustrate this with an example using a 2-dimensional ball. Let us consider the oriented 2-dimensional ball in Figure 3.7 and let \mathfrak{b} represent this ball. There are two 0-cells $\{p, q\}$,

two 1-cells $\{a, b\}$ and one 2-cell $\{c\}$.

We have $\dots = C_4(\mathbf{b}) = C_3(\mathbf{b}) = \{0\}$ since $\dim x < n$, where $n = 3, 4, 5, \dots$. Also we have

$$C_2(\mathbf{b}) = \mathbb{Z}[c] \cong \mathbb{Z}, C_1(\mathbf{b}) = \mathbb{Z}[a, b] \cong \mathbb{Z}^2 \text{ and } C_0(\mathbf{b}) = \mathbb{Z}[p, q] \cong \mathbb{Z}^2.$$

The cellular chain complex is: $\{0\} \xrightarrow{0} \mathbb{Z}[c] \xrightarrow{d_2} \mathbb{Z}[a, b] \xrightarrow{d_1} \mathbb{Z}[p, q] \xrightarrow{d_0} \{0\}$. Now we obtain the differentials (boundary maps) d_1 and d_2 by observing what they do on the generators.

When we act d_2 on c by considering the direction shown by the dashed arrow, we get $d_2(c) = a - b$. Observe that d_2 is injective and so $\ker d_2 = \{0\}$. Thus the homology at $\mathbb{Z}[c]$ is $H_2C_*(\mathbf{b}) = \ker d_2 / \text{im}(0) = \{0\}$

Also when we act d_1 on each of a and b , we get $p - q$. We have that $d_1(a) = d_1(b)$ implies $d_1(a - b) = 0$. Thus $a - b$ generates the kernel of d_1 and so $\ker d_1 = \mathbb{Z}[a - b]$. The image of d_2 is $\text{im}(d_2) = \mathbb{Z}[a - b]$. The homology at $\mathbb{Z}[a, b]$ is therefore $H_1C_*(\mathbf{b}) = \mathbb{Z}[a - b] / \mathbb{Z}[a - b] = \{0\}$.

Finally, $\ker d_0 = \mathbb{Z}[p, q]$ and $\text{im}(d_1) = \mathbb{Z}[p - q]$. An element in $\ker(d_0)$ is of the form $tp + sq$, $t, s \in \mathbb{Z}$ and so an element in the quotient $\ker d_0 / \text{im}(d_1)$ is of the form $(tp + sq) + \text{im}(d_1)$ which we can rewrite as $(t + s)q + \text{im}(d_1)$, $(t + s) \in \mathbb{Z}$. The homology at $\mathbb{Z}[p, q]$ is therefore $H_0C_*(\mathbf{b}) = \{(t + s)q + \text{im}(d_1)\} \cong \mathbb{Z}$.

In conclusion to the example, we see that

$$H_kC_*(\mathbf{b}) = \begin{cases} 0 & \text{if } k \neq 0 \\ \mathbb{Z} & \text{if } k = 0. \end{cases}$$

Now let x be a cell in \mathbf{Q} . Since x is homeomorphic to the $\dim(x)$ -dimensional ball ($\mathbf{B}^{\dim(x)}$) by definition of a cell, we have that $P_*(x) = C_*(\mathbf{B}^{\dim(x)})$ and so

$$H_k((P_*(x))) = \begin{cases} 0 & \text{if } k \neq 0 \\ \mathbb{Z} & \text{if } k = 0, \end{cases}$$

which tells us that $P_*(x)$ is exact. Thus P_* is also exact by Proposition 2.2.17 in degree $k > 0$.

We can now construct a projective resolution of $\Delta\mathbb{Z}$. Let $\epsilon : P_0 \rightarrow \text{coker}(\delta_1)$ be the canonical surjection where $\text{coker}(\delta_1) = P_0 / \text{im}(\delta_1)$ is the cokernel of δ_1 .

Observe from the calculation of $P_*(x)$ that $\ker(0) = P_0(x)$ for $x \in \mathbf{Q}$ and so

$$\text{coker}(\delta_{1,x}) = P_0(x) / \text{im}(\delta_{1,x}) = H_0((P_*(x))) \cong \mathbb{Z} = \Delta\mathbb{Z}(x).$$

This shows that $\text{coker}(\delta_1) = P_0/\text{im}(\delta_1) \cong \Delta\mathbb{Z}$. Thus we can have the following projective resolution of $\Delta\mathbb{Z}$:

$$\dots \xrightarrow{\delta_2} P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\epsilon} \Delta\mathbb{Z} \xrightarrow{0} \{0\}.$$

Applying the functor $\text{Hom}_{\mathbf{pSh}(C)}(-, F)$ to the complex

$$P_* = \dots \xrightarrow{\delta_2} P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{0} \{0\}$$

we get the following cochain complex

$$\{0\} \xrightarrow{0} \text{Hom}_{\mathbf{pSh}(C)}(P_1, F) \xrightarrow{\delta_1^*} \text{Hom}_{\mathbf{pSh}(C)}(P_2, F) \xrightarrow{\delta_2^*} \text{Hom}_{\mathbf{pSh}(C)}(P_3, F) \xrightarrow{\delta_3^*} \dots,$$

from which we compute the right derived functors $\varprojlim^i(F) \cong R^i \text{Hom}_{\mathbf{pSh}(C)}(-, F)(\Delta\mathbb{Z})$ (see Equation 3.5). Letting $F = F_{KH}$ in the preceding chain complex, we get the right derived functors $\varprojlim^i F_{KH}$, that is the higher limits of F_{KH} .

3.5.3 The Relation between Khovanov homology and the higher limits of F_{KH}

Now the following theorem given by Everitt and Turner in [5] shows that the right derived functors $\varprojlim^i F_{KH}$ describe the unnormalised Khovanov homology.

Theorem 3.5.4. *Let D be a link diagram and let $F_{KH} : \mathbf{Q}^{op} \rightarrow \mathbf{Ab}$ be the Khovanov presheaf. Then $\overline{KH}^i(D) \cong \varprojlim_{\mathbf{Q}^{op}}^i F_{KH}$.*

Proof. We need to show that there is an isomorphism $f : \text{Hom}_{\mathbf{pSh}(\mathbf{Q})}(P_*, F_{KH}) \rightarrow K^*$, where K^* is the Khovanov cochain complex.

We will show that f is a morphism of cochain complexes which means that the following diagram commutes. That is $f^n \delta_n^* = d f^{n-1}$.

$$\begin{array}{ccc} \text{Hom}_{\mathbf{pSh}(\mathbf{Q})}(P_{n-1}, F_{KH}) & \xrightarrow{\delta_n^*} & \text{Hom}_{\mathbf{pSh}(\mathbf{Q})}(P_n, F_{KH}) \\ f^{n-1} \downarrow & & \downarrow f^n \\ K^{n-1} & \xrightarrow{d} & K^n \end{array}$$

Recall that $K^n = \bigoplus_{|x|=n} F_{KH}(x)$ and $d = \sum [x, y] F_{KH}(x \prec y)$ as defined before. Also define $f^n : \text{Hom}_{\mathbf{pSh}(\mathbf{Q})}(P_n, F_{KH}) \rightarrow K^n$ by $f^n(\mu) = \sum_{\dim x=n} \mu_x(x)$ as in Proposition 3.5.2.

Let $\mu \in \text{Hom}_{\mathbf{pSh}(\mathbf{Q})}(P_{n-1}, F_{KH})$ and let x be an n -cell in \mathbf{Q} .

First of all we compute $f^n \delta_n^*(\mu)$. Applying the induced map δ_n^* to μ , we get $\delta_n^*(\mu) = \mu \delta_n$ and so

$$\begin{aligned}
 f^n(\mu \delta_n) &= \sum_{\dim x=n} (\mu \delta_n)_x(x) \\
 &= \sum_{\dim x=n} \mu_x \delta_{n,x}(x), \text{ by composition of natural transformations} \\
 &= \sum_{\dim x=n} \mu_x \left(\sum_{x \prec z} [x, z] z \right), z \text{ is an } (n-1)\text{-cell} \\
 &= \sum_{\dim x=n} \sum_{x \prec z} [x, z] \mu_x(x) \\
 &= \sum_{\dim x=n} \sum_{x \prec z} [x, z] F_{KH}(x \prec z)(\mu_z(z)) \text{ by naturality.}
 \end{aligned}$$

Secondly we compute $df^{n-1}(\mu)$. Let z be an $(n-1)$ -cell. Then

$$\begin{aligned}
 df^{n-1}(\mu) &= d \left(\sum_{\dim z=n-1} \mu_z(z) \right) \\
 &= \sum_{\dim x=n} [x, z] F_{KH}(x \prec z) \left(\sum_{\dim z=n-1} \mu_z(z) \right) \\
 &= \sum_{\dim x=n} \sum_{\dim z=n-1} [x, z] F_{KH}(x \prec z)(\mu_z(z)) \\
 &= \sum_{\dim x=n} \sum_{x \prec z} [x, z] F_{KH}(x \prec z)(\mu_z(z)).
 \end{aligned}$$

The above computations show that $f^n \delta_n^* = df^{n-1}$ and so f is a morphism of the cochain complexes $\text{Hom}_{\mathbf{pSh}(\mathbf{Q})}(P_*, F_{KH})$ and K^* . This, together with the fact that

$$f^n : \text{Hom}_{\mathbf{pSh}(\mathbf{Q})}(P_n, F_{KH}) \longrightarrow K^n$$

is an isomorphism for all n by Proposition 3.5.2 shows that f is an isomorphism of cochain complexes. Thus

$$\text{Hom}_{\mathbf{pSh}(\mathbf{Q})}(P_*, F_{KH}) \cong K^*.$$

When we compute the homology, we get $R^i \text{Hom}_{\mathbf{pSh}(\mathbf{Q})}(P_*, F_{KH}) \cong \overline{KH}^i$. But by Proposition 3.4.6, we have $\varprojlim_{\mathbf{Q}^{op}}^i F_{KH} \cong R^i \text{Hom}_{\mathbf{pSh}(\mathbf{Q})}(P_*, F_{KH}) \cong \overline{KH}^i(D)$, which completes the proof of Theorem 3.5.4. \square

Chapter 4

Khovanov homology and presheaves

This chapter is based on the paper [4] by Everitt and Turner.

