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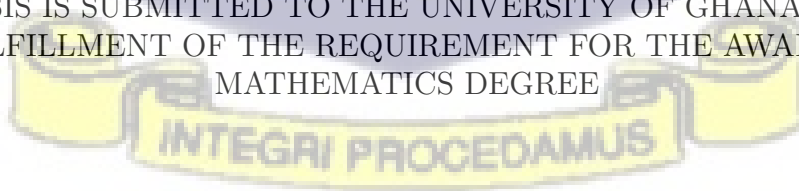
A CHARACTERIZATION OF THE ALTERNATING GROUP A_n , FOR $n \geq 8$, BY ITS
FUSION SYSTEM OVER A SYLOW 2-SUBGROUP

By

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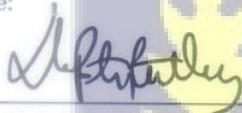


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DECLARATION

The work presented here was carried out under the supervision of Professor Ellen Henke, Institute of Algebra, Technische Universität Dresden-TU Dresden, Germany. I declare that except where due acknowledgment has been given, this work has not been previously submitted, in whole or in part, for an award of a degree in any university.

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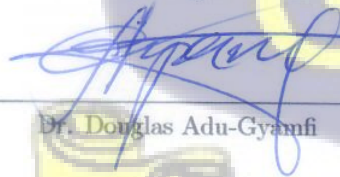
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ABSTRACT

This thesis aims to characterize the alternating group A_n , for $n \geq 8$, by its fusion system over a Sylow 2-subgroup. The results here will contribute to Aschbacher's programme which is aimed at providing a simplified proof of the theorem of classification of finite simple groups.



DEDICATION

I dedicate this thesis to, the memory of my father; Abubakar Ibrahim, my mother; Raihanah Iddrisu, my wife; Safiatu Shafiu-Alidu and kids; Abdul Hannan, Raihanah, Zakiyya, and my siblings; Mariama, A'ishatu, Mohammed, Jafaru, Tamimu, Habiba, Ibrahim, Hawawu.

Alhamdulillah Rabbil-'alamin.



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TABLE OF CONTENTS

Abstract	i
Acknowledgments	iii
List of Abbreviations	vi
List of figures	viii
Introduction	1
Chapter 1 Overview of finite simple groups	6
1.1 Basic terminology in finite group theory	6
1.2 Symmetric groups and alternating groups	12
1.3 The known simple groups	15
1.4 Quasisimple groups	18
Chapter 2 Overview of some group theoretical methods	19
2.1 Finite p -groups	19
2.2 Normal p -complements	23
2.3 The focal and hyperfocal subgroup	23
2.4 p -constrained groups	24
2.5 Strongly p -embedded subgroups	24
2.6 Closure and control of fusion	26
2.7 Group action and coprime action	27
2.8 Components and 2-components	29
2.9 Generalized Fitting subgroups	31
2.10 Goldschmidt groups	32
2.11 Signalizer functors	34
2.12 k -balanced groups and Signalizer functors	35
2.13 Generation	38
Chapter 3 Introduction to fusion systems	41
3.1 Fusion category of a finite group	41
3.2 The notion of abstract fusion system	43
3.3 Saturated fusion systems	48
3.4 Subsystems of fusion systems	50
3.5 Morphisms of fusion systems	52
3.6 Alperin-Goldschmidt fusion theorem	53
3.7 An application of Alperin-Goldschmidt fusion theorem	55
3.8 Strongly closed and normal subgroups of a fusion system	59
3.9 Factor/Quotient fusion systems	62

3.10	Constrained fusion systems	65
3.11	The focal and hyperfocal subgroups of fusion system	67
3.12	Normal subsystems	68
3.13	Simple fusion systems	71
3.14	Components of fusion systems	72
3.15	Tame fusion systems	73
Chapter 4	Some properties of the alternating groups	75
4.1	A presentation of the alternating group	75
4.2	Involutions of the alternating group	76
4.3	The structure of a centralizer of an involution	76
4.4	The Sylow 2-subgroups of the alternating groups	78
4.5	Some lemmas	79
Chapter 5	Finite groups realizing the fusion systems of A_{2m} , for $m \geq 6$ over its Sylow 2-subgroup: The structure of centralizers of involutions	84
5.1	General setup and remarks	85
5.2	The structure of the group $H = C_G(t)/O(C_G(t))$, for t a short involution	86
5.3	The structure of $H = C_G(t_q)/O(C_G(t_q))$ for any involution $t_q \neq t$	89
5.4	Proof of Theorem 5.0.1	96
Chapter 6	Finite groups realizing the fusion system of A_{2m} , for $m \geq 6$, over its Sylow 2-subgroup	97
6.1	The structure of G	97
6.2	Proof of Theorem A	105
Bibliography		106



LIST OF ABBREVIATIONS

Throughout this thesis, we let p denote a prime and p' the set of primes distinct from p . Maps are written on the right hand side and are composed from left to right.

Let G be a finite group with $x, y \in G$ and $Q, R \leq G$.

- $G^\#$ denote the set of all non-identity elements of G
- $x^y := y^{-1}xy$ is the conjugate of x by y .
- $R^x := \{r^x \mid r \in R\}$ is the conjugate of the group R by x .
- $[x, y] = x^{-1}x^y$ is the commutator of x and y .
- $[Q, R] = \langle [q, r] \mid q \in Q, r \in R \rangle$, is the commutator of Q and R .
- $G' = [G, G]$ is the commutator subgroup of G .
- $C_G(x)$ is the centralizer of x in G .
- $C_G(R)$ is the centralizer of R in G .
- $N_G(R)$ is the normalizer of R in G .
- $Z(G)$ is the center of G .
- $\langle Q \rangle$ is the subgroup generated by Q .
- $\Omega_i(G) := \langle g \in G \mid g^{p^i} = 1 \rangle$.
- $Syl_p(G)$ is the set of all Sylow p -subgroups of G .
- $|G|_p$ is the order of any $P \in Syl_p(G)$.
- $o(x)$ is the order of x .
- $O_p(G)$ is the largest normal p -subgroup of G .
- $O_{p'}(G)$ is the largest normal p' -subgroup of G .
- $O^p(G)$ is the smallest normal subgroup H of G such that G/H is a p -group.
- $O^{p'}(G)$ is the smallest normal subgroup H of G such that G/H is a p' -group.

- $O(G)$ is $O_{2'}(G)$.
- $O_{p',p}(G)$ is the normal subgroup R of G that contains $O_{p'}(G)$ and satisfies $R/O_{p'}(G) = O_p(G/O_{p'}(G))$.
- Q char G is a characteristic subgroup Q of G .
- $Aut(G)$ is the group of automorphisms of G .
- $Inn(G)$ is the group of inner automorphisms of G .
- $Out(G) := Aut(G)/Inn(G)$ is the group of outer automorphisms of G .
- $Aut_G(R) \cong N_G(R)/C_G(R)$ is the automizer of R in G .
- $F(G)$ is the fitting subgroup of G .
- $F^*(G)$ is the generalized fitting subgroup of G .

For specific groups we use the following notation;

- S_n denotes the symmetric group on n letters.
- A_n is the alternating group on n letters.
- C_n is the cyclic group of order n .
- E_{p^n} is an elementary abelian p -group of order p^n .
- D_{2n} is the dihedral group of order $2n$, for $n \geq 2$.
- Q_{2^n} is the generalized quaternion group of order 2^n , for $n \geq 3$.
- $GF(p^n)$ is the finite field of order p^n .
- $SL_n(p)$ is the special linear group of degree n over $GF(p)$.
- $SU_n(p)$ is the special unitary group of degree n over $GF(p)$.
- $PSL_n(p)$ is the projective special linear group of degree n over $GF(p)$.
- $PSU_n(p)$ is the projective special unitary group of degree n over $GF(p)$.
- $Sz(2^n)$ is the Suzuki group over $GF(2^n)$.
- ${}^2G_2(3^n)$ is the Ree group over $GF(3^n)$.

LIST OF FIGURES

31	Fusion category of S_4 over $T \cong D_8$	43
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INTRODUCTION

The classification of finite simple groups (hereafter abbreviated CFSG) is considered a monumental achievement in the history of mathematics, which provides a complete understanding of the basic building blocks of finite groups. The CFSG states that every finite simple group can be categorized as either a group of prime order, an alternating group, a finite simple group of Lie type, the Tits group, or one of the twenty-six (26) sporadic groups. The proof of the CFSG spanned approximately 15,000 pages and encompassed hundreds of articles published across various journals over three decades. Over one hundred mathematicians from around the globe contributed to this proof, with its final stages published by M. Aschbacher and S. Smith in 2004, following an earlier announcement of its completion in 1983 by D. Gorenstein. The proof also required the development of new mathematical techniques and tools. Due to the extensive length and complexity of the proof for the CFSG, there is a strong need for a simplified, condensed, and streamlined version. A concise and condensed presentation of the proof would be immensely advantageous. For instance, it will preserve some of the techniques and tools presented in the proof, making it highly accessible to researchers. Currently, there are three programmes dedicated to achieving this objective:

1. The CGLS programme (Capdeboscq, Gorenstein, Lyons, Solomon).
2. The MSS programme (Meierfrankenfeld, Stellmacher, Stroth).
3. Aschbacher's programme.

The CGLS programme aims to revise, restructure and improve the proof, consolidating it into a comprehensive and self-contained treatment. This effort will be presented through a series of monographs, starting with [27], which aims to deliver a second-generation proof of the CFSG.

The MSS programme focuses on understanding and classifying finite groups of local characteristic p , where p is a prime. If successful, this programme would significantly modify a substantial portion of the proof of the CFSG.

Aschbacher's programme aims to achieve a new proof of the CFSG by employing fusion systems. For an overview of the CFSG and these programmes, one may check the references [24, Introduction], [9, Preface], [23], [27] and [45].

This thesis primarily focuses on a specific aspect of Aschbacher's programme. Before diving into the description of this programme, let us first examine the concept of fusion

systems. Let X be a finite group and T be a p -subgroup of X , where p is a prime. In this context, we define the *fusion category* of X over T as $\mathcal{F}_T(X)$, which can be described as follows: The objects in $\mathcal{F}_T(X)$ correspond precisely to the subgroups of T . The morphisms in $\mathcal{F}_T(X)$ correspond precisely to the group homomorphisms between subgroups of T induced by conjugation in X . The composition of morphisms in $\mathcal{F}_T(X)$ follows the usual composition of group homomorphisms. The fusion categories of finite groups over p -subgroups serve as the conventional examples of fusion systems. Abstract fusion systems offer a broader framework for the concept of fusion category, where the p -subgroup is no longer required to be a subgroup of some larger group. A fusion system over a finite p -group T , where p is a prime, is a category where the subgroups of T serve as objects, and the morphisms exhibit properties akin to being induced by conjugation within a finite group that has T as a subgroup (See Definition 3.2.1). The concept of abstract fusion systems is considered as overly general. Therefore, there arises a need to impose certain restrictions to enhance its usefulness. This gave rise to a more specialized version known as saturated fusion systems. A fusion system is called *saturated* when it satisfies certain specific properties that draw inspiration from the properties of fusion categories associated with finite groups over Sylow subgroups (see Definition 3.3.1). With respect to a finite p -group T (where p is a prime), a saturated fusion system $\mathcal{F}$ is said to be *realizable* by a finite group X if $\mathcal{F}$ is isomorphic to $\mathcal{F}_T(X)$. Saturated fusion systems which are not realizable by a finite group are called *exotic*. Throughout the thesis, we shall assume all our fusion systems are saturated unless otherwise stated. This beautiful concept of saturated fusion systems can be traced back to L. Puig([52]), who employed them as a valuable tool in modular representation theory. Subsequently, the theory of fusion systems was further advanced by Broto, Levi, and Oliver([11]), who drew inspiration from homotopy theory. Today, the study of fusion systems is a thriving and active area of research encompassing finite group theory, representation theory, and topology. The references [8] and [14] provide more details on the subject.

Aschbacher's programme is a significant research endeavour in the classification of finite simple groups. The programme utilizes saturated fusion systems as a fundamental tool in its approach. With this approach, most of the proof takes place in the category of saturated fusion systems over a 2-group, contrary to the category of groups as it was done in the original proof of the CFSG. The use of a saturated fusion system is motivated by the following facts:

1. In the category of groups, p -local analysis (which deals with local finite group theory, which is the study of normalizers of non-trivial subgroups of order a power of a prime p) of finite groups played a significant role in the proof of the CFSG. It happens that most concepts from local group theory can be translated into the language of saturated fusion systems. Thus the local theory of saturated fusion systems can be built

similarly to that of local finite group theory. In particular, there are notions of normal subsystems, simple systems, components, normalizers, and centralizers analogous to the corresponding notions in finite group theory (See for example Section 3.12-3.14 and Definition 3.4.4).

2. In the category of groups, one major challenge to the 2-local analysis is the core of a 2-local subgroup Y of a group X , where the core of Y is the largest normal subgroup $O(Y)$ of odd order in Y . A part of the 2-local analysis of the proof of the CFSG requires that the core of a certain subgroup be trivial or at worst have a very simple structure. The proof of this requirement came with great effort and was not very attractive. This challenge is avoided in the category of fusion systems over a 2-group, as cores do not exist in fusion systems as indicated in Corollary 3.9.5.

With these facts, it is anticipated that the use of saturated fusion systems will lead to a significant simplification of a certain part of the proof of the CFSG. Aschbacher's programme can be broadly divided into two main steps as follows:

Step 1: Classify a certain class of simple saturated fusion systems on finite 2-groups. This takes inspiration from the original proof of the CFSG as a guiding framework.

Step 2: This entails transitioning from saturated fusion systems to finite groups.

A key problem in step 2 is for a given simple fusion system $\mathcal{F}$ over a 2-group, how does one find the simple groups X such that $\mathcal{F}_T(X) = \mathcal{F}$? In simpler words, the objective is to identify a finite simple group from its fusion system over a Sylow 2-subgroup. This particular aspect forms the core focus of this thesis. For a comprehensive account of Aschbacher's programme, one may check the reference [7].

In order to state our main result, we set up as follows:

Hypothesis A. *Let G be a finite group such that $O(G) = 1$ and $O^2(G) = G$. Let $S \in \text{Syl}_2(A_{2m})$ where $m \geq 4$, and assume S is contained in G as a Sylow 2-subgroup such that $\mathcal{F}_S(G) = \mathcal{F}_S(A_{2m})$. Suppose furthermore that for every 2-local subgroup H of G and every component K of $H/O(H)$ we have that $K/Z(K)$ is a known finite simple group.*

The following theorem is our main result.

Theorem A. *Assume Hypothesis A. Then $G \cong A_{2m}$ or $G \cong A_{2m+1}$.*

We will prove Theorem A by induction on m . We note that the theorem is true for $m = 4, 5$ as presented in [25]. So we will work under the following hypothesis.

Hypothesis B. Assume Hypothesis A and $m \geq 6$. Suppose furthermore that Theorem A is true with m' in place of m for all $4 \leq m' < m$.

The thesis is structured as follows: The initial four chapters primarily focus on gathering the necessary background material essential for proving Theorem A. The purpose of these chapters is to provide a comprehensive understanding of foundational concepts and results, which will serve as a basis for the subsequent discussions and proofs in the thesis. We acknowledge that many of the results covered in these chapters are already established. The last three chapters contain results achieved in the PhD research leading to this thesis.

In Chapter 1, the focus is on providing the necessary background material regarding standard terminologies and results related to finite group theory. The chapter covers the symmetric and alternating groups, quasisimple groups and the known finite simple groups, which are essential for this thesis.

Chapter 2 aims to provide a comprehensive overview of definitions, results, and concepts relevant to finite groups, with a particular focus on finite p -groups, constrained groups, strongly p -embedded subgroups, closure of subgroups, group action, generalized Fitting subgroups, and Goldschmidt groups. The chapter also delves into the theory of signalizer functors which will play an important role in the proof of our main result.

Chapter 3 begins by introducing the concept of a fusion category, which is subsequently expanded to the notion of an abstract fusion system. The focus is primarily on saturated fusion systems, which are of particular importance. The chapter aims to provide a foundational understanding of fusion systems, particularly for their application in Chapter 5.

In Chapter 4, the focus is on presenting various properties of alternating groups including the balance and generation properties. The chapter also covers the description of involutions in the alternating group and explores the general structure of a centralizer of involutions and its associated properties.

In Chapter 5, our goal is to analyze the structure of the quotient groups of 2-local subgroups within an arbitrary finite group G , which realizes the fusion system of the alternating group A_{2m} over a Sylow 2-subgroup, where $m \geq 6$. To achieve this objective, we utilize various results from finite group theory (Chapter 1-2) and the theory of fusion systems (Chapter 3). Most importantly, we show that there exists some group X such that the quotient groups of the 2-local subgroups modulo cores are isomorphic to that of the the quotient groups of the 2-local subgroups of G modulo cores.

In Chapter 6, we establish that the core of a centralizer of an involution in a finite group G , which realizes the fusion system of A_{2m} over a Sylow 2-subgroup, is largely trivial or has order 3 in few cases. To accomplish this, we use results from Chapter 2, which are proved

using signalizer functor theory. We further show that G is isomorphic to some group X where $X \in \{A_{2m}, A_{2m+1}\}$. We finally present a proof for Theorem A.



Chapter 1 Overview of finite simple groups

This chapter deals with background material on finite simple groups, needed in this thesis. In Section 1.1, we collect some standard terminologies and results about finite group theory. Section 1.2 introduces symmetric groups and alternating groups with their associated properties while Sections 1.3-1.4 deal with quasisimple groups and the known finite simple groups.

1.1 Basic terminology in finite group theory

In this section we will collect some standard terminologies, and statements of a number of well known results in finite group theory, which are fairly treated in most undergraduate group theory courses. The material presented here can be found in most group theory textbooks such as [4], [21], [35], [41], [56] and [57]. We begin with a basic algebraic structure called a group.

Definition 1.1.1. A non-empty set X together with an associative binary operation is a group, if

- (i) it has an identity, and
- (ii) every element of X has an inverse.

Furthermore, the group is called *abelian* if the binary operation is commutative. The *order* of a group X is the cardinality of the associated set X . We write $|X|$ to denote the order of X . If the cardinality of X is finite, then X is said to be a *finite* group, and *infinite* otherwise. For a group X , we let $X^\#$ denote the set of all non-identity elements of X .

Throughout this thesis, we shall assume all our groups to be finite unless otherwise stated.

Definition 1.1.2. A *subgroup* of X is a non-empty subset Y of X such that for all $a, b \in Y$,

- (i) $ab \in Y$, and
- (ii) $b^{-1} \in Y$.

Thus, the binary operation on X restricts to a binary operation on Y , making Y a group with the same inverses and identity as X . We write $Y \leq X$ to denote that Y is a subgroup of X .

Definition 1.1.3. For a non-empty subset Y of X ,

$$\langle Y \rangle = \{a_1^{n_1} \cdots a_j^{n_j} \mid a_j \in Y, n_j \in \mathbb{Z}, j \in \mathbb{N}\}$$

is the subgroup generated by Y . The elements a_j are called the generators of the group $\langle Y \rangle$.

As a special case of Definition 1.1.3 we have the following.

Definition 1.1.4. A group X is said to be *cyclic* if it is generated by some element $x \in X$.

Example 1.1.5. For any positive integer n , the factor group $\mathbb{Z}/n\mathbb{Z}$ is a cyclic group of order n . Any cyclic group of order n is isomorphic to $\mathbb{Z}/n\mathbb{Z}$.

For $Y, Z \leq X$ and $x, a \in X$, we set the following:

1. $x^a = a^{-1}xa$, and ${}^ax = axa^{-1}$, is the conjugate of x by a . The element x^a or ax is said to be a conjugate of x .
2. $Y^a = \{y^a \mid y \in Y\}$, is the set of conjugates of Y by a .
3. $Y^Z = \{Y^a \mid a \in Z\}$, is the set of conjugates of Y by Z .

Definition 1.1.6. A subgroup Y of X is said to be *normal* if $Y^x \subseteq Y$ for each $x \in X$. We write $Y \trianglelefteq X$ to denote that Y is a normal subgroup of X .

Definition 1.1.7. A non-trivial group X is *simple* if it has no proper non-trivial normal subgroup. Equivalently, X is simple if the number of its normal subgroups is 2, that is the trivial subgroup $\{1\}$ and X itself.

Example 1.1.8. The cyclic group $\mathbb{Z}/p\mathbb{Z}$ for a prime p is simple, as its subgroups are exactly the trivial subgroup $\{1\}$ and $\mathbb{Z}/p\mathbb{Z}$ itself. Any abelian simple group of prime order is isomorphic to $\mathbb{Z}/p\mathbb{Z}$ for some prime p .

Definition 1.1.9. For a finite group X , and $Y, R \subseteq X$, we define

1. $N_X(Y, R) = \{x \in X \mid Y^x \leq R\}$ as the *X-transporter set* of Y into R .
2. $N_X(Y) = \{x \in X \mid Y^x = Y\}$ as the *normalizer* of Y in X . Observe that $N_X(Y)$ is a subgroup of X . In particular, if $Y \leq X$, then $N_X(Y)$ is the largest subgroup of X in which Y is normal.
3. $C_X(Y) = \{x \in X \mid y^x = y \text{ for all } y \in Y\}$ as the *centralizer* of Y in X . It is also a subgroup of X .

4. $Z(X) = \{x \in X \mid x^a = x \text{ for all } a \in X\}$ as the *centre* of X . It is a normal subgroup of X .

Definition 1.1.10. A subgroup U of a group X is said to be a *TI-subgroup* provided that $U \cap U^x = 1$ for all $x \in X - N_X(U)$.

For the following definition and the rest of the thesis, we will usually write the operation of maps from right to left.

Definition 1.1.11. Let X and Y be groups.

- A group *homomorphism* from X into Y is a function

$$\phi : X \rightarrow Y$$

from the set X into the set Y such that for all a, b in X , $(ab)\phi = a\phi b\phi$. The set of all group homomorphisms from X to Y is denoted as $Hom(X, Y)$. We write $Inj(X, Y)$ for the set of injective group homomorphisms from X to Y . Also if $X = Y$, then the homomorphism ϕ is called an *endomorphism* of X , and the set of all endomorphisms of X is denoted as $End(X)$. Furthermore, the homomorphism ϕ is an *isomorphism* if ϕ is a bijection. We usually write $X \cong Y$ to denote that X is isomorphic to Y . In particular, if $X = Y$, then the isomorphism ϕ is called an *automorphism* of X . The set of all automorphisms of X is denoted as $Aut(X)$. The set $Aut(X)$ is a group with respect to functional composition.

- The group Y is said to be a *homomorphic image* of X if there is a surjective homomorphism of X onto Y .

We shall also adopt the following notation throughout the thesis. For groups X, Y and a given homomorphism $\phi : X \rightarrow Y$, if $P \leq X$ and $Q \leq Y$ such that $P\phi \leq Q$, then let $\phi|_{P,Q}$ be the restriction of ϕ to P . In cases where $P = X$ or $Q = Y$, the corresponding subscript is omitted. The presence or absence of the comma in the subscript will indicate that the only subscript is the new source or the new target respectively. The *core* of ϕ is defined as the restriction $\phi|_{X\phi}$. Thus, the core of any monomorphism is an isomorphism.

Definition 1.1.12. Let X be a group, and let r be any element of X . The mapping

$$c_r : X \rightarrow X \text{ defined as } x \mapsto x^r$$

is a bijection. The map c_r is an automorphism of X , called the *inner automorphism* induced by r (or induced by conjugation by r).

The mapping

$$c : X \rightarrow \text{Aut}(X) \text{ defined as } r \mapsto c_r$$

is a homomorphism from X to $\text{Aut}(X)$. The set of inner automorphisms of X , denoted by

$$\text{Inn } X := \{ c_r \mid r \in X \}$$

is a normal subgroup of $\text{Aut}(X)$. For $Z(X)$, the center of X , we have

$$X/Z(X) \cong \text{Inn } X.$$

For an isomorphism $\phi: X \rightarrow Y$, where $\alpha \in \text{Aut}(X)$ and $\beta \in \text{Aut}(Y)$, we write $\alpha c_\phi = \alpha^\phi = \phi^{-1}\alpha\phi \in \text{Aut}(Y)$ and $\beta c_\phi = \beta^\phi = \phi^{-1}\beta\phi \in \text{Aut}(X)$.

Definition 1.1.13. An automorphism of a group X which is not inner is referred to as *non-inner*. The quotient group

$$\text{Out}(X) := (\text{Aut}(X))/(\text{Inn } X)$$

is called the *outer automorphism group* of X .

Let Q, R be subgroups of X such that $Q^x \leq R$ for an element x of X . Then the map

$$c_x : Q \rightarrow R$$

is an injective group homomorphism. The set of all such maps is denoted as $\text{Hom}_X(Q, R)$. Therefore

$$\text{Hom}_X(Q, R) := \{ c_x|_{Q,R} \mid x \in X, Q^x \leq R \},$$

and we set $\text{Aut}_X(Q) := \text{Hom}_X(Q, Q)$. In particular, we can see $\text{Inn}(X) := \text{Aut}_X(X)$.

Lemma 1.1.14. Let X be a group with R as a subgroup. Then $\text{Aut}_X(R)$ is a subgroup of $\text{Aut}(R)$ and the mapping

$$c : N_X(R) \rightarrow \text{Aut}_X(R) \text{ defined as } x \mapsto c_x$$

is a surjective group homomorphism with kernel $C_X(R)$. In particular,

$$N_X(R)/C_X(R) \cong \text{Aut}_X(R).$$

Proof. See [41, 3.1.9]. □

Remark 1.1.15. We note that the map c in Lemma 1.1.14 maps $N_X(R)$ onto $\text{Aut}_X(R)$, and such maps are usually denoted by c_R .

Definition 1.1.16. Let X be a group. A subgroup U of X is a *characteristic subgroup* of X if $U\phi = U$ for all $\phi \in \text{Aut}(X)$. We usually write $U \text{ char } X$ in this case. Characteristic subgroups are normal in X .

Example 1.1.17. For any finite group X , the trivial subgroup $\{1\}$, the center $Z(X)$, and the group X itself are all characteristic subgroups of X .

Definition 1.1.18. Let X and Y be groups.

- If the mapping

$$\phi : Y \rightarrow \text{Aut}(X) \text{ defined as } y \mapsto \phi_y$$

is a homomorphism from Y to $\text{Aut}(X)$, then we say Y *acts* on X with respect to ϕ .

- Suppose Y acts on X via ϕ . A subgroup B of X is *Y -invariant* if for all $y \in Y$,

$$B^y := B\phi_y = B.$$

In particular, if B is a Y -invariant subgroup of X , then Y acts on B with respect to the homomorphism

$$Y \rightarrow \text{Aut}(B) \text{ given by } y \mapsto \phi_y|_{B,B}.$$

Definition 1.1.19. Let X be a group with elements a and b .

- We define the *commutator* of a and b as

$$[a, b] := a^{-1}b^{-1}ab = (b^{-1})^a b = a^{-1}a^b.$$

- The subgroup generated by all commutators

$$\langle [a, b] \mid a, b \in X \rangle$$

is called the *commutator subgroup* of X . We write $[X, X]$ or X' as the commutator subgroup of X .

- The group X is said to be *perfect* if $X = [X, X]$.
- The group X is said to be *solvable* if for all non-trivial subgroups U of X ,

$$U \neq [U, U].$$

- The group X is said to be *nilpotent* if for all non-trivial subgroup Q of X , $Q < N_X(Q)$.

For subgroups A, B and C of X , we set $[A, B, C] = [[A, B], C]$.

The following lemma is useful.

Lemma 1.1.20 (Three-Subgroups Lemma). *Let A, B, C be subgroups of a group X . Suppose $[A, B, C] = [B, C, A] = 1$. Then $[C, A, B] = 1$.*

Proof. See for example [35, Lemma 4.9] □

One of the ways of building a group out of smaller ones is by taking products. The following definition gives the notion of one such case.

Definition 1.1.21. Let X and Y be groups with $X_1 \leq Z(X)$ and $Y_1 \leq Z(Y)$, and suppose there is a group isomorphism $\phi : X_1 \rightarrow Y_1$. The *central product* is the quotient of the direct product $X \times Y$ by the normal subgroup $N = \{(x, y) \mid x \in X_1, y \in Y_1, \text{ and } \phi(x)y = 1, \text{ where } 1 \text{ is the identity element of } Y\}$. We write $X * Y$ as the central product of X and Y .

The following three fundamental results will come in handy later.

Lemma 1.1.22 (Dedekind Identity). *Let X be a group with subgroups A, B such that $X = AB$. Then for every subgroup C of X with $A \leq C \leq X$, the factorization $C = A(B \cap C)$ holds.*

Proof. Clearly A and $B \cap C$ are subgroups of C , and so $A(B \cap C) \subseteq C$. On the other hand, let $c \in C \leq X$. Then $c = ab$ where $a \in A$ and $b \in B$. As $A \leq C$ we have that $b \in B \cap C$, and so $c \in A(B \cap C)$. Thus $C \subseteq A(B \cap C)$ and the result follows. □

Theorem 1.1.23 (Gaschütz's Theorem). *Let X be a group with an abelian normal subgroup A . Suppose C is a subgroup of X such that $A \leq C$ with $|A|$ and $|X : C|$ being coprime. If A has a complement in C , then A has a complement in X .*

Proof. See for example [41, 3.3.2]. □

Theorem 1.1.24 (Schur-Zassenhaus' Theorem). *Suppose X is a group with a normal subgroup Q such that $|Q|$ and $|X : Q|$ are coprime. Then Q has a complement in X . Furthermore, if Q or X/Q is solvable, then all complements of Q are conjugate in X .*

Proof. See for example [41, 6.2.1]. □

We end this section with the following lemma which we will need later.

Lemma 1.1.25. *Let X and Y be groups, both of the form*

$$X = (N \times K) \rtimes \langle t \rangle \text{ and } Y = (\tilde{N} \times \tilde{K}) \rtimes \langle \tilde{t} \rangle$$

where t and $\tilde{t}$ are involutions such that N and K are $\langle t \rangle$ -invariant, and $\tilde{N}$ and $\tilde{K}$ are $\langle \tilde{t} \rangle$ -invariant. Assume there are isomorphisms

$$\alpha: N \langle t \rangle \rightarrow \tilde{N} \langle \tilde{t} \rangle \text{ and } \delta: K \langle t \rangle \rightarrow \tilde{K} \langle \tilde{t} \rangle$$

with $N\alpha = \tilde{N}$, $K\delta = \tilde{K}$ and $\tilde{t} = t\alpha = t\delta$. Then the map $\phi: X \rightarrow Y$ such that $nkt^i \mapsto (n\alpha)(k\delta)\tilde{t}^i$ for $n \in N$, $k \in K$ and $i \in \{0, 1\}$ is an isomorphism.

Proof. For $n_1, n_2 \in N$, $k_1, k_2 \in K$ and $i, j \in \{0, 1\}$, we have

$$\begin{aligned} (n_1 k_1 t^i n_2 k_2 t^j) \phi &= (n_1 k_1 t^i t^{-i} t^i n_2 t^{-i} t^i k_2 t^{-i} t^i t^j) \phi \\ &= (n_1 k_1 n_2^{t^{-i}} k_2^{t^{-i}} t^{i+j}) \phi \\ &= ((n_1 n_2^{t^{-i}})(k_1 k_2^{t^{-i}}) t^{i+j}) \phi \\ &= (n_1 n_2^{t^{-i}}) \alpha(k_1 k_2^{t^{-i}}) \delta \tilde{t}^{i+j} \\ &= (n_1 \alpha)(n_2 \alpha)^{\tilde{t}^{-i}} (k_1 \delta)(k_2 \delta)^{\tilde{t}^{-i}} \tilde{t}^{i+j} \\ &= (n_1 \alpha)(k_1 \delta)(n_2 \alpha)^{\tilde{t}^{-i}} (k_2 \delta)^{\tilde{t}^{-i}} \tilde{t}^{i+j} \\ &= (n_1 \alpha)(k_1 \delta) \tilde{t}^i \tilde{t}^{-i} (n_2 \alpha)^{\tilde{t}^{-i}} \tilde{t}^i \tilde{t}^{-i} (k_2 \delta)^{\tilde{t}^{-i}} \tilde{t}^i \tilde{t}^{-i} \tilde{t}^{i+j} \\ &= (n_1 \alpha)(k_1 \delta) \tilde{t}^i (n_2 \alpha)(k_2 \delta) \tilde{t}^j \\ &= ((n_1 k_1 t^i) \phi)((n_2 k_2 t^j) \phi) \end{aligned}$$

Therefore ϕ is a homomorphism. One easily sees that ϕ is surjective. We are left to show that ϕ is injective. Let $nkt^i \in \ker \phi$ where $n \in N$, $k \in K$, $i \in \{0, 1\}$. Then $(n\alpha)(k\delta)\tilde{t}^i = 1$ and as $\tilde{t} \notin N \times K$, we have $i = 0$. This implies $(n\alpha)(k\delta) = 1$ and so $n\alpha = k\delta = 1$. As α and δ are isomorphisms, we get $n = 1$ and $k = 1$. Therefore $nkt^i = 1$. Hence ϕ is injective. $\square$

1.2 Symmetric groups and alternating groups

We introduce probably the most familiar non-abelian simple groups called alternating groups and denoted by A_n . These are subgroups of index 2 in the symmetric group S_n . Our main aim is to define these groups and present some of their basic properties. The main references for this section are [41], [56] and [59].

We begin with the term permutation. A *permutation* is a bijection from a set to itself. Let ω be a set. The *symmetric group* denoted $\text{Sym}(\omega)$ on ω is the group of all permutations

of ω . If the set ω has cardinality n , then we will take $\omega = \{1, \dots, n\}$. The corresponding symmetric group is denoted S_n and it is said to be the symmetric group of degree n . The symmetric group S_n has order $n!$.

For a permutation $\sigma \in S_n$, it can be written as a list of images $\sigma(1), \dots, \sigma(n)$ of the points in order, and fixes a point $j \in \omega$ if $\sigma(j) = j$. The permutation σ is said to be a m -cycle or a *cycle of length m* if it permutes m points cyclically and fixes all other points. More precisely, a m -cycle permutation $\sigma \in S_n$ can be written as $(b_1, b_2, \dots, b_m)$ if there exist m different elements $b_1, \dots, b_m \in \omega$ such that $\sigma(b_i) = b_{i+1}$ for every $1 \leq i < m$, $\sigma(b_m) = b_1$ and $\sigma(b) = b$ for all $b \in \omega \setminus \{b_1, \dots, b_m\}$. Two cycles are said to be *disjoint* if they have no element in common. Every permutation can be written uniquely as a product of pairwise disjoint cycles. This way of representation of a permutation is referred to as the *cycle decomposition* of the permutation. For any permutation $\sigma \in S_n$, the 1-cycles describe the fixed points of σ in S_n . We often omit 1-cycles in the cycle decomposition. A permutation is said to have a *cycle type* $(1^{k_1}, 2^{k_2}, \dots, n^{k_n})$ if its cycle decomposition consists of k_i distinct i -cycles for all $1 \leq i \leq n$. If $k_i = 0$, then we omit i^{k_i} , and if $k_i = 1$, then we simply write i in place of i^{k_i} . The following proposition is a well known result in finite group theory.

Proposition 1.2.1. *Two permutations of S_n are said to be conjugate if and only if they have the same cycle type.*

Proof. A proof can be found in [41, 4.3.1] and [59, 2.3.1]. Let $\sigma = (a_1, a_2, \dots, a_k)$. Then for all $\pi \in S_n$ we have

$$\pi^{-1}(a_1, a_2, \dots, a_k)\pi = (a_1\pi, a_2\pi, \dots, a_k\pi).$$

So the conjugate of σ is also a k -cycle. As every permutation can be written uniquely as a product of disjoint cycles, it follows that the cycle types of the conjugate permutations are the same. Conversely, let

$$\begin{aligned}\sigma &= (a_{11} \dots a_{1k_1})(a_{21} \dots a_{2k_2}) \dots (a_{s1} \dots a_{sk_s}) \text{ and} \\ \beta &= (b_{11} \dots b_{1k_1})(b_{21} \dots b_{2k_2}) \dots (b_{s1} \dots b_{sk_s}).\end{aligned}$$

So σ and β are of the same cycle type. As the cycles in σ and that of β have the same lengths, we can define a permutation $\pi \in S_n$ satisfying the relation $\pi : a_{ij} \mapsto b_{ij}$. Then

$$\begin{aligned}\pi^{-1}\sigma\pi &= \pi^{-1}(a_{11} \dots a_{1k_1})\pi \pi^{-1}(a_{21} \dots a_{2k_2})\pi \dots \pi^{-1}(a_{s1} \dots a_{sk_s})\pi \\ &= (a_{11}\pi \dots a_{1k_1}\pi)(a_{21}\pi \dots a_{2k_2}\pi) \dots (a_{s1}\pi \dots a_{sk_s}\pi) \\ &= (b_{11} \dots b_{1k_1})(b_{21} \dots b_{2k_2}) \dots (b_{s1} \dots b_{sk_s}) = \beta.\end{aligned}$$

□

The 2-cycle permutations of the symmetric group S_n are also referred to as *transpositions*. A *double transposition* is a permutation that interchanges two pairs of elements. We note that every permutation of S_n can be written as a product of transpositions. The representation of a permutation as a product of transpositions is not uniquely determined, however in whichever case the number of the transpositions in the product is always even or always odd. A permutation is said to be *even* if it can be written as a product of an even number of transpositions, otherwise it is referred to as *odd*. It follows that every permutation can be described as even or odd and not both. For $m \geq 2$, every m -cycle is the product of $m - 1$ transpositions. Thus a m -cycle permutation is even if and only if m is odd. For any permutation σ of S_n and s the number of transpositions in the representation of σ as the product of transpositions, the mapping

$$\text{sgn} : \sigma \mapsto (-1)^s$$

is well defined, and it is a homomorphism from S_n into the multiplicative group $\{-1, 1\}$ of order 2. The kernel of this homomorphism is the set of all even permutations of S_n , which is referred to as the *alternating group* A_n of degree n . In particular, A_n is a normal subgroup of S_n and has index 2 in S_n . The alternating group A_n has order $\frac{1}{2}n!$.

An *involution* in a symmetric group S_n is any permutation σ such that $\sigma^2 = 1$. So the number of involutions of S_n will be the number of permutations σ in S_n such that σ has order 2. For any permutation of order 2, it follows that either σ is a transposition or a product of disjoint transpositions. For a given involution σ of a symmetric group S_n , if σ is a product of l transpositions, then there are $\binom{n}{2l}$ ways of choosing $2l$ elements. It follows that there are $\frac{(2l)!}{l!2^l}$ ways of pairing the selected elements to form disjoint transpositions. Notice that l cannot exceed $\frac{n}{2}$, and so the total number of involutions in S_n is given as

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} \frac{(2k)!}{k!2^k}.$$

An involution permutation σ of S_n is said to be *long* if σ fixes exactly one point of n . If the involution σ centralizes a Sylow 2-subgroup of S_n , then σ is said to be *2-central*.

We end the section with some basic properties of the symmetric group and the alternating group. These properties are well known in finite group theory.

Proposition 1.2.2. *The alternating group A_n is the commutator subgroup of the symmetric group S_n .*

Proof. We will follow the closely the arguments provided in [41, 4.3.3]. Let J be the commutator subgroup of S_n . For $n = 1$, notice that $S_n = A_n = J$. So we will assume $n \geq 2$.

As $S_n/A_n \cong C_2$ which is abelian, we have that $J \leq A_n$. As S_n/J is abelian and for any transposition x of S_n such that $x \notin J$, we have that $y^{-1}xyJ = xJ$ for all $y \in S_n$. Thus $\langle x \rangle J$ is a normal subgroup of S_n . As transpositions of S_n are conjugate by Proposition 1.2.1 and all elements of S_n are products of transpositions, we have $S_n = \langle x \rangle J$ and so $J = A_n$. $\square$

Theorem 1.2.3. *If $n \neq 4$, then the alternating group A_n is simple.*

Proof. See for instance [56, Chapter 3, Theorem 2.11]. Part of the proof can also be found in [41, 4.3.5] and [59, 2.3.3]. $\square$

Theorem 1.2.4. *If $n \geq 3$ and $n \neq 6$, then $\text{Aut}(A_n) = \text{Aut}(S_n) \cong S_n$. Moreover, $\text{Aut}(A_6)$ contains S_6 as a subgroup of index 2.*

Proof. See for instance [56, Chapter 3, 2.17, 2.18], [41, 4.3.5] and [59, Theorem 2.3]. $\square$

1.3 The known simple groups

This section gives a brief description of the known simple groups. We begin by stating the classification of the finite simple group (CFSG) theorem.

Theorem 1.3.1. *Any finite simple group is isomorphic to one of the following:*

1. *a group of prime order.*
2. *an alternating group of degree at least 5.*
3. *a finite simple group of Lie type.*
4. *one of 26 sporadic groups.*
5. *the Tits group*

The alternating group of degree at least 5 which is the object of interest, is discussed in Section 1.2. The description of the rest of the groups provided here were drawn from [9], [27] and [29]. The groups of prime order in the first case of Theorem 1.3.1 are clearly abelian, and cyclic simple groups. For any prime p , these groups are unique and isomorphic to the cyclic group $\mathbb{Z}_p$ of integers modulo p under the group operation of addition of congruence classes. In fact, any finite abelian simple group is isomorphic to $\mathbb{Z}_p$.

The finite simple groups of Lie type are regarded as fixed points of suitable endomorphisms of infinite simple algebraic groups defined over the algebraic closure of a finite field of characteristic p , where p is a prime. Basically they are matrix groups defined over finite

fields. They are further categorised as untwisted and twisted groups of Lie type. The untwisted groups of Lie type, also known as Chevalley groups, arise as automorphism groups of the corresponding simple Lie algebras. Fix n as a positive integer and q a finite field. The untwisted groups of Lie type are listed as follows:

$$A_n(q) \cong L_{n+1}(q) \ n > 1, \ B_n(q) \cong P\Omega_{2n+1}(q) \ n > 1, \ C_n(q) \cong PSp_{2n}(q) \ n > 2,$$

$$D_n(q) \cong P\Omega_{2n+1}^+(q) \ n > 3, E_6(q), E_7(q), E_8(q), F_4(q), G_2(q).$$

The twisted groups of Lie type arise as fixed points of certain automorphisms of the Chevalley groups and they are listed as follows:

$${}^2A_n(q) \cong U_{n+1}(q) \ n > 1, \ {}^2D_n(q) \cong P\Omega_{2n}^-(q) \ n \geq 3, \ {}^3D_4(q), {}^2E_6(q),$$

$${}^2B_2(2^{2n+1}) \cong Sz(2^{2n+1}), {}^2F_4(2^{2n+1}), {}^2G_2(2^{2n+1}) \cong Ree(3^{2n+1}).$$

The groups on the first row of the list of twisted groups of Lie type are also called Steinberg groups while the ones on the second row are referred to as Suzuki-Ree groups. In all there are 16 families of finite simple groups of Lie type.

The 26 sporadic groups are as follows:

- the five Mathieu groups M_{11}, M_{12}, M_{23} , and M_{24} ;
- the four Janko groups J_1, J_2, J_3 , and J_4 ;
- the three Conway groups Co_1, Co_2 , and Co_3 ;
- the three Fischer groups Fi_{22}, Fi_{23} , and Fi'_{24} ;
- the McLaughlin group McL ;
- the Higman-sims group HS ;
- the Held group He ;
- the Lyons group Ly ;
- the O'Nan group $O'N$;
- the Rudvalis group Ru ;
- the sporadic Suzuki group Suz ;
- the Harada-Norton group HN (or F_5);

- the Thompson group $Th(or F_3)$;
- the Baby Monster $BM(or F_2)$; and
- the Monster $M(or F_1)$.

Unlike the other groups listed in Theorem 1.3.1, these 26 sporadic groups are not members of a natural infinite family of simple groups. The first five of the 26 sporadic groups were constructed as highly transitive permutation groups by Mathieu in the 1860's. The rest were discovered between the mid-1960's and 1980.

The Tits group ${}^2F_4(2)'$ is the derived subgroup of the Ree group ${}^2F_4(2)$. It also occurs as a maximal subgroup of the Fischer group Fi_{22} .

We note that there are no coincidences of isomorphism types between any of the sporadic groups and the alternating groups or any of the groups of Lie type. However, the following theorem indicates there are coincidences of isomorphism types between the groups of Lie type and the alternating groups.

Theorem 1.3.2. *For the alternating groups A_n where $n \geq 5$ and the finite simple group of Lie type in all characteristics, the following are the only coincidences of isomorphism types:*

$$D_3(q) \cong A_3(q); \quad {}^2D_3(q) \cong {}^2A_3(q); \quad {}^2D_2(q) \cong A_1(q^2),$$

$$B_2(q) \cong C_2(q); \quad B_n(2^m) \cong C_n(2^m), \quad n \geq 1,$$

$$A_1(4) \cong A_1(5) \cong A_5 \cong {}^2D_2(2), \text{ of order } 60; \quad A_1(7) \cong A_2(2) \text{ of order } 168,$$

$$A_1(9) \cong (B_2(2))' \cong A_6 \cong (C_2(2))', \text{ of order } 360; \quad A_3(2) \cong A_8 \text{ of order } 20160,$$

$$A_1(8) \cong ({}^2G_2(3))', \text{ of order } 504; \quad {}^2A_2(3) \cong (G_2(2))', \text{ of order } 6048,$$

$${}^2A_3(2) \cong B_2(3) \cong C_2(3) \cong {}^2D_3(2) \text{ of order } 25920,$$

where q is a finite field.

Proof. See [29, Theorem 2.2.10]. □

We denote by $\mathcal{K}$ the set of (isomorphism classes of) simple groups listed in Theorem 1.3.1.

Definition 1.3.3. A finite simple group is said to be *known* provided that it is an element of $\mathcal{K}$.

The following theorem which is due to Feit-Thompson guarantees that every non-abelian simple group contains an involution.

Theorem 1.3.4. [15, Theorem] *Any finite group of odd order is solvable.*

1.4 Quasisimple groups

This section introduces an important class of groups which play a fundamental role in the study of finite groups, providing a crucial link between simple groups and general group structures.

Definition 1.4.1. A group X is called *quasisimple* provided that X is perfect and $X/Z(X)$ is simple.

Lemma 1.4.2. Let X be a quasisimple group. Suppose Q is a normal subgroup of X . Then the following hold:

- (a) either $Q = X$ or $Q \leq Z(X)$.
- (b) $X/Z(X)$ is nonabelian.
- (c) If $Q \leq Z(X)$, then $Z(X/Q) = Z(X)/Q$.

Proof. A proof can be found in [35, Lemma 9.2] and [57, Chapter 6, 6.2].

- (a) Set $Z := Z(X)$, so Z is a maximal normal subgroup of X as $X/Z(X)$ is simple. Let Q be a proper normal subgroup of X . Assume $Q \not\leq Z$. Then $Z < QZ$, and so $QZ = X$ as Z is maximal. It follows that $X/Q = QZ/Q \cong Z/(Q \cap Z)$ is abelian. Thus $X = X' \leq Q < X$, a contradiction. So $Q \leq Z = Z(X)$.
- (b) This follows from the fact that $(X/Z(X))' = X'Z(X)/Z(X) = X/Z(X)$.
- (c) Let M be a subgroup of X containing Q such that $M/Q = Z(X/Q)$. As M/Q is an abelian normal subgroup of X/Q , we have that $M/Z(X) \cong (M/Q)/(Z(X)/Q)$ is an abelian normal subgroup of $(X/Q)/(Z(X)/Q) \cong X/Z(X)$. It follows that $M = Z(X)$ since $X/Z(X)$ is non-abelian and simple. Thus $Z(X)/Q = Z(X/Q)$. □

Proposition 1.4.3. Let X be a quasisimple group. Then every non-identity homomorphic image of X is quasisimple.

Proof. The proof can be found in [35, Lemma 9.2] and [57, Chapter 6, 6.2] as well. Let Q be a proper normal subgroup of X . We will show that X/Q is quasisimple. It follows from Lemma 1.4.2 that X/Q is perfect and $((X/Q)/Z(X/Q)) = (X/Q)/(Z(X)/Q) \cong X/Z(X)$ is simple. □

Chapter 2 Overview of some group theoretical methods

In this chapter we will collect a number of definitions and well known results about finite groups which we will refer to frequently in this thesis. Sections 2.1-2.7 contain foundational material on finite p -groups, constrained groups, strongly p -embedded subgroups, closure of subgroups, control of fusion in Sylow p -subgroups, and group actions. Sections 2.8-2.9 deal with notions that provide insights into the structure of a group. The next section presents a class of groups called Goldschmidt groups. The rest of the chapter describes some signalizer functor theory.

2.1 Finite p -groups

We present some basic facts and results about p -groups and some other classes of groups which will be useful subsequently. The material presented in this section is drawn from [8], [14], [31], [41] and [57].

Definition 2.1.1. Let π be a finite set of prime numbers, and X a finite group.

- (i) A π -subgroup of X is a subgroup whose order is divisible only by primes in π . For $\pi = \{p\}$ we write π -subgroup as p -subgroup.
- (ii) $O_\pi(X)$ is defined as the largest normal π -subgroup of X . For each prime p , we write $O_{\{p\}}(X)$ as $O_p(X)$.
- (iii) $O^\pi(X)$ is defined as the smallest normal subgroup H of X such that X/H is a π -group. For each prime p , we write $O^{\{p\}}(X)$ as $O^p(X)$.

Subsequently we let p' to denote the set of primes distinct from p .

Lemma 2.1.2. Let J be a quasisimple group. Then $O^p(J) = J$ for any prime p .

Proof. We have that $J/O^p(J)$ is perfect as J is perfect, and nilpotent as $J/O^p(J)$ is a p -group. Thus $J/O^p(J)$ is trivial and $J = O^p(J)$. $\square$

Definition 2.1.3. Let X be a finite group, and p a prime. A subgroup T of X is said to be a Sylow p -subgroup of X if $|T|$ is a power of p and the index $|X : T|$ is not divisible by p . We denote the set of Sylow p -subgroups of X by $Syl_p(X)$.

Remark 2.1.4. We note that if the order of X is not divisible by p , then the trivial subgroup is a Sylow p -subgroup of X , and if X is a p -group, then X is a Sylow p -subgroup of itself.

The following theorem guarantees the existence of Sylow p -subgroups, and the fact that they are conjugate.

Theorem 2.1.5 (Sylow's Theorem). *Let X be a finite group. For a fixed prime p , and a positive integer r , let p^r be the largest p -power dividing the order of X .*

- (a) *The Sylow p -subgroups of X are precisely the subgroups of order p^r .*
- (b) *The Sylow p -subgroups of X are conjugate in X .*

Proof. See for example [41, 3.2.3]. □

Lemma 2.1.6. *Let X be a finite group and T a Sylow p -subgroup of X .*

- (a) *For each $P \leq T$, there is $Q \leq T$ such that Q is X -conjugate to P and*
 - (i) $N_T(Q) \in \text{Syl}_p(N_X(Q))$
 - (ii) $C_T(Q) \in \text{Syl}_p(C_X(Q))$
 - (iii) $\text{Aut}_T(Q) \in \text{Syl}_p(\text{Aut}_X(Q))$
- (b) *Let $P, Q \leq T$ and $x \in X$ such that $P^x = Q$. Suppose $N_T(Q) \in \text{Syl}_p(N_X(Q))$ and let $N := \{y \in N_T(P) \mid y^x \in C_X(Q) N_T(Q)\}$. Then there is $m \in C_X(Q)$ such that $N^{xm} \leq T$.*

Proof. See [8, Lemma 1.2]. □

Definition 2.1.7. Let X be a finite group and p a prime. Then $O_{p',p}(X)$ denotes the normal subgroup N of X that contains $O_{p'}(X)$ and satisfies $N/O_{p'}(X) = O_p(X/O_{p'}(X))$.

Definition 2.1.8. For any finite group X , the group $O(X) := O_{2'}(X)$ is called the *core* of X . A group X is said to be *core-free* provided $O(X)$ is trivial.

Definition 2.1.9. For any finite group X , the preimage of $Z(X/O(X))$ in X is denoted as $Z^*(X)$.

Definition 2.1.10. A *p -local subgroup* of X is the normalizer $N_X(R)$ of some non-trivial p -subgroup R of X .

Lemma 2.1.11. *For any p -group X , the following are equivalent.*

- (a) *Any abelian subgroup of X is cyclic.*
- (b) *X has a unique subgroup of order p .*

(c) X is either a cyclic or a generalized quaternion group Q_8 .

Proof. See [57, Chapter 4, 4.4]. □

Lemma 2.1.12 (Frattini argument). *Suppose X is a finite group with a normal subgroup R , and $T \in \text{Syl}_p(R)$ for some prime p . Then $X = RN_X(T)$*

Proof. See [41, 3.2.7]. □

As an application of the Frattini argument, we have the following lemma.

Lemma 2.1.13. *Let X be a finite group with a normal p' -subgroup N for some prime p . Let Q be a p -subgroup of X , and set $\bar{X} = X/N$. Then $N_{\bar{X}}(\bar{Q}) = \overline{N_X(Q)}$ and $C_{\bar{X}}(\bar{Q}) = \overline{C_X(Q)}$.*

Proof. See [41, 3.2.8]. □

Proposition 2.1.14. *Let X be a finite group. Assume R is a normal subgroup of X , and set $\bar{X} = X/R$. Then for every 2-subgroup Q of X , we have*

$$\overline{O(C_X(Q))} \leq O(C_{\bar{X}}(\bar{Q})).$$

Proof. A proof can be found in [31, Proposition 3.11]. □

Lemma 2.1.15. *Let R be a p -subgroup of a group X . If R is a Sylow p -subgroup of $N_X(R)$, then R is a Sylow p -subgroup of X .*

Proof. We prove by contrapositive. Assume R is not a Sylow p -subgroup of X . Then by Sylow's theorem, R is properly contained in a Sylow p -subgroup T of X . If R is normal in T , then $N_T(R) = T \neq R$. Moreover, if R is not normal in T , then $N_T(R)$ contains conjugates of R which are not equal to R , and thus $N_T(R) \neq R$. Therefore $R \subset N_T(R)$. As $N_T(R)$ is a p -subgroup of $N_X(R)$, we get that R cannot be a Sylow p -subgroup of $N_X(R)$. □

Definition 2.1.16. Let p be a prime.

- An abelian p -group F is said to be *elementary abelian* provided every non-identity element of F has order p .
- For an abelian group F , the rank of F denoted by $m(F)$ is the minimal number of elements in any set of generators of F . If F is an elementary abelian p -group, then the rank of F is denoted by $m_p(F)$, which is the number of p -cyclic factors in the decomposition of F into a direct product of p -cyclic groups.
- An elementary abelian 2-group of rank 2 is also referred to as *Klein four group* or simply *four group*.

Lemma 2.1.17. *If V is a Klein four group, then $\text{Aut}(V) \cong GL_2(2) \cong S_3$.*

Proof. See for instance [14, Example 4.50]. □

Definition 2.1.18. Let F be a 2-group.

- F is called a *dihedral group* if

$$F = \langle x, y \mid x^2 = y^n = (xy)^2 = 1 \rangle.$$

Dihedral groups of order $2n$ are usually denoted as D_{2n} .

- F is called a *quaternion group* if

$$F = \langle x, y \mid x^{2n-1} = y^4 = x^{y-1}x = 1, x^{2n-2} = y^2 \rangle.$$

Quaternion groups of order 2^n are usually denoted as Q_{2^n} .

The following theorem gives some properties of the dihedral group of order 8. These properties will be useful in Section 3.7.

Theorem 2.1.19. *Suppose S is a dihedral group of order 8. Then*

- $\text{Aut}(S)$ is a 2-group.*
- S has exactly two subgroups that are elementary abelian of order 4.*
- Suppose T is another dihedral group of order 8. Let V_1 and V_2 be the two elementary abelian subgroups of S of order 4, and U_1 and U_2 be the two elementary abelian subgroups of T of order 4. Then there exists an isomorphism $\alpha : S \rightarrow T$ such that $V_1\alpha = U_1$ and $V_2\alpha = U_2$.*

Proof. See [21, Lemma 7.2(vi) and (iii)]. □

We end this section with a theorem of Gorenstein and Walter [30] which classify all finite simple groups with dihedral Sylow 2-subgroups.

Theorem 2.1.20. [30, Theorem 2] *Suppose X is a finite simple group with a dihedral Sylow 2-subgroup. Then X is isomorphic to either A_7 or $PSL_2(q)$, q odd and $q > 3$.*

2.2 Normal p -complements

In this short section we present a concept that has implications for the structure and properties of a finite group, such as the existence of normal subgroups and the behaviour of Sylow subgroups. Our main reference is [41].

Definition 2.2.1. Let X be a group with a normal subgroup N , and let p be a prime. Then N is said to be a normal p -complement of X if X is the semidirect product of N with a Sylow p -subgroup of X . Equivalently, we have that $O^p(X) = N = O_{p'}(X)$.

The following theorem of Burnside comes in handy when establishing the existence of a normal p -complement of a group.

Theorem 2.2.2. *Let X be a group with a Sylow p -subgroup R . If $N_X(R) = C_X(R)$, then X has a normal p -complement.*

Proof. See for instance [41, 7.2.1]. □

We remark that if R in Theorem 2.2.2 is cyclic and p is the smallest prime divisor of the order of X , then one observes that $N_X(R) = C_X(R)$. Thus one easily gets the following corollary:

Corollary 2.2.3. *Let X be a group with p as the smallest prime divisor of the order of X . If X has no p -complement, then any Sylow p -subgroup of X is not cyclic.*

2.3 The focal and hyperfocal subgroup

In this section we will introduce two very important concepts in group theory. These concepts help in determining whether a finite group has a quotient group of order a power of p , where p is a prime number. The main references are [21] and [51].

Definition 2.3.1. Let X be a finite group with Sylow p -subgroup T where p is a prime number. The intersection $T \cap X'$ is called the *focal subgroup* of T in X .

The following theorem gives a description of the focal subgroup.

Theorem 2.3.2 (Focal subgroup theorem). *Suppose X is a finite group with Sylow p -subgroup T where p is a prime number. Then*

$$T \cap X' = \langle x^{-1}y \mid x, y \in T, y = x^a \text{ for some } a \in X \rangle.$$

Proof. See [21, Chapter 7, Theorem 3.4]. □

Definition 2.3.3. Let X be a finite group with Sylow p -subgroup T where p is a prime number. The intersection $T \cap O^p(X)$ is called the *hyperfocal subgroup* of T in X .

The following theorem enable one to calculate the hyperfocal subgroup.

Theorem 2.3.4 (Puig's hyperfocal subgroup theorem). *Let X be a finite group with Sylow p -subgroup T where p is a prime number. Then*

$$T \cap O^p(X) = \langle x^{-1}(x\phi) \mid x \in R \leq T, \phi \in O^p(\text{Aut}_X(R)) \rangle.$$

Proof. See [51, 1.1]. □

2.4 p -constrained groups

In this section we present a brief account of a certain class of groups, and provide a framework for studying groups with specific constraints on the sizes of centralizers of elements. The main references are [28] and [41].

Definition 2.4.1. Let X be a finite group, p a prime and T a Sylow p -subgroup of X .

- i. We say that X is *p -reduced* if $O_p(X) = 1$.
- ii. X is said to be *p -constrained* if

$$C_T(O_p(X)) \leq O_p(X).$$

If $O_{p'}(X) = 1$, then X is *p -constrained* if $C_X(O_p(X)) \leq O_p(X)$.

Definition 2.4.2. Let p be a prime number. A finite group X is said to be of *characteristic p* if

$$C_X(O_p(X)) \leq O_p(X).$$

Theorem 2.4.3. *Let Q be a p -subgroup of a p -constrained group X . Then $O_{p'}(C_X(Q)) \subset O_{p'}(X)$.*

Proof. See for example [41, 8.2.12]. □

2.5 Strongly p -embedded subgroups

The following is another significant concept about groups which plays a major role in determining the structure of the core of a group. Our main references are [28] and [57].

Definition 2.5.1. Let X be a finite group and let p be a fixed prime. A proper subgroup M of X is *strongly p -embedded* if

- (i) p divides $|M|$, and
- (ii) for every $x \in X \setminus M$, $M \cap M^x$ has order coprime to p .

If $p = 2$, then the subgroup M of X is said to be *strongly embedded*.

Proposition 2.5.2. Let X be a finite group and let p be a prime dividing the order of X . Let T be a Sylow p -subgroup of X . Suppose that M is a proper subgroup of X such that

- (i) $C_X(x) \leq M$ for all $x \in T$ of order p ,
- (ii) $N_X(T) \leq M$.

Then M is a strongly p -embedded subgroup of X .

Proof. See [28, Proposition 17.11]. □

Lemma 2.5.3. The symmetric group S_3 has a strongly embedded subgroup.

Proof. Observe that the symmetric group S_3 has three subgroups of order 2 which do not commute. Let M be generated by an involution u of S_3 . Then M is a Sylow 2-subgroup of S_3 , and $C_{S_3}(u) \leq M$. By Proposition 2.5.2, M is a strongly embedded subgroup. □

Proposition 2.5.4. Let X be a finite group and let $T \in \text{Syl}_p(X)$. Then the following hold.

- (a) Every strongly p -embedded subgroup $M < X$ contains some $T \in \text{Syl}_p(X)$.
- (b) If X has a strongly p -embedded subgroup, then $O_p(X) = 1$.

Proof. (a) Let $M < X$ be a strongly p -embedded subgroup. Then there exists some $m \in M$ of order p , since p divides $|M|$. Let $T \in \text{Syl}_p(X)$ such that $m \in T$ and let $z \in Z(T)$ of order p . Then $m \in M \cap M^z$, and so $z \in M$. Thus, $Z(T) \leq M$. Therefore for all $t \in T$, $1 \neq Z(T) \leq M \cap M^t$ and so $t \in M$. Hence $T \leq M$.

(b) Let $R = O_p(X)$. Suppose $M < X$ is strongly p -embedded. For a fixed $x \in X \setminus M$, we have $R \leq M \cap M^x$. So R has order coprime to p . Thus $R = 1$. □

Proposition 2.5.5. Suppose X is a finite group with a strongly embedded subgroup. Then all involutions of X are conjugate.

Proof. A proof can be found in [57, Chapter 6, 4.4(i)]. □

2.6 Closure and control of fusion

This section presents the notion of control of fusion in Sylow p -subgroups of a finite group, and a reformulation of a famous theorem of Burnside. The material presented here is mainly drawn from [14].

We begin with the following concepts whose analogues in fusion systems are defined later.

Definition 2.6.1. Let X be a finite group with subgroups $Q \leq R$.

1. Q is said to be *weakly closed* in R with respect to X provided that for all x in X , $Q^x \leq R$, implies $Q^x = Q$
2. Q is said to be *strongly closed* in R with respect to X provided that for all x in X , and any subgroup $\tilde{Q}$ of Q , $\tilde{Q}^x \leq R$, implies $\tilde{Q} \leq Q$.

Notice that strongly closed in X implies weakly closed in X .

Definition 2.6.2. Let X be a finite group with subgroups Q and R such that $Q \leq R$. Let a and b be elements of Q .

1. If a and b are not conjugate in Q , then a and b are said to be *fused* in R provided that they are conjugate by an element in R . Similarly, two subgroups (two conjugacy classes) of Q are said to be *fused* in R provided that they are not conjugate in Q but they are conjugate by an element in R .
2. The subgroup R is said to *control weak fusion* in Q with respect to X provided that whenever $a, b \in Q$ are fused in X , they are fused in R .
3. The subgroup R is said to *control X -fusion* in Q with respect to X provided that whenever two subgroups Q_1 and Q_2 of Q are conjugate via a conjugation map

$$c_x : Q_1 \rightarrow Q_2$$

for some $x \in X$, there is some $y \in R$ such that

$$ac_x = ac_y$$

for all $a \in Q_1$.

Clearly, one sees that control X -fusion in Q implies control weak fusion in Q with respect to X . However, the converse in general is not true, check for instance [14, Example 1.3].

The following is a restatement of a famous theorem of Burnside which appears in [12].

Theorem 2.6.3. *Suppose X is a finite group with T as a Sylow p -subgroup of X , such that T is abelian. Then the normalizer in X of T controls X -fusion in T .*

Proof. Let Q and R be subgroups of T . Assume that there exists $x \in X$ such that

$$c_x : Q \rightarrow R.$$

As T is abelian, we have that

$$T^x \leq C_X(Q)^x = C_X(Q^x) = C_X(R).$$

It follows that T and T^x are both Sylow p -subgroups of $C_X(R)$. By Theorem 2.1.5, there exists $y \in C_X(R)$ such that $T^{xy} = T$. This implies $xy \in N_X(T)$. As $y \in C_X(R)$, we have that

$$a^{xy} = (a^x)^y = a^x$$

for every $a \in Q$ and so $c_{xy} = c_x$, as required. □

Theorem 2.6.3 indicates that for any abelian Sylow p -subgroup T , its fusion in T must take within its normalizer. In general, this does not hold for any non-abelian Sylow p -subgroup, see for instance [14, Example 1.6].

2.7 Group action and coprime action

This section introduces the basic ideas of group actions and a result about coprime actions. The material presented here were mainly drawn from [28] and [41].

We first give the definition of a group action.

Definition 2.7.1. A *group action* of a group X on a non-empty set Ω is a map

$$\Omega \times X \rightarrow \Omega \text{ defined as } (\alpha, x) \mapsto \alpha x$$

satisfying the following conditions:

1. $\alpha 1 = \alpha$ for all $\alpha \in \Omega$, where 1 is the identity element of X .
2. $(\alpha x)y = \alpha(xy)$ for all $\alpha \in \Omega$ and all $x, y \in X$.

Remark 2.7.2. Equivalently, a group action of X on a non-empty set Ω can be written as a group homomorphism ϕ from X to the symmetric group S_Ω of Ω , namely $(\alpha)(x)\phi = \alpha x$ for all $\alpha \in \Omega$ and $x \in X$. If the kernel of ϕ is trivial, then X acts *faithfully* on Ω ; and if the kernel of ϕ is X , then X acts *trivially* on Ω .

Example 2.7.3. Let X be a group and let $\Omega = X$. Define

$$\Omega \times X \rightarrow \Omega \text{ as } (\alpha, x) \mapsto \alpha^x.$$

One easily checks that this map is a group action. This action is called the conjugation action of X on itself. Observe that $\alpha^x = \alpha$ if and only if $\alpha x = x\alpha$, and so the kernel of the action is the centre $Z(X)$.

Definition 2.7.4. A non-empty set Ω , together with an action of X defined on it, is referred to as an X -set.

For a given X -set Ω , the action of X partitions Ω into *orbits*: two elements α and β are said to be in the same orbit if and only if there exists $x \in X$ such that $\alpha x = \beta$. The quotient of Ω by X , usually denoted as Ω/X , is the set of orbits.

Definition 2.7.5. A group X is said to act *transitively* on a non-empty set Ω provided Ω/X consists of only one element.

Definition 2.7.6. A group X is said to act *2-transitively* on a non-empty set Ω provided it acts transitively on the set of distinct ordered pairs $\{(x, y) \in \Omega \times \Omega : x \neq y\}$.

Definition 2.7.7. Let X be a group that act on another group Y . The action of X on Y is said to be *coprime* provided

1. $(|X|, |Y|) = 1$, and
2. X or Y is solvable.

The following result will be useful subsequently.

Theorem 2.7.8. Suppose A is a finite non-cyclic elementary abelian p -group which acts on a p' -group X , where p is a prime. Then

$$X = \langle C_X(F) \mid F \leq A \text{ and } [A : F] = p \rangle.$$

In particular, $X = \langle C_X(b) \mid 1 \neq b \in A \rangle$.

Proof. See for example [28, Proposition 11.23], and [41, 8.3.4((a) and (b))]. □

We close this section with the following definition.

Definition 2.7.9. Fix a prime p . Let X be a group acting on an elementary abelian p -group E .

- An element x of X is said to act quadratically on E if $[E, x, x] = 1$.
- The group X act quadratically on E if $[E, X, X] = 1$.

2.8 Components and 2-components

The concepts of component and 2-component provide insights into the structure and decomposition of groups. They aid in analyzing the properties and relationships of groups. Our main references are [4], [41] and [57].

Definition 2.8.1. For a group X with J as a subgroup, J is said to be *subnormal* in X if there is a chain

$$J = J_1 \leq J_2 \leq \cdots \leq J_k = X$$

such that $J_i \trianglelefteq J_{i+1}$ for all $1 \leq i \leq k - 1$.

Lemma 2.8.2. Let X be a finite group such that $O(X) = 1$. Suppose R is a subnormal subgroup of X . Then $O(R) = 1$.

Proof. Suppose $R = R_0 \trianglelefteq R_1 \trianglelefteq \cdots \trianglelefteq R_k = X$. By induction on k we assume $O(R_1) \leq O(X)$. So $O(R) \text{ char } R \trianglelefteq R_1$ and thus $O(R) \trianglelefteq R_1$. It follows that $O(R) \leq O(R_1) \leq O(X) = 1$. $\square$

Definition 2.8.3. A subgroup J of a finite group X is called a *component* of X provided J is quasisimple and subnormal in X . We denote $\text{Comp}(X)$ as the set of components of X .

The subgroup of X generated by the components of X is called *Layer* of X and it is denoted by $E(X)$. If there is no component, then $E(X) = 1$.

Proposition 2.8.4. Let X be a finite group with a component J . Suppose U is a subnormal subgroup of X . Then $J \in \text{Comp}(U)$ or $[J, U] = 1$.

Proof. See for example [4, 31.4] and [41, 6.5.2]. $\square$

An immediate consequence of the Proposition 2.8.4 is the following corollary.

Corollary 2.8.5. Suppose J_1 and J_2 are components of a finite group X . Then $J_1 = J_2$ or $[J_1, J_2] = 1$. In particular, the layer of X is the central product of its components.

The following lemmas will subsequently be useful.

Lemma 2.8.6. Let X be a finite group. Then $\text{Comp}(O^p(X)) = \text{Comp}(X)$ for any prime p .

Proof. Notice that $\text{Comp}(O^p(X)) \subseteq \text{Comp}(X)$. Assume J is a component of X . Then $J = O^p(J) \subseteq O^p(X)$. As $J/Z(J)$ is simple, in particular $J \neq Z(J)$, we have $[J, O^p(X)] \neq 1$. Therefore by Proposition 2.8.4, $J \in \text{Comp}(O^p(X))$ completing the proof. $\square$

Lemma 2.8.7. Let X be a finite group. Suppose J is a component of a subnormal subgroup of X . Then J is a component of X .

Proof. Let R be a subnormal subgroup of X , and J a component of R . Then J is quasisimple, and J is subnormal in X by transitivity. Thus J is a component of X . $\square$

Lemma 2.8.8. *Let X be a group with subgroups Z and J such that $Z \leq Z(X)$. If JZ/Z is a component of X/Z , then J' is a component of X .*

Proof. Set $\bar{X} = X/Z$. Notice that $\overline{JZ'} = \overline{(JZ)'} \cong \overline{J'Z}$ as $Z \leq Z(X)$ and so $J' = (JZ)'$. As $\overline{JZ}$ is subnormal in $\bar{X}$ we have that JZ is subnormal in X and so J' is subnormal in X .

Notice $(JZ/Z)/Z(JZ/Z) = (JZ/Z)/(Z(J)Z/Z) \cong J/Z(J)$ is a simple group. If $J/Z(J)$ is abelian, then $J/Z(J)$ is cyclic. It follows that J is abelian and so $J = Z(J)$, a contradiction. Therefore $J/Z(J)$ is non-abelian simple group.

Observe that $J'Z(J)/Z(J)$ is normal in $J/Z(J)$. So either $J'Z(J)/Z(J) = 1$ or $J'Z(J)/Z(J) = J/Z(J)$. In the formal case we have $J'Z(J) = Z(J)$ and so $J' = 1$. This means J is abelian, a contradiction. Therefore $J'Z(J)/Z(J) = J/Z(J)$ and thus $J = J'Z(J)$.

We have that $J'/Z(J') = J'/Z(J) \cap J' \cong J/Z(J)$ is non-abelian simple group. So $J'/Z(J) \cap J' = (J'/Z(J) \cap J')' \cong J''(Z(J) \cap J')/Z(J) \cap J'$. This implies $J' = J''(Z(J) \cap J')$ and so $J = J''Z(J)$. Observe that $J/J'' \cong Z(J)/Z(J) \cap J''$ is abelian and so $J' = J''$. Thus J' is perfect. $\square$

Lemma 2.8.9. *Let X be a finite group. If $X = X_1 \times X_2$, then any component of X is a component of X_1 or X_2 .*

Proof. Let Q be a component of X . As X_1 is normal in X , we have that $Q \leq X_1$ or $[Q, X_1] = 1$. If $[Q, X_1] = 1$, then $Q = [Q, Q] \leq [X, Q] = [X_1, Q][X_2, Q] = [X_2, Q] \leq X_2$ since X_2 is normal in X . As X_1 and X_2 are both normal in X , it follows that $\text{Comp}(X) = \text{Comp}(X_1) \sqcup \text{Comp}(X_2)$. $\square$

Next is the concept of 2-component.

Definition 2.8.10. A 2-component J of a finite group X is a subnormal subgroup of X such that $J = O^{2'}(J)$ and $J/O(J)$ is quasisimple. The subgroup $L_{2'}(X)$ of X generated by the 2-components of X is called the 2-layer of X .

The 2-components of a group can be characterized as follows.

Theorem 2.8.11. *Let X be a finite group with J as a subgroup. Then the following are equivalent.*

- (i) J is a 2-component of X .
- (ii) J is perfect, subnormal in X , and $J/O(J)$ is quasisimple.

Proof. Assume J is a 2-component of X . Then the image of J in $X/O(X)$ is a component of $X/O(X)$, and so the perfectness of the image of J implies that $J = J'O(X)$. It follows that $J' \geq O^{2'}(J) = J$. Thus (ii) holds. Conversely, assume that (ii) holds. Then $J/O^{2'}(J)$ is perfect as J is perfect. As $J/O^{2'}(J)$ has odd order, then it is solvable by Theorem 1.3.4 (Feit-Thompson theorem). Therefore $J/O^{2'}(J) = 1$ and so $J = O^{2'}(J)$. The result follows. $\square$

Theorem 2.8.11 implies that in the definition of 2-component the condition $J = O^{2'}(J)$ could be replaced by $J = J'$.

Proposition 2.8.12. *Suppose X is a finite group. Then any component of X is a 2-component.*

Proof. Let J be a component of X . Then J is perfect, and subnormal in X . So by Theorem 1.3.4 J can not have an odd order and thus $O(J)$ is a proper normal subgroup of J . By Lemma 1.4.2, $O(J) \leq Z(J)$ and $J/O(J)$ is quasisimple. By Theorem 2.8.11, J is a 2-component. $\square$

Proposition 2.8.13. *Suppose X is a finite group such that $O(X) = 1$. Then any 2-component of X is a component of X .*

Proof. Let J be a 2-component of X . Then J is perfect, subnormal in X , and $J/O(J)$ is quasisimple. By Lemma 2.8.2, $O(J) = 1$ and so J is quasisimple. $\square$

2.9 Generalized Fitting subgroups

This section presents two characteristic subgroups of any finite group. They play a key role in determining the structure of any finite group. The main references for this section are [35] and [41].

Definition 2.9.1. Let X be a finite group. The *Fitting subgroup* of X , denoted $F(X)$, is the largest nilpotent normal subgroup of X . Equivalently, it is the direct product of all groups $O_p(X)$ where $p \in \pi(X)$. In particular, if $O_{p'}(X) = 1$, then $F(X) = O_p(X)$.

We recall that for any finite group, the *layer* $E(X)$, is the subgroup generated by the components of X , and notice that $[F(X), E(X)] = 1$.

Definition 2.9.2. For any finite group X , the *generalized Fitting subgroup* $F^*(X)$ of X is defined as the central product

$$F^*(X) = F(X)E(X).$$

The following is a fundamental property of $F^*(X)$ which indicates that the generalized Fitting subgroup has the self-centralizing property.

Theorem 2.9.3. *Suppose X is a finite group. Then $C_X(F^*(X)) = Z(F^*(X))$.*

Proof. See for example [35, 31.13] and [41, Theorem 6.5.8]. □

2.10 Goldschmidt groups

We present the definition of a Goldschmidt group and a famous theorem of Goldschmidt [19, Theorem A] in this section. We prove an elementary Lemma 2.10.7 which plays an important role in this work. Our main reference is [19]. We begin with the following definitions.

Definition 2.10.1. A group G is called a Goldschmidt group if it is a non-abelian finite simple group having a non-trivial strongly closed abelian subgroup.

Definition 2.10.2. A quasisimple group K is said to be of type I whenever $K/Z(K)$ is isomorphic to one of the following groups:

- i. $SL_2(2^n)$ for $n \geq 3$;
- ii. $PSU_3(2^n)$ for $n \geq 2$;
- iii. $Sz(2^{2n+1})$ for $n \geq 1$.

Definition 2.10.3. A finite group K is said to be of type JR (Janko-Ree) if $Z(K)$ has an odd order and $K = K/Z(K)$ is a simple group containing an involution t such that

1. $|\overline{K} : C_{\overline{K}}(t)|$ is odd.
2. $C_{\overline{K}}(t) = \langle t \rangle \times L$.
3. L contains a normal subgroup Q of odd index such that $C_L(Q) = 1$ and $Q \cong PSL_2(q)$ for $w \cong 3, 5 \pmod{8}$.

Definition 2.10.4. A finite group K is said to be of type II if K is isomorphic to $PSL_2(w)$ for $w \cong 3, 5 \pmod{8}$ or a group of type JR.

It is known that groups of type JR are either isomorphic to the Ree groups ${}^2G_2(3^{2n+1})$ for $n \geq 2$ or to the Janko group J_1 . This does however not play a role in this thesis. The groups of type I and of type II are known to be Goldschmidt groups. More precisely, the following holds.

Lemma 2.10.5. *Let G be a group of type I or of type II and $T \in \text{Syl}_2(G)$. Then the following hold:*

- (a) *If $G \cong SL_2(2^n)$ for $n \geq 3$ or G is of type II, then T is elementary abelian. Moreover, T itself is the only abelian subgroup of T , which is strongly closed in T with respect to G .*
- (b) *If $G \cong PSU_3(2^n)$ for $n \geq 2$ or $G \cong Sz(2^{2n+1})$ for $n \geq 1$, then $A := \Omega_1(T) = Z(T)$ is strongly closed in T with respect to G and $\text{Aut}_G(T)$ acts regularly on $A^\#$.*

In particular, in either case, $\text{Aut}_G(\Omega_1(T))$ is a non-trivial group of odd order.

Proof. (a) If $G \cong SL_2(2^n)$ for $n \geq 3$, then this is (3.2)(2) in [19]. If $G \cong PSL_2(w)$ with $w \equiv 3, 5 \pmod{8}$ or G is of type JR, then see (3.4)(1)-(3) in [19].

(b) See (3.2)(3),(4) in [19]. □

Goldschmidt showed indeed that the finite simple groups of type I and of type II are the only Goldschmidt groups. More precisely he proved the following theorem.

Theorem 2.10.6. [19, Theorem A] *Let G be a finite group and let T be a Sylow 2-subgroup of G . Suppose T contains an abelian subgroup A which is strongly closed in T with respect to G . If $J = \langle A^G \rangle$ with $\bar{J} := J/O(J)$, then the following holds.*

- (a) *$\bar{J}$ is a central product of an Abelian 2-group and quasisimple groups of type I and type II.*
- (b) *$\bar{A} = O_2(\bar{J})\Omega_1(\bar{W})$ for some Sylow 2-subgroup W of J containing A .*

When applying Theorem 2.10.6, we need the following elementary lemma.

Lemma 2.10.7. *Assume the hypothesis of Theorem 2.10.6 and $O(G) = 1$. Write J as a central product $J = U * K_1 * \cdots * K_r$ with U abelian, $K_1, \dots, K_r$ of type I or II, and $r \geq 0$. Then $\text{Aut}_J(A)$ is a group of odd order, which is non-trivial if and only if $r \geq 1$.*

Proof. If $r = 0$, then clearly $\text{Aut}_J(A) = 1$. So we may assume $r \geq 1$. Choose the notation such that $U = O_2(J)$. Notice that U is central in J . By part (b) of Theorem 2.10.6, we have $A = U\Omega_1(W)$ for $A \leq W \in \text{Syl}_2(G)$. Write $W = U * T_1 * \cdots * T_r$ with $T_i \in \text{Syl}_2(K_i)$ and set $A_i := \Omega_1(T_i)$ for each i . Then

$$A = U * A_1 * \cdots * A_r.$$

One checks now easily that the map

$$\text{Aut}_J(A) \rightarrow \text{Aut}_{K_1}(A_1) \times \cdots \times \text{Aut}_{K_r}(A_r), \alpha \mapsto (\alpha|_{A_1}, \dots, \alpha|_{A_r})$$

is a well-defined group isomorphism. By Lemma 2.10.5, $\text{Aut}_{K_i}(A_i)$ is non-trivial of odd order for each $i = 1, 2, \dots, r$. This yields the assertion. $\square$

2.11 Signalizer functors

In this section, we look at the concept of signalizer functor which was first defined in [22]. This concept will help us to study the largest odd order normal subgroups also known as cores, of centralizers of involutions in a finite group. Our main references are [23] and [57].

We assume throughout this section that A is non-cyclic abelian p -group, where p is a prime.

Definition 2.11.1 (Signalizer Functor). Let G be a group, p a prime, A an elementary abelian p -subgroup of G . An A -signalizer functor θ assigns the subgroup $\theta(C_G(a))$ of $C_G(a)$ to each non-identity element $a \in A$ such that

- i. $\theta(C_G(a))$ is an A -invariant p' -subgroup of $C_G(a)$, and
- ii. $\theta(C_G(a)) \cap C_G(b) \subseteq \theta(C_G(b))$ for any non-identity elements $a, b \in A$.