4.1 Khovanov homology and the classifying space BQ

A *simplicial set* X over the category of sets **Sets**, consists of

- (i) for every integer $n \geq 0$ a set $X^n \in \mathbf{Sets}$ and
- (ii) for every pair of integers (i, n) with $0 \leq i \leq n$, *face* and *degeneracy maps*

$$d_i : X^n \rightarrow X^{n-1} \text{ and } s_i : X^n \rightarrow X^{n+1}$$

satisfying the following identities:

$$d_i d_j = d_{j-1} d_i \text{ for } i < j.$$

$$d_i s_j = s_{j-1} d_i \text{ for } i < j.$$

$$d_i s_j = \begin{cases} id & \text{for } i = j, j + 1 \\ s_j d_{i-1} & \text{for } i > j + 1. \end{cases}$$

$$s_i s_j = s_j s_{i-1} \text{ for } i > j.$$

The elements of X^n are called n -simplicies. Let $x \in X_n$, then x is called *degenerate* if $x = s_i y$ for some $y \in X^{n-1}$ and for some i , otherwise it is called *non-degenerate*. It turns out that the simplicial set is the same as the covariant functor $X : \mathbf{SCat}^{op} \rightarrow \mathbf{Sets}$ in Example 2.2.10(ii).

Let \mathcal{C} be a small category. For every object $[n]$ in the simplex category \mathbf{SCat} (see Example 2.2.2(v)) the *nerve* $N^*\mathcal{C}$ of \mathcal{C} is the simplicial set $N^*\mathcal{C} : \mathbf{SCat}^{op} \rightarrow \mathbf{Sets}$ defined by $[n] \mapsto N^n\mathcal{C}$ and for $f : [n] \rightarrow [m]$ in \mathbf{SCat} , the induced map is $N^*\mathcal{C}(f) = f_* : N^m\mathcal{C} \rightarrow N^n\mathcal{C}$. An n -simplex $a \in N^n\mathcal{C}$ is of the form $a = a_n \rightarrow a_{n-1} \rightarrow \dots \rightarrow a_0$ where $a_i \in \mathcal{C}$ for all i .

Let $\delta^i : [n-1] \rightarrow [n]$ and $\sigma^i : [n+1] \rightarrow [n]$ in \mathbf{SCat} be defined by

$$\delta^i(\{0, \dots, n-1\}) = \{0, \dots, \widehat{i}, \dots, n\} \text{ and } \sigma^i(\{0, \dots, n+1\}) = \{0, \dots, i, i, \dots, n\}$$

respectively, where $0 \leq i \leq n$ and \widehat{i} means omit i . For example if $n = 3$ and $i = 1$, then we have $\delta^1(\{0, 1, 3\}) = \{0, \widehat{1}, 2, 3\} = \{0, 2, 3\}$. Also $\sigma^1(\{0, 1, 2, 3, 4\}) = \{0, 1, 1, 2, 3\} = \{0, 1, 2, 3\}$. The face maps, $d_i := N^*\mathcal{C}(\delta^i) : N^n\mathcal{C} \rightarrow N^{n-1}\mathcal{C}$ and the degeneracy maps, $s_i := N^*\mathcal{C}(\sigma^i) : N^n\mathcal{C} \rightarrow N^{n+1}\mathcal{C}$ are defined respectively by

$$d_i(a) = a_n \rightarrow \dots \rightarrow \widehat{a_i} \rightarrow \dots \rightarrow a_0 \text{ and } s_i(a) = a_n \rightarrow \dots \rightarrow a_i \rightarrow a_i \rightarrow \dots \rightarrow a_0.$$

We can define for $F \in \mathbf{pSh}(\mathcal{C})$ a cochain complex $S^*(\mathcal{C}; F)$ with the n th cochain groups given by

$$S^n(\mathcal{C}; F) = \prod_{a \in N^n\mathcal{C}} F(a_n).$$

Let $c \in S^n(\mathcal{C}; F)$ and $a \in N^n\mathcal{C}$. Then we write $c \cdot a$ for the component of c indexed by a and so if $a = a_n \rightarrow a_{n-1} \rightarrow \dots \rightarrow a_0$ then $c \cdot a \in F(a_n)$. The differential operator $d : S^{n-1}(\mathcal{C}; F) \rightarrow S^n(\mathcal{C}; F)$ is defined by

$$dc \cdot a = \sum_{i=0}^{n-1} (-1)^i c \cdot d_i(a) + (-1)^n F_{a_n}^{a_{n-1}}(c \cdot d_n(a)).$$

The cohomology, $HS^*(\mathcal{C}; F)$, of the complex, $S^*(\mathcal{C}; F)$, is the *cohomology of the category \mathcal{C} with coefficient in F* .

Let (\mathbf{P}, \leq) be a poset and consider the Yoneda presheaf $\Upsilon_x A$, $x \in \mathbf{P}$ (see Example 2.2.10(iii)). If $x \in \mathbf{P}$, then there is a functor $\Upsilon_x : \mathbf{Ab} \rightarrow \mathbf{pSh}(\mathbf{P})$ called the **Yoneda functor** defined by $A \mapsto \Upsilon_x A$. Let $x \leq y$ be in \mathbf{P} and let $f : A \rightarrow B$ be in \mathbf{Ab} . Then there is an induced morphism (a natural transformation) of presheaves $\Upsilon_x A \rightarrow \Upsilon_y B$ in $\mathbf{pSh}(\mathbf{P})$. For $x \in \mathbf{P}$, the functor $x^e : \mathbf{pSh}(\mathbf{P}) \rightarrow \mathbf{Ab}$ defined by $x^e(F) = F(x)$ is called the **evaluation functor**.

Proposition 4.1.1. *For $A \in \mathbf{Ab}$ and $F \in \mathbf{pSh}(\mathbf{P})$, there is a natural bijection*

$$\tau_{A,F} : \text{Hom}_{\mathbf{pSh}(\mathbf{P})}(\Upsilon_x A, F) \rightarrow \text{Hom}_{\mathbf{Ab}}(A, x^e(F))$$

defined by $\alpha \mapsto \alpha_x$.

The proof of Proposition 4.1.1 is similar to the proof of Proposition 3.4.1. We can immediately see that the map $\text{Hom}_{\mathbf{pSh}(\mathbf{P})}(\Upsilon_y B, F) \longrightarrow \text{Hom}_{\mathbf{pSh}(\mathbf{P})}(\Upsilon_x A, F)$ is equal to $\text{Hom}_{\mathbb{Z}}(B, F(y)) \longrightarrow \text{Hom}_{\mathbb{Z}}(A, F(x))$ by the natural bijection and this map sends $\alpha_y \in \text{Hom}_{\mathbb{Z}}(B, F(y))$ to $\alpha_x = F_x^y \alpha_y \eta_x \in \text{Hom}_{\mathbb{Z}}(A, F(x))$, where $\eta : \Upsilon_x A \rightarrow \Upsilon_y B$. From this we see that $\Upsilon_x A$ is projective if and only if A is projective.

Corollary 4.1.2. *Let $F \in \mathbf{pSh}(\mathbf{P})$. Then there is an isomorphism*

$$\theta : \text{Hom}_{\mathbf{pSh}(\mathbf{P})}(\Upsilon_x A, F) \longrightarrow F(x)$$

defined by $\theta(\eta) = \eta_x(1)$.

The *classifying space*, BC , of a small category, \mathcal{C} , is a simplicial complex and it is the geometric realisation, $|N^*\mathcal{C}|$ (see Section A.4), of the nerve of \mathcal{C} . In the following proposition we will show that the cohomology, $H^*(B\mathbf{P}; F)$, of the classifying space $B\mathbf{P}$ is isomorphic to the higher limit $\varprojlim_{\mathbf{P}^{op}}^* F$ for $F \in \mathbf{pSh}(\mathbf{P})$, from which we deduce a relation between the Khovanov homology and the classifying space $B\mathbf{Q}$.

Proposition 4.1.3. *Let $F \in \mathbf{pSh}(\mathbf{P})$. Then $H^*(B\mathbf{P}; F) \cong \varprojlim_{\mathbf{P}^{op}}^* F$.*

Proof. We use the idea that a projective resolution can be constructed from the simplicial structure of the classifying space $B\mathbf{P}$, that is the simplicial structure of the geometric realisation of the nerve of \mathbf{P} .