If $\theta(C_G(a)) = 1$ for all non-identity element a in A , then θ is called the *trivial A -signalizer functor*.

Definition 2.11.2. The closure of an A -signalizer functor θ on G is defined as

$$\theta(G; A) := \langle \theta(C_G(a)) \mid a \in A, a \neq 1 \rangle.$$

Definition 2.11.3. A subgroup X of G is called a θ -subgroup if X is an A -invariant p' -subgroup such that $C_X(a) \subseteq \theta(C_G(a))$ for each non-identity element $a \in A$.

Let $\aleph_\theta(A)$ denote the set of all θ -subgroups of G , and $\aleph_\theta^*(A)$ denote the set of maximal elements of $\aleph_\theta(A)$.

Definition 2.11.4. An A -signalizer functor θ on G is complete provided

- i. $\theta(G; A)$ is a p' -group and
- ii. $C_{\theta(G; A)}(a) = \theta(C_G(a))$ for all $a \in A^\#$.

An A -signalizer functor θ on G is complete, precisely if $\aleph_\theta^*(A)$ contains exactly one element. This unique maximal element is denoted as $\theta(G)$.

Lemma 2.11.5. If an A -signalizer functor θ on G is complete, then $\theta(G; A) = \theta(G)$.

Proof. Let $E = \theta(G; A) = \langle \theta(C_G(a)) \mid a \in A, a \neq 1 \rangle$. As θ is complete, E is a p' -group, and $C_E(a) = \theta(C_G(a))$ for each non-identity element $a \in A$. Thus $E \in \mathfrak{N}_\theta(A)$. As A is non-cyclic elementary abelian p -group, we have by Theorem 2.7.8 that $\theta(G) = \langle C_{\theta(G)}(a) \mid 1 \neq a \in A \rangle$. Since $\theta(G) \in \mathfrak{N}_\theta(A)$, we get $C_{\theta(G)}(a) \subseteq \theta(C_G(a))$ for each non-identity element $a \in A$. So $\theta(G) \subseteq E$. By maximality of $\theta(G)$, we get $E = \theta(G)$. $\square$

Definition 2.11.6. An A -signalizer functor θ is solvable if $\theta(C_G(a))$ is solvable for each non-identity element $a \in A$.

Lemma 2.11.7. Let G be a group, and A an elementary abelian 2-subgroup of G . Then any A -signalizer functor on G is solvable.

Proof. For any A -signalizer functor θ on G , $\theta(C_G(a))$ is odd for every non-identity element $a \in A$. Therefore for each non-identity element $a \in A$, $\theta(C_G(a))$ is solvable by Theorem 1.3.4. Hence, every A -signalizer functor θ on G is solvable. $\square$

Lemma 2.11.8. For a finite group G , p a prime and A an elementary abelian p -group, let $\theta := O_{p'}(C_G(a))$ for each non-identity element $a \in A$. If $C_G(a)$ is p -constrained for each non-identity element $a \in A$, then θ is an A -signalizer functor on G .

Proof. We first show that θ as defined above, is an A -invariant p' -subgroup of $C_G(a)$. As A is abelian, $C_G(a)$ is an A -invariant. Thus $O_{p'}(C_G(a))$ is an A -invariant p' -subgroup of $C_G(a)$. So it suffices to show that $\theta(C_G(a)) \cap C_G(b) \subseteq \theta(C_G(b))$ for any non-identity elements $a, b \in A$. Let $Q := C_G(b)$ for some fixed non-identity element $b \in A$. Then $C_G(a) \cap C_G(b) = C_Q(a)$ and so $O_{p'}(C_G(a)) \cap C_G(b) \subseteq O_{p'}(C_Q(a))$. As Q is p -constrained we get by Theorem 2.4.3 that $O_{p'}(C_G(a)) \cap C_G(b) \subseteq O_{p'}(C_Q(a)) \subset O_{p'}(Q) = O_{p'}(C_G(b))$ for any non-identity elements $a, b \in A$ as required. $\square$

Theorem 2.11.9. [17, Theorem] Let G be a group, p a prime, A an elementary abelian p -subgroup of G of rank at least 3. If θ is a solvable A -signalizer functor on G , then θ is complete.

2.12 k -balanced groups and Signalizer functors

In this section we will look at the notion of k -balanced groups which is a generalization of the notion of balanced groups defined in previous section. We will also give a construction of a signalizer functor on k -balanced groups. The material presented here is due to [31] and we shall follow closely the presentation in [37].

We begin this section by recalling from Definition 2.1.16 that for any elementary abelian p -group F , the rank of F is denoted by $m_p(F)$, which is the number of p -cyclic factors in

the decomposition of F into a direct product of p -cyclic groups. We introduce the following notations: For any finite group G , $\mathcal{E}(G)$ denotes the set of all elementary abelian 2-subgroups of G , and $\mathcal{E}_k(G) := \{F \mid F \in \mathcal{E}(G) \text{ and } m_2(F) = k\}$.

Definition 2.12.1. Let G be a finite group, and k a natural number. Fix $A \in \mathcal{E}(G)$.

(i) For any non-trivial $F \in \mathcal{E}(G)$, we define

$$\Delta_G(F) := \bigcap_{1 \neq b \in F} O(C_G(b)).$$

(ii) The group G is said to be *k-balanced with respect to A* if for all $F \in \mathcal{E}_k(A)$ and for every non-trivial element b of A , we have

$$\Delta_G(F) \cap C_G(b) \leq O(C_G(b)).$$

(iii) The group G is said to be *k-balanced* if for all $F \in \mathcal{E}_k(G)$ and for every involution b of G centralizing F , we have

$$\Delta_G(F) \cap C_G(b) \leq O(C_G(b)).$$

We note that 1-balanced (or 1-balanced with respect to A) is also referred to as *balanced* (or *balanced with respect to A*).

For any finite group with $A \in \mathcal{E}(G)$ of rank at least $k + 2$ for some natural number k , and if G is k -balanced with respect to A , then the following theorem makes precise how to define an A -signalizer functor on G .

Theorem 2.12.2. Let G be a finite group with $A \in \mathcal{E}(G)$ of rank at least $k + 2$ for some natural number k . Assume G is k -balanced with respect to A and for each non-trivial element b of A , we define

$$\theta(C_G(b)) := \langle \Delta_G(F) \cap C_G(b) \mid F \in \mathcal{E}_k(G) \rangle.$$

Then θ is an A -signalizer functor on G .

Proof. See [31, Theorem 6.9]. □

Adopting the notation of the above theorem, the following proposition says that any subgroup $E \in \mathcal{E}_{k+1}(A)$ determines the closure of θ .

Proposition 2.12.3. Let the notation be as in Theorem 2.12.2, and assume $E \in \mathcal{E}(A)$ is of rank at least $k + 1$. Then

$$\theta(G; A) = \langle \Delta_G(F) \mid F \in \mathcal{E}_k(E) \rangle.$$

Proof. We follow the argument presented in the proof of [37, Proposition 2.7.8]. We recall that

$$\theta(G; A) = \langle \theta(C_G(b)) \mid b \in A, b \neq 1 \rangle,$$

and let

$$W_E := \langle \Delta_G(F) \mid F \in \mathcal{E}_k(E) \rangle.$$

We will demonstrate that $\theta(G; A) = W_E$. Before that, notice θ is solvable by Lemma 2.11.7 and so by Theorem 2.11.9, θ is complete. It follows that $\theta(G; A) = \theta(G)$ has odd order. On one hand we show now $W_E \leq \theta(G; A)$. It suffices to show that $\Delta_G(F) \leq \theta(G; A)$ for any $F \in \mathcal{E}_k(E)$. Observe that $\Delta_G(F)$ is A -invariant 2'-subgroup as A is abelian. It follows from Theorem 2.7.8 that

$$\Delta_G(F) = \langle \Delta_G(F) \cap C_G(b) \mid b \in A^\# \rangle.$$

By the definition of θ , $\Delta_G(F) \cap C_G(b) \leq \theta(C_G(b))$ for any $b \in A^\#$. Thus $\Delta_G(F)$ is a θ -subgroup. By the maximality of $\theta(G; A) = \theta(G)$, we have $\Delta_G(F) \leq \theta(G; A)$. On the other hand, by Theorem 2.7.8, we have

$$\theta(G; A) = \langle C_{\theta(G; A)}(F) \mid F \in \mathcal{E}_k(E) \rangle.$$

As θ is complete, $C_{\theta(G; A)}(b) = \theta(C_G(b))$ for all $b \in A^\#$. As G is k -balanced with respect to A and by the definition of θ , we have $\theta(C_G(b)) \leq O(C_G(b))$ for all $b \in A^\#$. So for all $F \in \mathcal{E}_k(E)$, we get

$$C_{\theta(G; A)}(F) = \bigcap_{b \in F^\#} C_{\theta(G; A)}(b) = \bigcap_{b \in F^\#} \theta(C_G(b)) \leq \bigcap_{b \in F^\#} O(C_G(b)) = \Delta_G(F) \leq W_E.$$

It follows that

$$\theta(G; A) = \langle C_{\theta(G; A)}(F) \mid F \in \mathcal{E}_k(E) \rangle \leq W_E$$

and this completes the proof. $\square$

Definition 2.12.4. For a finite group G with $A \in \mathcal{E}(G)$ of rank at least $k + 1$ for some natural number k , we define

$$W_A = \langle \Delta_G(F) \mid F \in \mathcal{E}_k(A) \rangle.$$

Proposition 2.12.5. Suppose G is a finite group with $A \in \mathcal{E}(G)$ of rank at least $k + 1$ for some natural number k . Then for all $g \in G$, $(W_A)^g = W_{A^g}$.

Proof. Let $A \in \mathcal{E}(G)$ of rank at least $k + 1$ for some natural number k , and let $F \in \mathcal{E}_k(A)$. For all $g \in G$ and for each $1 \neq b \in A$, we note that

$$(O(C_G(b)))^g = O((C_G(b))^g) = O(C_G(b^g)).$$

It follows that

$$\Delta_G(F)^g = \left(\bigcap_{b \in F^\#} O(C_G(b)) \right)^g = \bigcap_{b \in F^\#} (O(C_G(b)))^g = \bigcap_{b \in F^\#} O(C_G(b^g)) = \Delta_G(F^g)$$

for all $g \in G$. Thus for all $g \in G$,

$$(W_A)^g = \langle \Delta_G(F)^g \mid F \in \mathcal{E}_k(A) \rangle = \langle \Delta_G(F^g) \mid F \in \mathcal{E}_k(A) \rangle = W_{A^g},$$

as required. $\square$

Proposition 2.12.6. *Let G be a finite group with $A \in \mathcal{E}(G)$ of rank at least $k + 2$ for some natural number k . Assume G is k -balanced with respect to A . Then W_A has odd order. Furthermore, if $E \in \mathcal{E}(A)$ is of rank at least $k + 1$, then $W_A = W_E$ and $N_G(E) \leq N_G(W_A)$.*

Proof. We follow the proof found in [37, Proposition 2.7.9(ii)]. Let $A \in \mathcal{E}(G)$ of rank at least $k + 2$ for some natural number k . Suppose G is k -balanced with respect to A . Then with θ as in Theorem 2.12.2, we have that θ is an A -signalizer functor on G . By Proposition 2.12.3 applied twice, we have $W_A = \theta(G; A) = \theta(G) = W_E$. Also if $g \in N_G(E)$, then $(W_A)^g = (W_E)^g = W_{E^g} = W_E = W_A$. Thus $g \in N_G(W_A)$ and so $N_G(E) \leq N_G(W_A)$. $\square$

2.13 Generation

This section basically looks at how to obtain information about the structure of cores of the centralizers of involutions in finite groups.

Definition 2.13.1. Let G be a finite group with T as a p -subgroup of G . For any natural number k , we define

$$\Gamma_{T,k}(G) := \langle N_G(R) \mid R \leq T, m(R) \geq k \rangle.$$

We note that for all $g \in G$, $(\Gamma_{T,k}(G))^g = \Gamma_{T^g,k}(G)$.

Lemma 2.13.2. *Let X be a finite group, k a natural number and $A \leq X$ a p -group of rank at least $k + 1$. Set $\bar{X} := X/O_{p'}(X)$. Then $X = \Gamma_{A,k}(X)$ if and only if $\bar{X} = \Gamma_{\bar{A},k}(\bar{X})$.*

Proof. It follows from Lemma 2.1.13 that $\Gamma_{\bar{A},k}(\bar{X}) = \overline{\Gamma_{A,k}(X)}$. Therefore, $X = \Gamma_{A,k}(X)$ implies $\bar{X} = \Gamma_{\bar{A},k}(\bar{X})$, and $\bar{X} = \Gamma_{\bar{A},k}(\bar{X})$ implies $X = \Gamma_{A,k}(X)O_{p'}(X)$. Moreover, we have that A has rank at least $k + 1$, so there exists an elementary abelian p -subgroup B of A of rank $k + 1$. Invoking Theorem 2.7.8 we have that

$$O_{p'}(X) = \langle C_{O_{p'}(X)}(F) : F \leq B, [B : F] = p \rangle \leq \Gamma_{A,k}(X).$$

Thus $\bar{X} = \Gamma_{\bar{A},k}(\bar{X})$ and hence $X = \Gamma_{A,k}(X)$. $\square$

Lemma 2.13.3. *Let $s \geq 3$ be a natural number, and V be a permutation module for S_{2s} with S_{2s} -invariant basis $\{v_1, \dots, v_{2s}\}$. Assume $X = W \rtimes S_{2s}$ where $W \subseteq V$ is the submodule of “even sums”, and set $A = \langle v_1 + v_2, v_3 + v_4, \dots, v_{2s-1} + v_{2s}, (12), (34), \dots, (2s-1, 2s) \rangle$. Then*

$$X = \Gamma_{A,2}(X) = \langle C_X(v_{2i-1} + v_{2i}) : i = 1, 2, \dots, s \rangle.$$

Proof. Since W is elementary 2-abelian of rank at least 2, we get that $W \leq \Gamma_{A,2}(X)$ and also $W \leq X_0 := \langle C_X(v_{2i-1} + v_{2i}) : i = 1, 2, \dots, s \rangle$. Therefore, it is sufficient to argue that S_{2s} is contained in $\Gamma_{A,2}(X)$ and in X_0 . In fact, as $S_{2s-2} \leq C_X(\langle v_{2s-1} + v_{2s}, (2s-1, 2s) \rangle) \leq X_0$ and $S_{\{3,4,\dots,2s\}} \leq C_X(\langle v_1 + v_2, (12) \rangle) \leq X_0$ are both contained in X_0 , and thus $\Gamma_{A,2}(X)$, it is sufficient to argue that $X = \langle S_{2s-2}, S_{\{3,4,\dots,2s\}} \rangle$. Since $\langle S_{2s-2}, S_{\{3,4,\dots,2s\}} \rangle$ is 2-transitive on $\{1, 2, \dots, 2s\}$ and contains a transposition, it contains every transposition in S_{2s} and is thus indeed equal to S_{2s} . $\square$

The following proposition can be used to obtain information concerning the structure of the cores of the centralizers of involutions in finite groups. We note that the proposition and its proof was suggested to me by Prof. Ellen Henke.

Proposition 2.13.4. *Let G be a finite group with T as a Sylow 2-subgroup of G , and $A \in \mathcal{E}(T)$. Let k be a natural number and assume the following hold:*

- (i) $O(G) = 1$,
- (ii) G has at least two conjugacy classes of involutions,
- (iii) G is k -balanced with respect to A ,
- (iv) for each $x \in A^\#$, we have $C_G(x) \leq \Gamma_{A,k+1}(G)$,
- (v) A has rank at least $k+2$,
- (vi) $A \leq N_G(T)$,
- (vii) every involution in T is conjugate to an element of A under an element of $\langle C_G(x) \mid x \in A^\# \rangle N_G(A)$.

Then $\Delta_G(F) = 1$ for all $F \in \mathcal{E}_k(A)$.

Proof. We recall that

$$W_E := \langle \Delta_G(F) \mid F \in \mathcal{E}_k(E) \rangle$$

for each $E \in \mathcal{E}(A)$ of rank at least $k + 1$. Set $W := W_A$. By Proposition 2.12.6, it follows from assumptions (iii) and (v) that

$$W = W_E \text{ and } N_G(E) \leq N_G(W) \text{ for all } E \leq A \text{ of rank at least } k + 1, \quad (2.1)$$

and W is of odd order. Set now

$$H := N_G(W).$$

By definition of $W = W_A$ it is sufficient to show that $W = 1$. Indeed, as W is of odd order, it is by (i) sufficient to show that $H = G$.

So assume from now on $H \neq G$. Using (vi) and (2.1) with A in place of E , we see that

$$N_G(T) \leq N_G(A) \leq H. \quad (2.2)$$

Indeed, (2.1) gives that $\Gamma_{A,k+1}(G) \leq H$. Hence, it follows from (iv) that

$$C_G(x) \leq H \text{ for all } x \in A^\#. \quad (2.3)$$

Note that (2.2) and (2.3) yield in particular that

$$\langle C_G(x) \mid x \in A^\# \rangle N_G(A) \leq H.$$

Hence, if we consider an arbitrary involution $y \in T$, then (vii) yields that there exists $h \in$ such that $y^h \in A$. Then $C_G(y)^h = C_G(y^h) \leq H$ by (2.3). Hence,

$$C_G(y) \leq H \text{ for every involution } y \in T.$$

This property together with (2.2) yields by Proposition 2.5.2 that H is strongly 2-embedded, which in turn implies by Proposition 2.5.5 that G has only one conjugacy class of involutions, contradicting assumption (ii). Hence, $H = G$ and as argued before, this yields the assertion. $\square$

Lemma 2.13.5. *Let G be a finite group, k a natural number and $A \in \mathcal{E}(G)$. Assume there is a subset $\mathcal{I} \subseteq A^\#$ such that $C_G(x) = \Gamma_{A,k+1}(C_G(x))$ for all $x \in \mathcal{I}$ and $C_G(y) = \langle C_{C_G(y)}(x) \mid x \in \mathcal{I} \rangle$ for all $y \in A^\# \setminus \mathcal{I}$. Then Property (iv) in Proposition 2.13.4 holds.*

Proof. As $N_{C_G(x)}(R) = N_G(R) \cap C_G(x) \leq N_G(R)$ for any $R \leq A$ of rank at least $k + 1$, we have

$$C_G(x) = \Gamma_{A,k+1}(C_G(x)) = \langle N_{C_G(x)}(R) \mid R \in A, m(R) \geq k + 1 \rangle \leq \Gamma_{A,k+1}(G)$$

for all $x \in \mathcal{I}$. As $C_{C_G(y)}(x) = C_G(x) \cap C_G(y) \leq C_G(x) \leq \Gamma_{A,k+1}(G)$ for all $x \in \mathcal{I}$ and for all $y \in A^\# \setminus \mathcal{I}$, we have that

$$C_G(y) = \langle C_{C_G(y)}(x) \mid x \in \mathcal{I} \rangle \leq \Gamma_{A,k+1}(G).$$

$\square$

Chapter 3 Introduction to fusion systems

This chapter is a brief general introduction to the basic theory of fusion systems. We describe some properties of fusion systems which will be useful in this thesis especially in Chapter 6. We begin with the notion of a fusion category and extend it to the notion of abstract fusion system, most importantly saturated fusion systems. We give a version of Alperin's fusion theorem in fusion systems setting. The Sections 3.8-3.14 mainly deal with the local theory of fusion systems where concepts such as strongly closed and normal subgroup of a fusion systems, quotient fusion systems, constrained fusion systems, normal subsystems, simple fusion systems, and components of a fusion systems are discussed. The main references for this chapter are [6], [8], [14], and [53], however the following references [5], [32], [42], [43], [52], and [54] were also very useful.

3.1 Fusion category of a finite group

In this section, we look at the concept of category, from which we define the fusion category of a finite group.

Definition 3.1.1. A category $\mathcal{C}$ consists of

- (i) a collection $Ob(\mathcal{C})$ of objects $P, Q, R, \dots$,
 - (ii) a collection $Mor(\mathcal{C})$ of morphisms $\alpha, \beta, \gamma, \dots$,
 - (iii) for each morphism α there are objects:
 $P = dom(\alpha)$ and $Q = cod(\alpha)$ called the domain and codomain of α respectively with $\alpha : P \rightarrow Q$,
 - (iv) for any given morphisms $\alpha : P \rightarrow Q$ and $\beta : Q \rightarrow R$ with $cod(\alpha) = dom(\beta)$ there is a morphism $\alpha \circ \beta : P \rightarrow R$ called the composite of α and β ,
 - (v) for each object P , there is a morphism $1_P : P \rightarrow P$ called the identity morphism of P ,
- such that the following laws are satisfied:

- (C1) the composition of morphisms is associative,
- (C2) there exist units 1_P and 1_Q such that $1_P \circ \alpha = \alpha = \alpha \circ 1_Q$ for all $\alpha : P \rightarrow Q$.

Example 3.1.2. 1. The category **Sets** whose objects are sets and morphism are functions. Here the units are the identity maps and the composition of morphisms is the usual function composition.

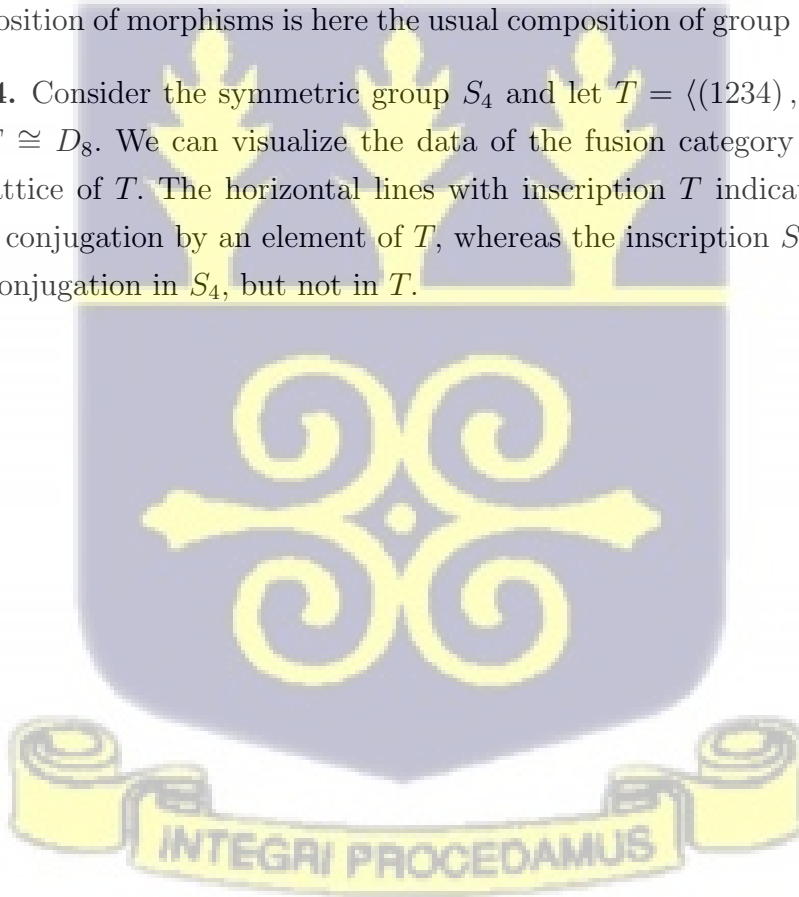
2. The category **Grp** with groups as objects and group homomorphisms as morphisms. The composition of morphisms is the usual composition of group homomorphisms and the units are the usual identity maps.

Definition 3.1.3. Let G be a finite group and T a Sylow p -subgroup of G . The fusion category of G over T is the category $\mathcal{F}_T(G)$, whose objects are all subgroups of T and whose morphism sets are

$$\text{Hom}_{\mathcal{F}_T(G)}(P, Q) := \text{Hom}_G(P, Q) \quad \text{for all } P, Q \leq T,$$

that is, the set of all group homomorphism from P to Q induced by conjugation with elements in G . The composition of morphisms is here the usual composition of group homomorphisms.

Example 3.1.4. Consider the symmetric group S_4 and let $T = \langle (1234), (13) \rangle$. Then $T \in \text{Syl}_2(S_4)$ and $T \cong D_8$. We can visualize the data of the fusion category $\mathcal{F}_T(S_4)$ through the subgroup lattice of T . The horizontal lines with inscription T indicate the subgroups are fused under conjugation by an element of T , whereas the inscription S_4 means that the subgroups are conjugation in S_4 , but not in T .



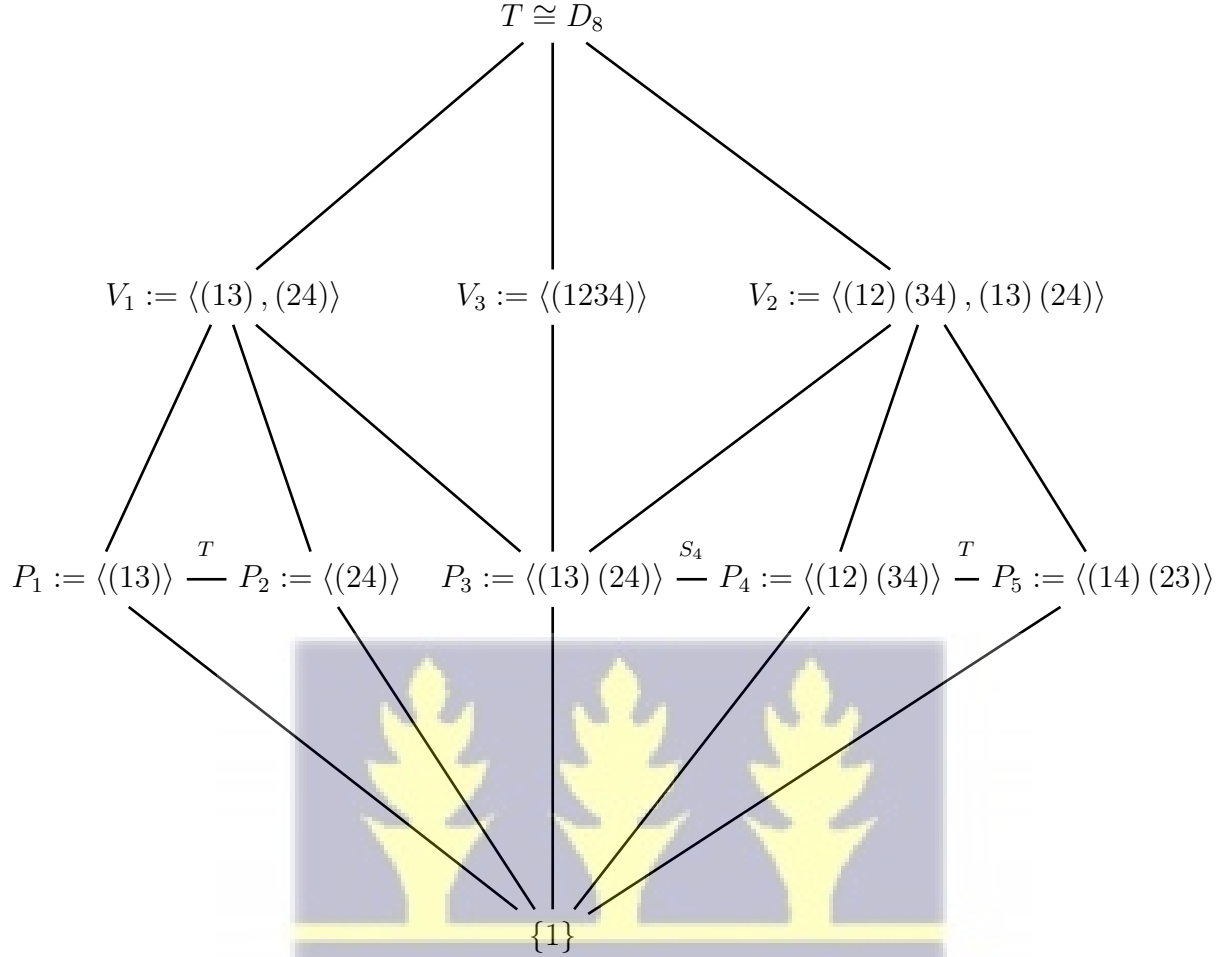


Figure 31: Fusion category of S_4 over $T \cong D_8$

Notice that the fusion category $\mathcal{F}_T(T)$ is embedded in the fusion category $\mathcal{F}_T(S_4)$.

The fusion category of G over T is also referred to as the fusion system of G on T and hence will be referred to as such.

3.2 The notion of abstract fusion system

In this section we will introduce a more abstract notion of fusion system. In such an abstract setting, we will not require anymore that T is a subgroup of some larger group G and also loosen the requirement that every morphism must be induced by conjugation. But we will assume that every morphism induced by conjugation in T is in the fusion system.

Definition 3.2.1. A fusion system over a finite p -group T is a category $\mathcal{F}$, whose objects are all subgroups of T and whose morphism sets $Hom_{\mathcal{F}}(P, Q)$ for any pair of objects P and Q , satisfy the following:

- (i) $Hom_T(P, Q) \subseteq Hom_{\mathcal{F}}(P, Q) \subseteq Inj(P, Q)$ for all $P, Q \leq T$; and
- (ii) for each $\phi \in Hom_{\mathcal{F}}(P, Q)$, its core $\phi|_{P\phi}$ is an $\mathcal{F}$ -isomorphism.

We note that, by condition (i), a fusion system over T contains all inclusion maps and condition (ii) implies that for any $\mathcal{F}$ -morphism ϕ , the core of ϕ and the inverse of the core are $\mathcal{F}$ -morphisms.

Example 3.2.2. Let T be a finite p -group. Then

1. $\mathcal{F}_T(T)$ is a fusion system contained in any fusion system over T . If $T = \{1\}$, then we call $\mathcal{F}_T(T)$ the trivial fusion system, denoted as $\mathcal{F}_1(1)$.
2. The universal fusion system over T is the fusion system over T with $Hom_{\mathcal{F}}(P, Q) = Inj(P, Q)$ for all $P, Q \leq T$. It is usually denoted by $\mathcal{U}(T)$.

From now on in this section, $\mathcal{F}$ is assumed to be a fusion system over T .

Lemma 3.2.3. If $\phi \in Hom_{\mathcal{F}}(P, Q)$, then the restricted map $\phi|_{P'\phi} \in Hom_{\mathcal{F}}(P', Q')$ for all $P' \leq P$ and $P'\phi \leq Q' \leq Q$.

Proof. Let $\phi \in Hom_{\mathcal{F}}(P, Q)$. By Definition 3.2.1, the core $\phi|_{P\phi}$ is an $\mathcal{F}$ -isomorphism. For all $P' \leq P$ and $P'\phi \leq Q' \leq Q$, let $i_{P',P} : P' \rightarrow P$ and $i_{P'\phi,Q'} : P'\phi \rightarrow Q'$ be the inclusion maps. Then the map $\psi := i_{P',P} \circ \phi$ is in $Hom_{\mathcal{F}}(P', Q)$ since composition of $\mathcal{F}$ -morphisms is an $\mathcal{F}$ -morphism. Therefore, $\phi|_{P'\phi} = \psi|_{P'\phi} \circ i_{P'\phi,Q'}$ is in $Hom_{\mathcal{F}}(P', Q')$. $\square$

The following lemma follows immediately from Definition 3.2.1.

Lemma 3.2.4. Suppose $\phi \in Hom_{\mathcal{F}}(P, Q)$. Then ϕ is an isomorphism in $\mathcal{F}$ if and only if $|P| = |Q|$.

Definition 3.2.5. Two subgroups P, Q of T are said to be $\mathcal{F}$ -conjugate if $Hom_{\mathcal{F}}(P, Q)$ contains an isomorphism. The set of all subgroups of T which are $\mathcal{F}$ -conjugate to P is denoted as $P^{\mathcal{F}}$.

Corollary 3.2.6. Every $\phi \in Hom_{\mathcal{F}}(P, P)$ is an automorphism.

Definition 3.2.7. Let $\mathcal{F}$ be a fusion system over a p -group T . For each $P \leq T$, $Aut_{\mathcal{F}}(P)$ is defined as $Hom_{\mathcal{F}}(P, P)$ and $Aut_T(P)$ is defined as $Hom_T(P, P)$.

Consequently from Definition 3.2.1, we have that

$$Aut_T(P) \leq Aut_{\mathcal{F}}(P) \leq Aut(P).$$

It is clear from Lemma 3.2.3 that all meaningful restrictions of morphisms are again morphisms in $\mathcal{F}$. It is natural to ask if the reverse operation, that is extension of morphism is possible. In what follows, we discuss the extension of morphisms and the condition under which such extensions are possible.

Definition 3.2.8. For a given $\phi \in Hom_{\mathcal{F}}(P, Q)$, a morphism $\tilde{\phi} \in Hom_{\mathcal{F}}(\tilde{P}, \tilde{Q})$ is an extension of ϕ if

- (i) $\tilde{P} \geq P, Q \geq P\tilde{\phi}$ and
- (ii) $\tilde{\phi}|_{P,Q} = \phi$.

Remark 3.2.9. It is obvious that every extension of a morphism ϕ is at the same time an extension of the core of ϕ . Therefore, in considering extensions, we can restrict to the case where ϕ is an isomorphism. We also note that if $\phi \in Hom_{\mathcal{F}}(P, Q)$ is an isomorphism in $\mathcal{F}$ and has an extension $\tilde{\phi} \in Hom_{\mathcal{F}}(\tilde{P}, \tilde{Q})$, then ϕ extends to the subgroup $N_{\tilde{P}}(P) \leq N_T(P)$. Thus, if $P < \tilde{P}$, then $P < N_{\tilde{P}}(P) \leq N_T(P)$. To be precise, ϕ extends to a larger subgroup in T if and only if it extends to a larger subgroup in $N_T(P)$.

We recall that c_P denotes the homomorphism from $N_T(P)$ onto $Aut_T(P)$ defined by $ac_P = c_a|_{P,P}$ for all $a \in N_T(P)$. As mentioned before, this map c_P is surjective and its kernel is $C_T(P)$.

Lemma 3.2.10. Let P, Q be subgroups of T such that $Q \in P^{\mathcal{F}}$. If $\phi \in Hom_{\mathcal{F}}(P, Q)$ extends to a morphism $\tilde{\phi} \in Hom_{\mathcal{F}}(\tilde{P}, \tilde{Q})$ with $\tilde{P} \leq N_T(P)$, then $\tilde{P}\tilde{\phi} \leq N_T(Q)$. Moreover, for $a \in \tilde{P}$, we have $(ac_P)^{\phi} = (a\tilde{\phi})c_Q$.

Proof. Let $a \in \tilde{P}$ and b be an arbitrary element of Q . Then

$$b^{a\tilde{\phi}} = (a\tilde{\phi})^{-1} b (a\tilde{\phi}) = (a^{-1}\tilde{\phi}) b (a\tilde{\phi})$$

But $b = c\phi = c\tilde{\phi}$ for some $c \in P$. So,

$$(b)^{a\tilde{\phi}} = (a^{-1}\tilde{\phi}) c\tilde{\phi} (a\tilde{\phi}) = (c^a)\tilde{\phi}.$$

Since $c^a \in P$, $(b)^{a\tilde{\phi}} \in Q$. So, $a\tilde{\phi} \in N_T(Q)$ and $\tilde{P}\tilde{\phi} \leq N_T(Q)$.

Also,

$$\begin{aligned}
 b\left((ac_P)^\phi\right) &= \left((b\phi^{-1})(ac_P)\right)\phi \\
 &= (c(ac_P))\phi \\
 &= c^a\phi \\
 &= c^a\tilde{\phi} \quad \text{since } c^a \in P \\
 &= b^{a\tilde{\phi}} \\
 &= b\left((a\tilde{\phi})c_Q\right) \quad \text{since } a\tilde{\phi} \in N_S(Q).
 \end{aligned}$$

Thus, $(ac_P)^\phi = (a\tilde{\phi})c_Q$. □

Lemma 3.2.11. [53, Lemma 3.2] Suppose that an isomorphism $\phi \in \text{Hom}_{\mathcal{F}}(P, Q)$ has an extension $\tilde{\phi} \in \text{Hom}_{\mathcal{F}}(\tilde{P}, \tilde{Q})$. Then $(\tilde{P} \cap N_T(P))c_P \leq \text{Aut}_T(P) \cap \text{Aut}_T(Q)^{\phi^{-1}}$.

Proof. Without loss of generality we assume that $\tilde{P} = \tilde{P} \cap N_T(P)$. Thus, $\tilde{P} \leq N_T(P)$. Let $a \in \tilde{P}$, then $ac_P \in \text{Aut}_T(P)$. By Lemma 3.2.10, $(ac_P)^\phi \in \text{Aut}_T(Q)$, and $ac_P \in \text{Aut}_T(P) \cap \text{Aut}_T(Q)^{\phi^{-1}}$. □

The subgroup $\text{Aut}_T(P) \cap \text{Aut}_T(Q)^{\phi^{-1}}$ of $\text{Aut}_{\mathcal{F}}(P)$ is usually called the extension control subgroup (or the control of ϕ). Thus, the extension in $N_T(P)$ is strictly controlled. The full preimage under c_P of the control $\text{Aut}_T(P) \cap \text{Aut}_T(Q)^{\phi^{-1}}$ of ϕ is denoted by N_ϕ . Therefore, N_ϕ is the largest subgroup in $N_T(P)$ to which it is possible that one may extend the morphism ϕ . The following definition indicates a form in which the subgroup N_ϕ can be written.

Definition 3.2.12. Let $\mathcal{F}$ be a fusion system over a finite p -group T . For an isomorphism $\phi \in \text{Hom}_{\mathcal{F}}(P, Q)$, we define

$$N_\phi = \{t \in N_T(P) \mid c_t|_P \in \text{Aut}_T(Q)^{\phi^{-1}}\}.$$

Definition 3.2.13. A subgroup $Q \leq T$ is receptive if every isomorphism $\phi \in \text{Hom}_{\mathcal{F}}(P, Q)$ extends to a morphism $\Phi \in \text{Hom}_{\mathcal{F}}(N_\phi, T)$.

The following example gives an indication that it is not always true that any isomorphism in $\mathcal{F}$ can be extended to an overgroup of its domain.

Example 3.2.14. Consider the fusion system $\mathcal{F}_T(S_4)$ as in Example 3.1.4. The morphism set $\text{Hom}_{\mathcal{F}_T(S_4)}(P_3, P_4)$ consists of conjugation maps induced by elements in the set $\{(23), (14), (123), (142), (134), (243), (1243), (1342)\}$. If $\phi \in \text{Hom}_{\mathcal{F}_T(S_4)}(P_3, P_4)$, then

$$\begin{aligned}
 N_\phi &= \{s \in N_T(P_3) \mid c_s|_{P_3} \in \text{Aut}_T(P_4)^{\phi^{-1}}\} \\
 &= \{s \in T \mid c_s|_{P_3} \in \{Id\}\} \quad \text{since } C_T(P_4) = N_T(P_4) \\
 &= T \quad \text{since } P_3 \text{ is the center of } T.
 \end{aligned}$$

So, if Φ was an extension of ϕ then $\Phi \in \text{Aut}(T)$. But none of these conjugations fixes T . Therefore, they cannot be extended to an automorphism of T . Hence P_4 is not receptive. Similarly, P_5 is also not receptive.

However, P_3 is receptive since for every $\phi \in \text{Hom}_{\mathcal{F}_T(S_4)}(P_4, P_3)$ where ϕ is induced by conjugation with elements in the set $\{(23), (14), (132), (124), (143), (234), (1423), (1324)\}$;

$$\begin{aligned} N_\phi &= \{s \in N_T(P_4) \mid c_s|_{P_4} \in \text{Aut}_T(P_3)^{\phi^{-1}}\} \\ &= \{s \in V_2 \mid c_s|_{P_4} \in \{Id\}\} \\ &= V_2 \quad \text{since } V_2 \text{ is an elementary abelian group.} \end{aligned}$$

Since V_2 is normal in S_4 , we get that every $\phi \in \text{Hom}_{\mathcal{F}_T(S_4)}(P_4, P_3)$ extends to an automorphism of V_2 . Similar argument works for all $\phi \in \text{Hom}_{\mathcal{F}_T(S_4)}(P_5, P_3)$. Therefore, P_3 is receptive.

Lemma 3.2.15. *Suppose that $\phi \in \text{Hom}_{\mathcal{F}}(P, Q)$ is an isomorphism. Then $PC_T(P) \leq N_\phi$. Moreover, if Q is receptive, then ϕ extends to $\tilde{\phi} \in \text{Hom}_{\mathcal{F}}(PC_T(P), T)$.*

Proof. Let $\phi \in \text{Hom}_{\mathcal{F}}(P, Q)$ be an isomorphism. We want to show that $PC_T(P) \leq N_\phi$. We have that $P \leq N_T(P)$ and $P\phi = Q \leq N_T(Q)$. By Lemma 3.2.10, for all $a \in P$, $(ac_P)^\phi = (a\phi)c_Q$. This implies that $ac_P \in \text{Aut}_T(P) \cap \text{Aut}_T(Q)^{\phi^{-1}}$. Thus, $P \leq N_\phi$. Moreover, since N_ϕ is the full preimage of $\text{Aut}_T(P) \cap \text{Aut}_T(Q)^{\phi^{-1}}$ under c_P , the kernel of c_P which is $C_T(P)$ is in N_ϕ . Since $PC_T(P) = C_T(P)P$, we get that $PC_T(P) \leq N_\phi$. Assume that Q is receptive. Then ϕ extends to say, $\psi \in \text{Hom}_{\mathcal{F}}(N_\phi, T)$. Therefore the restricted map $\psi|_{PC_T(P)}$ is also in $\mathcal{F}$ and it is an extension of ϕ . $\square$

Definition 3.2.16. Let R be a subgroup of T .

- (a) R is called fully automized in $\mathcal{F}$ if $\text{Aut}_T(R)$ is a Sylow p -subgroup of $\text{Aut}_{\mathcal{F}}(R)$.
- (b) R is fully centralized if $|C_T(R)| \geq |C_T(Q)|$ for all $Q \in R^{\mathcal{F}}$.
- (c) R is fully $\mathcal{F}$ -normalized if $|N_T(R)| \geq |N_T(Q)|$ for all $Q \in R^{\mathcal{F}}$.

We denote $\mathcal{F}^f$ as the set of all subgroups of T which are fully $\mathcal{F}$ -normalized.

Proposition 3.2.17. [53, Proposition 3.7] *Every receptive subgroup is fully centralized.*

Proof. Suppose that Q is receptive and let $P \in Q^{\mathcal{F}}$. Let $\phi \in \text{Hom}_{\mathcal{F}}(P, Q)$, then ϕ extends to some $\Phi \in \text{Hom}_{\mathcal{F}}(N_\phi, \tilde{Q})$ for some $Q \leq \tilde{Q}$. By Lemma 3.2.15, we have that $C_T(P) \leq N_\phi$ and $C_T(P)\Phi \leq C_T(P\Phi) = C_T(Q)$ since Φ is a homomorphism. Thus, $|C_T(P)| \leq |C_T(Q)|$. Therefore, Q is fully centralized. $\square$

For the fusion systems of finite groups, the following proposition makes precise what fully centralized and fully normalized subgroups are.

Proposition 3.2.18. *Let G be a finite group. Suppose T is a Sylow p -subgroup of G and $R \leq T$. Then the following holds:*

- (i) R is fully $\mathcal{F}_T(G)$ -centralized if and only if $C_T(R) \in \text{Syl}_p(C_G(R))$.
- (ii) R is fully $\mathcal{F}_T(G)$ -normalized if and only if $N_T(R) \in \text{Syl}_p(N_G(R))$.

Proof. (i) Suppose R is a subgroup of T . Let $M \in \text{Syl}_p(C_G(R))$ such that $C_T(R) \leq M$. Then RM is a p -subgroup of G ; there is $g \in G$ such that $(RM)^g \leq S$. So, setting $Q = R^g$, we have $C_T(R)^g \leq M^g \leq C_T(Q)$. It follows that $|C_T(R)| \leq |M| \leq |C_T(Q)|$. If R is fully $\mathcal{F}_T(G)$ -centralized then $|C_T(R)| = |C_T(Q)| = |M|$. This implies $C_T(R) \in \text{Syl}_p(C_G(R))$.

Conversely, let $R \leq T$ and assume $C_T(R) \in \text{Syl}_p(C_G(R))$. Let $Q \in R^{\mathcal{F}}$. Thus, there is some $g \in G$ with $Q^g = R$. Then $C_T(Q)^g$ is a p -subgroup of $C_G(R)$ and thus has order less than or equal to the order of a Sylow p -subgroup of $C_G(R)$. Therefore, $|C_T(Q)| = |C_T(Q)^g| \leq |C_T(R)|$. Hence, R is fully $\mathcal{F}_T(G)$ -centralized.

(ii) Similar arguments as in (i) hold. □

Proposition 3.2.19. *Let G be a finite group and let T be a Sylow p -subgroup of G . If $Q \leq T$ such that $N_T(Q) \in \text{Syl}_p(N_G(Q))$, then Q is receptive.*

Proof. Let $\phi \in \text{Hom}_{\mathcal{F}_T(G)}(R, Q)$ be an isomorphism. Then

$$N_\phi = \{s \in N_T(R) \mid c_s|_R \in \text{Aut}_T(Q)^{\phi^{-1}}\}$$

Thus, if $s \in N_\phi$, then there exist $y \in N_T(Q)$ such that $c_s = \phi c_y \phi^{-1}$ and $c_s \phi = \phi c_y$. So we have that $\phi c_y \phi^{-1} c_{s^{-1}}$ centralizes R . Let $g \in G$ such that $\phi = c_g|_R$. Then from the definition of N_ϕ , we have $c_{s^g}|_Q = (c_s|_R)^\phi \in \text{Aut}_T(Q)$. Thus $s^g \in N_T(Q) C_G(Q)$ for any $s \in N_\phi$. Therefore $(N_\phi)^g \leq N_T(Q) C_G(Q)$. As $N_T(Q)$ is a Sylow p -subgroup of $N_G(Q)$, it is also a Sylow p -subgroup of $N_T(Q) C_G(Q)$. So by Sylow's theorem, there exists $a \in C_G(Q)$ such that $(N_\phi)^{ga} \leq N_T(Q)$. Then $c_{ga} \in \text{Hom}_{\mathcal{F}_T(G)}(N_\phi, T)$ extends $\phi = c_g|_R$. Therefore Q is receptive. □

3.3 Saturated fusion systems

The concept of abstract fusion system as defined before is considered to be too broad for much use. Therefore, the need to focus on what we referred to as saturated fusion system.

In this section we look at the concept of saturated fusion systems. We let $\mathcal{F}$ be a fusion system on the p -group T .

Definition 3.3.1. A fusion system $\mathcal{F}$ over a p -subgroup T is called *saturated* if every $\mathcal{F}$ -conjugacy class of subgroups of T contains a subgroup which is fully automized and receptive.

Specifically, we have the following for a fusion system over a finite group.

Theorem 3.3.2. [8, Theorem 2.3] Let G be a finite group and T a Sylow p -subgroup of G . The fusion system $\mathcal{F}_T(G)$ is saturated.

Proof. We will show that every $\mathcal{F}_T(G)$ -conjugacy class of T contains a subgroup which is fully automized and receptive. Let $P \leq T$, then by Lemma 2.1.6 there is some $Q \leq T$ which is $\mathcal{F}_T(G)$ -conjugate to P and $N_T(Q) \in \text{Syl}_p(N_G(Q))$. By Proposition 3.2.19, Q is receptive. We are left to show that Q is fully automized. Now, $N_T(Q)$ normalizes $C_G(Q)$, and $N_T(Q)C_G(Q) \leq N_G(Q)$. Thus, $N_T(Q)C_G(Q)/C_G(Q) \in \text{Syl}_p(N_G(G)/C_G(Q))$. By the second Isomorphism theorem we have

$$N_T(Q)C_G(Q)/C_G(Q) \cong N_T(Q)/(N_T(Q) \cap C_G(Q)) = N_T(Q)/C_T(Q) \cong \text{Aut}_T(Q).$$

Thus, $\text{Aut}_T(Q) \in \text{Syl}_p(\text{Aut}_G(Q))$ and so, Q is fully $\mathcal{F}_T(G)$ -automized. Hence Q is a saturated subgroup. $\square$

Definition 3.3.3. A saturated fusion system over a p -subgroup T is said to be *realizable* if $\mathcal{F} = \mathcal{F}_T(G)$ for some finite group G with $T \in \text{Syl}_p(G)$.

However, not all saturated fusion systems are realizable.

Definition 3.3.4. A saturated fusion system over a p -subgroup T is said to be *exotic* if there is no finite group G realizing it.

A. Ruiz and A. Viruel [54] discovered three exotic saturated fusion systems over Sylow 7-subgroups of $SL_3(7)$. R. Solomon [55] found a family of exotic saturated fusion systems over T , where $T \in \text{Syl}_2(\text{Spin}_7(q))$ with $q \cong \pm 3 \pmod{8}$.

Theorem 3.3.5. [53, Theorem 5.2] Suppose that the fusion system $\mathcal{F}$ over a p -group T is saturated. Then

- (a) any fully normalized subgroup in T is fully automized and fully centralized and
- (b) any fully centralized subgroup is receptive.

We state without proof an equivalent definition of a saturated fusion system which is due to [53].

Theorem 3.3.6. [53, Theorem 5.3] *A fusion system $\mathcal{F}$ over T is saturated if and only if*

- (a) *T is fully automized and*
- (b) *every $Q \leq T$ is $\mathcal{F}$ -conjugate to a fully normalized subgroup that is receptive.*

We saw in Example 3.2.14 that extension of morphism to proper overgroup of its domain is not always possible. However, the following lemma gives a situation where all morphisms extend to automorphism of T .

Lemma 3.3.7. *Let $\mathcal{F}$ be a saturated fusion system over T . If T is abelian then every morphism in $\mathcal{F}$ extends to an $\mathcal{F}$ -automorphism of T .*

Proof. Let $\phi \in \text{Hom}_{\mathcal{F}}(R, Q)$ be an isomorphism. Since T is abelian, every $Q \leq T$ is fully $\mathcal{F}$ -normalized and by Theorem 3.3.5, Q is receptive. Then ϕ extends to say, $\psi \in \text{Hom}_{\mathcal{F}}(RC_T(R), T)$ by Lemma 3.2.15. Since T is abelian, $C_T(R) = T$ and $RC_T(R) = T$. Thus, $\psi \in \text{Aut}_{\mathcal{F}}(T)$. $\square$

If the fusion system $\mathcal{F}$ is realizable, then Lemma 3.3.7 can be seen as a reformulation of Theorem 2.6.3 (Burnside's Theorem).

3.4 Subsystems of fusion systems

Just as we can talk about substructure of any mathematical structure being it groups, rings, etc, we do the same for fusion systems. In this section we give the definition for a subsystem of a fusion system. In particular, we give the notion of a normalizer and centralizer of a fusion system, which serves as the foundation of the local theory of fusion systems. The local theory of fusion systems play a similar role to the local theory of finite groups.

Definition 3.4.1. Let $\mathcal{F}$ be a fusion system over a finite p -group T . A *subsystem* $\mathcal{E}$ of $\mathcal{F}$ on a subgroup $P \leq T$ is a subcategory that is itself a fusion system on P . We write $\mathcal{E} \subseteq \mathcal{F}$.

Definition 3.4.2. Let T be a finite p -group for some prime p , and $\mathcal{F}_i$ be fusion systems over $T_i \leq T$ for $1 \leq i \leq n$. The intersection $\bigcap_{i=1}^n \mathcal{F}_i$ is the subsystem of $\mathcal{F}_i$ for each $1 \leq i \leq n$, with

- (i) objects as all subgroups of $\bigcap_{i=1}^n T_i$,
- (ii) morphisms as all morphisms that lie in $\mathcal{F}_i$ for all $1 \leq i \leq n$.

Definition 3.4.3. Let T be a finite p -group for some prime p . Let $\mathfrak{F}$ be a collection of fusion systems over T . The fusion system generated by $\mathfrak{F}$, denoted as $\langle \mathfrak{F} \rangle_T$, is the intersection of all fusion systems over T which contain each member of $\mathfrak{F}$. Here, the morphisms in $\langle \mathfrak{F} \rangle_T$ are the composites of restrictions of homomorphisms in the set $\mathfrak{F} \cup \text{Inn}(T)$ and their inverses.

We note here that $\langle \mathfrak{F} \rangle_T$ is the smallest fusion system over T containing the members of $\mathfrak{F}$. The subsystem $\langle \mathfrak{F} \rangle_T$ is not necessarily saturated.

Definition 3.4.4. Let $\mathcal{F}$ be a fusion system over a p -group T , and fix $P \leq T$.

- (a) The *normalizer* of P in $\mathcal{F}$, denoted by $N_{\mathcal{F}}(P)$, is defined to be the subsystem of $\mathcal{F}$, over $N_T(P)$ where for $Q, R \leq N_T(P)$, the morphisms

$$\text{Hom}_{N_{\mathcal{F}}(P)}(Q, R) = \{ \phi \in \text{Hom}_{\mathcal{F}}(Q, R) \mid \exists \Phi \in \text{Hom}_{\mathcal{F}}(QP, RP) \text{ such that } \Phi|_Q = \phi \text{ and } \Phi|_P \in \text{Aut}(P) \}.$$

- (b) The *centralizer* of P in $\mathcal{F}$, denoted by $C_{\mathcal{F}}(P)$, is defined to be the subsystem of $\mathcal{F}$, over $C_T(P)$ where for $Q, R \leq C_T(P)$, the morphisms

$$\text{Hom}_{C_{\mathcal{F}}(P)}(Q, R) = \{ \phi \in \text{Hom}_{\mathcal{F}}(Q, R) \mid \exists \Phi \in \text{Hom}_{\mathcal{F}}(QP, RP) \text{ such that } \Phi|_Q = \phi \text{ and } \Phi|_P = \text{Id}_P \}.$$

We note that the definitions in Definition 3.4.4 are two special cases of a more general notion which we will not cover in this thesis. For details on this general notion see for example [8, Part I.5] and [14, Chapter 4.4]. Moreover, $N_{\mathcal{F}}(P)$ and $C_{\mathcal{F}}(P)$ are not necessarily saturated. The following theorem tells us situations where $N_{\mathcal{F}}(P)$ and $C_{\mathcal{F}}(P)$ are saturated.

Theorem 3.4.5. [52, Proposition 2.15] Let $\mathcal{F}$ be a saturated fusion system over a p -group T , and fix $R \leq T$.

- (i) If R is fully normalized, then $N_{\mathcal{F}}(R)$ is a saturated fusion system over $N_T(R)$.
- (ii) If R is fully centralized, then $C_{\mathcal{F}}(R)$ is a saturated fusion system over $C_T(R)$.

Proof. See for example [8, Part I, Theorem 5.5]. □

We end this section with a characterization of fully normalized and fully centralized subgroups of a finite group.

Theorem 3.4.6. Let G be a finite group with Sylow p -subgroup T , and fix $R \leq T$. Let $\mathcal{F} = \mathcal{F}_T(G)$.

(i) If R is fully normalized, then $N_{\mathcal{F}}(R) = \mathcal{F}_{N_T(R)}(N_G(R))$.

(ii) If R is fully centralized, then $C_{\mathcal{F}}(R) = \mathcal{F}_{C_T(R)}(C_G(R))$.

Proof. See [14, Theorem 4.27]. □

3.5 Morphisms of fusion systems

We define the notion of morphism between two fusion systems in this section. As a fusion system is a category, the morphism between them is a functor. This functor is defined as follows:

Definition 3.5.1. Let $\mathcal{F}$ and $\mathcal{T}$ be fusion systems over finite p -groups S and T respectively. A morphism $\Omega : \mathcal{F} \rightarrow \mathcal{T}$ of fusion systems is a family $(\alpha, \alpha_{P,Q} : P, Q \leq S)$ such that

- (i) $\alpha : S \rightarrow T$ is a group homomorphism, and
- (ii) $\alpha_{P,Q} : \text{Hom}_{\mathcal{F}}(P, Q) \rightarrow \text{Hom}_{\mathcal{T}}(P\alpha, Q\alpha)$ is a function,

such that for all P, Q and $\phi \in \text{Hom}_{\mathcal{F}}(P, Q)$, the relation $\phi\alpha = \alpha(\phi\alpha_{P,Q})$ is satisfied.

We note that the maps $\alpha_{P,Q}$ are uniquely determined by α and the last relation.

Definition 3.5.2. Let $\mathcal{F}$ and $\mathcal{T}$ be fusion systems over finite p -groups S and T respectively. Let $\Omega : \mathcal{F} \rightarrow \mathcal{T}$ be a morphism of fusion systems with the underlying group homomorphism $\alpha : S \rightarrow T$.

- (a) The kernel of the morphism Ω denoted as $\ker \Omega$, defined to be the kernel of the underlying group homomorphism α .
- (b) The morphism Ω is said to be injective if α is injective.
- (c) The morphism Ω is surjective if
 - i. α is surjective, and
 - ii. for all $R, Q \leq S$, $\alpha_{R_0, Q_0} : \text{Hom}_{\mathcal{F}}(R_0, Q_0) \rightarrow \text{Hom}_{\mathcal{T}}(R\alpha, Q\alpha)$ is surjective, where for $X \leq S$, X_0 is the preimage in S of $X\alpha$ under α . Equivalently, for all $R', Q' \in T$ and $\phi' \in \text{Hom}_{\mathcal{T}}(R', Q')$, there exist $R, Q \in S$ and $\phi \in \text{Hom}_{\mathcal{F}}(R, Q)$, such that $\phi\alpha_{R,Q} = \phi'$
- (d) The morphism Ω is an isomorphism if it both injective and surjective.

Lemma 3.5.3. *Let $\mathcal{F}$ and $\mathcal{T}$ be fusion systems over finite p -groups. Suppose $\Omega : \mathcal{F} \longrightarrow \mathcal{T}$ is a surjective morphism of fusion systems and $\mathcal{F}$ is saturated. Then $\mathcal{T}$ is saturated.*

Proof. See [8, Part II, Lemma 5.4]. □

We end this section with a definition which will be useful.

Definition 3.5.4. Let $\mathcal{F}$ be a fusion system over a finite p -group T . An *automorphism* of $\mathcal{F}$ is an isomorphism $\Omega : \mathcal{F} \longrightarrow \mathcal{F}$. The group of all automorphisms of $\mathcal{F}$ is denoted by $\text{Aut}(\mathcal{F})$.

As a morphism of fusion systems is determined by the underlying group homomorphism, we have that $\text{Aut}(\mathcal{F})$ is subgroup of $\text{Aut}(T)$ for a fusion system $\mathcal{F}$ over a finite p -group T . Moreover, an automorphism $\phi \in \text{Aut}_{\mathcal{F}}(T)$ induces an automorphism of the fusion system $\mathcal{F}$. Thus we have that

$$\text{Aut}_{\mathcal{F}}(T) \leq \text{Aut}(\mathcal{F}) \leq \text{Aut}(T).$$

We define $\text{Out}(\mathcal{F}) = \text{Aut}(\mathcal{F})/\text{Aut}_{\mathcal{F}}(T)$.

3.6 Alperin-Goldschmidt fusion theorem

In this section, we state a version of the fundamental Alperin-Goldschmidt fusion theorem for abstract fusion systems, which basically tells us that any saturated fusion system is generated by certain automorphism groups, a result which is originally due to L. Puig([52], Corollary 5.10). We begin the section with some classes of subgroups in a given fusion system over a p -group T .

Definition 3.6.1. A subgroup R of T is $\mathcal{F}$ -centric if $C_T(Q) = Z(Q)$ for all $Q \in R^{\mathcal{F}}$. Equivalently, a subgroup R is $\mathcal{F}$ -centric if $Q \geq C_T(Q)$ for all $Q \in R^{\mathcal{F}}$.

Let $\mathcal{F}^c$ denote the set of $\mathcal{F}$ -centric subgroups of T .

Remark 3.6.2. It is obvious that if R is $\mathcal{F}$ -centric, then so are all subgroups in $R^{\mathcal{F}}$. It must be noted that any $\mathcal{F}$ -centric subgroup is fully centralized, since the centralizers of all $\mathcal{F}$ -conjugate subgroups have the same order as their centres.

The next lemma indicates that the overgroup of an $\mathcal{F}$ -centric subgroup is also $\mathcal{F}$ -centric.

Lemma 3.6.3. *Let P, Q be subgroups of T with $Q \leq P$. If Q is $\mathcal{F}$ -centric, then so is P and $Z(P) \leq Z(Q)$.*

Proof. Let $\phi : P \longrightarrow T$ be an isomorphism in $\mathcal{F}$. Since Q is $\mathcal{F}$ -centric, $C_T(P\phi) \leq C_S(Q\phi) \leq Q\phi \leq P\phi$. So, P is $\mathcal{F}$ -centric. In particular, $Z(P) = C_T(P) \leq C_T(Q) = Z(Q)$. Thus, $Z(P) \leq Z(Q)$. $\square$

Definition 3.6.4. A subgroup $R \leq T$ is said to be $\mathcal{F}$ -radical if $O_p(\text{Aut}_{\mathcal{F}}(R)) = \text{Inn}(R)$.

Remark 3.6.5. (i) $\text{Inn}(R)$ is always normal in $\text{Aut}_T(R)$ and thus contained in $O_p(\text{Aut}_{\mathcal{F}}(R))$.

(ii) We observed in particular that if $\mathcal{F}$ is a saturated fusion system over a p -group T , then T is $\mathcal{F}$ -radical since $\text{Inn}(T) = \text{Aut}_T(T) \in \text{Syl}_p(\text{Aut}_{\mathcal{F}}(T))$.

(iii) Let $\mathcal{F} = \mathcal{F}_T(G)$ be the fusion system of a finite group G , where T is a Sylow p -subgroup of G . We note that $N_G(R)/RC_G(R) \cong \text{Aut}_{\mathcal{F}}(R)/\text{Inn}(R)$. Thus a subgroup R of T is $\mathcal{F}_T(G)$ -radical if and only if $O_p(N_G(R)/RC_G(R)) = 1$.

Definition 3.6.6. A subgroup $R \leq T$ is $\mathcal{F}$ -essential if R is

- (i) fully normalized in $\mathcal{F}$ and $\mathcal{F}$ -centric and
- (ii) $\text{Out}_{\mathcal{F}}(R) = \text{Aut}_{\mathcal{F}}(R)/\text{Inn}(R)$ contains a strongly p -embedded subgroup.

Again, for the fusion system $\mathcal{F}_T(G)$ of a finite group G , a subgroup R of T is essential if and only if R is $\mathcal{F}_T(G)$ -centric and fully normalized and $N_G(R)/RC_G(R)$ has a strongly p -embedded subgroup.

Proposition 3.6.7. *Every $\mathcal{F}$ -essential subgroup is $\mathcal{F}$ -radical.*

Proof. Let R be $\mathcal{F}$ -essential subgroup. Then $\text{Out}_{\mathcal{F}}(R)$ contains a strongly p -embedded subgroup. By Proposition 2.5.4(b), $O_p(\text{Out}_{\mathcal{F}}(R)) = 1$ and it follows that $O_p(\text{Aut}_{\mathcal{F}}(R)) = \text{Inn}(R)$. Therefore, R is $\mathcal{F}$ -radical. $\square$

We now state the Alperin-Goldschmidt fusion theorem.

Theorem 3.6.8 (Alperin-Goldschmidt fusion theorem). *Let $\mathcal{F}$ be a saturated fusion system over a p -group T . Then $\mathcal{F} = \langle \text{Aut}_{\mathcal{F}}(R) \mid R = T \text{ or } R \text{ is } \mathcal{F}\text{-essential} \rangle_T$.*

Proof. See [8, Theorem 3.5]. $\square$

Corollary 3.6.9. *Let $\mathcal{F}$ be a saturated fusion system over a p -group T . Suppose that $\text{Aut}_{\mathcal{F}}(R)$ is a p -group for each $R \leq T$. Then $\mathcal{F}$ is the fusion system of the group T .*

Proof. As all the automorphism groups are p -groups, $O_p(\text{Aut}_{\mathcal{F}}(R)) \neq \text{Inn}(R)$ for $R < T$. Therefore, there are no $\mathcal{F}$ -essential subgroups. Hence by Theorem 3.6.8, $\mathcal{F}$ is generated by $\text{Aut}_{\mathcal{F}}(T) = \text{Inn}(T)$. Thus, $\mathcal{F} = \mathcal{F}_T(T)$. $\square$

We let $\mathcal{F}^e$ denote the set of essential subgroups of $\mathcal{F}$. The following lemma determines when two fusion systems are isomorphic.

Lemma 3.6.10. *Suppose $\mathcal{F}$ is a saturated fusion system over a p -group S and $\mathcal{T}$ is a saturated fusion system over a p -group T . Let $\alpha : S \rightarrow T$ be a group isomorphism such that $\{P\alpha : P \in \mathcal{F}^e\} = \mathcal{T}^e$ and, for every $P \in \mathcal{F}^e \cup \{S\}$, $\text{Aut}_{\mathcal{F}}(P)^\alpha := \{\beta^\alpha : \beta \in \text{Aut}_{\mathcal{F}}(P)\} = \text{Aut}_{\mathcal{T}}(P\alpha)$. Then α induces an isomorphism from $\mathcal{F}$ to $\mathcal{T}$.*

Proof. This follows from Theorem 3.6.8. □

3.7 An application of Alperin-Goldschmidt fusion theorem

In this section we will demonstrate the importance of Theorem 3.6.8 in classifying the fusion systems on a given p -group. For the rest of this section, we assume S is a dihedral group of order 8 and $\mathcal{F}$ is a saturated fusion system over S . We will be interested in finding all saturated fusion systems over the dihedral group of order 8.

Lemma 3.7.1. *For a fusion system $\mathcal{F}$ over S , R is an $\mathcal{F}$ -essential subgroup of S if and only if R is an elementary abelian group of order 4 and $\text{Aut}_{\mathcal{F}}(R) = \text{Aut}(R) \cong S_3$.*

Proof. Suppose $R \leq S$ is essential in $\mathcal{F}$. Then R as a subgroup of S is either dihedral, cyclic or a Klein 4-group. Since $\text{Out}_{\mathcal{F}}(R)$ has a strongly 2-embedded subgroup, $\text{Aut}(R)$ is not a 2-group. Therefore, R is not a dihedral or cyclic subgroup since their automorphism groups are 2-groups. Hence R is a Klein 4-group, which is an elementary abelian group. By Lemma 2.1.17, $\text{Aut}_{\mathcal{F}}(R) = \text{Aut}(R) \cong S_3$.

Conversely, suppose R is an elementary abelian subgroup of order 4 and $\text{Aut}_{\mathcal{F}}(R) = \text{Aut}(R) \cong S_3$. Then R is normal in S and so R is fully normalized. As R is abelian, $Z(S) = C_S(R) = R$ and so R is $\mathcal{F}$ -centric. Since $\text{Inn}R = \{(1)\}$, $\text{Out}_{\mathcal{F}}(R) \cong \text{Aut}_{\mathcal{F}}(R) \cong S_3$ and by Lemma 2.5.3 $\text{Out}_{\mathcal{F}}(R)$ contains a strongly p -embedded subgroup. Therefore R is $\mathcal{F}$ -essential. □

Proposition 3.7.2. *Let S_4 be the Symmetric group of degree 4 with subgroups;*

$$T := \langle (1234), (13) \rangle, V_1 := \langle (13), (24) \rangle, V_2 := \langle (12)(34), (13)(24) \rangle.$$

Let $\mathcal{F}_1 := \mathcal{F}_T(S_4)$. Then

- (a) T is dihedral of order 8, $T \in \text{Syl}_2(S_4)$ and V_1 and V_2 are the two elementary abelian subgroups of T of order 4.
- (b) V_2 is normal in S_4 and $\text{Aut}_{\mathcal{F}_1}(V_2)/\text{Inn}(V_2) \cong \text{Aut}_{\mathcal{F}_1}(V_2) = \text{Aut}(V_2) \cong S_3$.

(c) V_2 is the only essential subgroup of $\mathcal{F}_1$.

Proof. (a) The generators of T satisfy the following relations; $r^4 = s^2 = 1$ and $r^s = r^3$, where r and s are generators of D_8 . We have that

$$(1234)^2 = (13)(24) \quad \text{and} \quad (13)^2 = (1).$$

Moreover,

$$(1234)^3 = (13)(24)(1234) = (1432) \quad \text{and} \quad (1234)^4 = (1432)(1234) = (1).$$

Also,

$$(1234)^{(13)} = (3214) = (1432) = (1234)^3.$$

So, $T \cong D_8$ and $|T| = 8$. Since $|S_4| = 24 = 2^3 \cdot 3$, we get that $T \in \text{Syl}_2(S_4)$. As V_1 and V_2 are generated by elements of T , we have that they are subgroups of T . We note that V_1 and V_2 are generated by two distinct commuting involutions and thus, are Klein 4-groups. Therefore V_1 and V_2 are elementary abelian 2-groups of order 4.

(b) Two elements of S_4 are conjugate if they are of the same cycle type. Thus, conjugation in S_4 does not change the cycle type. So, we have that $V_2^g \subseteq V_2$ for all $g \in S_4$, since V_2 contains all the cycle types of $(2, 2)$. Thus, $V_2 \trianglelefteq S_4$ and notice that $\text{Aut}(V_2) \cong S_3$. Again, since $V_2 \trianglelefteq S_4$, we get that $N_{S_4}(V_2) = S_4$ and $C_{S_4}(V_2) = V_2$ for V_2 being abelian. So, $\text{Aut}_{\mathcal{F}_1}(V_2) = N_{S_4}(V_2)/C_{S_4}(V_2) \cong S_3$. Thus, $\text{Aut}_{\mathcal{F}_1}(V_2) = \text{Aut}(V_2)$. Also, $\text{Inn}(V_2) = \{(1)\}$, since V_2 is abelian. Therefore, $\text{Out}_{\mathcal{F}_1}(V_2) = \text{Aut}_{\mathcal{F}_1}(V_2)/\text{Inn}(V_2) \cong \text{Aut}_{\mathcal{F}_1}(V_2) = \text{Aut}(V_2) \cong S_3$.

(c) By (a), all essential subgroups in T are elementary abelian and T has two elementary abelian subgroups, V_1 and V_2 . By (b) and Lemma 3.7.1, V_2 is an essential subgroup of $\mathcal{F}_1$.

Claim. V_1 is not $\mathcal{F}_1$ -essential. Note that

$$N_{S_4}(V_1) = \{(1), (13), (24), (13)(24), (14)(23), (12)(34), (1234), (1432)\} = T \quad \text{and}$$

$$C_{S_4}(V_1) = \{(1), (13), (24), (13)(24)\} = V_1.$$

So, $\text{Aut}_{\mathcal{F}_1}(V_1) = N_{S_4}(V_1)/C_{S_4}(V_1) \cong \mathbb{Z}_2$. Therefore, $\text{Aut}_{\mathcal{F}}(V_1)$ is a 2-group and hence, cannot be an essential subgroup.

Thus, V_2 is the only essential subgroup in $\mathcal{F}_1$. □

Proposition 3.7.3. Let A_6 be the Alternating group of degree 6 with subgroups; $R := \langle (1234)(56), (24)(56) \rangle$, $U_1 := \langle (12)(34), (13)(24) \rangle$, $U_2 := \langle (24)(56), (13)(56) \rangle$, $X_1 := \langle (123), (23)(56) \rangle$, $X_2 := \langle (125)(346), (25)(46) \rangle$. Let $\mathcal{F}_2 := \mathcal{F}_R(A_6)$. Then

- (a) R is dihedral of order 8 and $R \in Syl_2(A_6)$.
- (b) For $i = 1, 2$; $X_i \leq N_{A_6}(U_i)$, $C_{X_i}(U_i) = 1$, and $Aut(U_i) = Aut_{\mathcal{F}_2}(U_i) \cong X_i \cong S_3$.
- (c) U_1 and U_2 are the only essentials in $\mathcal{F}_2$.

Proof. (a) Let $a = (1234)(56)$ and $b = (24)(56)$. Then

$$a^2 = (1234)(56)(1234)(56) = (13)(24) \text{ and } b^2 = (24)(56)(24)(56) = (1)$$

$$a^3 = (13)(24)(1234)(56) = (1432)(56) \text{ and } a^4 = ((13)(24))^2 = (1)$$

So, $|a| = 4$ and $|b| = 2$. Moreover $a^b = (1432)(56) = a^3$. Therefore,

$$R = \langle a, b \mid a^4 = b^2 = 1, a^b = a^3 \rangle$$

and $R \cong D_8$ is of order 8. Thus $R \in Syl_2(A_6)$.

(b) Since the generators of U_1 and U_2 are commuting involutions, we have that U_1, U_2 are elementary abelian 2-groups of order 4. We note that $((123))^3 = ((23)(56))^2 = (1)$ and $(123)^{(23)(56)} = (123)^2$. So $X_1 \cong S_3$ (twisted S_3 in A_6). Moreover, we have

$$((125)(346))^3 = ((25)(46))^2 = (1) \text{ and } ((125)(346))^{(25)(46)} = ((125)(346))^2.$$

Thus $X_2 \cong S_3$ (diagonal S_3 in A_6).

We want to show that X_i is a subgroup of $N_{A_6}(U_i)$ for each $i = 1, 2$. Observe

$$((12)(34))^{(123)} = (23)(14) \in U_1, ((12)(34))^{(23)(56)} = (13)(24) \in U_1,$$

$$((13)(24))^{(123)} = (12)(34) \in U_1, \text{ and } ((13)(24))^{(23)(56)} = (12)(34) \in U_1.$$

Since $(12)(34)$ and $(13)(24)$ are generators of U_i , we have that $(123), (23)(56) \in N_{A_6}(U_1)$. Again, since $(123), (23)(56)$ are generators of X_1 and $N_{A_6}(U_1)$ is a subgroup, the elements generated by (123) and $(23)(56)$ are in $N_{A_6}(U_1)$. Therefore, $X_1 \leq N_{A_6}(U_1)$.

Likewise, for X_2 , we have

$$((24)(56))^{(123)(346)} = (56)(13) \in U_2, ((24)(56))^{(25)(46)} = (56)(24) \in U_2,$$

$$((13)(56))^{(123)(346)} = (13)(24) \in U_2, \text{ and } ((13)(56))^{(23)(56)} = (13)(24) \in U_2.$$

Thus $(125)(346), (25)(46) \in N_{A_6}(U_2)$. Therefore, $X_2 \leq N_{A_6}(U_2)$, since the elements generated by $(125)(346)$ and $(25)(46)$ are also in $N_{A_6}(U_2)$.

Now, $C_{X_1}((1)) = X_1$

$C_{X_2}((1)) = X_2$

$$C_{X_1}((12)(34)) = \{(1), (12)(56)\}$$

$$C_{X_2}((24)(56)) = \{(1), (25)(46)\}$$

$$C_{X_1}((13)(24)) = \{(1), (13)(56)\}$$

$$C_{X_2}((13)(56)) = \{(1), (15)(36)\}$$

$$C_{X_1}((14)(23)) = \{(1), (23)(56)\}$$

$$C_{X_2}((13)(24)) = \{(1), (12)(34)\}$$

We have $C_{X_i}(U_i) \trianglelefteq X_i$ and every non-trivial normal subgroup of S_3 contains the unique subgroup of order 3. Hence, the above calculations show $C_{X_i}(U_i) = 1$ for $i = 1, 2$. So $X_i/C_{X_i}(U_i) \cong X_i \cong S_3$, for $i = 1, 2$ embeds into $Aut_{\mathcal{F}_2}(U_i) \leq Aut(U_i) \cong S_3$. So $Aut_{X_i}(U_i) = Aut_{\mathcal{F}_2}(U_i) = Aut(U_i) \cong S_3$.

(c) Following from (b) and Lemma 3.7.1, U_i for $i = 1, 2$ are the only $\mathcal{F}_2$ -essential subgroups. $\square$

Lemma 3.7.4. *Let $\mathcal{F}$ be a saturated fusion system over the dihedral group S . Then $\mathcal{F} = \mathcal{F}_S(S)$ or $\mathcal{F} \cong \mathcal{F}_1$ or $\mathcal{F} \cong \mathcal{F}_2$, where $\mathcal{F}_1$ and $\mathcal{F}_2$ are as in Proposition 3.7.2 and Proposition 3.7.3.*

Proof. We have that S has exactly two subgroups that are elementary abelian of order 4, say, K_1 and K_2 . By Lemma 3.7.1, K_i are the only subgroups which could be $\mathcal{F}$ -essential. That is, we can have both the K_i , exactly one of them or neither of them, being essential. Therefore, there are three possibilities for $\mathcal{F}$ being a saturated fusion system over S , depending on whether $\mathcal{F}$ is generated by automorphisms of both K_i , exactly one of the K_i or neither of the K_i . Also, we have that $Aut_S(K_i)$ has order 2 and $Aut_S(K_i) \leq Aut_{\mathcal{F}}(K_i) \leq Aut(K_i)$. Thus, $|Aut_{\mathcal{F}}(K_i)| = 2$ or 6.

If $|Aut_{\mathcal{F}}(K_i)| = 2$, for $i = 1, 2$, then $Aut_{\mathcal{F}}(K_i) = Aut_S(K_i)$ is a 2-group. Therefore there is no $\mathcal{F}$ -essential subgroup. By Theorem 3.6.8,

$$\mathcal{F} = \langle Aut_{\mathcal{F}}(S) \rangle = \langle Inn(S) \rangle \text{ and } \mathcal{F} \cong \mathcal{F}_S(S).$$

If $|Aut_{\mathcal{F}}(K_i)| = 6$, it could be for both or exactly one of the K_i .

Case 1: If $|Aut_{\mathcal{F}}(K_i)| = 6$, for exactly one of the K_i , without loss of generality say K_1 . Then by Theorem 3.6.8,

$$\mathcal{F} = \langle Aut_{\mathcal{F}}(K_1), Aut_{\mathcal{F}}(S) \rangle$$

By Theorem 2.1.19, there exist a group isomorphism $\alpha : S \rightarrow T$ such that $K_1\alpha = V_2$. We also have $Aut_{\mathcal{F}}(S) = Inn(S)$ and $Aut_{\mathcal{F}}(T) = Inn(T)$ as S, T are isomorphic to D_8 . Therefore, $Aut_{\mathcal{F}}(S)^{\alpha} = Aut_{\mathcal{F}_1}(T)$.

We have that the restriction $\alpha|_{K_1}$ induces an isomorphism $\phi : Aut_{\mathcal{F}}(K_1) \rightarrow Aut_{\mathcal{F}_1}(V_2)$ defined by $\beta\phi = \beta^{\alpha} = \alpha^{-1}\beta\alpha$. As $Aut_{\mathcal{F}}(K_1) = Aut(K_1)$ and $Aut_{\mathcal{F}_1}(V_2) = Aut(V_2)$, $Aut_{\mathcal{F}}(K_1)^{\alpha} := \{\beta^{\alpha} : \beta \in Aut_{\mathcal{F}}(K_1)\} = Aut_{\mathcal{F}_1}(V_2)$. By Lemma 3.6.10 and Proposition 3.7.2, $\mathcal{F} \cong \mathcal{F}_1$.

Case 2: If $|Aut_{\mathcal{F}}(K_i)| = 6$, for both of K_i . Then by Theorem 3.6.8

$$\mathcal{F} = \langle Aut_{\mathcal{F}}(K_1), Aut_{\mathcal{F}}(K_2), Aut_{\mathcal{F}}(S) \rangle$$

By Theorem 2.1.19, there exist $\alpha : S \longrightarrow T$ an isomorphism, such that $K_1\alpha = U_1$ and $K_2\alpha = U_2$. The isomorphism α induces an isomorphism $\Phi : \text{Aut}_{\mathcal{F}}(S) \longrightarrow \text{Aut}_{\mathcal{F}_2}(T)$ defined by $\beta\Phi = \beta^\alpha = \alpha^{-1}\beta\alpha$. Subsequently, the restricted isomorphism $\alpha|_{K_i}$ induces an isomorphism $\phi : \text{Aut}_{\mathcal{F}}(K_i) \longrightarrow \text{Aut}_{\mathcal{F}_2}(U_i)$ defined by $\beta_i\phi = \beta_i^\alpha = \alpha^{-1}\beta_i\alpha$, for $i = 1, 2$. As in case 1, we have that for every $K_i \in E(\mathcal{F})$, $\text{Aut}_{\mathcal{F}}(K_i)^\alpha := \{\beta^\alpha : \beta \in \text{Aut}_{\mathcal{F}}(K_i)\} = \text{Aut}_{\mathcal{F}_2}(U_i)$, for $i = 1, 2$. Hence by Lemma 3.6.10 and Proposition 3.7.3, $\mathcal{F} \cong \mathcal{F}_2$. $\square$

3.8 Strongly closed and normal subgroups of a fusion system

In this section we will introduce the concepts of weakly and strongly $\mathcal{F}$ -closed subgroups. We build on these notions to define central and normal subgroups of a fusion system. For any fusion system $\mathcal{F}$, we will demonstrate the relationship between all these subgroups.

Definition 3.8.1. Let $\mathcal{F}$ be a fusion system over a p -group T , and fix $P \leq T$.

- (a) P is *weakly closed* in $\mathcal{F}$ if $P^{\mathcal{F}} = \{P\}$.
- (b) P is *strongly closed* in $\mathcal{F}$ if, whenever $Q \leq P$, and $\phi \in \text{Hom}_{\mathcal{F}}(Q, T)$, then $Q\phi \leq P$.

It is clear from the above definition that every strongly closed subgroup in $\mathcal{F}$ is weakly closed in $\mathcal{F}$.

Example 3.8.2. Consider the fusion system $\mathcal{F}_1 = \mathcal{F}_T(S_4)$ as in Example 3.1.4 and Proposition 3.7.2. Then T, V_1, V_2 and V_3 are weakly $\mathcal{F}_1$ -closed subgroups since each is $\mathcal{F}_1$ -conjugate only to itself. Moreover, each subgroup P_i of order two of V_1, V_2 and V_3 respectively are $\mathcal{F}_1$ -conjugate to other subgroups P_j of order two of V_1, V_2 and V_3 respectively. Therefore T is strongly $\mathcal{F}_1$ -closed. In particular V_2 is strongly $\mathcal{F}_1$ -closed since V_2 is a normal subgroup of S_4 . However, V_1 and V_3 are not strongly $\mathcal{F}_1$ -closed since for P_3 of V_1 and V_3 , $P_3\phi$ is not in V_1 nor in V_3 for some non-trivial $\phi \in \text{Hom}_{\mathcal{F}_1}(P_3, T)$.