Let $P_n = \bigoplus_a \Upsilon_{a_n} \mathbb{Z}$, where the sum is over the n -simplices $a = a_n \leq \dots \leq a_0 \in N^n \mathbf{P}$. For a fixed $b \in \mathbf{P}$, we get that the abelian group $P_n(b) = \bigoplus_a \Upsilon_{a_n} \mathbb{Z}(b)$ is free on all composable strings $b \leq a_n \leq \dots \leq a_0$ which implies that $P_n(b)$ is projective and so is P_n .

Now let $f : \mathbb{Z} \rightarrow \mathbb{Z}$ be a homomorphism in \mathbf{Ab} . Then there is an induced morphism (a natural transformation) of presheaves $\eta : \Upsilon_{a_n} \mathbb{Z} \rightarrow \Upsilon_{a_{n-1}} \mathbb{Z}$ and so define $d : P_n \longrightarrow P_{n-1}$ by $ds \cdot d_n a = \sum_{i=0}^{n-1} (-1)^i \eta(s \cdot d_i a)$ where $s \in P_n$ and $s \cdot a$ is the component of s in $\Upsilon_{a_n} \mathbb{Z}$ indexed by a .

Observe that $(P_*(b), d)$ is the complex computing the simplicial homology of the nerve of the coslice category b/\mathbf{P} (see Example 2.2.2(vi)). We have that b/\mathbf{P} has an initial object which is the identity morphism $id : b \rightarrow b$ and so the nerve $N^*(b/\mathbf{P})$ is contractible (see Section A.5) which implies that P_* is exact. Thus we have $\dots \rightarrow P_1 \rightarrow P_0 \rightarrow \Delta \mathbb{Z} \rightarrow 0$ as a projective resolution of $\Delta \mathbb{Z}$. Applying $\text{Hom}_{\mathbf{pSh}(\mathbf{P})}(-, F)$ to $\dots \rightarrow P_1 \rightarrow P_0 \rightarrow 0$ we get

a complex $\text{Hom}_{\mathbf{pSh}(\mathbf{P})}(P_*, F)$, from which the cohomology $H^i(B\mathbf{P}; F)$ is computed for all i . But $H^i(B\mathbf{P}; F)$ is just the right derived functor $R^i\text{Hom}_{\mathbf{pSh}(\mathbf{P})}(P_*, F)$ which is isomorphic to $\varprojlim_{\mathbf{P}^{op}}^* F$ (see Remark 3.4.4). Thus $H^*(B\mathbf{P}; F) \cong \varprojlim_{\mathbf{P}^{op}}^* F$. \square

Remark 4.1.4. Corollary 4.1.2 gives us an isomorphism, $\text{Hom}_{\mathbf{pSh}(\mathbf{P})}(\Upsilon_x A, F) \cong F(x)$, and so we have that the complex, $\text{Hom}_{\mathbf{pSh}(\mathbf{P})}(P_*, F)$, is exactly the complex, $S^*(\mathbf{P}; F)$, with $S^n(\mathbf{P}; F) = \prod_{a \in N^n \mathcal{C}} F(a_n)$, where $a = a_n \leq a_{n-1} \leq \dots \leq a_0$ and $a_i \in \mathbf{P}$. It follows that $HS^*(\mathbf{P}; F) \cong H^*(B\mathbf{P}; F)$.

Remark 4.1.5. The preceding proposition tells us that the higher limit, $\varprojlim_{\mathbf{P}^{op}}^* F$, is isomorphic to the cohomology, $H^*(B\mathbf{P}; F)$, of the classifying space $B\mathbf{P}$. If F is the Khovanov presheaf, $F_{KH} : \mathbf{Q}^{op} \rightarrow \mathbf{Ab}$, then we can have $H^*(B\mathbf{Q}; F_{KH}) \cong \varprojlim_{\mathbf{Q}^{op}}^* F_{KH}$. Theorem 3.5.4 says that $\varprojlim_{\mathbf{Q}^{op}}^* F_{KH} \cong \overline{KH}^*(D)$ and so we have $\overline{KH}^*(D) \cong H^*(B\mathbf{Q}; F_{KH})$, where D is a link diagram.

4.2 Cellular cohomology and presheaves

In the preceding section, we looked at the cohomology of a category with coefficient in a presheaf. In this section we will look at the *cellular cohomology* of a poset with coefficients in a presheaf.

Let \mathbf{P} be the poset and $F \in \mathbf{pSh}(\mathbf{P})$ be a presheaf. $HS^*(\mathbf{P}; F)$ is called the cohomology of a poset with coefficient in the presheaf F .

Let $f : \mathbf{Q} \rightarrow \mathbf{P}$ be a map of posets. We can have the following induced maps and functors:

- The induced map of simplicial sets $N^*\mathbf{Q} \rightarrow N^*\mathbf{P}$ which sends $\sigma = \sigma_n \leq \dots \leq \sigma_0 \in N^n\mathbf{Q}$ to $f(\sigma) = f(\sigma_n) \leq \dots \leq f(\sigma_0) \in N^n\mathbf{P}$.
- The induced functor $\mathbf{pSh}(\mathbf{P}) \rightarrow \mathbf{pSh}(\mathbf{Q})$ which sends $F \in \mathbf{pSh}(\mathbf{P})$ to $f^*F := F \circ f$ and the natural transformation $\eta : F \rightarrow G$ to $f^*\eta : f^*F \rightarrow f^*G$ with $f^*\eta_a = \eta_{f(a)}$. If f is an inclusion then $f^*F = F$ since $f(a) = a$ for $a \in \mathbf{Q}$.
- The induced map $f^* : S^*(\mathbf{P}; F) \rightarrow S^*(\mathbf{Q}; f^*F)$ called the *pull-back* which is defined for $s \in S^*(\mathbf{P}; F)$ and $\sigma \in N^n\mathbf{Q}$ by

$$f^*s \cdot \sigma = s \cdot f(\sigma). \quad (4.1)$$

- If $f : \mathbf{Q} \rightarrow \mathbf{P}$ is finite-to-one, then there is an induced map of groups $f_* : S^*(\mathbf{Q}; f^*F) \rightarrow S^*(\mathbf{P}; F)$, the **push-forward**, defined for $s \in S^*(\mathbf{Q}; f^*F)$ and $\sigma \in N^n \mathbf{P}$ by

$$f_* s \cdot \sigma = \sum_{a \in f^{-1}\sigma} s \cdot a. \quad (4.2)$$

We now look at some Lemmas which will be useful in describing the cellular cohomology of a poset.

Lemma 4.2.1. (a) The pull-back f^* is a cochain map. (b) If $g : \mathbf{R} \rightarrow \mathbf{Q}$ is a poset map, then $(fg)^* = g^* f^* : S^*(\mathbf{P}; F) \rightarrow S^*(\mathbf{R}; (fg)^*F)$.

Proof. (a) We show that the following diagram commutes.

$$\begin{array}{ccc} S^{n-1}(\mathbf{P}; F) & \xrightarrow{d} & S^n(\mathbf{P}; F) \\ \downarrow f^* & & \downarrow f^* \\ S^{n-1}(\mathbf{Q}; f^*F) & \xrightarrow{d} & S^n(\mathbf{Q}; f^*F). \end{array}$$

That is $f^*d = df^*$.

Let $s \in S^{n-1}(\mathbf{P}; F)$ and $\sigma \in N^n \mathbf{P}$. We first compute f^*ds We have

$$ds \cdot \sigma = \sum_{i=0}^{n-1} (-1)^i s \cdot d_i(\sigma) + (-1)^n F_{\sigma_n}^{\sigma_{n-1}}(s \cdot d_n(\sigma))$$

and $ds \in S^n(\mathbf{P}; F)$. Let $\tau \in N^n \mathbf{Q}$, then $f^*ds \cdot \tau = ds \cdot f(\tau)$.

Since $f^*s \in S^{n-1}(\mathbf{Q}; f^*F)$, $df^*s \in S^n(\mathbf{Q}; f^*F)$ and so

$$\begin{aligned} df^*s \cdot \tau &= \sum_{i=0}^{n-1} (-1)^i f^*s \cdot d_i \tau + (-1)^n F_{f(\tau_n)}^{f(\tau_{n-1})}(f^*s \cdot d_n(\tau)) \\ &= \sum_{i=0}^{n-1} (-1)^i s \cdot d_i f \tau + (-1)^n F_{f(\tau_n)}^{f(\tau_{n-1})}(s \cdot d_n(f \tau)) \\ &= ds \cdot f(\tau). \end{aligned}$$

We therefore see that $f^*d = df^*$.

(b) We now show that $(fg)^* = g^* f^*$.