Definition 3.8.3. Let $\mathcal{F}$ be a fusion system over a p -group T , and fix $R \leq T$.

- (a) R is *central* in $\mathcal{F}$ if $R \trianglelefteq T$ and for all $P, Q \leq T$ and all $\phi \in \text{Hom}_{\mathcal{F}}(P, Q)$, ϕ extends to a morphism $\tilde{\phi} \in \text{Hom}_{\mathcal{F}}(PR, QR)$ such that $\tilde{\phi}|_R = \text{Id}_R$.
- (b) R is *normal* in $\mathcal{F}$ denoted as $R \trianglelefteq \mathcal{F}$, if $R \trianglelefteq S$ and for all $P, Q \leq T$ and all $\phi \in \text{Hom}_{\mathcal{F}}(P, Q)$, ϕ extends to a morphism $\tilde{\phi} \in \text{Hom}_{\mathcal{F}}(PR, QR)$ such that $\tilde{\phi}(R) = R$.

Thus R being central in $\mathcal{F}$ implies R is normal in $\mathcal{F}$.

Proposition 3.8.4. *Let $\mathcal{F}$ be a fusion system over a finite p -group T , and let R be a subgroup of T . If R is normal in $\mathcal{F}$, then R is strongly closed in $\mathcal{F}$.*

Proof. Let $P \leq R$, and let $\phi \in \text{Hom}_{\mathcal{F}}(P, Q)$ be an isomorphism. Then ϕ extends to $\tilde{\phi} \in \text{Hom}_{\mathcal{F}}(PR, QR)$ such that $\tilde{\phi}(R) = R$. So $\tilde{\phi}(P) = \phi(P) = Q \leq R$. Thus R is strongly closed in $\mathcal{F}$. $\square$

Proposition 3.8.5. *Let $\mathcal{F}$ be a saturated fusion system over a finite p -group T . Then for any subgroup Q of T , Q is normal in $\mathcal{F}$ if and only if there exist a series $1 = Q_0 \leq Q_1 \leq \dots \leq Q_n = Q$ of subgroups such that*

- (a) Q_i is strongly closed in $\mathcal{F}$ for every $1 \leq i \leq n$, and
- (b) $[Q, Q_i] \leq Q_{i-1}$ for every $1 \leq i \leq n$.

Proof. See [8, Part I, Proposition 4.6]. $\square$

Following from Definitions 3.8.1, 3.8.3, and Proposition 3.8.4, we have the following implication for any fusion system $\mathcal{F}$ on a finite p -group S :

$$R \text{ central} \implies R \text{ normal} \implies R \text{ strongly closed} \implies R \text{ weakly closed} \implies R \trianglelefteq T.$$

Lemma 3.8.6. *Let G be a finite group and let T be a Sylow p -subgroup of G . If R is a normal p -subgroup of G , then $R \trianglelefteq \mathcal{F}_T(G)$.*

Proof. Let R be a normal p -subgroup of G . Let $P, Q \leq T$ and $\phi \in \text{Hom}_{\mathcal{F}_T(G)}(P, Q)$ be an isomorphism. Then there exist $g \in G$ such that $P^g = Q$ and $\phi = c_g|_{P, Q}$. Let $\tilde{\phi} := c_g|_{PR, QR} \in \text{Hom}_{\mathcal{F}_T(G)}(PR, QR)$. Then $\tilde{\phi}|_{P, Q} = \phi$ and $R\tilde{\phi} = R^g = R$ since R is normal in G . Thus R is normal in $\mathcal{F}_T(G)$. $\square$

Proposition 3.8.7. *Let G be a finite group and let T be a Sylow p -subgroup of G . Let Q be a subgroup of T .*

- (a) Q is weakly closed in $\mathcal{F}_T(G)$ if and only if it is weakly closed in T with respect to G .
- (b) Q is strongly $\mathcal{F}_T(G)$ -closed if and only if it is strongly closed in T with respect to G .

Proof. (a) Suppose Q is weakly closed in $\mathcal{F}_T(G)$. Then Q is the only $\mathcal{F}_T(G)$ conjugate of itself. Thus, $Q^g = Q$ for any $g \in G$ with $Q^g \leq T$. Therefore Q is weakly closed in T with respect to G . Conversely, suppose Q is weakly closed in T with respect to G . Then $Q^g = Q$ for any $g \in G$ with $Q^g \leq T$. This implies that Q is the only $\mathcal{F}_T(G)$ conjugate of itself. So, Q is weakly closed in $\mathcal{F}_T(G)$.

(b) Suppose Q is strongly closed in $\mathcal{F}_T(G)$. Then for any $R \leq Q$ and $\phi \in \text{Hom}_{\mathcal{F}_T(G)}(R, T)$, $R\phi \leq Q$. But ϕ is induced by conjugation with element in G . Thus, ϕ is strongly closed in T with respect to G . Conversely, if Q is strongly closed in T with respect to G , then for any $R \leq Q$, $R^g \leq Q$ for any $g \in G$ with $R^g \leq T$. Therefore Q is strongly $\mathcal{F}_T(G)$ -closed since the morphisms in $\mathcal{F}_T(G)$ are induced by conjugation with elements in G . $\square$

Lemma 3.8.8. [14, Lemma 4.59] *Let $\mathcal{F}$ be the fusion system over a finite p -group T and let P, Q be subgroups of T .*

(a) *If P and Q are weakly closed in $\mathcal{F}$, then so is PQ .*

(b) *If P and Q are strongly $\mathcal{F}$ -closed, then so is $P \cap Q$.*

Proof. (a) Suppose P and Q are weakly closed in $\mathcal{F}$. Let $\Phi \in \text{Hom}_{\mathcal{F}}(PQ, S)$. Then the restrictions $\phi := \Phi|_P$ and $\psi := \Phi|_Q$ are morphisms in $\mathcal{F}$. Since P and Q are weakly closed in $\mathcal{F}$, we have that $P\phi = P$ and $Q\psi = Q$. Therefore, since Φ is a homomorphism, $(PQ)\Phi = (P\Phi)(Q\Phi) = PQ$. Hence PQ is weakly closed in $\mathcal{F}$.

(b) Suppose P and Q are strongly closed in $\mathcal{F}$. Let $R \leq P \cap Q$ and $\phi \in \text{Hom}_{\mathcal{F}}(R, S)$. Then $R\phi \leq P$ and $R\phi \leq Q$ since P and Q are strongly closed in $\mathcal{F}$. Therefore $R\phi \leq P \cap Q$. Hence $P \cap Q$ is strongly closed in $\mathcal{F}$. $\square$

Theorem 3.8.9. *For every saturated fusion system $\mathcal{F}$, the product of two strongly closed subgroups in $\mathcal{F}$ is strongly closed in $\mathcal{F}$.*

Proof. See [14, Theorem 5.22]. $\square$

Lemma 3.8.10. *Let $\mathcal{F}$ and $\mathcal{T}$ be fusion systems over finite p -groups. Suppose $\Omega : \mathcal{F} \rightarrow \mathcal{T}$ is a morphism of fusion systems. Then the kernel of Ω is strongly $\mathcal{F}$ -closed.*

Proof. Let $\Omega = (\alpha, \alpha_{P,Q})$ be the morphism between $\mathcal{F}$ and $\mathcal{G}$ where $\alpha : S \rightarrow T$ is a group homomorphism and $\alpha_{P,Q} : \text{Hom}_{\mathcal{F}}(P, Q) \rightarrow \text{Hom}_{\mathcal{G}}(P\alpha, Q\alpha)$ is the corresponding map on morphisms for all $P, Q \leq S$. Let $M = \ker \Omega = \ker \alpha$ and so $M \leq S$. Suppose $P' \leq M$ and $\phi \in \text{Hom}_{\mathcal{F}}(P', Q')$ is an isomorphism. Since $P' \leq \ker \alpha$, the composite $\phi\alpha : P' \rightarrow Q'\alpha$ is the map sending every element of P' to the identity of $Q'\alpha$. As ϕ is an isomorphism, we have $Q'\alpha = 1$. Thus $P'\phi = Q' \leq M$. $\square$

Definition 3.8.11. Let $\mathcal{F}$ be a fusion system over a finite p -group T ,

(a) the largest subgroup of T which is normal in $\mathcal{F}$ is denoted as $O_p(\mathcal{F})$,

(b) the largest subgroup of T which is central in $\mathcal{F}$ is denoted as $Z(\mathcal{F})$.

We note that $O_p(\mathcal{F}) \trianglelefteq T$ and $Z(\mathcal{F}) \leq Z(T)$

The following theorem directly follows from the famous Glauberman's Z^* -theorem [16].

Theorem 3.8.12. *Suppose G is a finite group with $O(G) = 1$, and let T be a Sylow 2-subgroup of G . Then $Z(\mathcal{F}_T(G)) = Z(G)$.*

Proof. Notice that $Z(G)$ is a 2-group as $O(G) = 1$. It follows that $Z(G) \leq T$. By applying [16, Corollary 1] we get that $Z(\mathcal{F}_T(G)) = Z(G) \cap T = Z(G)$. $\square$

Proposition 3.8.13. *Let $\mathcal{F}$ be a saturated fusion system over a p -group T .*

- (a) *An abelian subgroup $R \leq T$ is normal in $\mathcal{F}$ if and only if R is strongly $\mathcal{F}$ -closed.*
- (b) *$O_p(\mathcal{F}) \neq 1$ if and only if there is a non-trivial abelian subgroup of T which is strongly $\mathcal{F}$ -closed.*

Proof. See [8, Corollary 4.7]. $\square$

Lemma 3.8.14. *Let G be a finite group and T a Sylow p -subgroup of G such that $\mathcal{F} = \mathcal{F}_T(G)$. Suppose one of the following holds:*

- i. *T is abelian or*
- ii. *T is a TI-subgroup of G .*

Then $\mathcal{F} = \mathcal{F}_T(N_G(T))$ and $T = O_p(\mathcal{F})$.

Proof. See [8, Part II, Lemma 11.4]. $\square$

3.9 Factor/Quotient fusion systems

For any fusion system $\mathcal{F}$ over a p -group T with some strongly closed $P \leq T$, we can construct a subsystem $\mathcal{F}/P$ as follows:

Definition 3.9.1. Let $\mathcal{F}$ be a fusion system over a p -group T and P a strongly $\mathcal{F}$ -closed subgroup of T . The factor/quotient system $\mathcal{F}/P$ is the category with objects all subgroups of T/P and for any two subgroups Q and R containing P , we have that $\text{Hom}_{\mathcal{F}/P}(Q/P, R/P)$ is the set of homomorphisms ϕ induced from the set $\text{Hom}_{\mathcal{F}}(Q, R)$.

Proposition 3.9.2. *Let $\mathcal{F}$ be a fusion system over a finite p -group T . If P is a strongly $\mathcal{F}$ -closed subgroup of T , then the category $\mathcal{F}/P$ is a fusion system over T/P .*

Proof. The first axiom of a fusion system is satisfied, since conjugation by Pg is induced from conjugation with $g \in T$, taken modulo P . Secondly, if $\Phi \in \text{Hom}_{\mathcal{F}/P}(Q/P, R/P)$ then there exists $\phi \in \text{Hom}_{\mathcal{F}}(Q, R)$ which induces Φ and the corresponding core $\phi|_{Q\phi} \in \text{Hom}_{\mathcal{F}}(Q, Q\phi)$ which is an $\mathcal{F}$ -isomorphism induces an $\mathcal{F}$ -isomorphism in $\mathcal{F}/P$ corresponding to the core of Φ . Finally, if $\Phi \in \text{Hom}_{\mathcal{F}/P}(Q/P, R/P)$ is an $\mathcal{F}/P$ -isomorphism, then it is induced by some isomorphism $\phi \in \text{Hom}_{\mathcal{F}}(Q, R)$, and the inverse of ϕ also induces the inverse of Φ . Therefore $\mathcal{F}/P$ is a fusion system over T/P . $\square$

Proposition 3.9.3. *Let $\mathcal{F}$ be a saturated fusion system over a finite p -group T . If P is a strongly $\mathcal{F}$ -closed subgroup of T , then the fusion system $\mathcal{F}/P$ is saturated.*

Proof. See [8, Part II, Lemma 5.5] $\square$

Theorem 3.9.4. *Let G be a finite group, p is a prime, and T a Sylow p -subgroup of G . Suppose N is a normal subgroup of G and let $\bar{G} = G/N$. Then $\mathcal{F}_T(G)/(N \cap T) \cong \mathcal{F}_{\bar{T}}(\bar{G})$.*

Proof. Fix $\mathcal{F} := \mathcal{F}_T(G)$ and observe that $N \cap T$ is strongly closed in $\mathcal{F}$. Set $P := N \cap T$, $\mathcal{F}_1 := \mathcal{F}/P$ and $\mathcal{F}_2 = \mathcal{F}_{\bar{T}}(\bar{G})$. We have that the map

$$\alpha : T/P \rightarrow \bar{T} \text{ defined as } tP \mapsto tN = \bar{t}$$

is an isomorphism. So both $\mathcal{F}_1$ and $\mathcal{F}_2$ are fusion systems on T/P . Let Q and R be subgroups of T containing P , and let $\phi \in \text{Hom}_{\mathcal{F}}(Q, R)$ where $\phi = c_g$ for some $g \in G$. Then we need to argue that for induced map $\bar{\phi} \in \text{Hom}_{\mathcal{F}_1}(Q/P, R/P)$ we have $\alpha^{-1}\bar{\phi}\alpha \in \text{Hom}_{\mathcal{F}_2}(\bar{Q}, \bar{R})$. Let $q \in Q$. Then $\bar{q} = (qP)\alpha$. So

$$\bar{q}\alpha^{-1}\bar{\phi}\alpha = (qP)\bar{\phi}\alpha = (q^gP)\alpha = \bar{q}^g = \bar{q}^{\bar{g}} = \bar{q}c_{\bar{g}}.$$

Thus $\alpha^{-1}\bar{\phi}\alpha = c_{\bar{g}} \in \mathcal{F}_2$. On the other hand if $\psi \in \text{Hom}_{\mathcal{F}_2}(\bar{Q}, \bar{R})$, then $\psi = c_{\bar{h}}$ for some $h \in G$. It follows that $(QN)^h \leq RN$. As $P \leq R$, we have that $R \in \text{Syl}_p(RN)$. As Q^h is a p -group, by Sylow's theorem (see Theorem 2.1.5) there exist $x \in RN = NR$ with $Q^{hx} \leq R$. We may assume $x \in N$. So $\bar{h} = \bar{h}x$ and we may assume without loss of generality $Q^h \leq R$. Then as seen above for g , $\alpha^{-1}c_{\bar{h}}\alpha = c_{\bar{h}} = \psi$, showing that the map from $\text{Hom}_{\mathcal{F}_1}(Q/P, R/P)$ to $\text{Hom}_{\mathcal{F}_2}(\bar{Q}, \bar{R})$ defined by $\bar{\phi} \mapsto \alpha^{-1}\bar{\phi}\alpha$ is well-defined and surjective. Clearly it is injective. As α is an isomorphism so is the map from $\mathcal{F}_1$ to $\mathcal{F}_2$. $\square$

Corollary 3.9.5. *Let G be a finite group, p is a prime, and T a Sylow p -subgroup of G . Let $\bar{G} = G/O_{p'}(G)$. Then the natural map $T \rightarrow \bar{T}$ induces an isomorphism from $\mathcal{F}_T(G)$ to $\mathcal{F}_{\bar{T}}(\bar{G})$.*

Proof. This follows from Theorem 3.9.4. □

Lemma 3.9.6. *Let $\mathcal{F}$ be a fusion system over a finite p -group T . Let R be a strongly $\mathcal{F}$ -closed subgroup of T such that $R \leq P \leq T$. Then*

- (a) $(P/R)^{\mathcal{F}/R} = \{Q/R : Q \in P^{\mathcal{F}}\}$.
- (b) P is fully normalized in $\mathcal{F}$ if and only if P/R is fully normalized in $\mathcal{F}/R$.
- (c) If P/R is centric in $\mathcal{F}/R$ then P is centric in $\mathcal{F}$.

Proof. (a) Let $M = \{Q/R \mid Q \in P^{\mathcal{F}}\}$. Suppose $Q/R \leq T/R$, is $\mathcal{F}/R$ -conjugate to P/R . Then there exists an isomorphism Φ in $\text{Hom}_{\mathcal{F}/R}(P/R, Q/R)$, such that Φ is induced by a morphism, $\phi \in \text{Hom}_{\mathcal{F}}(P, Q)$ with $R\phi = R$. Moreover, $Q = P\phi$ since Φ is an isomorphism. Therefore, $Q \in P^{\mathcal{F}}$ and $(P/R)^{\mathcal{F}/R} \subseteq M$.

Conversely, suppose $Q/R \in M$. Since $Q \in P^{\mathcal{F}}$ and $R \leq Q$, there exist an isomorphism $\phi \in \text{Hom}_{\mathcal{F}}(P, Q)$ such that $R\phi = R$. Then ϕ induces an isomorphism $\Phi \in \text{Hom}_{\mathcal{F}/R}(P/R, Q/R)$. Therefore, $M \subseteq (P/R)^{\mathcal{F}/R}$. Hence $(P/R)^{\mathcal{F}/R} = \{Q/R \mid Q \in P^{\mathcal{F}}\}$.

(b) We will show that $N_{T/R}(P/R) = N_T(P)/R$ for every subgroup $P \leq T$ with $R \leq P$. To see this, note that

$$\begin{aligned} N_{T/R}(P/R) &= \{aR \in T/R \mid (aR)^{-1}(bR)(aR) \in P/R, \forall b \in P\} \\ &= \{aR \in T/R \mid (a^{-1}baR) \in P/R, \forall b \in P\}. \end{aligned}$$

This implies that

$$\begin{aligned} N_{T/R}(P/R) &= \{aR \in T/R \mid a^{-1}ba \in P, \forall b \in P\} \\ &= \{aR \in T/R \mid a \in N_T(P), \} \\ &= N_T(P)/R. \end{aligned}$$

Therefore $|N_{T/R}(P/R)| = |N_T(P)|/|R|$.

Suppose P is fully normalized in $\mathcal{F}$. Let $Q/R \leq T/R$, be $\mathcal{F}/R$ -conjugate to P/R . Then we have that, $|N_{T/R}(Q/R)| = |N_T(Q)|/|R|$. Therefore, $|N_{T/R}(P/R)| \geq |N_{T/R}(Q/R)|$, since $|N_T(P)| \geq |N_T(Q)|$ as Q is $\mathcal{F}$ -conjugate to P . Hence P/R is fully normalized in $\mathcal{F}/R$.

Conversely, suppose P/R is fully normalized in $\mathcal{F}/R$. Then for all $Q/R \in (P/R)^{\mathcal{F}/R}$,

$$|N_{T/R}(P/R)| \geq |N_{T/R}(Q/R)|.$$

This implies that for all $Q/R \in (P/R)^{\mathcal{F}/R}$,

$$\frac{|N_T(P)|}{|R|} \geq \frac{|N_T(Q)|}{|R|}.$$

Thus, $|N_T(P)| \geq |N_T(Q)|$ for all $Q \in P^{\mathcal{F}}$ by (a). Therefore P is fully normalized in $\mathcal{F}$.

(c) Suppose P/R is centric in $\mathcal{F}/R$. Then for all $Q/R \in (P/R)^{\mathcal{F}/R}$, $C_{T/R}(Q/R) \leq Q/R$ and $Q \in P^{\mathcal{F}}$. Now, let $a \in C_T(Q)$, then for every $bR \in Q/R$,

$$\begin{aligned} (aR)^{-1} (bR) (aR) &= (a^{-1}ba) R \\ &= bR. \end{aligned}$$

It follows that $C_T(Q)R/R \leq C_{T/R}(Q/R)$. This implies that $C_T(Q)R/R \leq Q/R$ and $C_T(Q)R \leq Q$. Thus, $C_T(Q) \leq Q$. Therefore P is centric in $\mathcal{F}$ by (a). □

In general, the converse of Lemma 3.9.6(c) is not true. Take for instance the following example.

Example 3.9.7. Let $\mathcal{F}$ be a fusion system of S_4 over a Sylow 2-subgroup. Let $G = S_4 \times C_2$, $R := O_2(S_4)$, $P := R \times C_2$ and $S' \in \text{Syl}_2(G)$. Thus, $S' \cong D_8 \times C_2$. Let $\mathcal{F}' := \mathcal{F}_{S'}(G)$. Taking $R' = R \times \{1\}$, we get that $R' \trianglelefteq G$ since $R \trianglelefteq S_4$, and $R' \leq P$.

Claim 1. R' is strongly closed in $\mathcal{F}'$.

Since R' is normal in G and by Lemma 3.8.6, R' is normal in $\mathcal{F}'$. In particular R' is strongly $\mathcal{F}'$ -closed by Proposition 3.8.4.

Claim 2. P is $\mathcal{F}'$ -centric.

We have that $P^{\mathcal{F}'} = \{P\}$, since P is normal in G . Therefore, $C_{S'}(P) \leq P$. So, P is $\mathcal{F}'$ -centric.

Claim 3. P/R' is not $\mathcal{F}'/R'$ -centric.

Since D_8/R and $C_2/\{1\}$ are abelian, we get that $S'/R' \cong (D_8 \times C_2)/(R \times \{1\})$ is abelian. This implies that the centralizer of any subgroup in S'/R' is S'/R' itself. Therefore, P/R' which is a proper subgroup of S'/R' does not contain its own centralizer. Hence, P/R' is not $\mathcal{F}'/R'$ -centric.

3.10 Constrained fusion systems

This section deals with an analogue of p -constrained groups in fusion systems.

Definition 3.10.1. Let $\mathcal{F}$ be a saturated fusion system over a finite p -group T . Then $\mathcal{F}$ is said to be *constrained* if there is a normal subgroup $P \trianglelefteq \mathcal{F}$ which is $\mathcal{F}$ -centric. Equivalently, $\mathcal{F}$ is *constrained* if $C_T(O_p(\mathcal{F})) \leq O_p(\mathcal{F})$.

The Definition 3.10.1 is motivated by its analogue in finite group theory, see 2.4.1.

Lemma 3.10.2. Let G be a finite group, $T \in \text{Syl}_p(G)$ and $\mathcal{F} = \mathcal{F}_T(G)$. If $C_G(O_p(G)) \leq O_p(G)$, then $\mathcal{F}$ is constrained.

Proof. By Lemma 3.8.6, $O_p(G) \trianglelefteq \mathcal{F}$ and so $O_p(G) \leq O_p(\mathcal{F})$. It follows that $C_G(O_p(\mathcal{F})) \leq C_G(O_p(G)) \leq O_p(G) \leq O_p(\mathcal{F})$. Thus $\mathcal{F}$ is constrained. $\square$

Definition 3.10.3. A *model* for a constrained $\mathcal{F}$ is a finite group G such that $T \in \text{Syl}_p(G)$, $\mathcal{F}_T(G) = \mathcal{F}$, and $C_G(O_p(G)) \leq O_p(G)$. We denote the class of all finite groups which are models for $\mathcal{F}$ as $\mathcal{G}(\mathcal{F})$.

The following theorem says every constrained fusion system has a model.

Theorem 3.10.4. *If $\mathcal{F}$ is constrained, then $\mathcal{G}(\mathcal{F})$ is non-empty.*

Proof. See [8, Part II, Theorem 4.2]. $\square$

Lemma 3.10.5. *Let $\mathcal{F}$ be constrained. Suppose G is a finite group such that $G \in \mathcal{G}(\mathcal{F})$. Then $O_p(\mathcal{F}) = O_p(G)$.*

Proof. It follows from Lemma 3.8.6 that $O_p(G) \leq O_p(\mathcal{F})$. To prove the other direction, it suffices to show that $O_p(\mathcal{F}) \trianglelefteq G$. Assume $T \in \text{Syl}_p(G)$, and let $g \in G$. As $O_p(\mathcal{F}) \trianglelefteq \mathcal{F}$, we have that $c_g|_{O_p(G)} \in \text{Aut}_{\mathcal{F}}(O_p(G))$ extends to $\psi \in \text{Aut}_{\mathcal{F}}(O_p(\mathcal{F}))$. Since $\mathcal{F} = \mathcal{F}_T(G)$ there exist $h \in G$ such that $\psi = c_h|_{O_p(\mathcal{F})}$. Thus $h \in N_G(O_p(\mathcal{F}))$. As $c_g|_{O_p(G)} = \psi|_{O_p(G)} = c_h|_{O_p(G)}$ we have $gh^{-1} \in C_G(O_p(G)) \leq O_p(G) \leq T \leq N_G(O_p(\mathcal{F}))$. Thus $g \in N_G(O_p(\mathcal{F}))$. As $g \in G$ was arbitrary, we have $O_p(\mathcal{F}) \trianglelefteq G$. $\square$

The next theorem says that models of a constrained fusion systems are isomorphic.

Theorem 3.10.6. *Suppose $\mathcal{F}$ is constrained and let G_1, G_2 be members of $\mathcal{G}(\mathcal{F})$. Then there exists an isomorphism $\Phi : G_1 \rightarrow G_2$ which is an identity on the underlying Sylow p -subgroup T .*

Proof. See [8, Part II, Lemma 4.4]. $\square$

Lemma 3.10.7. *Let $\mathcal{F}$ be a saturated fusion system over an abelian p -group T . Then $\mathcal{F}$ is constrained. In particular, every model of $\mathcal{F}$ is a unique p -constrained group.*

Proof. By Lemma 3.3.7, every morphism in $\mathcal{F}$ extends to an element of $\text{Aut}(T)$, which also lies in $\mathcal{F}$. It follows that $T \trianglelefteq \mathcal{F}$ and $O_p(\mathcal{F}) = T$. Then one easily sees that $C_T(O_p(\mathcal{F})) = C_T(T) \leq T$, and so $\mathcal{F}$ is constrained. The rest of the results follow from Theorem 3.10.4 and Theorem 3.10.6. $\square$

Example 3.10.8. Consider the saturated fusion system $\mathcal{F}$ over the Klein four group T . Then we have $\mathcal{F} = \mathcal{F}_T(A_5) = \mathcal{F}_T(A_4)$. By Lemma 3.10.7, $\mathcal{F}$ is constrained. One easily notice that A_5 is not 2-constrained and so A_5 is not the unique 2-constrained model for $\mathcal{F}$. However, A_4 is 2-constrained and so the unique 2-constrained model for $\mathcal{F}$.

In general, if $\mathcal{F}$ is a saturated fusion systems over an abelian p -group T , and $R = \text{Aut}_{\mathcal{F}}(T)$, then the model of $\mathcal{F}$ is $T \rtimes R$. See for instance [14, Exercise 3.7].

Lemma 3.10.9. *Let G be a non-abelian finite simple group and T a Sylow 2-subgroup of G such that $\mathcal{F} = \mathcal{F}_T(G)$. Then the following are equivalent:*

- (a) $\mathcal{F}$ is constrained.
- (b) $T \leq \mathcal{F}$.
- (c) G is a Goldschmidt group.

Proof. See [8, Part II, Lemma 11.5]. □

3.11 The focal and hyperfocal subgroups of fusion system

This section deals with a saturated subsystem of a saturated fusion system which plays a role analogous to that of a subgroup of a finite group X containing $O^p(G)$. We begin with focal and hyperfocal subgroups of a fusion system whose definitions are motivated by those in Section 2.3 for a finite group. The main reference is [8].

Definition 3.11.1. Let $\mathcal{F}$ be a saturated fusion system over a Sylow p -group T .

- (a) The *focal subgroup* of $\mathcal{F}$ is the subgroup

$$\text{foc}(\mathcal{F}) := \langle g^{-1}h \mid g, h \in T \text{ are } \mathcal{F}\text{-conjugate} \rangle.$$

- (b) The *hyperfocal subgroup* of $\mathcal{F}$ is the subgroup

$$\text{hfp}(\mathcal{F}) := \langle g^{-1}(g\phi) \mid g \in R \leq T, \phi \in O^p(\text{Aut}_{\mathcal{F}}(R)) \rangle.$$

The following lemma describes the relationship between the subgroups $\text{foc}(\mathcal{F})$ and $\text{hfp}(\mathcal{F})$ of any saturated fusion system $\mathcal{F}$.

Lemma 3.11.2. *Let $\mathcal{F}$ be saturated fusion system over a p -group T . Then*

$$\text{foc}(\mathcal{F}) = \text{hfp}(\mathcal{F})[T, T].$$

In particular, $\text{foc}(\mathcal{F}) = T$ if and only if $\text{hfp}(\mathcal{F}) = T$.

Proof. See [8, Lemma 7.2]. □

Definition 3.11.3. Let $\mathcal{F}$ be a saturated fusion system over a Sylow p -group T , and let $\mathcal{E}$ be a subsystem in $\mathcal{F}$ over $P \leq T$. Then $\mathcal{E}$ is said to have *p -power index in $\mathcal{F}$* provided that $\text{hnp}(\mathcal{F}) \leq P$, and $O^p(\text{Aut}_{\mathcal{F}}(Q)) \leq \text{Aut}_{\mathcal{E}}(Q)$ for every $Q \leq P$.

The following theorem describes saturated subsystems of p -power index.

Theorem 3.11.4. *Let $\mathcal{F}$ be a saturated fusion system over a Sylow p -group T . Then for every subgroup $R \leq T$ containing $\text{hnp}(\mathcal{F})$, there is a unique saturated fusion subsystem*

$$\mathcal{F}_R = \langle \text{Inn}(R), O^p(\text{Aut}_{\mathcal{F}}(Q)) \mid Q \leq R \rangle$$

over R of p -power index in $\mathcal{F}$. In particular, there is a unique minimal saturated fusion subsystem $O^p(\mathcal{F}) \trianglelefteq \mathcal{F}$ over $\text{hnp}(\mathcal{F}) \trianglelefteq T$ of p -power index in $\mathcal{F}$.

Proof. See [8, Theorem 7.4]. □

It follows immediately from Theorem 3.11.4 and Lemma 3.11.2 that:

Corollary 3.11.5. *Let $\mathcal{F}$ be a saturated fusion system over a Sylow p -group T . Then*

$$O^p(\mathcal{F}) = \mathcal{F} \Leftrightarrow \text{hnp}(\mathcal{F}) = T \Leftrightarrow \text{foc}(\mathcal{F}) = T.$$

3.12 Normal subsystems

For any saturated fusion system $\mathcal{F}$ we define properties of a subsystem $\mathcal{E} \subseteq \mathcal{F}$ analogous to a normal subgroup of a finite group. We note that there are several different definitions of a normal subsystem of a fusion system. Here we adopt a more restrictive definition due to M. Aschbacher[5]. The definitions due to B. Oliver[48], and M. Linckelmann[42], are equivalent to what we refer to here as weakly normal subsystem. The definition due to L. Puig[52] is also equivalent to what we call here as invariant subsystem.

Definition 3.12.1. Let $\mathcal{F}$ be a fusion system over a finite p -group T . A subsystem $\mathcal{E} \subseteq \mathcal{F}$ over $P \trianglelefteq T$ is said to be *$\mathcal{F}$ -invariant* provided:

- (i) P is strongly closed in $\mathcal{F}$,
- (ii) (Invariance condition) $\mathcal{E}^\alpha = \mathcal{E}$ for all $\alpha \in \text{Aut}_{\mathcal{F}}(P)$, and
- (iii) (Frattni condition) for all $Q \leq P$ and for all $\psi \in \text{Hom}_{\mathcal{F}}(Q, P)$ there exists $\gamma \in \text{Aut}_{\mathcal{F}}(P)$ and $\psi_o \in \text{Hom}_{\mathcal{E}}(Q, P)$ such that $\psi = \psi_o \circ \gamma$

Definition 3.12.2. Let $\mathcal{F}$ be a saturated fusion system over a finite p -group T , and $\mathcal{E} \subseteq \mathcal{F}$ a fusion subsystem over $P \trianglelefteq T$.

- (a) $\mathcal{E}$ is said to be a *weakly normal* in $\mathcal{F}$ if $\mathcal{E}$ is saturated and $\mathcal{F}$ -invariant.
- (b) $\mathcal{E}$ is a *normal* fusion subsystem of $\mathcal{F}$, denoted by $\mathcal{E} \trianglelefteq \mathcal{F}$, provided:
 - i. $\mathcal{E}$ is *weakly normal* in $\mathcal{F}$, and
 - ii. (Extension condition) any $\phi \in \text{Aut}_{\mathcal{E}}(P)$ extends to some $\Phi \in \text{Aut}_{\mathcal{F}}(PC_T(P))$ such that $[C_T(P), \Phi] \leq Z(P)$.

We now state a number of results about normal fusion subsystem of a fusion system.

Proposition 3.12.3. *Let G be a finite group and $T \in \text{Syl}_p(G)$. Suppose N is a normal subgroup of G and set $U = T \cap N \in \text{Syl}_p(N)$. Then $\mathcal{F}_U(N) \trianglelefteq \mathcal{F}_T(G)$.*

Proof. See for example [8, Part I, Proposition 6.2]. □

Proposition 3.12.4. *Let $\mathcal{F}$ be a saturated fusion system over a finite p -group T . Suppose R is a subgroup of T , which is abelian and strongly closed in $\mathcal{F}$. Then $\mathcal{F}_R(R)$ is normal in $\mathcal{F}$.*

Proof. As R is abelian, we have that the morphisms in the subsystem $\mathcal{F}_R(R)$ are inclusions. Since R is strongly closed in $\mathcal{F}$, the result follows. □

Theorem 3.12.5. *Let G be a finite group and let $T \in \text{Syl}_2(G)$. Suppose N is a normal subgroup of G with $O(N) = 1$. Then $C_T(\mathcal{F}_{T \cap N}(N)) = C_T(N)$.*

Proof. See [32, Theorem B]. □

Theorem 3.12.6. *Let G be a finite group, and let T be a Sylow p -subgroup of G , with U a subgroup of T . Suppose $C_G(O_p(G)) \leq O_p(G)$, and $\mathcal{E}$ a normal subsystem of $\mathcal{F}_T(G)$ on U . Then there exists a unique normal subgroup N of G such that $U \in \text{Syl}_p(N)$ and $\mathcal{E} = \mathcal{F}_U(N)$.*

Proof. See [8, Part II, Theorem 7.5]. □

The following lemma will be useful in the subsequent chapters.

Lemma 3.12.7. *Let G_1 and G_2 be groups which share a common Sylow p -subgroup S such that $\mathcal{F}_S(G_1) = \mathcal{F}_S(G_2)$. Suppose furthermore, for $i = 1, 2$, we are given $N_i \trianglelefteq G_i$ such that $T := S \cap N_1 = S \cap N_2$ and $\mathcal{F}_T(N_1) = \mathcal{F}_T(N_2)$. Then the following hold:*

- (a) For $R \leq S$ we have $\mathcal{F}_{TR}(N_1 R) = \mathcal{F}_{TR}(N_2 R)$.

(b) Assume $O^p(C_{G_i}(T)) \leq C_{G_i}(N_i)$ for $i = 1, 2$. Then

$$\mathcal{F}_{C_S(N_1)}(C_{G_1}(N_1)) = \mathcal{F}_{C_S(N_2)}(C_{G_2}(N_2)).$$

For $R \leq S$, we have

$$\mathcal{F}_{C_S(N_1)R}(C_{G_1}(N_1)R) = \mathcal{F}_{C_S(N_2)R}(C_{G_2}(N_2)R).$$

Proof. (a) By [33, Proposition 3.1], for each $i = 1, 2$, the fusion system $\mathcal{F}_{T R}(N_i R)$ equals a certain product fusion system $\mathcal{F}_T(N_i)R$ defined in that paper. As $\mathcal{F}_T(N_1) = \mathcal{F}_T(N_2)$, this implies the assertion.

(b) Assume $O^p(C_{G_i}(T)) \leq C_{G_i}(N_i)$ for $i = 1, 2$. Then $O^p(C_{G_i}(T)) = O^p(C_{G_i}(N_i))$. Set $\mathcal{F} := \mathcal{F}_S(G_1) = \mathcal{F}_S(G_2)$. By Theorem 3.4.6, $C_{G_i}(T)$ realizes $C_{\mathcal{F}}(T)$. It follows moreover from Theorem 3.12.5 that $C := C_S(N_1) = C_S(\mathcal{F}_T(N_1)) = C_S(\mathcal{F}_T(N_2)) = C_S(N_2)$. By the hyperfocal subgroup theorem of Puig (see Theorem 2.3.4)

$$\text{hyp}(C_{\mathcal{F}}(T)) = O^p(C_{G_i}(T)) \cap C_S(T) \leq C_S(N_i) = C.$$

In particular, for each $i = 1, 2$, $\mathcal{F}_C(C_{G_i}(N_i))$ is saturated subsystem of $C_{\mathcal{F}}(T)$ of p -power index as in Definition 3.11.3. At the same time by Theorem 3.11.4 there is a unique saturated subsystem of $C_{\mathcal{F}}(T)$ over C of p -power index. As everything we said is true for each $i = 1, 2$, it follows that $\mathcal{F}_{C_S(N_1)}(C_{G_1}(N_1)) = \mathcal{F}_{C_S(N_2)}(C_{G_2}(N_2))$. This proves the first statement in (b). The second statement in (b) follows now from (a) applied with $C_{G_i}(N_i) \trianglelefteq G_i$ instead of N_i . $\square$

Lemma 3.12.8. *Let $\mathcal{F}$ be a saturated fusion system over a p -group T . Assume $\mathcal{E}$ is a saturated subsystem of $\mathcal{F}$ over a strongly closed subgroup $U \leq T$. Then $O_2(\mathcal{F}) \cap U$ is normal in $\mathcal{E}$.*

Proof. Suppose $\mathcal{E} \trianglelefteq \mathcal{F}$ and note that this assertion is true by Proposition 3.12.3. Let $Q := O_2(\mathcal{F})$. It follows from Proposition 3.8.5 that there exist a series $1 = Q_0 \leq Q_1 \leq \dots \leq Q_n = Q$ such that for every $1 \leq i \leq n$, Q_i is strongly closed in $\mathcal{F}$ and $[Q, Q_{i-1}] \leq Q_{i-1}$. Set $P := Q \cap U$. We have that $P_i := Q_i \cap U$ is strongly closed in $\mathcal{E}$ for each $0 \leq i \leq n$, and $[P, P_i] \leq U \cap [Q, Q_i] \leq U \cap Q_{i-1} = P_{i-1}$. It follows immediately from Proposition 3.8.5 that $Q \cap U = O_2(\mathcal{F}) \cap U \trianglelefteq \mathcal{E}$. $\square$

There is also the notion of a subnormal fusion subsystem analogous to that of group theory.

Definition 3.12.9. Let $\mathcal{F}$ be a saturated fusion system over a finite p -group T . A saturated fusion subsystem $\mathcal{E} \subseteq \mathcal{F}$ over $P \trianglelefteq T$ is *subnormal* in $\mathcal{F}$ if there exists a series

$$\mathcal{E} = \mathcal{E}_0 \trianglelefteq \mathcal{E}_1 \trianglelefteq \cdots \trianglelefteq \mathcal{E}_k = \mathcal{F}.$$

Lemma 3.12.10. *Let $\mathcal{F}$ be a constrained fusion system. Then every subnormal subsystem of $\mathcal{F}$ is constrained.*

Proof. See [6, 14.6]. □

3.13 Simple fusion systems

Following from the previous section, we can now define a simple saturated fusion system and its related concepts.

Definition 3.13.1. A saturated fusion system $\mathcal{F}$ over a finite p -group T is called *simple* if it has no proper non-trivial normal fusion subsystems. Equivalently, the fusion system $\mathcal{F}$ itself and the trivial fusion subsystem $\mathcal{F}_1(1)$ are the only normal fusion subsystems of $\mathcal{F}$.

We note that the saturated fusion system $\mathcal{F}$ realizable by a finite simple group in general is not simple.

Example 3.13.2. Let $\mathcal{F}$ be a saturated fusion system over a finite 2-group realizable by the finite simple group $G = A_5$. Thus $\mathcal{F} = \mathcal{F}_T(G)$ where $T \in \text{Syl}_2(G)$. As T is abelian, we have by Theorem 2.6.3 that $N_G(T)$ control fusion in T . Thus, $\mathcal{F}_T(N_G(T)) = \mathcal{F}_T(G) = \mathcal{F}$. As T is normal in $N_G(T)$, we have by Proposition 3.12.3 that $\mathcal{F}_T(T) \trianglelefteq \mathcal{F}$. Therefore $\mathcal{F}$ is not simple. Alternatively, one could use Proposition 3.12.4 to establish the results.

However, the following proposition indicates that if a saturated simple fusion is realizable by a finite group with some properties, then that group must be simple.

Proposition 3.13.3. [43, Proposition 8.2] *Let $\mathcal{F}$ be a simple saturated fusion system over a finite p -group T realizable by some finite group G , with $O_{p'}(G) = 1$. Suppose $\mathcal{F}_T(H) \neq \mathcal{F}_T(G)$ for any proper subgroup $T \leq H \leq G$. Then G is simple.*

Proof. Suppose $N \leq G$ is a non-trivial normal subgroup. Then $U := N \cap T \in \text{Syl}_p(N)$. By Proposition 3.12.3, $\mathcal{F}_U(N) \trianglelefteq \mathcal{F}_T(G)$. As $O_{p'}(G) = 1$, we have $U \neq 1$ and so the subsystem $\mathcal{F}_U(N)$ is non-trivial. Since $\mathcal{F}_T(G)$ is simple, we have $\mathcal{F}_U(N) = \mathcal{F}_T(G)$ and thus $U = N \cap T = T$. By our assumption, we get $N = G$. So G is simple. □

Lemma 3.13.4. *The 2-fusion system of A_n is simple if and only if $n \geq 6$.*

Proof. See [6, 16.5] □

Lemma 3.13.5. *Let S_n be a symmetric group of degree n where $n \geq 6$, and let $T \in \text{Syl}_2(S_n)$. Then $O_2(\mathcal{F}_T(S_n)) = 1$.*

Proof. Set $U := T \cap A_n$, $\mathcal{F} := \mathcal{F}_T(S_n)$, $\mathcal{E} := \mathcal{F}_U(A_n)$. By Proposition 3.12.3, $\mathcal{E} \trianglelefteq \mathcal{F}$. From Lemmas 3.12.8 and 3.13.4, we get that $\mathcal{E}$ is simple and $O_2(\mathcal{F}) \cap U = 1$. As U is non-trivial $O_2(\mathcal{F}) = 1$. □

3.14 Components of fusion systems

This section deals with the analogues of quasisimple groups and components of a finite group in fusion systems.

Definition 3.14.1. Let $\mathcal{F}$ be a saturated fusion system over a finite p -group T .

- (a) $\mathcal{F}$ is called *quasisimple* if $\mathcal{F}/Z(\mathcal{F})$ is simple and $\text{foc}(\mathcal{F}) = T$.
- (b) A *component* $\mathcal{C}$ of $\mathcal{F}$ is a subnormal quasisimple subsystem of $\mathcal{F}$.

We denote the set of all components of $\mathcal{F}$ by $\text{Comp}(\mathcal{F})$.

Lemma 3.14.2. *Let H be a finite group with $T \in \text{Syl}_2(H)$ and K a component of H such that $K/Z(K)$ has a simple 2-fusion system. Then $\mathcal{F}_{T \cap K}(K)$ is a component of $\mathcal{F}_T(H)$.*

Proof. We will demonstrate that $\mathcal{F}_{T \cap K}(K)$ is both quasisimple and subnormal. Following the focal subgroup theorem as in Theorem 2.3.2, $\text{foc}(\mathcal{F}_{T \cap K}(K)) = \langle g^{-1}\alpha(g) \mid g \in P \leq T \cap K, \alpha \in \text{Aut}_{\mathcal{F}_{T \cap K}(K)}(P) \rangle = T \cap K' = T \cap K$. Then from Corollary 3.11.5 we have $O^p(\mathcal{F}_{T \cap K}(K)) = \mathcal{F}_{T \cap K}(K)$. Let $\bar{K} := K/Z(K)$. Set $\mathcal{F} = \mathcal{F}_{T \cap K}(K)/T \cap Z(K)$. By Theorem 3.9.4 $\mathcal{F} \cong \mathcal{F}_{\bar{T} \cap \bar{K}}(\bar{K})$ which by assumption is simple. We have that $Z(\mathcal{F}) \geq T \cap Z(K)$ and so $Z(\mathcal{F})/(T \cap Z(K)) \trianglelefteq \mathcal{F}/(T \cap Z(K))$ which is also simple. Therefore $Z(\mathcal{F})/(T \cap Z(K))$ is trivial and thus $Z(\mathcal{F}) = T \cap Z(K)$. Hence $\mathcal{F}/Z(\mathcal{F}) \cong \mathcal{F}_{\bar{T} \cap \bar{K}}(\bar{K})$ is simple. Moreover, since K is subnormal we have the subnormal series $K = K_0 \trianglelefteq K_1 \trianglelefteq \dots \trianglelefteq H$ and $T \cap K = T \cap K_0 \trianglelefteq T \cap K_1 \trianglelefteq \dots \trianglelefteq T$. Then by using Proposition 3.12.3, we form a subnormal series $\mathcal{F}_{T \cap K}(K) = \mathcal{F}_{T \cap K_0}(K_0) \trianglelefteq \mathcal{F}_{T \cap K_1}(K_1) \trianglelefteq \dots \trianglelefteq \mathcal{F}_T(H)$. Thus $\mathcal{F}_{T \cap K}(K)$ is subnormal of $\mathcal{F}_T(H)$. Hence $\mathcal{F}_{T \cap K}(K)$ is a component of $\mathcal{F}_T(H)$. □

Lemma 3.14.3. *Let G be a finite group such that $O(G) = 1$ and for each component L of G , $L/Z(L)$ is a known finite simple group. Let $T \in \text{Syl}_2(G)$ and let $\mathcal{C}$ be a component of $\mathcal{F}_T(G)$. Then there exist a component L of G such that $\mathcal{C} = \mathcal{F}_{T \cap L}(L)$.*

Proof. See [34, Lemma 2.49]. □

Lemma 3.14.4. *Let $\mathcal{F}$ be a saturated fusion system. Then the following statements are equivalent:*

- (a) $\mathcal{F}$ is constrained,
- (b) $F^*(\mathcal{F}) = O_p(\mathcal{F})$,
- (c) $E(\mathcal{F}) = 1$.

Proof. See [6, 14.2]. □

3.15 Tame fusion systems

In the proof and statement of Lemma 3.15.2 below, we will need some advanced concepts to cite results from the literature. Firstly, for a finite group G with Sylow p -subgroup S , there is a centric linking system $\mathcal{L} = \mathcal{L}_S^c(G)$ of G defined (see e.g. [8, Definition III.3.1]). There is moreover a certain group of outer automorphisms of $\mathcal{L}$ defined, denoted by $\text{Out}_{\text{typ}}(\mathcal{L})$ (see for example [8, pp.152-153]), and a group homomorphism $\kappa_G : \text{Out}(G) \rightarrow \text{Out}_{\text{typ}}(\mathcal{L})$ (see for example p.109 in [1]). One says that G *tamely realizes* a fusion system $\mathcal{F}$ if $\mathcal{F} = \mathcal{F}_S(G)$ and κ_G is split surjective. There is also a map $\mu_{\mathcal{L}} : \text{Out}_{\text{typ}}(\mathcal{L}) \rightarrow \text{Out}(\mathcal{F}_S(G))$ (see Lemma 4.9 and top of p.155 in [8]). It follows from [8, Theorem III.5.12] and the existence and uniqueness of centric linking systems ([49] and [18]) that $\mu_{\mathcal{L}}$ is surjective.

Lemma 3.15.1. *Let G be a finite group with $S \in \text{Syl}_p(G)$ such that G tamely realizes $\mathcal{F}_S(G)$. Then every element of $\text{Aut}(\mathcal{F}_S(G))$ extends to an automorphism of G .*

Proof. For every finite group G with $S \in \text{Syl}_p(G)$, the map $\alpha_G : N_{\text{Aut}(G)}(S) \rightarrow \text{Aut}(\mathcal{F}_S(G))$, $\phi \mapsto \phi|_S$ is well-defined. We need to argue that α_G is surjective. By the Frattini argument (see Lemma 2.1.12), $\text{Aut}(G) = \text{Inn}(G)N_{\text{Aut}(G)}(S)$. Thus, α_G induces a map $\bar{\alpha}_G : \text{Out}(G) \rightarrow \text{Out}(\mathcal{F}_S(G))$, which is surjective precisely when α_G is surjective. Indeed, $\bar{\alpha}_G$ is the composition of κ_G with $\mu_{\mathcal{L}}$ for $\mathcal{L} = \mathcal{L}_S^c(G)$ as can be seen from the construction of these maps. Recall from above that $\mu_{\mathcal{L}}$ is surjective. If G tamely realizes $\mathcal{F}_S(G)$, then also κ_G is surjective. Hence, in that case $\bar{\alpha}_G$ and thus α_G is surjective as required. □

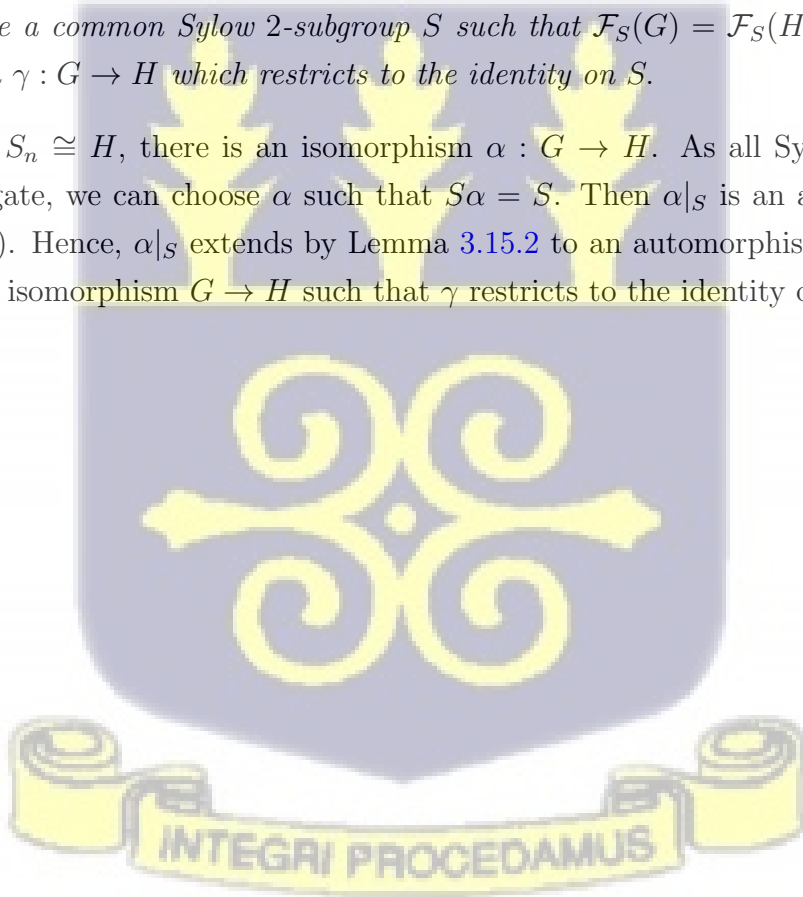
Lemma 3.15.2. *Let $S \in \text{Syl}_2(S_n)$ for some $n \geq 5$. Then $\mathcal{F}_S(S_n)$ is tamely realized by S_n . In particular, every automorphism of $\mathcal{F}_S(S_n)$ extends to an automorphism of S_n .*

Proof. By Lemma 3.15.1, it is sufficient to argue that $\mathcal{F} := \mathcal{F}_S(S_n)$ is tamely realized by S_n . Set $T := S \cap A_n$. By [1, Proposition 4.8], $\mathcal{F}_T(A_n)$ is tamely realized by $G_0 := A_n$.

Note that $\mathcal{F}_0 = \mathcal{F}_T(A_n) = O^2(\mathcal{F})$ and $Z(\mathcal{F}) = 1$. Hence, by [1, Proposition 1.31] (using the language introduced there), there is a normal pair of linking systems $\mathcal{L}_0 \trianglelefteq \mathcal{L}$ associated to $\mathcal{F}_0 \trianglelefteq \mathcal{F}$ such that $\text{ob}(\mathcal{L}_0) = \mathcal{F}_0^c$ is $\text{Aut}(\mathcal{F}_0)$ -invariant, $\text{ob}(\mathcal{L})$ is $\text{Aut}(\mathcal{F})$ -invariant and $\mathcal{L}_0$ is $\text{Aut}_{\text{typ}}(\mathcal{L})$ -invariant. Given the uniqueness of centric linking systems, we have $\mathcal{L}_0 \cong \mathcal{L}_T^c(G_0)$. Note moreover $Z(G_0) = 1 = Z(\mathcal{F}_0)$. Thus, by [1, Proposition 2.16], $\mathcal{F}$ is tamely realized by a finite group G with $A_n = G_0 \trianglelefteq G$, $S \in \text{Syl}_2(G)$ and $G/G_0 \cong \mathcal{L}/\mathcal{L}_0$. Here $\mathcal{L}/\mathcal{L}_0 := \text{Aut}_{\mathcal{L}}(T)/\text{Aut}_{\mathcal{L}_0}(T) = N_{S_n}(T)/N_{A_n}(T) \cong C_2$ as $S_n/A_n \cong C_2$. Thus, G_0 has index 2 in G . Note that $G = G_0 C_G(G_0)$ would imply $G \cong G_0 \times C_2$ contradicting $\mathcal{F} = \mathcal{F}_S(G)$. Hence, $G_0 = F^*(G)$ and G embeds into $\text{Aut}(G_0)$. If $n \neq 6$ it follows immediately that $G \cong \text{Aut}(G_0) \cong S_n$. For $n = 6$ note that S is isomorphic to $T \times C_2$ and thus G needs to be isomorphic to S_n as well. Hence, in either case $\mathcal{F}$ is tamely realized by $G \cong S_n$ and thus by S_n . $\square$

Corollary 3.15.3. *Let G and H be finite groups which are both isomorphic to S_n for some $n \geq 5$ and share a common Sylow 2-subgroup S such that $\mathcal{F}_S(G) = \mathcal{F}_S(H)$. Then there is an isomorphism $\gamma : G \rightarrow H$ which restricts to the identity on S .*

Proof. As $G \cong S_n \cong H$, there is an isomorphism $\alpha : G \rightarrow H$. As all Sylow 2-subgroups of H are conjugate, we can choose α such that $S\alpha = S$. Then $\alpha|_S$ is an automorphism of $\mathcal{F}_S(G) = \mathcal{F}_S(H)$. Hence, $\alpha|_S$ extends by Lemma 3.15.2 to an automorphism β of H . Then $\gamma := \alpha\beta^{-1}$ is an isomorphism $G \rightarrow H$ such that γ restricts to the identity on S . $\square$



Chapter 4 Some properties of the alternating groups

The chapter presents some properties of the alternating groups. We first present a presentation of the alternating group in Section 4.1. We will describe some involutions of the alternating group and the general structure of a centralizer of involution in Section 4.2 and Section 4.3. In Section 4.4, we present some properties of Sylow 2-subgroups of the alternating groups. We end the chapter with some useful lemmas in Section 4.5. The main reference is [39].

4.1 A presentation of the alternating group

In this section, we briefly describe the alternating group via generators and relations.

We begin with a definition.

Definition 4.1.1. Let S be a subset of a group X .

- (a) A *presentation* for X is of the form

$$\langle S \mid R \rangle$$

where R is a set of relations among the elements of S . The elements of S are called the generators of G .

- (b) X is said to be *finitely generated* provided that there exist a presentation

$$\langle S \mid R \rangle$$

such that S is a finite set.

- (c) X is said to be *finitely presented* provided that there exist a presentation

$$\langle S \mid R \rangle$$

such that both S and R are finite sets.

The following theorem gives a presentation of the alternating group.

Theorem 4.1.2. Suppose the group X has the presentation

$$\langle x_3, \dots, x_n \mid x_i^3 = 1, (x_i x_j)^2 = 1, 3 \leq i \neq j \leq n \rangle.$$

Then X is isomorphic to A_n .

Proof. See for instance [13, Chapter VII, Section 48(IV)]. □

4.2 Involutions of the alternating group

This section describes some involutions of the alternating group.

We first recall the following: If $\omega = \{1, 2, \dots, 2m\}$, then $Sym(\omega)$, the symmetric group on $2m$ objects is denoted by S_{2m} . So for the set $\omega' \subseteq \{1, 2, \dots, 2m\}$ of order l , the symmetric group $Sym(\omega') \leq Sym(\omega)$ is isomorphic to the symmetric group S_l on l objects.

For the rest of the thesis, we let A_{2m} be the alternating group of degree $2m$, for $m \geq 4$. Set $2m = 4s + d$, where s is a positive integer and $d = 0$ or 2 . We define some involutions of A_{2m} as follows:

$$\sigma_q = (4q - 3 \ 4q - 2)(4q - 1 \ 4q) \quad \text{for } 1 \leq q \leq s, \quad (4.1)$$

$$\tilde{\sigma}_q = (4q - 3 \ 4q - 1)(4q - 2 \ 4q) \quad \text{for } 1 \leq q \leq s, \quad (4.2)$$

$$t_q = \sigma_1 \sigma_2 \cdots \sigma_q \quad \text{for } 1 \leq q \leq s, \quad (4.3)$$

$$\nu_q = (1 \ 2)(4q + 1 \ 4q + 2) \quad \text{for } 1 \leq q \leq s - 1, \text{ and} \quad (4.4)$$

$$\nu_s = \begin{cases} (1 \ 2)(4s + 1 \ 4s + 2) & \text{if } d = 2, \\ 1 & \text{if } d = 0. \end{cases} \quad (4.5)$$

The element t_q is usually called the generalized involution of A_{2m} , and the involution t_s is 2-central.

4.3 The structure of a centralizer of an involution

In this section we present the structure of centralizers of involutions in the alternating group A_{2m} for a positive integer $m \geq 2$. We fix s, d as in Section 4.2, and the involutions $\sigma_q, \tilde{\sigma}_q, t_q, \nu_q, \nu_s$ as described in equations 4.1-4.5.

Fix $t = t_1 = (1 \ 2)(3 \ 4)$ the short involution. For $P = \{(1 \ 2), (3 \ 4)\}$ we have that

$$C_{S_{2m}}(t)/C_{S_{2m}}(\langle P \rangle) \cong S_{|P|} \cong S_2.$$

So

$$C_{S_{2m}}(t) \cong (\langle P \rangle \rtimes S_2) \times S_{2m-4} \cong (\langle (1 \ 2)(3 \ 4), (1 \ 3)(2 \ 4) \rangle \rtimes S_2) \times S_{2m-4}.$$

Therefore,

$$C_{S_{2m}}(t) \cong D_8 \times S_{2m-4},$$

and so

$$C_{A_{2m}}(t) = (\langle (1 \ 2)(3 \ 4), (1 \ 3)(2 \ 4) \rangle \times A_{\{5,6,\dots,2m\}}) \rtimes \nu_1,$$

where $A_{\{5,6,\dots,2m\}} \cong A_{2m-4}$.

For arbitrary $1 \leq q \leq s$ the structure of $C_{A_{2m}}(t_q)$ is for example described in Kondo[39, Section 1.1]. Namely, setting $k := m - 2q$ we have

$$C_{A_{2m}}(t_q) = (\beta_q \times L) \langle \nu_q \rangle \quad \text{where } 2 \leq q \leq s, \quad (4.6)$$

where β_q is the centralizer of t_q in the alternating group A_{4q} , and $\beta \langle \nu_q \rangle$ is isomorphic to the centralizer of t_q in the symmetric group S_{4q} . Also $L = A_{\{4q+1,\dots,2m\}}$ is isomorphic to A_{2k} for $k \neq 0$ and $L = 1$ if $k = 0$, and $L \langle \nu_q \rangle$ is isomorphic to the symmetric group S_{2k} .

Furthermore, for $t_q = (1\ 2)(3\ 4)(5\ 6)(7\ 8) \cdots (4q-1\ 4q)$ we let

$$Q = \{(1\ 2), (3\ 4), (5\ 6), (7\ 8) \cdots (4q-1, 4q)\}.$$

Then

$$\langle Q \rangle = \langle (1\ 2), (3\ 4), (5\ 6), (7\ 8), \dots, (4q-1, 4q) \rangle \cong E_{2^{2q}}$$

where $E_{2^{2q}}$ is an elementary abelian 2-group of rank $2q$. So $C_{S_{4q}}(t_q)/C_{S_{4q}}(Q)$ embeds into $S_Q \cong S_{2q}$ and $\hat{V} := C_{S_{4q}}(Q) = \langle Q \rangle \cong E_{2^{2q}}$. Indeed, for the subgroup of S_{4q} acting on Q , which we denote by $N_{S_{4q}}(Q)$, we have $N_{S_{4q}}(Q)/\hat{V} \cong S_Q \cong S_{2q}$ and intersect trivially with $\hat{V}$. Here $\hat{V}$ can be thought of as the permutation module for $S_Q \cong S_{2q}$. Let $R := \{(1\ 2), (3\ 4), \dots, (4q-1, 4q)\}$ be set of pairs. We therefore have that

$$C_{S_{4q}}(t_q) \cong \hat{V} \rtimes N_{S_{4q}}(R) \cong E_{2^{2q}} \rtimes S_{2q}.$$

A closer look at $C_{A_{4q}}(t_q)$ indicates that $\hat{V} \not\leq A_{4q}$ and $V := \hat{V} \cap A_{4q}$ can be thought of as the even sums in the permutation module and $V/\langle t_q \rangle$ is irreducible. If $k \neq 0$, then

$$A_{4q} \langle \nu_q \rangle \cong S_{4q} \text{ and } \beta_q \langle \nu_q \rangle = C_{A_{4q} \langle \nu_q \rangle}(t_q) \cong C_{S_{4q}}(t_q) \cong E_{2^{2q}} \rtimes S_{2q}.$$

Indeed for $\tilde{V} := V \langle \nu_q \rangle$, we have $\beta_q \langle \nu_q \rangle = \tilde{V} \rtimes N_{A_{4q} \langle \nu_q \rangle}(R)$ where $\tilde{V}$ is thought of as permutation module for $N_{A_{4q} \langle \nu_q \rangle}(R) \cong S_{2q}$. In particular ν_q does not normalize $N_{A_{4q} \langle \nu_q \rangle}(R)$.

As $|C_{S_{4q}}(t_q) : C_{A_{4q}}(t_q)| = 2$, we get that

$$|\beta_q| = |C_{A_{4q}}(t_q)| = \frac{|C_{S_{4q}}(t_q)|}{2}. \quad (4.7)$$

So

$$|\beta_q|_2 = |C_{A_{4q}}(t_q)|_2 = \frac{2^{2q}|S_{2q}|_2}{2} = 2^{2q}|A_{2q}|_2 = |A_{4q}|_2. \quad (4.8)$$

The following lemma describes the conjugacy classes of the centralizer in A_{2m} of the involution t_s .

Lemma 4.3.1. *The conjugacy classes of involutions of $C_{A_{2m}}(t_s)$ are represented as follows:*

- (a) $\sigma_1\sigma_2\cdots\sigma_q\tilde{\sigma}_{q+1}\cdots\tilde{\sigma}_{q+i}$ where $1 \leq q+i \leq s$ and $\tilde{\sigma}_1\tilde{\sigma}_2\cdots\tilde{\sigma}_s\sigma_s$, for $d = 0$.
- (b) $\sigma_1\sigma_2\cdots\sigma_q\tilde{\sigma}_{q+1}\cdots\tilde{\sigma}_{q+i}$ where $1 \leq q+i \leq s$ and $\sigma_1\sigma_2\cdots\sigma_q\tilde{\sigma}_{q+1}\cdots\tilde{\sigma}_{q+i}\nu_{s-1}\nu_s$ where $1 \leq q \leq s-1, 1 \leq i \leq s-1-q$, for $d = 2$.

Proof. See [39, 1.3 Lemma]. □

4.4 The Sylow 2-subgroups of the alternating groups

This brief section presents some properties of the Sylow 2-subgroup of the alternating group and the centralizer of involutions.

We begin with the Sylow 2-subgroup of a centralizer of involution.

Lemma 4.4.1. *Let R be a Sylow 2-subgroup of the centralizer $C_{A_{2m}}(t_s)$ of the involution t_s . Then R is a Sylow 2-subgroup of the Alternating group A_{2m} .*

Proof. Recall that $2m = 4s + d$, where s is a natural number and $d = 0$ or 2 . We proceed by proof by cases: If $d = 0$, then $A_{2m} = A_{4s}$, $L = 1$, and $\nu_s = 1$. It follows that $C_{A_{2m}}(t_s) \cong C_{A_{4s}}(t_s)$ and thus by equation 4.8 we have that

$$|C_{A_{2m}}(t_s)|_2 = |C_{A_{4s}}(t_s)|_2 = |A_{4s}|_2 = |A_{2m}|_2.$$

The result follows in this case.

If $d = 2$, then $L = A_2$ and $|\nu_s| = 2$. It follows that

$$|C_{A_{2m}}(t_s)| = |(C_{A_{4s}}(t_s) \times A_2)\langle\nu_s\rangle| = |C_{A_{4s}}(t_s)\langle\nu_s\rangle| = |C_{S_{4s}}(t_s)|.$$

Thus $|A_{2m}|_2 = |S_{4s}|_2$. It also follows that

$$|C_{A_{2m}}(t_s)|_2 = 2|C_{A_{4s}}(t_s)|_2 = 2^{2s}|S_{2s}|_2 = |S_{4s}|_2 = |A_{2m}|_2$$

proving the result in this case. □

Lemma 4.4.2. *Suppose T is a Sylow 2-subgroup of the Alternating group A_{2m} . Then T is a Sylow 2-subgroup of the Alternating group A_{2m+1} .*

Proof. Assume $T \in \text{Syl}_2(A_{2m})$. As $|A_{2m+1}| = (2m+1)|A_{2m}|$, we have that $|A_{2m+1}|_2 = |A_{2m}|_2$ and so $T \in \text{Syl}_2(A_{2m+1})$. □

As a consequence of Lemma 4.4.2, we have the following corollary.

Corollary 4.4.3. *For any involution t_q for $1 \leq q \leq s$, a Sylow 2-subgroup of $C_{A_{2m}}(t_q)$ is a Sylow 2-subgroup of $C_{A_{2m+1}}(t_q)$.*

Proof. We have that

$$C_{A_{2m+1}}(t_q) \cong (\beta_q \times A_{2k+1}) \langle \nu_q \rangle \quad \text{and} \quad C_{A_{2m}}(t_q) \cong (\beta_q \times A_{2k}) \langle \nu_q \rangle.$$

By Lemma 4.4.2, a Sylow 2-subgroup of A_{2k+1} is a Sylow 2-subgroup of A_{2k} and so the result follows. $\square$

4.5 Some lemmas

In this section we prove several results on a centralizer of an involution, which will be useful in Chapter 6.

Consider now the groups A_{2m} or A_{2m+1} where $2m = 4s + d$ with $d \in \{0, 2\}$. We assume here that $m \geq 6$. Moreover, we regard A_n always as a subgroup of $A_{n'}$ for $n \leq n'$. A *long involution* is an involution in A_{2m} or A_{2m+1} which is the product of $2s$ pairwise disjoint transpositions.

Notation 4.5.1. We recall the definitions of the double transpositions σ_q and $\tilde{\sigma}_q$ for $1 \leq q \leq s$ from Section 4.2. Set

$$A = \langle \sigma_1, \dots, \sigma_s, \tilde{\sigma}_1, \dots, \tilde{\sigma}_s \rangle$$

and

$$B_0 = \langle \sigma_1, \dots, \sigma_{s-1}, \tilde{\sigma}_1, \dots, \tilde{\sigma}_{s-1} \rangle.$$

We write $\mathcal{B}$ for the set of $N_G(A)$ -conjugates of B_0 .

Notice that A is elementary abelian of rank $2s$. Furthermore, the elements of $\mathcal{B}$ are subgroups of A of index 4.

Lemma 4.5.2. (a) $N_{A_{2m}}(A)$ acts transitively on all elements of A of a given cycle type, and $N_{A_{2m}}(B_0)$ acts transitively on involutions in B_0 of a given cycle type.

(b) A is weakly closed in S with respect to A_{2m} and A_{2m+1} , where $S \in \text{Syl}_2(A_{2m})$.

(c) Let $S \in \text{Syl}_2(A_{2m})$ and $\mathcal{F} = \mathcal{F}_S(A_{2m})$. For every $1 \neq a \in A$, every fully $\mathcal{F}$ -centralized conjugate $\langle b \rangle$ of $\langle a \rangle$ is contained in A .

Proof. Part (a) is easy to check. Part (b) follows from the fact that A is by [26, Lemma 3.18(i)] the unique elementary abelian subgroup of S of rank $2s$.

For the proof of (c) choose now $\langle b \rangle \in \langle a \rangle^{\mathcal{F}}$ fully $\mathcal{F}$ -centralized. Then there exists $\alpha \in \text{Hom}_{\mathcal{F}}(C_S(a), S)$ with $a\alpha = b$. As A is abelian, $A \leq C_S(a)$ and $b \in A\alpha = A$ by (b). $\square$

Lemma 4.5.3. (a) Let $u, v \in A^\#$. Then

$$O(C_{A_{2m}}(u) \cap C_{A_{2m}}(v)) = 1.$$

Moreover, if $d = 0$, then

$$O(C_{A_{2m+1}}(u) \cap C_{A_{2m+1}}(v)) = 1.$$

(b) Suppose $d = 2$ and let $u \in A$. Then there exists $B \in \mathcal{B}$ such that for every $v \in B$ we have

$$O(C_{A_{2m+1}}(u) \cap C_{A_{2m+1}}(v)) \leq O(C_{A_{2m+1}}(u)).$$

If $u = t_q$ for some $1 \leq q \leq s$, then we can take $B = B_0$.

(c) Let $d = 2$ and $v \in A$. Then

$$O(C_{A_{2m+1}}(v) \cap C_{A_{2m+1}}(t_s)) = A_{\{2m-1, 2m, 2m+1\}}$$

has order 3.

Proof. We prove parts (a), (b), (c) simultaneously, and so let $X \in \{A_{2m}, A_{2m+1}\}$. As all elements of A of a given cycle type are by Lemma 4.5.2(a) conjugate under $N_G(A)$, we can assume for the proof of (a) and (b) that $u = t_q$ for some $1 \leq q \leq s$. Moreover, for the proof of (c), we set $q = s$ and $u = t_s$. If $X = A_{2m+1}$, $d = 2$ and $q < s$, then we assume $v \in B_0$ to prove (b). Otherwise, $v \in A$ is chosen arbitrary.

Let $\tilde{L} = A_{\{4q+1, \dots, 2m\}} \cong A_{2m-4q}$ if $X = A_{2m}$ and $\tilde{L} = A_{\{4q+1, \dots, 2m+1\}} \cong A_{2m-4q+1}$ if $X = A_{2m+1}$. In either case, we have

$$C_X(u) = C_X(t_q) = (\beta_q \times \tilde{L}) \langle \nu_q \rangle.$$

We can write $v = v_1 v_2$, where v_1 is an element of $\langle \sigma_1, \dots, \sigma_q, \tilde{\sigma}_1, \dots, \tilde{\sigma}_q \rangle \leq \beta_q$ and v_2 is an element of $\langle \sigma_{q+1}, \dots, \sigma_s, \tilde{\sigma}_{q+1}, \dots, \tilde{\sigma}_s \rangle \leq \tilde{L}$. Then we have that $C_{\beta_q \times \tilde{L}}(v) = C_{\beta_q}(v_1) \times C_{\tilde{L}}(v_2)$. Observe also that $O(C_X(u) \cap C_X(v)) \leq O^2(C_X(u)) \leq \beta_q \times \tilde{L}$ and thus

$$O(C_X(u) \cap C_X(v)) = O(C_{\beta_q \times \tilde{L}}(v)) = O(C_{\beta_q}(v_1) \times C_{\tilde{L}}(v_2)) = O(C_{\beta_q}(v_1)) \times O(C_{\tilde{L}}(v_2)).$$

Since β_q is of characteristic 2, $C_{\beta_q}(v_1)$ is of characteristic 2 by [44, Lemma 1.2(c)] and in particular 2'-reduced. Therefore,

$$O(C_X(u) \cap C_X(v)) = O(C_{\tilde{L}}(v_2)).$$

Take $0 \leq q' \leq s - q$ such that v_2 is a product of q' pairwise disjoint double transpositions and setting $t_0 = 1$, we get that the element v_2 is of the same cycle type as $t_{q'}$. We note that $2m - 4q = 4(s - q) + d$. Observe that

$$C_{\tilde{L}}(v_2) \cong C_{A_{2m-4q}}(t_{q'}) \cong (\beta_{q'} \times A_{2m-4q-4q'}) \langle \nu_{q'} \rangle \text{ if } X = A_{2m}$$

and

$$C_{\tilde{L}}(v_2) \cong C_{A_{2m-4q+1}}(t_{q'}) \cong (\beta_{q'} \times A_{2m-4q-4q'+1}) \langle \nu_{q'} \rangle \text{ if } X = A_{2m+1}.$$

We note that $O(\beta_{q'}) = 1$. As $2m - 4q - 4q'$ is even, we can conclude that $O(C_X(u) \cap C_X(v)) = O(C_{\tilde{L}}(v_2)) \cong O(A_{2m-4q-4q'}) = 1$ if $X = A_{2m}$. Similarly, if $d = 0$, then $2m - 4q - 4q' + 1 \equiv 1 \pmod{4}$ and so $O(C_X(u) \cap C_X(v)) = O(C_{\tilde{L}}(v_2)) \cong O(A_{2m-4q-4q'+1}) = 1$ if $X = A_{2m+1}$. This proves (a).

We complete the proof of (b) and (c), by considering now the case $d = 2$ and $X = A_{2m+1}$. If $s = q$, then $q' = 0$ and so

$$C_{\tilde{L}}(t_2) = \tilde{L} = A_{\{2m-1, 2m, 2m+1\}} = O(C_X(t_s)) = O(C_X(u))$$

proving (b) and (c) in this case. Henceforth, we assume that $q < s$ and prove (b) in this case. In this case, it is important that we have chosen $v \in B_0$. It follows that $q' \leq s - q - 1$. Therefore, $2m - 4q - 4q' + 1 = 4(s - q - q') + 3$ with $s - q - q' \geq 1$. Hence, we get that $O(C_X(u) \cap C_X(v)) = O(C_{\tilde{L}}(v_2)) \cong O(A_{2m-4q-4q'+1}) = 1$, proving (b). □

Lemma 4.5.4. *We have $\beta_s = \Gamma_{A,2}(\beta_s) = \langle C_{\beta_s}(\sigma_i) : i = 1, 2, \dots, s \rangle$.*

Proof. We write W for the submodule of even sums of the permutation module for S_{2s} . As argued before we have that $\beta_s \cong W \rtimes S_{2s}$. Moreover, the isomorphism $\beta_s \rightarrow W \rtimes S_{2s}$ can be chosen such that $\sigma_i \mapsto v_{2i-1} + v_{2i}$ for $i = 1, 2, \dots, s$ and $\tilde{\sigma}_i \mapsto (2i - 1, 2i) \in S_{2s}$. Therefore, the assertion follows from Lemma 2.13.3. □

Lemma 4.5.5. *Let $u \in A^\#$ and let $X \in \{A_{2m}, A_{2m+1}\}$.*

- (a) *We have $C_X(u) = \Gamma_{A,2}(C_X(u))$.*
- (b) *If $d = 2$, $X = A_{2m+1}$ and u is a double transposition, then $C_X(u) = \Gamma_{A,4}(C_X(u))$.*
- (c) *We have $C_X(u) = \langle C_{C_X(u)}(\tau) : \tau \in A \text{ double transposition} \rangle$.*

Proof. Since all elements of A of a given cycle type are by Lemma 4.5.2(a) conjugate under $N_X(A)$, one may assume always that $u = t_q$ for some $1 \leq q \leq s$. Then

$$\tilde{M} := C_X(u) = (\beta_q \times \tilde{L}) \langle \nu_q \rangle$$

with $\tilde{L} := A_{\{4q+1, \dots, 2m\}} \cong A_{2m-4q}$ if $X = A_{2m}$ and $\tilde{L} := A_{\{4q+1, \dots, 2m+1\}} \cong A_{2m-4q+1}$ if $X = A_{2m+1}$. Observe that ν_q normalizes A and is thus contained in $\Gamma_{A,4}(\tilde{M}) \leq \Gamma_{A,2}(\tilde{M})$.

(a) We note that $\langle \sigma_1, \tilde{\sigma}_1 \rangle \leq \beta_q \cap A$ and so $\beta_q \cap A$ has rank at least 2. Therefore, in any case $\tilde{L} \leq \Gamma_{A,2}(\tilde{M})$. If $q \leq s-1$, then $\tilde{L} \cap A \geq \langle \sigma_q, \tilde{\sigma}_q \rangle$ has rank at least 2 and thus $\beta_q \leq \Gamma_{A,2}(\tilde{M})$. If $q = s$, then $\beta_q = \Gamma_{A,2}(\beta_q) \leq \Gamma_{A,2}(\tilde{M})$ follows from Lemma 4.5.4, proving (a).

(b) For the proof of (b) we assume that u is a double transposition, i.e. $u = t_1$. As $q = 1$, $A \cap \tilde{L}$ has rank at least 4 and so $\beta_q \leq \Gamma_{A,4}(\tilde{M})$. Thus, it is enough to argue that $\tilde{L} \leq \Gamma_{A,4}(\tilde{M})$. We use that $\beta_q = \langle \sigma_1, \tilde{\sigma}_1 \rangle \leq A$ has rank 2. Notice that $\tilde{L} = A_{\{5,6, \dots, 2m+1\}} \cong A_{2m-3}$. We recall that $m \geq 6$ and thus $s \geq 3$. Moreover, as $d = 2$, we have $2m+1 = 4s+3 \geq 15$. We note that $N_{\tilde{L}}(\langle \sigma_2, \tilde{\sigma}_2 \rangle)$ contains $L_0 := A_{\{9,10, \dots, 2m+1\}}$ and $N_{\tilde{L}}(\langle \sigma_3, \tilde{\sigma}_3 \rangle)$ contains $L_1 := A_{\{5,6,7,8\} \cup \{13,14, \dots, 2m+1\}}$. Thus, $\langle L_0, L_1 \rangle \leq \Gamma_{A \cap \tilde{L},2}(\tilde{L}) \leq \Gamma_{A,4}(\tilde{M})$. Observe now that $\langle L_0, L_1 \rangle$ acts doubly transitively on $\{5, 6, \dots, 2m+1\}$. Since $\nu_1 = (12)(56) \in \Gamma_{A,4}(\tilde{M})$, it follows that $(12)(kj) \in \Gamma_{A,4}(\tilde{M})$ for any $k, j \in \{5, 6, \dots, 2m+1\}$ with $k \neq j$. So any product of an even number of transpositions in $S_{\{5,6, \dots, 2m+1\}}$ is in $\Gamma_{A,4}(\tilde{M})$. Thus $\tilde{L} \leq \Gamma_{A,4}(\tilde{M})$. This shows $\tilde{M} = \Gamma_{A,4}(\tilde{M})$ as required.

(c) Let now again $u = t_q$ for an arbitrary $1 \leq q \leq s$. Let $M_0 := \langle C_{\tilde{M}}(\tau) : \tau \in A \text{ double transposition} \rangle$. Notice that $\tilde{L}\langle \nu_1 \rangle \leq C_{\tilde{M}}(\sigma_1) \leq M_0$ as $\sigma_1 \in A \cap \beta_q$ is a double transposition. Thus, it is sufficient to show that $\beta_q \leq M_0$. If $q \leq s-1$, then $\sigma_q \in \tilde{L} \cap A$ is a double transposition and $\beta_q \leq C_{\tilde{M}}(\sigma_q) \leq M_0$. Thus, we may assume $q = s$. However, by Lemma 4.5.4, we have $\beta_s \leq M_0$. $\square$

Lemma 4.5.6. *Let $u \in A_{2m}$ be an involution of the form $u = tu_0$ for a double transposition $t \in A$ and an involution $u_0 \in A_{2m}$ disjoint from t . Then u is conjugate to an element of A under $C_{A_{2m}}(t)$.*

Proof. Let L_t be the subset of A_{2m} containing the elements of A_{2m} which are disjoint from t . Then L_t is a subgroup of $C_{A_{2m}}(t)$ isomorphic to A_{2m-4} and contained in a subgroup of $C_{A_{2m}}(t)$ isomorphic to S_{2m-4} . Note that $u_0 \in L_t$. Moreover, u is a product of at most s pairwise disjoint double transpositions and so u_0 is a product of at most $s-1$ pairwise disjoint double transpositions. Furthermore, $L_1 \cap A$ contains a product of r pairwise disjoint double transpositions for every $1 \leq r \leq s-1$. So there is an element in $L_1 \cap A$ of the same cycle type as u_0 . Since all elements of the same cycle type are conjugate in S_{2m-4} , it follows that $u_0^g \in A$ for some $g \in C_{A_{2m}}(t)$. Then $u^g = t^g u_0^g = t u_0^g \in A$. $\square$

Lemma 4.5.7. *For every double transposition $t \in C_{A_{2m}}(t_s)$ there exists a double transposition $x \in A$ centralizing t such that t is conjugate into A under $C_{A_{2m}}(x)$ (and thus under $C_{A_{2m+1}}(x)$).*

Proof. Let $T \in \text{Syl}_2(A_{2m})$. As $t \in T \leq C_{A_{2m}}(t_s)$, t acts on the set of orbits of t_s of length 2, namely on

$$\{\{1, 2\}, \{3, 4\}, \dots, \{4s-1, 4s\}\}$$

as well as on the set of fixed points of t_s , which is empty for $d = 0$, and equal to $\{2m-1, 2m\}$ for $d = 2$. As t is a double transposition, we have therefore only the following possibilities:

- t swaps two orbits $\{i, i+1\}$ and $\{j, j+1\}$ of length one and fixes all elements of $\{1, 2, \dots, 2m\} \setminus \{i, i+1, j, j+1\}$.
- t acts non-trivially on two orbits $\{i, i+1\}$ and $\{j, j+1\}$ and fixes all elements of $\{1, 2, \dots, 2m\} \setminus \{i, i+1, j, j+1\}$.
- $d = 2$, the involution t acts non-trivially on one orbit $\{i, i+1\}$ of length 2 and on the set of fixed points $\{2m-1, 2m\}$.