Let $s \in S^n(\mathbf{P}; F)$ and $\sigma \in N^n \mathbf{R}$, then

$$(fg)^* s \cdot \sigma = s \cdot (fg)\sigma = f^* s \cdot (g\sigma) = g^* f^* s \cdot \sigma.$$

Thus $(fg)^* = g^* f^*$. □

Lemma 4.2.2. *If f is injective then $f^* f_* = id$, so that f^* is surjective and f_* is injective.*

Proof. Let $s \in S^n(\mathbf{Q}; f^* F)$, then by (4.2) we have $f_* s \in S^n(\mathbf{P}; F)$. Now let $\sigma \in N^n \mathbf{Q}$, then

$$\begin{aligned} f^* f_* s \cdot \sigma &= \sum_{a \in f^{-1} f \sigma} s \cdot a \\ &= \sum_{a \in id \sigma} s \cdot a, \text{ since } f \text{ is injective} \\ &= s \cdot \sigma. \end{aligned}$$

Thus $f^* f_* = id$. This shows that f^* is surjective since for each $s \in S^n(\mathbf{Q}; f^* F)$, there exists $f_* s \in S^n(\mathbf{P}; F)$ such that $f^* f_* s = s$. Also if $m, n \in S^n(\mathbf{P}; F)$ and $f_*(n) = f_*(m)$, then $f^* f_* n = f^* f_* m$ implies $n = m$. Thus f_* is injective. □

4.2.3 Relative cohomology

Definition 4.2.4. Let $f : \mathbf{Q} \rightarrow \mathbf{P}$ be a poset map and let F be a presheaf on \mathbf{P} . Then we define $S^*(\mathbf{P}, \mathbf{Q}; F)$ to be the kernel of the pull-back f^* , that is $S^*(\mathbf{P}, \mathbf{Q}; F) := \ker f^*$.

For each $s \in S^*(\mathbf{P}; F)$, and f an inclusion, we have $s \in S^*(\mathbf{P}, \mathbf{Q}; F)$ if and only if $f^* s \cdot \sigma = s \cdot f\sigma = s \cdot \sigma = 0$, for all $\sigma \in f(N^* \mathbf{Q}) \subset N^* \mathbf{P}$. The differential d on the complex $S^*(\mathbf{P}; F)$ restricts to a differential on $S^*(\mathbf{P}, \mathbf{Q}; F)$. The cohomology $HS^*(\mathbf{P}, \mathbf{Q}; F)$ of the complex $S^*(\mathbf{P}, \mathbf{Q}; F)$ is $HS^*(\mathbf{P}, \mathbf{Q}; F) := H(S^*(\mathbf{P}, \mathbf{Q}; F), d)$. It is called the **relative cohomology** of the pair (\mathbf{P}, \mathbf{Q}) with coefficient in the presheaf F . We will consider the case where the poset maps are inclusions.

Let $f : \mathbf{Q} \rightarrow \mathbf{P}$ be injective. By Lemma 4.2.2, we get a short exact sequence

$$0 \rightarrow S^*(\mathbf{P}, \mathbf{Q}; F) \rightarrow S^*(\mathbf{P}; F) \xrightarrow{f^*} S^*(\mathbf{Q}; f^* F) \rightarrow 0 \quad (4.3)$$

from which get the following long exact sequence by Theorem 2.1.14.

$$\dots \xrightarrow{\beta} HS^n(\mathbf{P}, \mathbf{Q}; F) \rightarrow HS^n(\mathbf{P}; F) \rightarrow HS^n(\mathbf{Q}; f^*F) \xrightarrow{\beta} HS^{n+1}(\mathbf{P}, \mathbf{Q}; F) \rightarrow \dots, \quad (4.4)$$

where β is called the connecting homomorphism of the pair.

Lemma 4.2.5. *Let $\mathbf{P}, \mathbf{Q}, \mathbf{R}$ be posets with $j : \mathbf{R} \hookrightarrow \mathbf{Q}$ and $i : \mathbf{Q} \hookrightarrow \mathbf{P}$ inclusions and let $F \in \mathbf{pSh}(\mathbf{P})$. Then there is a short exact sequence*

$$0 \longrightarrow S^*(\mathbf{P}, \mathbf{Q}; F) \longrightarrow S^*(\mathbf{P}, \mathbf{R}; F) \xrightarrow{i^*} S^*(\mathbf{Q}, \mathbf{R}; F) \longrightarrow 0 \quad (4.5)$$

and hence a long exact sequence

$$\dots \xrightarrow{\delta} HS^n(\mathbf{P}, \mathbf{Q}; F) \rightarrow HS^n(\mathbf{P}, \mathbf{R}; F) \rightarrow HS^n(\mathbf{Q}, \mathbf{R}; F) \xrightarrow{\delta} HS^{n+1}(\mathbf{P}, \mathbf{Q}; F) \rightarrow \dots, \quad (4.6)$$

where δ is the connecting homomorphism.

Proof. We show that (4.5) is an exact sequence.

(i) Exactness at $S^*(\mathbf{P}, \mathbf{Q}; F)$:

By Definition 4.2.4, we have $S^*(\mathbf{P}, \mathbf{Q}; F) = \ker i^*$. The map from $S^*(\mathbf{P}, \mathbf{Q}; F)$ to $S^*(\mathbf{P}, \mathbf{R}; F)$ is $(ij)^*$ and so we show that $\ker(ij)^* = \{0\}$. Let $s \in S^*(\mathbf{P}, \mathbf{Q}; F)$ and $\sigma \in N^*\mathbf{R}$, then $(ij)^*s \cdot \sigma = i^*s \cdot j\sigma = i^*s \cdot \sigma = s \cdot \sigma = 0$ implies $s \in \ker(ij)^*$. But $s \cdot \sigma = 0$ implies $s = 0$, so $\ker(ij)^* = \{0\} = \text{im}(0)$.

(ii) Exactness at $S^*(\mathbf{P}, \mathbf{R}; F)$:

We show that $\text{im}(ij)^* = \ker i^*$. By Definition 4.2.4, we have $S^*(\mathbf{P}, \mathbf{R}; F) = \ker(ij)^*$.

(a) First we show that $\text{im}(ij)^* \subseteq \ker i^*$.

Let $t \in \text{im}(ij)^*$, then there exists some $m \in S^*(\mathbf{P}, \mathbf{Q}; F)$ such that $(ij)^*m \cdot \sigma = t \cdot \sigma, \sigma \in N^*\mathbf{R}$, which implies that $m \cdot \sigma = t \cdot \sigma$ since i and j are inclusions. Thus $t = m$. Now $i^*t \cdot \beta = t \cdot \beta$ for $\beta \in N^*\mathbf{Q}$ but $t = m \in S^*(\mathbf{P}, \mathbf{Q}; F)$ and so $i^*t = i^*m = 0$ which implies $\text{im}(ij)^* \subseteq \ker i^*$.

(b) We now show that $\ker i^* \subseteq \text{im}(ij)^*$. Let $n \in \ker i^*$, then $i^*n = 0$. Now observe that for $\gamma \in N^*\mathbf{R}$, $n \cdot (ij)\gamma = n \cdot \gamma$ which implies $(ij)^*n \cdot \gamma = n \cdot \gamma$ and so we have $n \in \text{im}(ij)^*$. Thus $\ker i^* \subseteq \text{im}(ij)^*$.

From (a) and (b), we see that $\ker i^* = \text{im}(ij)^*$.

(iii) Exactness at $S^*(\mathbf{Q}, \mathbf{R}; F)$:

By Definition 4.2.4, we have $S^*(\mathbf{Q}, \mathbf{R}; F) = \ker j^*$. Let $a \in S^*(\mathbf{P}, \mathbf{R}; F)$, then $j^*(i^*a) = 0$ which implies $i^*a \in \ker j^* = S^*(\mathbf{Q}, \mathbf{R}; F)$, thus $\text{im } i^* = \ker j^*$.

From (i), (ii) and (iii) we that the sequence (4.5) is exact. By Theorem 2.1.14 we get the associated long exact sequence 4.6. \square

Lemma 4.2.6. *Let $(\mathbf{P}, \mathbf{Q}, \mathbf{R})$ be the triple of Lemma 4.2.5 with $\iota : HS^n(\mathbf{Q}, \mathbf{R}; F) \rightarrow HS^n(\mathbf{Q}; F)$ induced by the inclusion $S^n(\mathbf{Q}, \mathbf{R}; F) \rightarrow S^n(\mathbf{Q}; F)$ and $\beta : HS^n(\mathbf{Q}; F) \rightarrow HS^{n+1}(\mathbf{P}, \mathbf{Q}; F)$ the connecting homomorphism of the pair (\mathbf{P}, \mathbf{Q}) . If δ is the connecting homomorphism of Lemma 4.2.5 then $\delta = \beta\iota$.*

Proof. We show that the following diagram commutes.

$$\begin{array}{ccc}
 HS^n(\mathbf{Q}, \mathbf{R}; F) & \xrightarrow{\delta} & HS^{n+1}(\mathbf{P}, \mathbf{Q}; F) \\
 & \searrow \iota & \nearrow \beta \\
 & HS^n(\mathbf{Q}; F) &
 \end{array}$$

Let $c \in S^n(\mathbf{Q}, \mathbf{R}; F)$ be a cocycle. Then c is also a cocycle in $S^n(\mathbf{Q}; F)$ by the inclusion $S^n(\mathbf{Q}, \mathbf{R}; F) \rightarrow S^n(\mathbf{Q}; F)$. The induced map ι therefore sends the homology class of c to itself, $\iota([c]) = [c]$. β also sends the homology class of c to a homology class $[m]$, where m is a cocycle in $S^{n+1}(\mathbf{P}, \mathbf{Q}; F)$. The connecting homomorphism δ sends $[c]$ to a homology class in $HS^{n+1}(\mathbf{P}, \mathbf{Q}; F)$ and this must be $[m]$. Thus $\delta = \beta\iota$. \square

We are now in a good position to describe the cellular cohomology of a poset with coefficients in a presheaf. Let \mathbf{P} be a poset. If $x \leq y$ is a morphism in \mathbf{P} and for any $x \leq z \leq y$ we have either $z = x$ or $z = y$, then y is said to cover x written $x \prec y$. We say \mathbf{P} is **graded** if there exists a function $rk : \mathbf{P} \rightarrow \mathbb{Z}$ called the rank function such that:

- (i) $x < y$ implies $rk(x) < rk(y)$ and (ii) $x \prec y$ implies $rk(y) = rk(x) + 1$.

Let rk be a fixed rank function on graded \mathbf{P} and suppose rk is bounded above with $r = \max_{x \in \mathbf{P}} \{rk(x)\}$. Define the corank function $| \cdot | : \mathbf{P} \rightarrow \mathbb{Z}^{\geq 0}$ by $|x| = r - rk(x)$. Let

$$\mathbf{P}^k = \{x \in \mathbf{P} : |x| \leq k\}. \quad (4.7)$$

Then the collection $\{\mathbf{P}^k\}_{k \in \mathbb{Z}}$ is a filtration of \mathbf{P} by the corank function and from this we get a sequence of inclusions $\mathbf{P}^0 \subset \mathbf{P}^1 \subset \mathbf{P}^2 \subset \dots$

Now form a long exact sequence for the triples $(\mathbf{P}^{n+1}, \mathbf{P}^n, \mathbf{P}^{n-1})$ and $(\mathbf{P}^n, \mathbf{P}^{n-1}, \mathbf{P}^{n-2})$ by using Lemma 4.2.5. That is

$$\dots \rightarrow HS^n(\mathbf{P}^{n+1}, \mathbf{P}^{n-1}; F) \rightarrow HS^n(\mathbf{P}^n, \mathbf{P}^{n-1}; F) \xrightarrow{\delta^n} HS^{n+1}(\mathbf{P}^{n+1}, \mathbf{P}^n; F) \rightarrow \dots,$$

and

$$\dots \rightarrow HS^{n-1}(\mathbf{P}^{n-1}, \mathbf{P}^{n-2}; F) \xrightarrow{\delta^{n-1}} HS^n(\mathbf{P}^n, \mathbf{P}^{n-1}; F) \rightarrow HS^n(\mathbf{P}^n, \mathbf{P}^{n-2}; F) \rightarrow \dots,$$

respectively. Observe that portions of these long exact sequences fit into the following diagram by applying Lemma 4.2.6.

$$\begin{array}{ccccccc}
 & & 0 & & & & \\
 & & \uparrow & & & & \\
 & & HS^{n-1}(P^{n-2}; F) & & & & \\
 & & \uparrow & & & & \\
 & & HS^{n-1}(P^{n-1}; F) & & & & \\
 & & \uparrow & \searrow^{\beta^{n-1}} & & & \\
 \dots & \rightarrow & HS^{n-1}(P^{n-1}, P^{n-2}; F) & \xrightarrow{\delta^{n-1}} & HS^n(P^n, P^{n-1}; F) & \xrightarrow{\delta^n} & HS^{n+1}(P^{n+1}, P^n; F) \rightarrow \dots \\
 & & & & \downarrow \iota^n & \nearrow \beta^n & \\
 & & & & HS^n(P^n; F) & & \\
 & & & & \downarrow & & \\
 & & & & HS^n(P^{n-1}; F) & & \\
 & & & & \downarrow & & \\
 & & & & 0 & &
 \end{array}$$

When we consider the horizontal row in the preceding diagram, we see that $\delta^n \delta^{n-1} = \beta^n \iota^n \beta^{n-1} \iota^{n-1} = 0$ since $\iota^n \beta^{n-1} = 0$. This is because the long sequence (4.4) is exact. The horizontal sequence which we denote by $C^*(\mathbf{P}; F)$ therefore gives us a cochain complex.

Definition 4.2.7. Let \mathbf{P} be graded with corank function, $\{\mathbf{P}^k\}_{k \in \mathbb{Z}}$ the associated filtration and F a presheaf on \mathbf{P}^k . The cochain complex $C^*(\mathbf{P}; F)$ with cochain groups $C^n(\mathbf{P}; F) = HS^n(\mathbf{P}^n, \mathbf{P}^{n-1}; F)$ is called the *cellular cochain complex* of the poset \mathbf{P} . The differential (coboundary map) $\delta^n : HS^n(\mathbf{P}^n, \mathbf{P}^{n-1}; F) \rightarrow HS^{n+1}(\mathbf{P}^{n+1}, \mathbf{P}^n; F)$ is given by $\delta^n = \beta^n \iota^n$.

The *cellular cohomology* of \mathbf{P} with coefficients in the presheaf F is defined to be the

homology of the complex:

$$HC^*(\mathbf{P}; F) := H(C^*(\mathbf{P}; F), \delta).$$

4.3 Cellular cohomology and higher limits of F_{KH}

Let \mathbf{P} be a poset with corank function. Let $\mathbf{P}_{\geq x} = \{y \in \mathbf{P} : y \geq x\}$. \mathbf{P} is *locally finite* if for any $x \in \mathbf{P}$ there are only finitely many y with $x \prec y$. The set $N_{\circ}^n \mathbf{P}_{\geq x}$ is the collection of the non-degenerate n -simplicies in the nerve of $\mathbf{P}_{\geq x}$

Definition 4.3.1. Let $\tau = x \leq \tau_{n-1} \leq \dots \leq \tau_{j+1} \leq \tau_{j-1} \leq \dots \leq \tau_0$ be a fixed $(n-1)$ -simplex in $N_{\circ}^{n-1} \mathbf{P}_{\geq x}$ with $0 \leq j < n$ and $|\tau_i| = i$. The *compatible family* given by τ is the set B_{τ} of all n -simplicies in $\sigma \in N_{\circ}^n \mathbf{P}_{\geq x}$ of the form $x \leq \tau_{n-1} \leq \dots \leq \tau_{j+1} \leq y \leq \tau_{j-1} \leq \dots \leq \tau_0$ (where necessarily $|y| = j$).

Now fix $x \in \mathbf{P}$ with $|x| = n$ and let $\sigma \in N_{\circ}^n \mathbf{P}_{\geq x}$, then σ is of the form $\sigma = x \leq \sigma_{n-1} \leq \dots \leq \sigma_0$ with $|\sigma_i| = i$. Let A_x be the abelian group having presentation with generators the $\sigma \in N_{\circ}^n \mathbf{P}_{\geq x}$ and relations $\sum_{\sigma \in B_{\tau}} \sigma = 0$ for each B_{τ} in $N_{\circ}^{n-1} \mathbf{P}_{\geq x}$ and let F be a presheaf on \mathbf{P} . Then the n -th cellular cochain group is seen as the product over $x \in \mathbf{P}$ of the tensor products $A^x \otimes F(x)$ where $F(x) \in \mathbf{Ab}$. The following proposition given by Everitt and Turner emphasizes this description.

Proposition 4.3.2. ([4], Proposition 6) *Let \mathbf{P} be graded locally finite with corank function, F a presheaf on \mathbf{P} and A_x as described above. Then there are isomorphisms*

$$C^n(\mathbf{P}; F) \cong \prod_{|x|=n} A_x \otimes F(x).$$

If $|y| = n-1$ and $x \prec y$ then the matrix element $\delta_x^y : A^y \otimes F(y) \rightarrow A_x \otimes F(x)$ of the differential $\delta : C^{n-1}(\mathbf{P}; F) \rightarrow C^n(\mathbf{P}; F)$ is given by $\delta_x^y : \sigma \otimes a \mapsto x\sigma \otimes (-1)^n F_x^y(a)$ where $\sigma \in N_{\circ}^n \mathbf{P}_{\geq y}$ is a generator of A_y with $a \in F(y)$ and $x\sigma$ is the result of pre-appending x onto σ .

Recall from Remarks 4.1.4 and 4.1.5 that $HS^*(\mathbf{P}, F)$ computes the higher limit $\varprojlim_{\mathbf{P}^{op}}^* F$. We now want to compute the higher limit cellularly but it turns out in general that $HS^*(\mathbf{P}; F)$ and $HC^*(\mathbf{P}; F)$ are not isomorphic as we will see in the example given in Subsection 4.3.5 below. So one will ask, under what condition can $HS^*(\mathbf{P}; F)$ be computed cellularly? That is when are the groups $HS^*(\mathbf{P}; F)$ and $HC^*(\mathbf{P}; F)$ isomorphic? We give the following condition to address this problem.

Condition 4.3.3. (Cellular condition)

Let \mathbf{P} be graded with corank function. Then \mathbf{P} is cellular if and only if for every presheaf $F \in \mathbf{pSh}(\mathbf{P})$ we have

$$HS^i(\mathbf{P}^n, \mathbf{P}^{n-1}; F) = 0 \text{ for } i \neq n.$$

The following Theorem given by Everitt and Turner now tells us that the groups $HS^*(\mathbf{P}; F)$ and $HC^*(\mathbf{P}; F)$ are isomorphic.