As $m \geq 6$ we have always $s \geq 3$. So the set

$$\Omega = \{\{1, 2, 3, 4\}, \{5, 6, 7, 8\}, \dots, \{4s-3, 4s-2, 4s-1, 4s\}\}$$

has $s \geq 3$ elements.

Notice that the i, j in our case distinction above are always odd. Thus, in the first two cases, $\{i, i+1\}$ and $\{j, j+1\}$ are each contained in an element of Ω . Thus, there is at least one element $\{4k-3, 4k-2, 4k-1, 4k\} \in \Omega$ which intersects trivially with $\{i, i+1\}$ and $\{j, j+1\}$. Similarly, in the third case, there exists $\{4k-3, 4k-2, 4k-1, 4k\} \in \Omega$ intersecting trivially with $\{i, i+1\}$. In either case, we choose $x = (4k-3, 4k-2)(4k-1, 4k) \in A$. Then x is disjoint from t and commutes thus with t . We recall that the subset L_x of A_{2m} containing precisely the elements of A_{2m} disjoint from x is a subgroup of $C_{A_{2m}}(x)$ isomorphic to $A_{2m-4} = A_{4(s-1)+d}$ and contained in a subgroup isomorphic to S_{2m-4} . In particular, t is contained in L_x . Observe that $A \cap L_x$ also contains a double transposition. Since all double transpositions in S_{2m-4} are conjugate, it follows that t is conjugate to an element of A under $C_{A_{2m}}(t)$. $\square$

Chapter 5 Finite groups realizing the fusion systems of A_{2m} , for $m \geq 6$ over its Sylow 2-subgroup: The structure of centralizers of involutions

In this chapter, we aim to determine the structure of the quotient groups of the 2-local subgroups of the arbitrary finite group G which realizes the 2-fusion system of the alternating group A_{2m} for $m \geq 6$. We apply a number of results from both finite group theory(Chapter 1-2) and the theory of fusion systems(Chapter 3) to achieve our aim. We recall the following from the introduction of this thesis.

Hypothesis A. *Let G be a finite group such that $O(G) = 1$ and $O^2(G) = G$. Let $S \in \text{Syl}_2(A_{2m})$ where $m \geq 4$, and assume S is contained in G as a Sylow 2-subgroup such that $\mathcal{F}_S(G) = \mathcal{F}_S(A_{2m})$. Suppose furthermore that for every 2-local subgroup H of G and every component K of $H/O(H)$ we have that $K/Z(K)$ is a known finite simple group.*

Theorem A. *Assume Hypothesis A. Then $G \cong A_{2m}$ or $G \cong A_{2m+1}$.*

We will prove Theorem A by induction on m . We use that the theorem is true for $m = 4, 5$ by [25]. Thus, we will work from now on under the following hypothesis.

Hypothesis B. *Assume Hypothesis A and $m \geq 6$. Suppose furthermore that Theorem A is true with m' in place of m for all $4 \leq m' < m$.*

To prove Theorem A, we follow the steps:

1. Determine the structure of the local subgroup $H := C_G(u)/O(C_G(u))$ of G for any involution u of G .
2. With Step 1, we further show that $H \cong C_X(u)/O(C_X(u))$ for some group $X \in \{A_{2m}, A_{2m+1}\}$. This is a crucial step in establishing Theorem A.
3. Limit the structure of $O(C_G(u))$ for any involution u of G . In most cases $O(C_G(u)) = 1$ while in some cases $O(C_G(u))$ has order 3, and determine the structure of G . This will be in Chapter 6.
4. By using induction, we complete the proof of Theorem A, in Chapter 6.

The first two steps form the focus of this chapter. In fact we show the following theorem.

Theorem 5.0.1. *Assume Hypothesis B. Then there exists $X \in \{A_{2m}, A_{2m+1}\}$ such that, for every involution $u \in S$, there is an isomorphism $C_G(u)/O(C_G(u)) \rightarrow C_X(u)/O(C_X(u))$ sending $sO(C_G(u))$ to $sO(C_X(u))$ for every $s \in C_S(u)$.*

It should be remarked here that, with X as above, in most cases $O(C_X(u)) = 1$. The only exception is the case that $X = A_{2m+1}$ where m is odd and u is the product of $m - 1$ pairwise disjoint transpositions. In this case $O(C_X(u))$ has order 3.

From now on we assume Hypothesis B and set $\mathcal{F} := \mathcal{F}_S(A_{2m}) = \mathcal{F}_S(G)$. Since all Sylow 2-subgroups of A_{2m} are conjugate in A_{2m} , we can choose S as we wish. For this chapter we usually use the following setup: For any given involution $x \in S$, we assume that S is chosen such that $\langle x \rangle$ is fully $\mathcal{F}$ -centralized. Then $C_{\mathcal{F}}(x) = \mathcal{F}_{C_S(x)}(C_G(x)) = \mathcal{F}_{C_S(x)}(C_{A_{2m}}(x))$. Setting $H := C_G(x)/O(C_G(x))$, the goal of the next sections is to determine the structure of H by considering the generalized Fitting subgroup of H .

5.1 General setup and remarks

We recall that for the alternating group A_{2m} of degree $2m$, we let $2m = 4s + d$, where s is a positive integer and $d = 0$ or 2 . For $1 \leq q \leq s$, we define the generalized involution t_q , and the involution ν_q of A_{2m} as follows:

$$\begin{aligned} t_q &= (1\ 2)(3\ 4)\dots(4q-3\ 4q-2)(4q-1\ 4q), \\ \nu_q &= (1\ 2)(4q+1\ 4q+2) \text{ for } 1 \leq q \leq s-1, \text{ and} \\ \nu_s &= \begin{cases} (1\ 2)(4s+1\ 4s+2) & \text{if } d = 2, \\ 1 & \text{if } d = 0. \end{cases} \end{aligned}$$

When proving Theorem 5.0.1, the following lemma will come in useful.

Lemma 5.1.1. *Let $X \in \{A_{2m}, A_{2m+1}\}$ and let $u \in S$ be fully $\mathcal{F}$ -centralized. Suppose there is an isomorphism*

$$\beta : C_G(u)/O(C_G(u)) \rightarrow C_X(u)/O(C_X(u)) \text{ with } sO(C_G(u)) \mapsto sO(C_X(u)) \text{ for every } s \in C_S(u).$$

Then for every $v \in u^{\mathcal{F}}$, there exists an isomorphism

$$C_G(v)/O(C_G(v)) \rightarrow C_X(v)/O(C_X(v)) \text{ with } sO(C_G(v)) \mapsto sO(C_X(v)) \text{ for every } s \in C_S(v).$$

Proof. Recall that, by Hypothesis B, $\mathcal{F} = \mathcal{F}_S(G) = \mathcal{F}_S(A_{2m}) = \mathcal{F}_S(X)$. Pick $v \in u^{\mathcal{F}}$. As u is fully $\mathcal{F}$ -centralized, there exists $\alpha \in \text{Hom}_{\mathcal{F}}(C_S(v), C_S(u))$ with $v\alpha = u$. Pick $g \in G$ and

$x \in X$ with $\alpha = c_g|_{C_S(v)} = c_x|_{C_S(v)}$. Then in particular, $v^g = v^x = u$. Hence, conjugation by g induces an isomorphism

$$\overline{c}_g : C_G(v)/O(C_G(v)) \rightarrow C_G(u)/O(C_G(u)), hO(C_G(v)) \mapsto h^g O(C_G(u)),$$

and conjugation by x induces an isomorphism

$$\overline{c}_x : C_X(v)/O(C_X(v)) \rightarrow C_X(u)/O(C_X(u)), yO(C_X(v)) \mapsto y^x O(C_X(u)).$$

With β as in the statement of the lemma, set $\gamma := \overline{c}_g \beta \overline{c}_x^{-1}$. Note that γ is an isomorphism $C_G(v)/O(C_G(v)) \rightarrow C_X(v)/O(C_X(v))$. Moreover, for $s \in C_S(v)$, we have $(sO(C_G(v)))\overline{c}_g = s^g O(C_G(u))$ with $s^g = s\alpha = s^x \in C_S(u)$. Thus,

$$(sO(C_G(v)))\gamma = (s^g O(C_G(u)))\beta \overline{c}_x^{-1} = (s^g O(C_X(u)))\overline{c}_x^{-1} = s^{gx^{-1}} O(C_X(v)) = sO(C_X(v)).$$

This proves the assertion. $\square$

Remark 5.1.2. In light of Lemma 5.1.1, for a given Sylow p -subgroup S and $u \in S$ fully $\mathcal{F}$ -centralized, we want to show the existence of an isomorphism

$$\beta : C_G(u)/O(C_G(u)) \rightarrow C_X(u)/O(C_X(u)) \text{ with } sO(C_G(u)) \mapsto sO(C_X(u)) \text{ for every } s \in C_S(u).$$

To ease notation, we will only consider the cases that u is of the form t_q with $1 \leq q \leq s$. We can do this as u is always of the same cycle type as one t_q . If we just focus on one involution u , we can thus basically relabel the elements of $\{1, 2, \dots, n\}$ to reduce to the case $u = t_q$. Notice that this kind of relabelling will also lead to a switch of the Sylow p -subgroup S . However, we can always choose $S \in \text{Syl}_2(A_n)$ such that t_q is fully $\mathcal{F}_S(A_n)$ -normalized.

5.2 The structure of the group $H = C_G(t)/O(C_G(t))$, for t a short involution

In this section, we determine the structure of $H = C_G(t)/O(C_G(t))$ where $t = (12)(34)$. We assume that S is chosen such that $t \in S$ and $\langle t \rangle$ is fully $\mathcal{F}$ -centralized. Recall for a symmetric group of degree $2m$, S_{2m} , where $m \geq 6$, we get

$$C_{S_{2m}}(t) \cong D_8 \times S_{2m-4},$$

and

$$M := C_{A_{2m}}(t) \cong (\langle (12)(34), (13)(24) \rangle \times A_{2(m-2)}) \rtimes \langle (12)(56) \rangle.$$

We have the following lemma.

Lemma 5.2.1. *The group M has a unique component L , which is equal to $O^2(M)$ and isomorphic to $A_{2(m-2)}$.*

Proof. Notice that $L = \langle (1) \rangle \times A_{2(m-2)} \rtimes \langle (1) \rangle$ is a normal subgroup of M which is isomorphic to $A_{2(m-2)}$ and thus simple. Therefore, L is a component of M . Moreover, L has 2-power index in M and must thus be the only component of M and equal to $O^2(M)$. $\square$

Lemma 5.2.2. *The group H has a unique component K , which is isomorphic to $A_{2(m-2)}$ or A_{2m-3} . Moreover, $K \cap \overline{C_S(t)} = O^2(H) \cap \overline{C_S(t)}$, and identifying $C_S(t_q)$ with its image in H , we have $C_S(t_q) \cap L = C_S(t_q) \cap K$ and $\mathcal{F}_{C_S(t_q) \cap L}(L) = \mathcal{F}_{C_S(t_q) \cap K}(K)$.*

Proof. Let $L \cong A_{2(m-2)}$ be a component of M . Note that L exists by Lemma 5.2.1 and $L = O^2(M)$. By Lemma 3.13.4, L has a simple 2-fusion system. It follows from Lemma 3.14.2 that $\mathcal{C} := \mathcal{F}_{C_S(t) \cap L}(L)$ is a component of $\mathcal{F}_{C_S(t)}(C_{A_{2m}}(t)) = C_{\mathcal{F}_S(A_{2m})}(t) = C_{\mathcal{F}}(t)$. Let $\overline{C_G(t)} := C_G(t) / O(C_G(t))$ and $\overline{C_S(t)} := C_S(t) / O(C_G(t))$. Then $O(\overline{C_G(t)}) = 1$ and $C_{\mathcal{F}}(t) = \mathcal{F}_{C_S(t)}(C_G(t)) \cong \mathcal{F}_{\overline{C_S(t)}}(\overline{C_G(t)})$. As $\mathcal{C}$ is a component of $C_{\mathcal{F}}(t)$, its image $\overline{\mathcal{C}}$ under the natural isomorphism from $C_{\mathcal{F}}(t)$ to $\mathcal{F}_{\overline{C_S(t)}}(\overline{C_G(t)})$ is a component of $\mathcal{F}_{\overline{C_S(t)}}(\overline{C_G(t)})$. So by Lemma 3.14.3 there exist a component K of $H = \overline{C_G(t)}$ such that $\mathcal{F}_{\overline{C_S(t)} \cap K}(K) = \overline{\mathcal{C}}$. As $\overline{\mathcal{C}} \cong \mathcal{C}$ is isomorphic to the 2-fusion system of $A_{2(m-2)}$, by Hypothesis B, $K \cong A_{2(m-2)}$ or A_{2m-3} . In particular as $\overline{\mathcal{C}} = \mathcal{F}_{\overline{C_S(t)} \cap K}(K)$, $K \cap \overline{C_S(t)} = \overline{C_S(t)} \cap L = \text{hnp}(\mathcal{F}_{\overline{C_S(t)}}(H)) = \text{hnp}(H) = \overline{C_S(t)} \cap O^2(H)$.

Suppose there is another component K^* of H . Then $K^* \cap \overline{C_S(t)} = O^2(K^*) \cap \overline{C_S(t)} \leq O^2(H) \cap \overline{C_S(t)} = K \cap \overline{C_S(t)}$. Since $[K, K^*] = 1$, it follows $K^* \cap \overline{C_S(t)} \leq Z(K^*)$ and thus $K^*/Z(K^*)$ is of odd order, which contradicts the Feit-Thompson Theorem. Therefore, K is the only component of H . $\square$

In the following proposition we determine the structure of $H = C_G(t) / O(C_G(t))$. From now on let K be always the unique component of H , which according to Lemma 5.2.2 exists and is isomorphic to A_{2m-4} or A_{2m-3} .

Proposition 5.2.3. *The following holds.*

- (a) $F^*(H) = K \times V$, where $V := \langle (12)(34), (13)(24) \rangle / O(C_G(t)) \cong V_4$.
- (b) $H = K\overline{C_S(t)}$.
- (c) $H \cong M$ if $K \cong A_{2(m-2)}$ and $H \cong C_{A_{2m+1}}(t)$ if $K \cong A_{2m-3}$. *Indeed, identifying $C_S(t)$ with its image in H , there is an isomorphism from H to M or to $C_{A_{2m+1}}(t)$ which restricts to the identity on $C_S(t)$.*

Proof. (a) For $H = C_G(t) / O(C_G(t))$, $O(H) = 1$. It follows that $E(H) = K$. Notice that $V \trianglelefteq C_{\mathcal{F}}(t)$ as $V \trianglelefteq C_{A_{2m}}(t)$. Note $C_{A_{2m}}(t)/V \cong S_{2(m-2)}$ and so $C_{\mathcal{F}}(t)/V \cong$

$\mathcal{F}_{C_S(t)/V}(C_{A_{2m}}(t)/V)$. By Lemma 3.13.5, $O_2(C_{\mathcal{F}}(t)/V) = 1$. Moreover, $O_2(C_{\mathcal{F}}(t))/V \leq O_2(C_{\mathcal{F}}(t)/V) = 1$. This implies $O_2(C_{\mathcal{F}}(t)) = V$. As $C_{\mathcal{F}}(t) = \mathcal{F}_{C_S(t)}(C_G(t)) \cong \mathcal{F}_{\overline{C_S(t)}}(\overline{C_G(t)})$, we identify $C_S(t)$ with its homomorphic image $\overline{C_S(t)}$ in H . Then $O_2(H) \leq V$, and V is strongly closed in H as it is strongly closed in $C_{\mathcal{F}}(t)$.

Claim. $O_2(H) = V$.

Let $J := \langle V^H \rangle$. Note that $O(J) = 1$ as $O(H) = 1$. Thus, by Theorem 2.10.6, J is a central product of an abelian 2-group and of quasisimple groups of type I and type II. The latter would be components of J and thus components of H . However, the only component of H is by Lemma 5.2.2 isomorphic to $A_{2(m-2)}$ or A_{2m-3} , where $m \geq 6$ by assumption. In particular, a component of H is not of type I or II. This shows that J is just an abelian 2-group and so $V \trianglelefteq H$. We have seen above that $O_2(H) \leq V$, hence $V = O_2(H)$.

It follows immediately that $F^*(H) = K \times V$.

(b) Note that $C_H(F^*(H)) = Z(F^*(H)) = V$. Let $C := C_H(K)$. Then $C_C(V) = C_H(F^*(H)) = V$. As $C/V = C/C_C(V) \leq \text{Aut}_{C_G(t)}(V) = \text{Aut}_{C_{\mathcal{F}}(t)}(V) = \text{Aut}_{C_{A_{2m}}(t)}(V) \cong C_2$, we get $|C/V| \leq 2$. Thus $|C| \leq 8$ and C is a 2-group. As C is normal in H , we have $C \leq O_2(H) = V$ and thus $C = V$. Therefore $V = C = C_H(K)$. Furthermore, $K C_H(K)/C_H(K) \trianglelefteq H/C_H(K) = H/V \leq \text{Aut}(K)$. This implies $O^2(H) \leq K C_H(K)$. Thus we have that $O^2(H) = O^2(K C_H(K)) = K$. As $\overline{C_S(t)} \in \text{Syl}_2(H)$ we get $H = K \overline{C_S(t)}$.

(c) By description of M we have $u = (1\ 2)(5\ 6) \in C_S(t)$ such that $L\langle u \rangle \cong S_{2(m-2)}$, $V \rtimes \langle u \rangle \cong D_8$, and $((C_S(t) \cap L) \times V)\langle u \rangle = C_S(t)$. Setting $V = \langle (12)(34), (13)(24) \rangle / O(C_G(t))$, we have that $C_S(t) = ((K \cap C_S(t)) \times V)\langle u \rangle$. Therefore $H = K C_S(t) = (K \times V)\langle u \rangle$. Note that $V\langle \bar{u} \rangle \cong D_8$, we need to show that $K\langle \bar{u} \rangle \cong S_{2(m-2)}$ if $K \cong A_{2(m-2)}$ and $K\langle \bar{u} \rangle \cong S_{2m-3}$ if $K \cong A_{2m-3}$. Set $T := (C_S(t) \cap L)\langle u \rangle$. Then we get that $T = (K \cap C_S(t))\langle u \rangle \in \text{Syl}_2(K\langle u \rangle)$. Now, in either case we have $|\text{Aut}(K)/\text{Inn}(K)| = 2$. Thus either $c_u : K \rightarrow K, k \mapsto k^u$ is in $\text{Inn}(K)$ or $K\langle u \rangle \cong \text{Aut}(K)$ and we are done. Suppose $c_u = c_k$ for some $k \in K$. Then $uk^{-1} \in C_{K\langle u \rangle}(K) =: A$. As $A \cap K = Z(K) = 1$ we get $|A| = 2$, and so $A \trianglelefteq K\langle u \rangle$. Since $T \in \text{Syl}_2(K\langle u \rangle)$ we have $A \leq T$. Observe that $\mathcal{F}_{K \cap C_S(t)}(K) \subseteq C_{C_{\mathcal{F}}(t)}(A)$ and so, $\mathcal{F}_{L \cap C_S(t)}(L) \subseteq C_{C_{\mathcal{F}}(t)}(A)$. By Theorem 3.12.5, we have $[L, A] = 1$. Therefore $(C_S(t) \cap L)\langle u \rangle = (C_S(t) \cap L)A$. However $L\langle u \rangle \cong S_{2(m-2)}$, so $C_{L\langle u \rangle}(L) = 1$. This implies $A = 1$ and so, $u \in L$. So we have shown that $K\langle u \rangle \cong S_{2(m-2)}$ if $K \cong A_{2(m-2)}$ and $K\langle u \rangle \cong S_{2m-3}$ if $K \cong A_{2m-3}$. In the first case, $K\langle u \rangle \cong L\langle u \rangle$ and in the second case $K\langle u \rangle = \tilde{L}\langle u \rangle$ where $\tilde{L} := A_{\{5,6,\dots,2m+1\}} \cong A_{2m-3}$. In either case $K \cap C_S(t) = L \cap C_S(t) = \tilde{L} \cap C_S(t)$ and $\mathcal{F}_{K \cap C_S(t)}(K) = \mathcal{F}_{L \cap C_S(t)}(L) = \mathcal{F}_{\tilde{L} \cap C_S(t)}(\tilde{L})$. Hence, by Lemma 3.12.7(a),

$$\mathcal{F}_{K\langle u \rangle \cap C_S(t)}(K\langle u \rangle) = \mathcal{F}_{L\langle u \rangle \cap C_S(t)}(L\langle u \rangle) = \mathcal{F}_{\tilde{L}\langle u \rangle \cap C_S(t)}(\tilde{L}\langle u \rangle).$$

Thus, by Corollary 3.15.3, if $K \cong A_{2(m-2)}$, then there exists an isomorphism $\delta : K\langle u \rangle \rightarrow L\langle u \rangle$ which restricts to the identity on $(C_S(t) \cap K)\langle u \rangle$, and if $K \cong A_{2m-3}$, then there exists an isomorphism $\delta : K\langle u \rangle \rightarrow \tilde{L}\langle u \rangle$ which restricts to the identity on $(C_S(t) \cap K)\langle u \rangle$. Recall that

$$H \cong (V \times K) \rtimes \langle u \rangle, \quad M = C_{A_{2m}}(t) = (V \times L) \rtimes \langle u \rangle \quad \text{and} \quad C_{A_{2m+1}}(t) = (V \times \tilde{L}) \rtimes \langle u \rangle.$$

Hence, by Lemma 1.1.25 applied with V in place of N , u in place of t and $\tilde{t}$, L or $\tilde{L}$ in place of $\tilde{K}$, and $\text{id}_{V\langle u \rangle}$ in place of α , we see that there is an isomorphism from H to M if $K \cong L$, and from H to $C_{A_{2m+1}}(t)$ if $K \cong \tilde{L}$ which restricts to the identity on $C_S(t)$. □

5.3 The structure of $H = C_G(t_q)/O(C_G(t_q))$ for any involution $t_q \neq t$

In this section, we will determine the structure of H for the generalized involution t_q , for $t_q \neq t$ where t is the short involution $(1\ 2)(3\ 4)$. Again we assume that S is chosen such that $t_q \in S$ and $\langle t_q \rangle$ is fully $\mathcal{F}$ -centralized. We may assume furthermore that $t = (1\ 2)(3\ 4) \in S$. Notice that the involution t_q is equal to t when $q = 1$ and the structure of $H = C_G(t_q)/O(C_G(t_q))$ in that case has been determined in Section 5.2. We fix now q with $2 \leq q \leq s$. From Section 4.3, we have

$$C_{A_{2m}}(t_q) = (\beta_q \times L) \langle \nu_q \rangle \quad \text{where } 2 \leq q \leq s. \quad (5.1)$$

Here, β_q is the centralizer of t_q in the alternating group A_{4q} , and $\beta_q \langle \nu_q \rangle$ is the centralizer of t_q in the group $A_{4q} \langle \nu_q \rangle \cong S_{4q}$. Fix $k := m - 2q$. Then $L = A_{\{4q+1, \dots, 2m\}}$ is isomorphic to A_{2k} if $k \neq 0$ or $L = 1$ if $k = 0$ and so $L \langle \nu_q \rangle$ is isomorphic to the symmetric group S_{2k} . We deduce from equation 5.1 that $C_{A_{2m}}(t_q)$ has shape

$$C_{A_{2m}}(t_q) \cong ((C_2^{2q-1} \rtimes S_{2q}) \times A_{2k}) \langle \nu_q \rangle \quad \text{if } k \geq 0,$$

and in particular

$$C_{A_{2m}}(t_q) \cong C_2^{2q-1} \rtimes S_{2q} \quad \text{if } k = 0.$$

We define V and $\tilde{V}$ to be the subgroups of elements of A_{4q} and $A_{4q} \langle \nu_q \rangle \cong S_{4q}$ respectively which leave each of the subsets $\{1, 2\}, \{3, 4\}, \dots, \{4q-1, 4q\}$ invariant. Thus $V \cong C_2^{2q-1}$ and $\tilde{V} \cong C_2^{2q}$ for $k \geq 1$, whereas $V = \tilde{V} \cong C_2^{2q-1}$ if $k = 0$. If $q \geq 3$, then $V = O_2(\beta_q)$.

We set $M := C_{A_{2m}}(t_q) = (\beta_q \times L) \langle \nu_q \rangle$. In what follows we determine the components of M .

Lemma 5.3.1. *For $k < 3$ the fusion system $C_{\mathcal{F}}(t_q)$ is constrained and $M \in \mathcal{G}(C_{\mathcal{F}}(t_q))$. In particular M has no component.*

Proof. Recall $L\langle\nu_q\rangle \cong S_{2k}$. So for $k = 0, 1$ or 2 , $C_M(O_2(M)) \leq O_2(M)$. So by Lemma 3.10.2 and Theorem 3.10.4, $C_{\mathcal{F}}(t_q)$ is constrained and M is a model of $C_{\mathcal{F}}(t_q)$. Thus $F^*(M) = O_2(M)$ and so M has no component. $\square$

Lemma 5.3.2. *For $k \geq 3$, L is the unique component of M and L is isomorphic to A_{2k} .*

Proof. We note that $\text{Comp}(M) = \text{Comp}(\beta_q \times L) = \text{Comp}(\beta_q) \cup \text{Comp}(L)$ by Lemma 2.8.6 and Lemma 2.8.9. A component of β_q would centralize $O_2(\beta_q) \cong C_2^{2q-1}$ but one checks easily $C_{\beta_q}(O_2(\beta_q)) \leq O_2(\beta_q)$ and so $\text{Comp}(\beta_q) = \emptyset$. It follows that $\text{Comp}(M) = \text{Comp}(L)$. For $k \geq 3$, observe that $L \cong A_{2k}$ is simple. Therefore the set of components of M , $\text{Comp}(M) = \text{Comp}(L) = \{L\}$. $\square$

Set now $H := \overline{C_G(t_q)} := C_G(t_q)/O(C_G(t_q))$.

Lemma 5.3.3. *Identifying V with its image in H under the natural homomorphism, we have $V \trianglelefteq H$.*

Proof. We identify elements and subgroups of $S_0 := C_S(t_q)$ with their images in H under the natural homomorphism. As $V \trianglelefteq C_{A_{2m}}(t_q)$, the subgroup V is normal in $C_{\mathcal{F}}(t_q) = \mathcal{F}_{S_0}(H)$. Hence, V is strongly closed in H . Set

$$J := \langle V^H \rangle.$$

By Theorem 2.10.6, J can be written as a central product $J = U * K_1 * \cdots * K_r$, where U is an abelian 2-group, $r \geq 0$, and $K_1, \dots, K_r$ are of type I or II. By Lemma 2.10.7, $\text{Aut}_J(V)$ is of odd order and non-trivial for $r \geq 1$. Observe that $\text{Aut}_J(V) \trianglelefteq \text{Aut}_H(V)$, and that $\text{Aut}_H(V) = \text{Aut}_{C_{\mathcal{F}}(t_q)}(V) = \text{Aut}_M(V) \cong S_{2q}$ for $q \geq 2$ does not have a normal subgroup of odd order. This yields $r = 0$, i.e. $J = U = V$ is an abelian 2-group and V is normal in H . $\square$

Lemma 5.3.4. *For $k = 2$, the group H has at most one component K which is isomorphic to $\text{PSL}_2(w)$ for $w \equiv 3, 5 \pmod{8}$. If $k = 0$ or 1 , then the group H has no component. Moreover identifying each subgroup of $C_S(t_q)$ with its image in H , if there is a component K , then $V \cap K = 1$, $V \trianglelefteq H$, and $C_S(t_q) \cap K = C_S(t_q) \cap L$.*

Proof. Identify elements and subgroups of $S_0 := C_S(t_q)$ with their images in H under the natural homomorphism. Set $S_L := S_0 \cap L$ and $A := V \times S_L$. Notice that $A = C_{S_0}(V)$ and $S_L \cong C_2^k$. Suppose there is a component K of H . As $V \trianglelefteq H$ by Lemma 5.3.3, we have $[V, K] = 1$. In particular, $K \cap S_0 \leq A$ and $K \cap V \leq Z(K)$, so a Sylow subgroup of $K/Z(K)$

has order at most $|S_L|$. As $K/Z(K)$ is simple, it follows from Burnside's Theorem (See Theorem 2.2.2 and Corollary 2.2.3) that K does not have cyclic Sylow 2-subgroups. Hence, $k = 2$, $K/Z(K)$ has a Sylow 2-subgroup of order 4 and $A = V(K \cap S_0)$. If there were a second component E of H , then E would have to centralize V and K . Thus, $E \cap S_0 \leq A = C_{S_0}(V) = V(K \cap S_0)$ would be central in E , contradicting the Feit-Thompson Theorem (See Theorem 1.3.4). This shows that K is the only component.

As K is quasisimple, $K = O^{2'}(K) = \langle (K \cap S_0)^K \rangle$. As $K \cap S_0 \leq A$, this yields

$$K \leq J := \langle A^H \rangle.$$

So K is a component of J . Notice now that A is abelian and normal in $C_{A_{2m}}(t_q)$, hence normal in $C_{\mathcal{F}}(t_q) = \mathcal{F}_{S_0}(H)$ and therefore strongly closed in H . Thus, by Theorem 2.10.6, a component of J is of type I or II. As $K/Z(K)$ has Sylow 2-subgroups of order 4, it follows that K is of type II, $K/Z(K) \cong PSL_2(w)$ for $w \equiv 3, 5 \pmod{8}$. In particular, by definition of type II, $Z(K)$ is of odd order and thus trivial as $O(H) = 1$. Thus, $K \cong PSL_2(w)$, $K \cap S_0 \cong C_2^2$ and $N_K(K \cap S_0) \cong A_4$. In particular, K is simple and so $K \cap V = 1$. Moreover, $Aut_K(A) \cong C_3$ and $Aut_K(A) \trianglelefteq Aut_H(A)$ as $K \trianglelefteq H$. Notice that $Aut_H(A) = Aut_{C_{\mathcal{F}}(t_q)}(A) = Aut_M(A) \cong S_{2q} \cong S_3$ and so $C_3 \cong Aut_L(A) = O_3(Aut_H(A))$. This implies $Aut_K(A) = Aut_L(A)$ and thus

$$K \cap S_0 = [A, Aut_K(A)] = [A, Aut_L(A)] = S_L.$$

□

Lemma 5.3.5. *For $k \geq 3$, the group H has a unique component K . Moreover, K is simple, and identifying $C_S(t_q)$ with its image in H under the natural homomorphism, we have*

$$C_S(t_q) \cap L = C_S(t_q) \cap K \text{ and } \mathcal{F}_{C_S(t_q) \cap L}(L) = \mathcal{F}_{C_S(t_q) \cap K}(K).$$

Proof. Let $L \cong A_{2k}$ be a component of M , which exists by Lemma 5.2.1. Appealing to Lemma 3.13.4, L has a simple 2-fusion system. Define $\mathcal{C} = \mathcal{F}_{C_S(t_q) \cap L}(L)$. Then from Lemma 3.14.2, $\mathcal{C}$ is a component of $\mathcal{F}_{C_S(t_q)}(C_{A_{2m}}(t_q)) = C_{\mathcal{F}_S(A_{2m})}(t_q) = C_{\mathcal{F}}(t_q)$. Set $\overline{C_G(t_q)} := C_G(t_q)/O(C_G(t_q))$ and $\overline{C_S(t_q)} := C_S(t_q)/O(C_G(t_q))$. This implies that $O(\overline{C_G(t_q)}) = 1$, and $C_{\mathcal{F}}(t_q) = \mathcal{F}_{C_S(t_q)}(\overline{C_G(t_q)}) \cong \mathcal{F}_{\overline{C_S(t_q)}}(\overline{C_G(t_q)})$. Note that as $\mathcal{C}$ is a component of $C_{\mathcal{F}}(t_q)$, its image $\overline{\mathcal{C}}$ under the natural isomorphism from $C_{\mathcal{F}}(t_q)$ to $\mathcal{F}_{\overline{C_S(t_q)}}(\overline{C_G(t_q)})$ is a component of $\mathcal{F}_{\overline{C_S(t_q)}}(\overline{C_G(t_q)})$. It follows from Lemma 3.14.3 that there exist a component K of $H = \overline{C_G(t_q)}$ such that $\mathcal{F}_{\overline{C_S(t_q) \cap K}}(K) = \overline{\mathcal{C}}$. The fusion system $\overline{\mathcal{C}}$ is isomorphic to the 2-fusion system of A_{2k} since $\overline{\mathcal{C}}$ is isomorphic to $\mathcal{C}$. Furthermore as K is subnormal in H , we have by Lemma 2.8.2 that $O(K) = 1$. By Theorem 3.8.12 $Z(K) = Z(\mathcal{F}_{\overline{C_S(t_q) \cap K}}(K))$ which is isomorphic to the center of the 2-fusion system of A_{2k} which is trivial. It follows that K is simple. Identify now $C_S(t_q)$

with its image in H and note that it remains only to show that K is the unique component of H . Assume there is a component $R \neq K$ of H . Then R would centralize K and also V , as $V \trianglelefteq H$ by Lemma 5.3.3. Hence, $R \cap C_S(t_q) \leq C_{C_S(t_q)}(V) = V(C_S(t_q) \cap L) = V(C_S(t_q) \cap K)$, and so $R \cap C_S(t_q)$ is central in R . This contradicts the Feit-Thompson Theorem (see Theorem 1.3.4).

□

Proposition 5.3.6. *For $k = 0$ or 1 , $H \cong M$.*

Proof. By Lemma 5.3.4, H has no component and so $F^*(H) = O_2(H)$. This implies that $C_H(O_2(H)) = Z(O_2(H)) \leq O_2(H)$. As $C_{\mathcal{F}}(t_q) \cong \mathcal{F}_{\overline{C_S(t_q)}}(H)$ is constrained, $H \in \mathcal{G}(C_{\mathcal{F}}(t_q))$. Notice from Lemma 5.3.1 that $M \in \mathcal{G}(C_{\mathcal{F}}(t_q))$ and so by Theorem 3.10.6, $H \cong M$. □

Remark 5.3.7. Recall that $k = m - 2q$. Hence, if $k = 0$, then $m = 2q$ so that $A_{2m} = A_{4q}$ and $A_{2m+1} = A_{4q+1}$. One sees then that

$$C_{A_{2m}}(t_q) = C_{A_{2m+1}}(t_q) \text{ if } k = 0.$$

If $k = 1$, then $m = 2q + 1$, $C_{A_{2m}}(t_q) = \beta_q \langle \nu_q \rangle$ and $C_{A_{2m+1}}(t_q) = (\beta_q \times N) \langle \nu_q \rangle$ where

$$N := A_{\{2m-1, 2m, 2m+1\}} \cong A_3.$$

So $N = O(C_{A_{2m+1}}(t_q))$ and the map

$$C_{A_{2m}}(t_q) \rightarrow C_{A_{2m+1}}(t_q)/N, x \mapsto xN \text{ is an isomorphism for } k = 1.$$

As before we denote by $t = (12)(34)$ the short involution. The following lemma gives us a connection between the structure of $C_G(t_q)$ and that of $C_G(t)$.

Lemma 5.3.8. *Suppose H has a component K . Then one of the following holds:*

- (i) $k \geq 3$, $K \cong A_{2k}$, and $C_G(t)/O(C_G(t))$ is isomorphic to $C_{A_{2m}}(t)$;
- (ii) $k \geq 2$, $K \cong A_{2k+1}$, and $C_G(t)/O(C_G(t))$ is isomorphic to $C_{A_{2m+1}}(t)$;

Proof. Notice from Lemma 5.3.3 that $V \leq H$, and so $[V, K] = 1$. By Lemma 5.3.4 and Lemma 5.3.5 K is simple and $K \cap C_S(t_q) = L \cap C_S(t_q)$. Let K_0 be the full preimage of K in $C_G(t_q)$, and set $R := O(C_G(t_q))$. As usual $H = \overline{C_G(t_q)} = C_G(t_q)/R$. Since $t \in V$, we have $[t, K] = 1$ and so $[K_0, t] \leq R$. It follows that $\overline{K_0 \langle \bar{t} \rangle} = C_{\overline{K_0 \langle \bar{t} \rangle}}(\bar{t})$. By Lemma 2.1.13 we get $\overline{K_0 \langle \bar{t} \rangle} = \overline{C_{K_0 \langle \bar{t} \rangle}(t)} = \overline{C_{K_0}(t) \langle \bar{t} \rangle}$. As $\overline{K_0} = K$, $K = O^2(K \langle \bar{t} \rangle) \leq O^2(\overline{C_{K_0}(t)})$. Therefore $K_1 := O^2(C_{K_0}(t))$ is a preimage of K in $C_G(t_q)$ and $K_1 \leq C_G(t)$. Thus $K_1 \leq C_{C_G(t)}(t_q)$. Notice that $K = \overline{K_1} = K_1 R / R \cong K_1 / K_1 \cap R$. Let $R_t := O(C_G(t))$ and set $\overline{C_G(t)} = C_G(t) / R_t$.

Note $\widetilde{K}_1 = K_1 R_t / R_t \cong K_1 / K_1 \cap R_t$. As $K_1 \cap R_t \trianglelefteq K_1$ and $K_1 / K_1 \cap R \cong K$ is simple, we have that $K_1 \cap R_t \leq K_1 \cap R$. Let $D := K_1 \cap R$ and note that $D \cap R_t = K_1 \cap R_t$. Then we have

$$\widetilde{K}_1 / \widetilde{D} = (K_1 R_t / R_t) / (D R_t / R_t) \cong (K_1 / K_1 \cap R_t) / (D / K_1 \cap R_t) \cong K_1 / D \cong K.$$

As $O^2(\widetilde{K}_1) = \widetilde{K}_1$, we have $\widetilde{K}_1 \leq O^2(\widetilde{C_G(t)})$. By Lemma 5.2.2 and Proposition 5.2.3(b), we have $\widetilde{Y} := O^2(\widetilde{C_G(t)}) \cong A_{2(m-2)}$ if $H \cong C_{A_{2m}}(t)$ and $\widetilde{Y} \cong A_{2m-3}$ if $H = C_{A_{2m+1}}(t)$. Then we have $\widetilde{K}_1 \leq C_{\widetilde{Y}}(\widetilde{t_q})$. By Proposition 5.2.3(a), $F^*(\widetilde{C_G(t)}) = \widetilde{V}_4 \times \widetilde{Y}$ where $\widetilde{V}_4 = \langle (12)(34), (13)(24) \rangle O(C_G(t)) / O(C_G(t))$. Also realize that $\widetilde{t_q}$ can be written in a form $\widetilde{t_q} = \widetilde{t} \widetilde{t_0}$ where $\widetilde{t_0} = (5\ 6)(7\ 8) \dots (4q-1\ 4q)$. Note $\widetilde{t_0} \in \widetilde{Y}$ as $\widetilde{Y} \cap \widetilde{C_S(t)} = A_{\{5,6,7,\dots,2m\}} \cap S$. Thus $\widetilde{t_q} \in \widetilde{V}_4 \times \widetilde{Y}$. As $K_0 \trianglelefteq C_G(t_q)$, we have $K_1 = O^2(C_{K_0}(t)) \trianglelefteq C_{C_G(t)}(t_q)$. So $\widetilde{K}_1 \trianglelefteq C_{\widetilde{C_G(t)}}(\widetilde{t_q}) = C_{\widetilde{C_G(t)}}(\widetilde{t_q})$ by Lemma 2.1.13. As $\widetilde{K}_1 \leq \widetilde{Y}$, this implies $\widetilde{K}_1 \trianglelefteq C_{\widetilde{Y}}(\widetilde{t_q})$. It follows that $\widetilde{K}_1 \trianglelefteq C := C_{\widetilde{Y}}(\widetilde{t_0}) \cong ((C_2^{2q-3} \rtimes S_{2q-2}) \times \widetilde{Y}_1) \rtimes C_2$ where $\widetilde{Y}_1 \cong A_{2m-4q} = A_{2k}$ if $\cong A_{2m-4}$ and $\widetilde{Y}_1 \cong A_{2m-4q+1} = A_{2k+1}$ if $\cong A_{2m-3}$. Notice $O_2(\widetilde{K}_1) = 1$, and so $[\widetilde{K}_1, O_2(C)] \leq O_2(\widetilde{K}_1) = 1$. If $k = 2$ and $\widetilde{Y}_1 \cong A_4$, then $\widetilde{K}_1 \leq C_C(O_2(C)) \leq O_2(C)$, a contradiction. So if $k = 2$, $\widetilde{Y}_1 \cong A_5$. In particular $\widetilde{Y}_1$ is in either case simple and thus a component of C . Hence $[\widetilde{K}_1, \widetilde{Y}_1] = 1$ or $\widetilde{Y}_1 \leq \widetilde{K}_1$. If $[\widetilde{K}_1, \widetilde{Y}_1] = 1$, then $\widetilde{K}_1 \leq C_C(\widetilde{Y}_1 O_2(C)) \leq O_2(C)$, a contradiction. This implies $\widetilde{Y}_1 \leq \widetilde{K}_1$. So $\widetilde{K}_1 / \widetilde{D} \cong K$ has $\widetilde{Y}_1 \widetilde{D} / \widetilde{D} \cong \widetilde{Y}_1$ as a component. As K is simple, it follows $K \cong \widetilde{Y}_1$ and the assertion follows. □

Proposition 5.3.9. *If $k = 2$, and H has no component, then $M \cong H$ via an isomorphism which sends s to $sO(C_G(t_q))$ for $s \in C_S(t_q)$. Moreover $C_G(t)/O(C_G(t)) \cong C_{A_{2m}}(t)$.*

Proof. As $E(H) = 1$, we have $F^*(H) = O_2(H)$ and so $C_H(O_2(H)) = Z(O_2(H)) \leq O_2(H)$. By Lemma 3.10.2 $C_{\mathcal{F}}(t_q) \cong \mathcal{F}_{\widetilde{C_S(t_q)}}(H)$ is constrained, and so $H \in \mathcal{G}(C_{\mathcal{F}}(t_q))$. Observe $M \in \mathcal{G}(C_{\mathcal{F}}(t_q))$ by Lemma 5.3.1. It follows from Theorem 3.10.6 that $H \cong M$ via an isomorphism which restricts to the identity on $C_S(t_q)$ if we identify $C_S(t_q)$ with its image in H .