Theorem 4.3.4. (*[4], Theorem 1*) *Let \mathbf{P} be graded, cellular, locally finite with corank function and let F be a presheaf on \mathbf{P} . Then there is an isomorphism*

$$HS^*(\mathbf{P}; F) \cong HC^*(\mathbf{P}; F).$$

The above Theorem is a generalisation of Theorem 3.5.4 which we will see in the example given in Subsection 4.3.6.

4.3.5 Example with $HS^*(\mathbf{P}; F) \not\cong HC^*(\mathbf{P}; F)$

In this example we demonstrate that $HS^*(\mathbf{P}; F)$ is not isomorphic to $HC^*(\mathbf{P}; F)$.

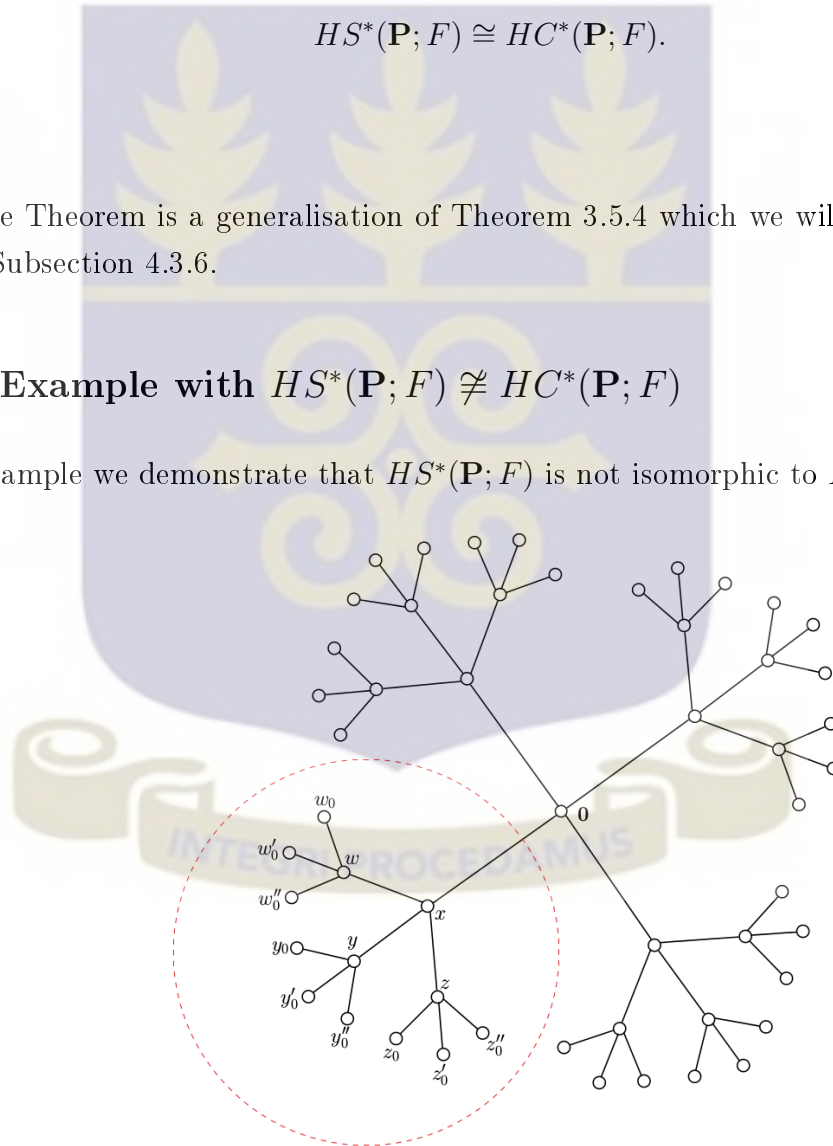


Figure 4.1: A finite tree

Let \mathbf{P} be a poset with elements the vertices of the finite tree Figure 4.1 above with a distinguished vertex $\mathbf{0}$. Let the vertices be ordered by $x \leq y$ when the unique path without passing through a vertex twice from $\mathbf{0}$ to y passes through x . Define the rank of $x \in \mathbf{P}$ to the number of edges between it and $\mathbf{0}$. For example in Figure 4.1, the vertices labelled x and w are of rank 1 and 2 respectively. The maximal elements are called leaves. For ease of computation, we suppose all the leaves have the same rank.

From the above description, we see that \mathbf{P} is graded, locally finite with corank function and so we can apply Proposition 4.3.2 to compute the cochain groups of the cellular complex $C^*(\mathbf{P}; F)$

The maximum rank of a vertex $x \in \mathbf{P}$ is $r = 3$ and so the corank of x is $|x| = 3 - rk(x)$.

We now compute the cochain groups $C^n(\mathbf{P}; F) \cong \prod_{|x|=n} A_x \otimes F(x)$ where $F = \Delta\mathbb{Z}$ and so $F(x) = \mathbb{Z}$.

(i) **Computing** $C^0(\mathbf{P}; \Delta\mathbb{Z}) \cong \prod_{|x|=0} A_x \otimes \mathbb{Z}$.

Let $x \in \mathbf{P}$ with $|x| = 0$, then x is a leaf. For example the vertex w_0 in Figure 4.1 is of rank 0. There are no compatible families for each leaf and so the abelian group A_x is generated by x , that is $A_x = \langle x \rangle \cong \mathbb{Z}$. Thus $C^0(\mathbf{P}; \Delta\mathbb{Z}) \cong \prod_{|x|=0} A_x \otimes \mathbb{Z}$ is free abelian on the leaves.

(ii) **Computing** $C^1(\mathbf{P}; \Delta\mathbb{Z}) \cong \prod_{|x|=1} A_x \otimes \mathbb{Z}$.

Let $x \in \mathbf{P}$ with $|x| = 1$. For example consider w in Figure 4.1. Fix $\tau = w \in N_{\circ}^0 \mathbf{P}_{\geq x}$. The only compatible family with respect to τ is

$$B_{\tau} = \{ww_0, ww'_0, ww''_0 \in N_{\circ}^1 \mathbf{P}_{\geq x} \mid |w_0| = |w'_0| = |w''_0| = 1\}$$

and so

$$A_w = \langle ww_0, ww'_0, ww''_0 \mid ww_0 + ww'_0 + ww''_0 = 0 \rangle.$$

Thus $C^1(\mathbf{P}; \Delta\mathbb{Z}) \cong \prod_{|x|=1} A_x \otimes \mathbb{Z}$ is free abelian on the corank 1 vertices.

(iii) **Computing** $C^i(\mathbf{P}; \Delta\mathbb{Z}) \cong \prod_{|x|=i} A_x \otimes \mathbb{Z}$ for $i > 1$. We will demonstrate with $i = 2$ to show that $A_x = \{0\}$ and so $C^i(\mathbf{P}; \Delta\mathbb{Z}) = \{0\}$ for all $i > 0$.

Let us consider the vertex labelled x in Figure 4.1. It has corank 2. In finding the

compatible family B_τ , the $\tau \in N^1\mathbf{P}_{\geq x}$ can be fixed in different ways:

- If $\tau = xw_0$, then $B_\tau = \{xww_0\}$.
- If $\tau = xw'_0$, then $B_\tau = \{xww'_0\}$.
- If $\tau = xw''_0$, then $B_\tau = \{xww''_0\}$.
- If $\tau = xy_0$, then $B_\tau = \{xyy_0\}$.
- If $\tau = xy'_0$, then $B_\tau = \{xyy'_0\}$.
- If $\tau = xy''_0$, then $B_\tau = \{xyy''_0\}$.
- If $\tau = xz$, then $B_\tau = \{xzz_0, xzz'_0, xzz''_0\}$.
- If $\tau = xz'_0$, then $B_\tau = \{xzz'_0\}$.
- If $\tau = xz''_0$, then $B_\tau = \{xzz''_0\}$.
- If $\tau = xw$, then $B_\tau = \{xww_0, xww'_0, xww''_0\}$.
- If $\tau = xz_0$, then $B_\tau = \{xzz_0\}$.
- If $\tau = xy$, then $B_\tau = \{xyy_0, xyy'_0, xyy''_0\}$.

Let $a = xww_0, b = xww'_0, c = xww''_0, d = xyy_0, e = xyy'_0, f = xyy''_0, g = xzz_0, h = xzz'_0$ and $i = xzz''_0$. Then A_x has a presentation

$$A_x = \langle a, b, c, d, e, f, g, h, i \mid a + b + c = 0, d + e + f = 0, g + h + i = 0, a = b = c = d = e = f = g = h = i = 0 \rangle = \{0\}.$$

for each compatible family B_τ where $|x| = 2$. Thus $C^2(\mathbf{P}; \Delta\mathbb{Z}) \cong \prod_{|x|=2} A_x \otimes \mathbb{Z} = 0$. Following the same idea for corank 2 elements, one can check that for $|x| = 3, 4$, we have $A_x = \{0\}$ and so $C^i(\mathbf{P}; \Delta\mathbb{Z}) \cong \prod_{|x|=i} A_x \otimes \mathbb{Z} = \{0\}$ for $i > 1$.

The cellular cochain complex for the tree is then

$$0 \rightarrow \prod_{|a|=0} A_a \otimes \mathbb{Z} \xrightarrow{\delta^0} \prod_{|x|=1} A_x \otimes \mathbb{Z} \xrightarrow{\delta^1} 0 \rightarrow 0 \rightarrow 0 \rightarrow 0.$$

The homology of the complex above:

- (a) The homology $HC^0(\mathbf{P}; \Delta\mathbb{Z})$ at $C^0(\mathbf{P}; \Delta\mathbb{Z}) \cong \prod_{|x|=0} A_x \otimes \mathbb{Z}$.

Let $\sigma \in N^0\mathbf{P}_{\geq a}$ be a generator of A_a then we have $\delta_x^a(a \otimes m) = x\sigma \otimes (-1)^1 m = -x\sigma \otimes m$, where $x\sigma \in A_x$ for some $m \in \mathbb{Z}$. Consider $\sigma = w_0, w'_0$ and w''_0 , and $x = w$ then we have $-(ww_0 + ww'_0 + ww''_0) \otimes m = -w(w_0 + w'_0 + w''_0) \otimes m = 0$ since $ww_0 + ww'_0 + ww''_0 = 0$. Thus if we consider all cases then the kernel of the δ^0 is generated by triples of leaves. In the diagram there are twelve triples of leaves and so the $\ker \delta^0$ is isomorphic to \mathbb{Z}^{12} . The homology at $C^0(\mathbf{P}; \Delta\mathbb{Z})$ is therefore $HC^0(\mathbf{P}; \Delta\mathbb{Z}) = \mathbb{Z}^{12}/\{0\} \cong \mathbb{Z}^{12}$.

- (b) The homology $HC^1(\mathbf{P}; \Delta\mathbb{Z})$ at $C^1(\mathbf{P}; \Delta\mathbb{Z}) \cong \prod_{|x|=1} A_x \otimes \mathbb{Z}$.

From (a), we have that the image of δ^0 is $C^0(\mathbf{P}; \Delta\mathbb{Z}) \cong \prod_{|x|=0} A_x \otimes \mathbb{Z}$. The map δ^1 takes all elements to the 0 element and so $\ker \delta^1 = C^1(\mathbf{P}; \Delta\mathbb{Z}) \cong \prod_{|x|=1} A_x \otimes \mathbb{Z}$. This shows that $HC^1(\mathbf{P}; \Delta\mathbb{Z}) = \{0\}$.

We therefore have $HC^0(\mathbf{P}; \Delta\mathbb{Z}) = \mathbb{Z}^{12}$ and $HC^i(\mathbf{P}; \Delta\mathbb{Z}) = \{0\}$ for $i > 0$

We now compute $HS^*(\mathbf{P}; F)$. \mathbf{P} is a small category with an initial object $\mathbf{0}$ and so we have that the nerve $|N^*\mathbf{P}|$ is contractible (see Proposition A.5.2). Thus $HS^0(\mathbf{P}; \Delta\mathbb{Z}) \cong \mathbb{Z}$ and $HS^i(\mathbf{P}; \Delta\mathbb{Z}) = 0$ for $i > 0$ (see Remark A.5.3).

The computations above shows that $HS^*(\mathbf{P}; F)$ is not isomorphic to $HC^*(\mathbf{P}; F)$.

4.3.6 Example with $HS^*(\mathbf{P}; F) \cong HC^*(\mathbf{P}; F)$

In this example we make use of Theorem 4.3.4 which is a generalisation of Theorem 3.5.4. This will tell us that Khovanov homology can be computed cellularly.

Recall the poset $\mathbf{Q} = \mathbf{Q}_\chi$ formed from resolving the crossings of a knot(link) with diagram D . \mathbf{Q} is

- graded with rank function $rk : \mathbf{Q} \rightarrow \mathbb{Z}$ defined by $rk(x) = |\chi| - |x|$, where $|\chi|$ is the number of crossings in the knot D and $|x|$ is the number of elements in x .
- cellular since it can be identified with the poset of cells of a regular CW-complex (see Subsection 3.5.1)
- locally finite since for any $x \in \mathbf{Q}$ there are only finitely many y with $x \prec y$.

If F is now the Khovanov presheaf, F_{KH} , then from the properties above, we see that \mathbf{Q} satisfies Theorem 4.3.4. Thus $HC^*(\mathbf{Q}; F_{KH}) \cong HS^*(\mathbf{P}; F)$.

Also since the poset \mathbf{Q} satisfies conditions of Theorem 4.3.4, we have $HC^*(\mathbf{Q}; F_{KH}) = \overline{KH}^*(D)$, since the cellular cochain complex $C^*(\mathbf{Q}; F_{KH})$ is now the Khovanov complex K^* .

This shows that the Khovanov homology can be computed cellularly under these conditions.

Chapter 5

Discussion

We saw in this thesis that the Khovanov homology of a knot can be identified with the right derived functors of the limits of the Khovanov presheaf which was our Theorem 3.5.4. We also saw that cohomology groups $HS^*(\mathbf{P}; F)$ of a poset \mathbf{P} , with coefficients in a presheaf F on the poset are related to the right derived functors of the limits of F via the classifying space of \mathbf{P} . Theorem 4.3.4 which was given by Everitt and Turner in [4] gave us an isomorphism between the cohomology $HS^*(\mathbf{P}; F)$ and the cellular cohomology $HC^*(\mathbf{P}; F)$ and this was seen to be a generalisation of Theorem 3.5.4. This leads to the fact that Khovanov homology can be computed cellularly.

A knot as we already know is a topological object. The idea used by M. Khovanov in the construction of the Khovanov homology of a knot was purely combinatorial and algebraic. Thus his idea lacked a topological and a geometric motivation. The fact that Khovanov homology of a knot can be computed cellularly is encouraging since it tells us that Khovanov homology can be given a topological and a geometric motivation. Everitt and Turner in their paper [5] also gave a homotopy theoretic interpretation of the Khovanov homology of a knot and this is also very interesting to look at.

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Appendix A

Some topological notions

A.1 Topology

Definition A.1.1. Let X be a non-empty set. A collection τ of subsets of X is said to be a topology on X if

- (i) X and the empty set, \emptyset , belong to τ ,
- (ii) the union of any (finite or infinite) number of sets in τ belongs to τ , and
- (iii) the intersection of any two sets in τ belongs to τ .

The pair (X, τ) is called a topological space.

Definition A.1.2. Let X be any non-empty set and let τ be the collection of all subsets of X . Then τ is called the *discrete topology* on the set X . The topological space (X, τ) is called a *discrete space*.

Definition A.1.3. Let S be a subset of a topological space X . A point $x \in X$ is called a *limit point* of S if every open set containing x contains a point of S different from x

Then *closure* \bar{S} of S is the union of S and all the limit points of S .

Definition A.1.4. A topological space X is said to be a *Hausdorff space* if given $x, y \in X$ with $x \neq y$, there exist open sets U, V such that $x \in U, y \in V$ and $U \cap V = \emptyset$.

A.2 CW complexes

Let $\overline{D}^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$ be the closed n -disk. Let $(D^n) = \{x \in \mathbb{R}^n : |x| < 1\}$ be the open disk. The boundary of D^n in \mathbb{R}^n is the standard $(n - 1)$ -sphere $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$.

An **n -cell** is a topological space homeomorphic to the open n -disk. A **cell decomposition** of a space X is a family $\Omega = \{e_\alpha : \alpha \in I\}$ of subspaces of X such that each e_α is a cell and

$$X = \coprod_{\alpha \in I} e_\alpha \text{ (disjoint union of subspaces.)}$$

The **n -skeleton** of X is the subspace

$$X^n = \coprod_{\alpha \in I: \dim(e_\alpha) \leq n} e_\alpha.$$

Definition A.2.1. A pair (X, Ω) consisting of a Hausdorff space X and a cell-decomposition Ω of X is called a **CW complex** if the following three axioms are satisfied.

- (i) For each n -cell $e_\alpha^n \in \Omega$, there is a map $\Phi_\alpha : \overline{D}^n \rightarrow X$ restricting to a homeomorphism $\Phi_\alpha|_{D^n} : D^n \rightarrow e_\alpha^n$ and taking sphere S^{n-1} into the skeleton X^{n-1} . The maps Φ_α are called the **characteristic maps**.
- (ii) For any cell $e_\alpha \in \mathcal{D}$, the closure \overline{e}_α intersects only a finite number of other cells in Ω .
- (iii) A subset $S \subseteq X$ is closed if and only if $A \cap \overline{e}_\alpha$ is closed in X for each $e_\alpha \in \Omega$.

Definition A.2.2. A CW complex (X, Ω) is called regular if for any cell $e_\alpha \in \Omega$, the characteristic map $\Phi_\alpha : (\overline{D}^n, S^{n-1}) \rightarrow (X^{n-1} \cup e_\alpha, X^{n-1})$ is a homeomorphism of \overline{D}^n into its image.

A.3 Quotient spaces

Definition A.3.1. Let X be a topological space. An equivalence relation \sim on X is a binary relation such that for all $x, y, z \in S$, we have

- (i) $x \sim x$ (reflexivity);

- (ii) $x \sim y$ if and only if $y \sim x$ (symmetry); and
- (iii) $x \sim y$ and $y \sim z$ implies $x \sim z$ (transitivity).

The equivalence class containing x , denoted by $[x]$, is defined by

$$[x] = \{u \in X : x \sim u\}.$$

Denote by S/\sim the set of equivalence classes $[x]$ and let the mapping $\pi : S \rightarrow S/\sim$ be defined by $x \mapsto [x]$. The collection of subsets U of S/\sim such that $\pi^{-1}(U)$ is open in S is a topology since

- (i) $\pi^{-1}(\emptyset) = \emptyset$ and $\pi^{-1}(S/\sim) = S$.
- (ii) $\pi^{-1}(U_1 \cap U_2) = \pi^{-1}(U_1) \cap \pi^{-1}(U_2)$, where $U_1, U_2 \in S/\sim$.
- (iii) $\pi^{-1}(\cup_{\alpha} U_{\alpha}) = \cup_{\alpha} \pi^{-1}(U_{\alpha})$.

Then the collection of sets $\tau = \{U \subset S/\sim : \pi^{-1}(U) \text{ is open in } S\}$ is called the quotient topology on S/\sim and $(S/\sim, \tau)$ is called the *quotient space*.

Definition A.3.2. Let X be a topological space. The *suspension* SX of X is the quotient space

$$SX = (X \times I) / \{(a, 0) \sim (b, 0) \text{ and } (a, 1) \sim (b, 1), \text{ for all } a, b \in X\}$$

where I is the unit interval $[0, 1]$.

The above definition says that SX is the quotient of $X \times I$ obtained by collapsing $X \times \{0\}$ to one point and $X \times \{1\}$ to another point.

For example we have the 1-simplex Δ^1 (see figure A.1a) and its suspension (see figure A.1b) with suspension points 1 and 1'.

A.4 Geometric realisation of a simplicial set

Definition A.4.1. The *standard n -simplex* is the set of all combinations

$$\Delta^n = [e_0, \dots, e_n] = \left\{ t_0 e_0 + \dots + t_n e_n : t_i \geq 0 \text{ and } \sum_{i=0}^n t_i = 1 \right\},$$

where $e_i = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{R}^n$ with 1 in the i th position.

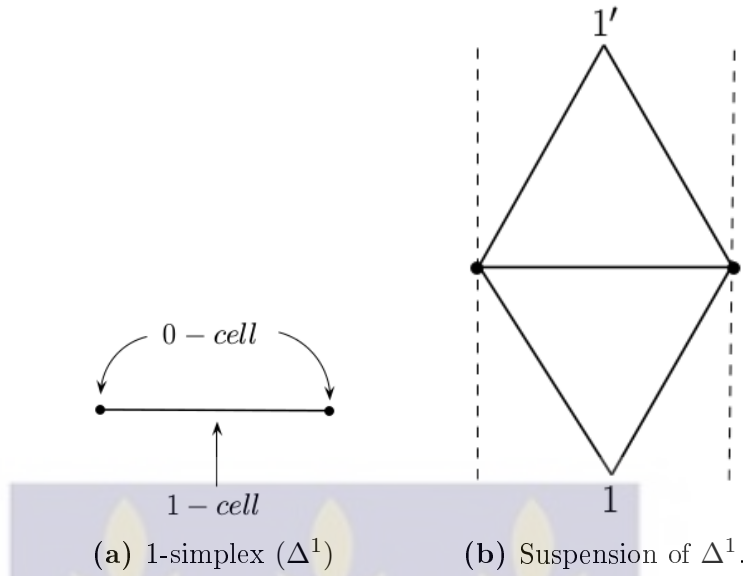


Figure A.1: A simplex and its suspension

There is a functor $T : \mathbf{SCat} \rightarrow \mathbf{Top}$ ⁷ such that for $[n] \in \mathbf{SCat}$ we have $T([n]) = \Delta^n$ and for $\eta : [n] \rightarrow [m]$ in \mathbf{SCat} we have the induced map $T(\eta) = \eta_* : \Delta^n \rightarrow \Delta^m$ defined by $\sum_{i=0}^n t_i e_i \mapsto \sum_{i=0}^n t_i e_{\eta(i)}$.

Let X be a simplicial set and give each X_n the discrete topology (see Definition A.1.2). We define the *geometric realisation* of X to be the quotient

$$|X| = \coprod_{n \geq 0} X_n \times \Delta^n / \sim$$

where the equivalence relation is given by $(d_i(x), p) \sim (x, \delta_*^i(p))$ and $(s_i(x), p) \sim (x, \sigma_*^i(p))$. We have $\delta_*^i(t_0, \dots, t_n) = (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_n)$ and $\sigma_*^i(t_0, \dots, t_n) = (t_0, \dots, t_{i-1}, t_i + t_{i+1}, \dots, t_n)$ where δ^i and σ^i are the maps defined in Section 4.1, and $T(\delta^i) = \delta_*^i$, $T(\sigma^i) = \sigma_*^i$

A.5 Contractible nerves

Let X and Y be two simplicial sets. A *simplicial map* is a natural transformation (see Definition 2.2.11) $\eta : X \rightarrow Y$ from X to Y .

Definition A.5.1. Let X and Y be simplicial sets. Two simplicial maps $f, g : X \rightarrow Y$

⁷ \mathbf{Top} is the category of topological spaces with topological spaces as objects and homeomorphisms as morphisms. See Example 2.2.2(v) for the definition of \mathbf{SCat} .

are homotopic if for each n there exist functions

$$h_n = \{h_i : X_n \longrightarrow Y_{n+1}\}$$

for each $i, 0 \leq i \leq n$ such that

1. $d_0 h_0 = f$ and $d_{n+1} h_n = g$.
2. $d_i h_j = h_{j-1} d_i$ if $i < j$,
 $d_{j+1} h_{j+1} = d_{j+1} h_j$,
 $d_i h_j = h_j d_{i-1}$ if $i > j + 1$.
3. $s_i h_j = h_{j+1} s_i$ if $i \leq j$,
 $s_i h_j = h_j s_{i-1}$ if $i > j$.

A simplicial set X is **contractible** if the identity map on X is homotopic to some constant map.

In the following proposition we will show that if \mathcal{C} is a small category having an initial object then its nerve, $N^*\mathcal{C}$, is contractible.

Proposition A.5.2. *Let \mathcal{C} be a small category with an initial object α . Then $N^*\mathcal{C}$ is contractible.*

Proof. We show that the identity map $id_{N^*\mathcal{C}} : N^*\mathcal{C} \longrightarrow N^*\mathcal{C}$ and the constant map $k_\alpha : N^*\mathcal{C} \longrightarrow N^*\mathcal{C}$ are homotopic. For $[n] \in \mathbf{SCat}$ and $\sigma \in N^n\mathcal{C}$ we have $(id_{N^*\mathcal{C}})_{[n]}(\sigma) = \sigma$ and $(k_\alpha)_{[n]}(\sigma) = \alpha$ where $\alpha \in N^0\mathcal{C}$.

Consider the identity map $id : \mathcal{C} \longrightarrow \mathcal{C}$ and the constant map $k : \mathcal{C} \longrightarrow \alpha$. Let $T : k \longrightarrow id$ be a natural transformation from k to id . Then T induces a simplicial homotopy by considering the following diagram for each n .

$$\begin{array}{ccccccc}
 \alpha = & & \alpha = & & \dots = & & \alpha \\
 \downarrow T_{\sigma_n} & & \downarrow T_{\sigma_{n-1}} & & & & \downarrow T_{\sigma_0} \\
 \sigma_n & \longrightarrow & \sigma_{n-1} & \longrightarrow & \dots & \longrightarrow & \sigma_0
 \end{array}$$

From the diagram, we can define for each n the functions

$$h_n = \{h_i : N^n\mathcal{C} \longrightarrow N^{n+1}\mathcal{C}, i = 0, \dots, n\}$$
 by

$$\begin{aligned}
 h_0(\sigma) &= \alpha \longrightarrow \alpha \longrightarrow \dots \longrightarrow \alpha \longrightarrow \alpha \longrightarrow \sigma_0 \\
 h_1(\sigma) &= \alpha \longrightarrow \alpha \longrightarrow \dots \longrightarrow \alpha \longrightarrow \sigma_1 \longrightarrow \sigma_0 \\
 &\vdots \\
 h_n(\sigma) &= \alpha \longrightarrow \sigma_n \longrightarrow \dots \longrightarrow \sigma_2 \longrightarrow \sigma_1 \longrightarrow \sigma_0
 \end{aligned}$$

Now observe that $d_0 h_0(\sigma) = \alpha \longrightarrow \dots \longrightarrow \alpha = \alpha$ and so $d_0 h_0 = k_\alpha$. Also $d_{n+1} h_n(\sigma) = \sigma_n \longrightarrow \dots \longrightarrow \sigma_0 = \sigma$ and so $d_{n+1} h_n = id_{N^*C}$. The other two conditions can be verified. Thus the identity map id_{N^*C} and the constant map k_α are homotopic, hence the nerve N^*C is contractible. \square

Remark A.5.3. The homology and cohomology of a contractible space vanish for degrees above zero but are \mathbb{Z} in degree zero since it has the same homotopy type of a point space.