Assume now that $\widetilde{C_G(t)} := C_G(t)/O(C_G(t))$ is not isomorphic to $C_{A_{2m}}(t)$. By Proposition 5.2.3(c) we have then $\widetilde{C_G(t)} \cong C_{A_{2m+1}}(t)$. Note that $C_{\widetilde{C_{A_{2m+1}}(t)}}(t_q)$ contains a subgroup isomorphic to $A_{2k+1} = A_5$ which is also centralized by β_q and thus by $O_2(\beta_q)$ of which t_q is a member. Hence, by Lemma 2.1.13,

$$C_{\widetilde{C_G(t)}}(\widetilde{O_2(\beta_q)}) = C_{\widetilde{C_G(t)}}(\widetilde{O_2(\beta_q)})$$

contains a subgroup isomorphic to A_5 . Thus, $C_G(O_2(\beta_q)) = C_{C_G(t_q)}(O_2(\beta_q))$ contains a subgroup R with $R/O(R) \cong A_5$. As $H \cong C_{A_{2m}}(t_q) = (\beta_q \times L)\langle \nu_q \rangle$ with $L \cong A_4$, we have

that $C_{\overline{C_G(t_q)}}(\overline{O_2(\beta_q)}) = C_H(\overline{O_2(\beta_q)}) \cong S_4$. As groups of odd order are solvable, it follows that $C_G(O_2(\beta_q))$ is solvable and has thus no section isomorphic to A_5 . This contradiction proves the assertion. $\square$

Proposition 5.3.10. *Let $k \geq 2$, and for $k = 2$ assume that H has a component.*

- (i) $H \cong C_{A_{2m}}(t_q)$ and $C_G(t)/O(C_G(t)) \cong C_{A_{2m}}(t)$.
- (ii) $H \cong C_{A_{2m+1}}(t_q)$ and $C_G(t)/O(C_G(t)) \cong C_{A_{2m+1}}(t)$.

Moreover, an isomorphism from H to $C_{A_{2m}}(t_q)$ or $C_{A_{2m+1}}(t_q)$ can be chosen such that it restricts to the identity on $C_S(t_q)$.

Proof. Identify subgroups of $S_0 := C_S(t_q)$ with their images in H under the natural homomorphism of $C_G(t_q)$ onto H . By Lemma 5.3.4 and Lemma 5.3.5 there exists a unique component K of H , and for such K , $S_0 \cap K = S_0 \cap L$, $\mathcal{F}_{S_0 \cap K}(K) = \mathcal{F}_{S_0 \cap L}(L)$ and K is simple. Indeed, by Lemma 5.3.8 one of the following holds:

- (i') $k \geq 3$, $K \cong A_{2k}$ and $C_G(t)/O(C_G(t)) \cong C_{A_{2m}}(t)$.
- (ii') $K \cong A_{2k+1}$ and $C_G(t)/O(C_G(t)) \cong C_{A_{2m+1}}(t)$.

Our goal will be thus to show that

$$H \cong C_{A_{2m}}(t_q) \text{ if } K \cong A_{2k}, \text{ and } H \cong C_{A_{2m+1}}(t_q) \text{ if } K \cong A_{2k+1},$$

with an isomorphism from H to $C_{A_{2m}}(t_q)$ or $C_{A_{2m+1}}(t_q)$ which restricts to the identity on $C_S(t_q)$.

Note that

$$M := C_{A_{2m}}(t_q) \cong (\beta_q \times L)\langle \nu_q \rangle \text{ with } L \cong A_{2k}, \text{ and}$$

$$\tilde{M} := C_{A_{2m+1}}(t_q) \cong (\beta_q \times \tilde{L})\langle \nu_q \rangle \text{ where } \tilde{L} = A_{\{4q+1, \dots, 2m+1\}} \cong A_{2k+1}.$$

Set $N := C_H(K)$. By Theorem 3.12.5 applied both in M and H , we have

$$S_0 \cap \beta_q = C_{S_0}(L) = C_{S_0}(\mathcal{F}_{S_0 \cap L}(L)) = C_{S_0}(\mathcal{F}_{S_0 \cap K}(K)) = C_{S_0}(K) = N \cap S_0.$$

Thus

$$S_0 \cap \beta_q = S_0 \cap N \in \text{Syl}_2(N) \tag{5.2}$$

In particular,

$$\begin{aligned} S_0 &= ((S_0 \cap \beta_q) \times (S_0 \cap L)) \rtimes \langle \nu_q \rangle \\ &= (S_0 \cap N) \times (S_0 \cap K) \rtimes \langle \nu_q \rangle. \end{aligned}$$

As $Out(K)$ is a 2-group, we have $H = NKS_0 = NK\langle\nu_q\rangle$. Since K is simple, we get that $K \cap N = 1$. So

$$H = (N \times K) \rtimes \langle\nu_q\rangle. \quad (5.3)$$

We argue next that

$$K\langle\nu_q\rangle \cong \begin{cases} L\langle\nu_q\rangle \cong S_{2k} & \text{if } K \cong A_{2k}, \\ \tilde{L}\langle\nu_q\rangle \cong S_{2k+1} & \text{if } K \cong A_{2k+1} \end{cases} \quad (5.4)$$

As $\nu_q \notin NK$, we have that ν_q induces an outer automorphism of $K \cong A_{2k}$ or A_{2k+1} . If $K \cong A_{2k}$ for $k \geq 4$ or if $K \cong A_{2k+1}$, then it follows immediately that

$$K\langle\nu_q\rangle \cong Aut(K) \cong \begin{cases} S_{2k} \cong L\langle\nu_q\rangle & \text{if } K \cong A_{2k}, \\ S_{2k+1} \cong \tilde{L}\langle\nu_q\rangle & \text{if } K \cong A_{2k+1}. \end{cases}$$

Consider now for a moment the case $k = 3$ and $K \cong A_6 \cong L_2(9)$. Note that $(K \cap S_0)\langle\nu_q\rangle = (L \cap S_0)\langle\nu_q\rangle$ is a Sylow 2-subgroup of $L\langle\nu_q\rangle \cong S_6$ and thus isomorphic to $D_8 \times C_2$. There is only one almost simple extension of A_6 with a Sylow 2-subgroup of this form, namely S_6 . Therefore, $K\langle\nu_q\rangle \cong S_6 \cong L\langle\nu_q\rangle$. This shows Equation 5.4.

By Equation 5.4, there is an isomorphism γ such that

$$\gamma : K\langle\nu_q\rangle \rightarrow L\langle\nu_q\rangle \text{ if } K \cong A_{2k}, \text{ and } \gamma : K\langle\nu_q\rangle \rightarrow \tilde{L}\langle\nu_q\rangle \text{ if } K \cong A_{2k+1}.$$

As $K = O^2(K\langle\nu_q\rangle)$, $O^2(L\langle\nu_q\rangle) = L$ and $O^2(\tilde{L}\langle\nu_q\rangle) = \tilde{L}$, we have $K\gamma = L$ if $K \cong A_{2k}$ and $K\gamma = \tilde{L}$ if $K \cong A_{2k+1}$. Recall that $\mathcal{F}_{S_0}(H) = \mathcal{F}_{S_0}(M) = \mathcal{F}_{S_0}(\tilde{M})$ and $\mathcal{F}_{S_0 \cap K}(K) = \mathcal{F}_{S_0 \cap L}(L) = \mathcal{F}_{S_0 \cap \tilde{L}}(\tilde{L})$ where $K \trianglelefteq H$, $L \trianglelefteq M$ and $\tilde{L} \trianglelefteq \tilde{M}$. Hence, by Lemma 3.12.7(a), we have

$$\mathcal{F}_{(S_0 \cap K)\langle\nu_q\rangle}(K\langle\nu_q\rangle) = \mathcal{F}_{(S_0 \cap L)\langle\nu_q\rangle}(L\langle\nu_q\rangle) = \mathcal{F}_{(S_0 \cap \tilde{L})\langle\nu_q\rangle}(\tilde{L}\langle\nu_q\rangle).$$

Hence, by Corollary 3.15.3, we can choose the isomorphism γ such that it restricts to the identity on $(S_0 \cap K)\langle\nu_q\rangle$.

Note that a Sylow 2-subgroup of A_{2k} or A_{2k+1} contains for $k \geq 2$ its own centralizer. This implies that $C_H(S_0 \cap K) \leq N = C_H(K)$, $C_M(S_0 \cap L) \leq \beta_q = C_M(L)$ and $C_{\tilde{M}}(S_0 \cap \tilde{L}) \leq \beta_q = C_{\tilde{M}}(\tilde{L})$. Hence, by Lemma 3.12.7(b) applied with (H, M, K, L) or $(H, \tilde{M}, K, \tilde{L})$ in place of (G_1, G_2, N_1, N_2) and with $\langle\nu_q\rangle$ in place of R , we have

$$\mathcal{F}_{(N \cap S_0)\langle\nu_q\rangle}(N\langle\nu_q\rangle) = \mathcal{F}_{(\beta_q \cap S_0)\langle\nu_q\rangle}(\beta_q\langle\nu_q\rangle).$$

As K is the only component of H , we have $E(N\langle\nu_q\rangle) = E(N) = 1$. Moreover, $O(N\langle\nu_q\rangle) = O(N) \leq O(H) = 1$. Thus $F^*(N\langle\nu_q\rangle) = O_2(N\langle\nu_q\rangle)$ and $N\langle\nu_q\rangle$ is of characteristic 2. Note that

also $\beta_q \langle \nu_q \rangle$ is of characteristic 2. Thus, $N \langle \nu_q \rangle$ and $\beta_q \langle \nu_q \rangle$ are models for the same saturated fusion system. Hence, by Theorem 3.10.6 there is an isomorphism $\delta : N \langle \nu_q \rangle \rightarrow \beta_q \langle \nu_q \rangle$ which restricts to the identity on $(N \cap S_0) \langle \nu_q \rangle$. Note that δ maps

$$N = O^2(N)(S_0 \cap N) = O^2(N \langle \nu_q \rangle)(S_0 \cap N)$$

to $\beta_q = O^2(\beta_q)(S_0 \cap \beta_q) = O^2(\beta_q \langle \nu_q \rangle)(S_0 \cap \beta_q)$. Hence, it follows from Lemma 1.1.25 that

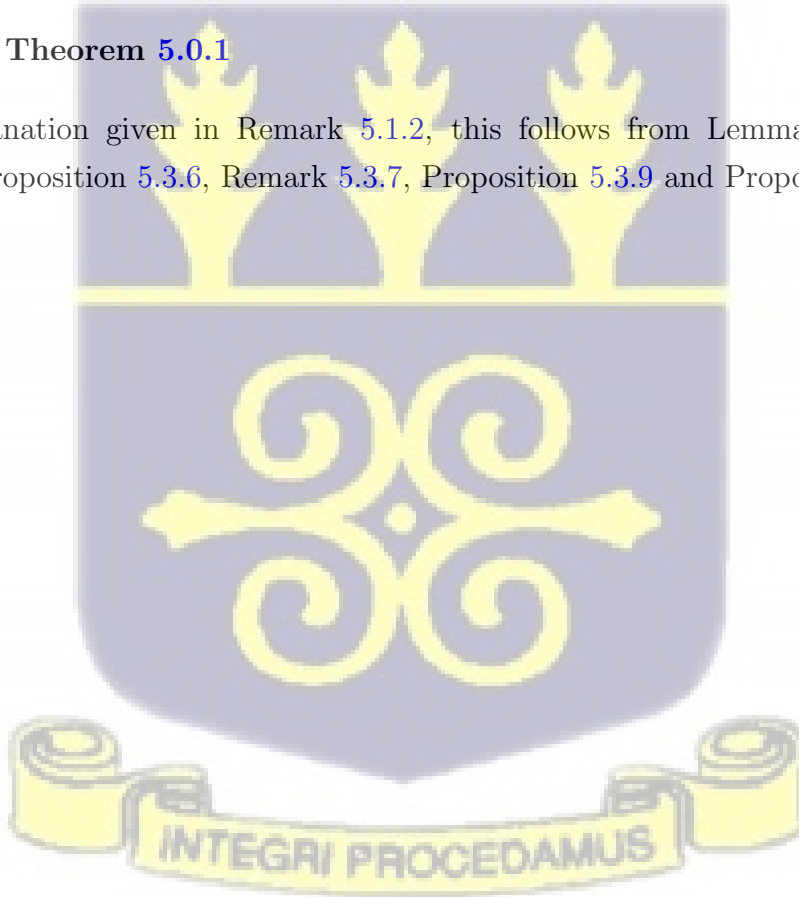
$$H \cong M = C_{A_{2m}}(t_q) = (\beta_q \times L) \rtimes \langle \nu_q \rangle \text{ if } K \cong A_{2k}, \text{ and}$$

$$H \cong \tilde{M} = C_{A_{2m+1}}(t_q) = (\beta_q \times \tilde{L}) \rtimes \langle \nu_q \rangle \text{ if } K \cong A_{2k+1}$$

with an isomorphism from H to M or $\tilde{M}$ which restricts to the identity on S_0 . As (i') or (ii') holds, this shows the assertion. □

5.4 Proof of Theorem 5.0.1

With the explanation given in Remark 5.1.2, this follows from Lemma 5.1.1, Proposition 5.2.3(c), Proposition 5.3.6, Remark 5.3.7, Proposition 5.3.9 and Proposition 5.3.10.



Chapter 6 Finite groups realizing the fusion system of A_{2m} , for $m \geq 6$, over its Sylow 2-subgroup

The main goal of this chapter is to prove Theorem A. We first show that any finite group G satisfying the conditions of Hypothesis B, G is isomorphic to X where X is as described in Theorem 5.0.1. We achieve this by first limiting the structure of $O(C_G(u))$ for any involution u of G . In most cases $O(C_G(u))$ is trivial while in some cases $O(C_G(u))$ has order 3. Then by using induction we achieve our aim.

6.1 The structure of G

Throughout this section, we will work under the conditions of Hypothesis B. Moreover, by Theorem 5.0.1, we may choose $X \in \{A_{2m}, A_{2m+1}\}$ such that for every involution $u \in S$, there is an isomorphism $C_G(u)/O(C_G(u)) \rightarrow C_X(u)/O(C_X(u))$ sending $sO(C_G(u))$ to $sO(C_X(u))$ for every $s \in C_S(u)$. Set $\mathcal{F} := \mathcal{F}_S(G)$ and recall that $\mathcal{F} = \mathcal{F}_S(A_{2m}) = \mathcal{F}_S(X)$. We recall from Section 4.2 that for the alternating group A_{2m} of degree $2m$ we let $2m = 4s + d$ where s is a positive integer and $d = 0$ or 2 , and defined the generalized involution t_q for $1 \leq q \leq s$. We now fix s, q and t_q as defined. We also adopt now Notation 4.5.1 so that A , B_0 and $\mathcal{B}$ are defined. By Lemma 4.4.1, every Sylow 2-subgroup of $C_{A_{2m}}(t_s)$ is a Sylow 2-subgroup of A_{2m} . As A is elementary abelian and $t_s \in A$, we can choose a Sylow 2-subgroup of $C_{A_{2m}}(t_s)$ containing A . Thus, we may assume from now on

$$A \leq S \leq C_{A_{2m}}(t_s).$$

Remark 6.1.1. We also remark here that as $\mathcal{F} = \mathcal{F}_S(A_{2m}) = \mathcal{F}_S(X)$, we get $\mathcal{F}_{C_S(x)}(C_G(x)) = \mathcal{F}_{C_S(x)}(C_X(x))$. This means that if some element of $C_X(x)$ conjugates some involution $u \in S$ into A , then also some element of $C_G(x)$ does the same.

Lemma 6.1.2. *Set $G_0 := \langle C_G(x) : x \in A^\# \rangle N_G(A)$. Then every involution of S is G_0 -conjugate to an element of A .*

Proof. We first demonstrate that

$$\text{If } t \in S \text{ is a double transposition, then } t \text{ is } G_0\text{-conjugate into } A. \quad (6.1)$$

For the proof note that, by Lemma 4.5.7, there is a double transposition $x \in A$ centralizing t such that t is $C_X(x)$ -conjugate into A . By Remark 6.1.1, we have that t is conjugate to a under $C_G(x) \leq G_0$. This proves (6.1).

Let now $u \in S$ be an arbitrary involution. By (6.1), we may assume that u is not a double transposition. We write

$$u = tu_0$$

where t is a double transposition and u_0 is an involution disjoint from t . By (6.1), t is G_0 -conjugate to an element $a' \in A$. By Lemma 4.5.2(c), there is a fully $\mathcal{F}$ -centralized conjugate a of a' in A . By Lemma 4.5.2(a), a' is conjugate to a under $\text{Aut}_{A_{2m}}(A) = \text{Aut}_{\mathcal{F}}(A) = \text{Aut}_G(A)$. As $N_G(A) \leq G_0$, this means that a and a' are G_0 -conjugate. Thus, t is G_0 -conjugate to the fully $\mathcal{F}$ -centralized element $a \in A$.

We note that $S \leq N_G(A) \leq G_0$ by Lemma 4.5.2(b). Therefore, S is a Sylow 2-subgroup of G_0 and $\mathcal{F}_0 := \mathcal{F}_S(G_0)$ is saturated. As $\mathcal{F}_0 \subseteq \mathcal{F}$ and a is fully $\mathcal{F}$ -centralized, a is also fully $\mathcal{F}_0$ -centralized. Therefore, there is $\alpha \in \text{Hom}_{\mathcal{F}_0}(C_S(t), C_S(a))$ such that $t\alpha = a$. Notice that $u, u_0 \in C_S(t)$ and $u\alpha = a(u_0\alpha)$ is G_0 -conjugate to u . As $\mathcal{F}_0 \subseteq \mathcal{F} = \mathcal{F}_S(A_{2m})$, α can also be realized by conjugation by an element of A_{2m} . Hence, $a = t\alpha$ is a double transposition and $u_0\alpha$ is an involution disjoint from a . Thus, replacing (u, t, u_0) by $(u\alpha, a, u_0\alpha)$, we may assume from now on that

$$t \in A.$$

Then by Lemma 4.5.6, u is conjugate to an element $b \in A$ under $C_X(t)$. Again by Remark 6.1.1, it follows that u is conjugate to $b \in A$ under $C_G(t) \leq G_0$. This proves the assertion. $\square$

Lemma 6.1.3. *The following hold:*

- (a) *If $X = A_{2m}$ or if $d = 0$, then G is 1-balanced with respect to A , that is*

$$C_G(u) \cap O(C_G(v)) \leq O(C_G(u)) \text{ for all } u, v \in A^\#.$$

- (b) *If $X = A_{2m+1}$ and $d = 2$, then for every $u \in A$ there exists $B \in \mathcal{B}$ such that, for every $v \in B$, we have*

$$C_G(u) \cap O(C_G(v)) \leq O(C_G(u)).$$

If $u = t_q$ for some $1 \leq q \leq s$, then we may take $B = B_0$.

- (c) *If $X = A_{2m+1}$ and $d = 2$, then G is 3-balanced with respect to A .*

Proof. **(a,b)** By Lemma 4.5.3, we can pick $u, v \in A^\#$ with $O(C_X(u) \cap C_X(v)) \leq O(C_X(u))$. Set $\widetilde{C_X(u)} = C_X(u)/O(C_X(u))$ and $\overline{C_G(u)} = C_G(u)/O(C_G(u))$. By Lemma 2.1.13 and Theorem 5.0.1 we have then

$$1 = O(\widetilde{C_{C_X(u)}(v)}) = O(\widetilde{C_{\widetilde{C_X(u)}}(\tilde{v})}) \cong O(C_{\overline{C_G(u)}}(\bar{v})) = O(\overline{C_{C_G(u)}(v)}) \geq \overline{O(C_G(u) \cap C_G(v))}$$

and thus $C_G(u) \cap O(C_G(v)) \leq O(C_G(u) \cap C_G(v)) \leq O(C_G(u))$. Thus, parts (a) and (b) follow from Lemma 4.5.3.

(c) For the proof of (c) take $F \in \mathcal{E}_3(A)$ and $1 \neq u \in A$. By (b), we can choose $B \in \mathcal{B}$ such that $C_G(u) \cap O(C_G(v)) \leq O(C_G(u))$ for all $v \in B$. As F has order 8 and B has index 4 in A , we can pick $1 \neq v \in B \cap F$. Then

$$C_G(u) \cap \Delta_G(F) \leq C_G(u) \cap O(C_G(v)) \leq O(C_G(u)),$$

Thus G is 3-balanced with respect to A . □

Lemma 6.1.4. *For every $u \in A^\#$, we have $C_G(u) = \Gamma_{A,2}(C_G(u)) \leq \Gamma_{A,2}(G)$.*

Proof. This follows from Lemmas 4.5.5(a), 2.13.2 and Theorem 5.0.1. □

Lemma 6.1.5. *Let $X = A_{2m+1}$ and $d = 2$. Write $\mathcal{I}$ for the set of double transpositions in A . Then the following hold:*

- (a) $C_G(u) = \Gamma_{A,4}(C_G(u))$ for every $u \in \mathcal{I}$.
- (b) For every $u \in A^\#$, we have $C_G(u) = \langle C_{C_G(u)}(\tau) : \tau \in \mathcal{I} \rangle$.
- (c) $C_G(u) \leq \Gamma_{A,4}(G)$ for every $u \in A^\#$.

Proof. (a) This follows from the choice of X and Lemmas 2.13.2 and 4.5.5(b).

(b) Let $u \in A^\#$. It follows from Lemma 2.1.13 and Lemma 4.5.5(c) that

$$C_G(u) = \langle C_{C_G(u)}(\tau) : \tau \in \mathcal{I} \rangle O(C_G(u)).$$

By Theorem 2.7.8, we have

$$O(C_G(u)) = \langle C_{O(C_G(u))}(F) : F \leq A, [A : F] = 2 \rangle.$$

For every $F \leq A$ of index 2, the subgroup F needs to intersect non-trivially with the subgroup $\langle \sigma_1, \tilde{\sigma}_1 \rangle = O_2(S_4)$ and contains thus a double transposition. Hence, $O(C_G(u)) \leq \langle C_{C_G(u)}(\tau) : \tau \in \mathcal{I} \rangle$ and so part (b) holds.

(c) This follows from parts (a) and (b) in combination with Lemma 2.13.5. □

Proposition 6.1.6. *The following hold:*

- (a) If $X = A_{2m}$ or if $d = 0$, then $O(C_G(u)) = 1$ for every $u \in A^\#$.
- (b) If $X = A_{2m+1}$ and $d = 2$, then $\Delta_G(F) = 1$ for every $F \in \mathcal{E}_3(A)$.

Proof. We will prove (a) and (b) simultaneously using Proposition 2.13.4. For that set $k = 1$ if $X = A_{2m}$ or if $d = 0$, and set $k = 3$ if $X = A_{2m+1}$ and $d = 0$. We check that the properties (i)-(vii) listed in Proposition 2.13.4 hold and refer to these properties in what follows.

As we assume that $O(G) = 1$, property (i) holds. Since $\mathcal{F}_S(G) = \mathcal{F}_S(A_{2m})$ and A_{2m} has more than one conjugacy class of involutions, we have that G has at least two conjugacy class of involutions, thus (ii) holds. Property (iii) that G is k -balanced with respect to A follows from Lemma 6.1.3(a),(c). Property (iv) follows from Lemma 6.1.4 if $k = 1$ and from Lemma 6.1.5 if $k = 3$. Notice $s \geq 3$, and so A has rank $2s \geq 6 \geq k + 2$. Thus property (v) holds.

By Lemma 4.5.2(b), A is weakly closed in S with respect to X . Thus, A is weakly closed in $\mathcal{F}_S(X) = \mathcal{F}_S(G)$ and thus weakly closed in S with respect to G . In particular, $N_G(S) \leq N_G(A)$. Thus (vi) holds. Property (vii) holds by Lemma 6.1.2. So by Proposition 2.13.4, we have $\Delta_G(F) = 1$ for every $F \in \mathcal{E}_k(A)$. For $k = 1$ this shows (a) and for $k = 3$ this shows (b). $\square$

Lemma 6.1.7. *Assume $X = A_{2m+1}$ and $d = 2$. Then the following hold:*

- (a) *If $v \in S$ is an involution, but not a long involution, then $O(C_G(v)) = 1$ and $C_G(v) \cong C_X(v)$ via an isomorphism which restricts to the identity on $C_S(v)$.*
- (b) *We have $|O(C_G(t_s))| \leq 3$.*

Proof. (a) By X , it is enough to show that $O(C_G(v)) = 1$. As v is conjugate into B_0 in $\mathcal{F}_S(A_{2m}) = \mathcal{F}_S(G)$ and thus under an element of G , we may assume $v \in B_0$. We argue first that

$$C_G(u) \cap O(C_G(v)) \leq O(C_G(u)) \text{ for every } u \in B_0^\#. \quad (6.2)$$

By Lemma 4.5.2(a), $N_X(B_0)$ and thus $N_G(B_0)$ acts transitively on the elements of B_0 of a given cycle type. Thus, for an arbitrary element $u \in B_0^\#$, there exists $g \in N_G(B_0)$ such that $u^g = t_q$ for some $1 \leq q \leq s - 1$. Then $v^g \in B_0$, so by Lemma 6.1.3(b), we have $C_G(u^g) \cap O(C_G(v^g)) \leq O(C_G(u^g))$. Conjugation with g^{-1} yields then (6.2).

Observe that B_0 acts on $O(C_G(v))$. Therefore, by Theorem 2.7.8, $O(C_G(v))$ is generated by subgroups of the form $C_{O(C_G(v))}(F)$ where $F \leq B_0$ has index 2 in B_0 . As B_0 has rank $2s - 2 \geq 4$, such a subgroup F has rank at least 3. Hence, it follows from (6.2) and Proposition 6.1.6(b) that

$$C_{O(C_G(v))}(F) = \bigcap_{u \in F^\#} C_G(u) \cap O(C_G(v)) \leq \bigcap_{u \in F^\#} O(C_G(u)) = \Delta_G(F) = 1,$$

proving (a).

(b) Let $N := O(C_G(t_s))$. We first claim that

Claim 1: If $1 \neq F \subseteq B_0$ with $C_N(F) \neq 1$, then

$$|C_N(F)| = 3 \text{ and } C_N(F) = O(C_{C_G(t_s)}(u))$$

for every $1 \neq u \in F$. For the proof of the claim, we note that for $1 \neq u \in F$,

$$C_N(F) \leq C_N(u) = O(C_G(t_s)) \cap C_G(u) \leq O(C_G(t_s) \cap C_G(u)) = O(C_{C_G(u)}(t_s)) = O(C_{C_G(t_s)}(u)).$$

By (a) $C_G(u) \cong C_X(u)$ and so by Lemma 4.5.3(c) $O(C_{C_G(t_s)}(u))$ has order 3. As $C_N(F)$ is non-trivial, we have that $C_N(F) = O(C_{C_G(t_s)}(u))$ has order 3, proving our claim.

We next claim that

Claim 2: If $F_1, F_2 \subseteq B_0$ are both of index 2 in B_0 and with $C_N(F_i) \neq 1$ for $i = 1, 2$, then $C_N(F_1) = C_N(F_2)$. To see that we note that $F_1 \cap F_2 \neq 1$. So for $1 \neq u \in F_1 \cap F_2$ we have by Claim 1 that

$$C_N(F_1) = O(C_{C_G(t_s)}(u)) = C_N(F_2).$$

This proves our claim.

Now by Theorem 2.7.8, we have that

$$N = \langle C_N(F) \mid F \leq B_0, |B_0 : F| = 2 \rangle.$$

One may assume that $N \neq 1$ and choose the $F \leq B_0$ with $C_N(F) \neq 1$. It follows immediately from Claim 2 that $N = C_N(F)$ and from Claim 1 that $N = C_N(F)$ has order 3. □

The following remark will be useful in proving the next lemma.

Remark 6.1.8. Let $n \in \{2m, 2m+1\}$ such that $X = A_n$. We know that

$$C_X((1\ 2)(3\ 4)) = (O_2(S_4) \times A_{\{5,6,\dots,n\}}) \langle (1\ 2)(5\ 6) \rangle.$$

In particular, we have that

$$O^2(C_X(1\ 2)(3\ 4)) = A_{\{5,6,\dots,n\}}.$$

For an arbitrary double transposition $(i\ j)(k\ l)$, it is conjugate to $(1\ 2)(3\ 4)$ and so we see that

$$O^2(C_X((i\ j)(k\ l))) = A_{\{1,\dots,n\} \setminus \{i,j,k,l\}}.$$

Particularly, we have that

$$O^2(C_X((5\ 6)(7\ 8))) = A_{\{1,2,3,4\} \cup \{9,\dots,n\}}.$$

Iterating this procedure we get that

$$\begin{aligned} O^2(C_X((5\ 6)(7\ 8))) \cap C_X((9\ 10)(11\ 12)) &= O^2(C_{A_{\{1,2,3,4\} \cup \{9, \dots, n\}}}((9\ 10)(11\ 12))) \\ &= A_{\{1,2,3,4\} \cup \{13, \dots, n\}}. \end{aligned}$$

Lemma 6.1.9. *There exists $H_0 \leq G$ such that $H_0 \cong X$.*

Proof. Let n be a positive integer such that $n \in \{2m, 2m+1\}$. We note that for every double transposition $t \in S$, we have that $O(C_G(t)) = 1$. As $O(C_X(t)) = 1$, it follows that we can fix an isomorphism

$$\phi_t : C_X(t) \rightarrow C_G(t) \text{ with } \phi_t|_{C_S(t)} = id_{C_S(t)}$$

for every such t . We set

$$t_1 := (5\ 6)(7\ 8), \quad t_2 := (9\ 10)(11\ 12).$$

Notice that $t_1, t_2 \in A \leq S$. Using the existence of ϕ_{t_1} and for $i \in \{3, 4\} \cup \{9, 10, \dots, n\}$, we set

$$v_i := (1\ 2\ i) \text{ and } d_i := v_i \phi_{t_1}.$$

It follows that $o(d_i) = 3$ and $o(d_i d_j) = 2$ for $i \neq j \in \{3, 4\} \cup \{9, 10, \dots, n\}$. Furthermore, notice that $(1\ 2)(5\ 6)$ and $(1\ 2)(7\ 8)$ invert every v_i and thus every d_i by conjugation.

We note that

$$\langle v_3, v_4, v_9, v_{10}, \dots, v_n \rangle = A_{\{1,2,3,4\} \cup \{9,10,\dots,n\}} = O^2(C_X(t_1)) \text{ and}$$

$$\langle v_3, v_4, v_{13}, v_{14}, \dots, v_n \rangle = A_{\{1,2,3,4\} \cup \{13,\dots,n\}} = O^2(C_X(t_1) \cap C_X(t_2)).$$

Moreover,

$$\begin{aligned} A_{\{1,2,3,4\} \cup \{13,\dots,n\}} \phi_{t_1} &= O^2(C_X(t_1) \cap C_X(t_2)) \phi_{t_1} \\ &= O^2(C_G(t_1) \cap C_G(t_2)) \\ &= O^2(C_X(t_1) \cap C_X(t_2)) \phi_{t_2} \\ &= A_{\{1,2,3,4\} \cup \{13,\dots,n\}} \phi_{t_2} \end{aligned}$$

In particular, we have that $O^2(C_G(t_1) \cap C_G(t_2)) = \langle d_3, d_4, d_{13}, \dots, d_n \rangle$.

Set $w_i := d_i \phi_{t_2}^{-1} \in A_{\{1,2,3,4\} \cup \{13,\dots,n\}}$ for $i \in \{3, 4\} \cup \{13, \dots, n\}$. Then we have that $o(w_i) = 3$, and $(1\ 2)(5\ 6)$ inverts each w_i as ϕ_{t_2} is the identity on $C_S(t_2)$ of which $(1\ 2)(5\ 6)$ is a member. Therefore for each i , w_i is of the form $(1\ 2\ k)$ or $(1\ k\ 2)$ such that $k \in \{1, 2, 3, 4\} \cup \{13, \dots, n\}$. Moreover, as $o(w_i w_j) = 2$ for $i \neq j$, we get that either

$$\{w_i \mid i = 3, 4, 13, \dots, n\} = \{(1\ 2\ i) \mid i = 3, 4, 13, \dots, n\} \text{ or}$$

$$\{w_i \mid i = 3, 4, 13, \dots, n\} = \{(1 \ i \ 2) \mid i = 3, 4, 13, \dots, n\}$$

and notice that $(1 \ 2 \ i)(1 \ j \ 2) = (2 \ i \ j)$ for $i \neq j$. For the first case, set

$$w_i := (1 \ 2 \ i) \text{ for } i = 5, 6, 7, 8$$

and in the second case, set

$$w_i := (1 \ i \ 2) \text{ for } i = 5, 6, 7, 8.$$

In either case, we set $d_i = w_i \phi_{t_2}$ for $i = 5, 6, 7, 8$. Then $d_i = w_i \phi_{t_2}$ for $i = 3, 4, \dots, 8$ and $i = 13, \dots, n$. We get that $o(d_i) = 3$ for $i = 5, 6, 7, 8$ and $o(d_i d_j) = 2$ for $i \neq j \in \{3, 4, \dots, 8\} \cup \{13, \dots, n\}$. Moreso, $(1 \ 2)(9 \ 10)$ and $(1 \ 2)(11 \ 12)$ invert d_i for $i = 5, 6, 7, 8$ via conjugation.

In summary we have that

$$o(d_i) = 3 \text{ for } i = 3, 4, \dots, n \text{ and}$$

$$o(d_i d_j) = 2 \text{ for } i < j \text{ if either } i, j \in \{3, 4\} \cup \{9, 10, \dots, n\} \text{ or } i, j \in \{3, 4, \dots, 8\} \cup \{13, \dots, n\}.$$

Now we want to show that

$$o(d_i d_j) = 2 \text{ for } 3 \leq i < j \leq n. \quad (6.3)$$

Thus we need to verify that

$$o(d_i d_j) = 2 \text{ if } i \in \{5, 6, 7, 8\} \text{ and } j \in \{9, 10, 11, 12\}.$$

We first argue that this holds for $i \in \{5, 6\}$ and $j \in \{9, 10\}$. We set $t_3 := (7 \ 8)(11 \ 12)$ and note that d_i centralizes t_3 for $i \in \{3, \dots, 6\} \cup \{9, 10\} \cup \{13, \dots, n\}$. As $(1 \ 2)(11 \ 12)$ inverts d_5, d_6 via conjugation and t_3 centralizes d_5, d_6 , we have that

$$(1 \ 2)(7 \ 8) = (1 \ 2)(11 \ 12)t_3$$

inverts d_5, d_6 via conjugation. We note that previously, $(1 \ 2)(7 \ 8)$ inverts

$$d_3, d_4, d_9, d_{10}, d_{13}, d_{14}, \dots, d_n$$

via conjugation. Set

$$u_i := d_i \phi_{t_3}^{-1} \text{ for } i \in \{3, \dots, 6\} \cup \{9, 10\} \cup \{13, \dots, n\} \text{ and } J := \{7, 8, 11, 12\}.$$

Notice that the u_i are pairwise distinct elements of order 3 in $O^2(C_X(t_3)) = A_{\{1, \dots, n\} \setminus J}$ which are inverted by $(1 \ 2)(7 \ 8)$. Furthermore, $o(u_3 u_j) = 2$ for $j \in \{4, 5, \dots, n\} \setminus J$. Therefore either

$$\{u_i \mid i \in \{3, 4, \dots, n\} \setminus J\} = \{(1 \ 2 \ i) \mid i \in \{3, 4, \dots, n\} \setminus J\} \text{ or}$$

$$\{u_i \mid i \in \{3, 4, \dots, n\} \setminus J\} = \{(1 \ i \ 2) \mid i \in \{3, 4, \dots, n\} \setminus J\}.$$

In any of the cases we have that $o(u_i u_j) = 2$ for $i \neq j \in \{3, 4, \dots, n\} \setminus J$. In particular we get that

$$o(u_i u_j) = 2 \text{ for } i \in \{5, 6\} \text{ and } j \in \{9, 10\}.$$

Therefore,

$$o(d_i d_j) = 2 \text{ for } i \in \{5, 6\} \text{ and } j \in \{9, 10\}.$$

Similarly, one is able to show that

$$o(d_i d_j) = 2 \text{ for } i \in \{5, 6\}, j \in \{11, 12\}$$

by considering $(7 \ 8)(9 \ 10)$ in place of t_3 , and

$$o(d_i d_j) = 2 \text{ for } i \in \{7, 8\}, j \in \{9, 10, 11, 12\}$$

by considering $(5 \ 6)(9 \ 10)$ in place of t_3 . In summary we have that equation 6.3 holds and thus

$$o(d_i d_j) = 2 \text{ for } 3 \leq i < j \leq n.$$

Hence, by Theorem 4.1.2, we have that

$$H := \langle d_3, d_4, \dots, d_n \rangle \cong A_n \cong X,$$

as required. □

Theorem 6.1.10. *G is isomorphic to X.*

Proof. By Lemma 6.1.9, there exists $H_0 \leq G$ such that $H_0 \cong X$. As

$$|H_0|_2 = |X|_2 = |G|_2,$$

we have that H_0 contains a Sylow 2-subgroup of G . Replacing H_0 by a suitable conjugate of H_0 if necessary, one may assume that $S \leq H_0$. Then

$$\mathcal{F}_S(X) \cong \mathcal{F}_S(H_0) \subseteq \mathcal{F}_S(G) = \mathcal{F}_S(X).$$

Thus $\mathcal{F}_S(H_0) = \mathcal{F}_S(X)$. In particular, we have that

$$C_{\mathcal{F}_S(H_0)}(u) = C_{\mathcal{F}_S(X)}(u)$$

for any involution $u \in S$ and thus by the structure of such involution centralizers (See Section 4.3)

$$C_{H_0}(u) \cong C_X(u).$$

Therefore, by Proposition 6.1.6(a) and Lemma 6.1.7, we get that

$$|C_G(u)| \leq |C_X(u)| = |C_{H_0}(u)|$$

for every $u \in A^\#$. As $C_{H_0}(u) \leq C_G(u)$, it follows that

$$C_G(u) = C_{H_0}(u) \leq H_0$$

for every $u \in A^\#$. Since every involution in S is conjugate to an element of A in $\mathcal{F}_S(X) = \mathcal{F}_S(H_0)$, we get that

$$C_G(u) \leq H_0$$

for every involutions $u \in S$. Moreover,

$$\text{Aut}_{\mathcal{F}_S(G)}(S) = \text{Aut}_{\mathcal{F}_S(X)}(S) = \text{Inn}(S).$$

It follows that $N_G(S) = SC_G(S)$. We choose a fully $\mathcal{F}_S(G)$ -centralized double transposition $u \in A$, and note that $C_S(u) \in \text{Syl}_2(C_G(u))$. By Proposition 6.1.6 and Lemma 6.1.7(a) we have that $C_G(u) \cong C_X(u)$. Therefore,

$$C_G(S) \leq C_{C_G(u)}(C_S(u)) \leq C_S(u) \leq S.$$

Thus $N_G(S) = S \leq H_0$. If $H_0 \neq G$, then it would follow immediately that H_0 is strongly 2-embedded and thus by Proposition 2.5.5, G has precisely one conjugacy class of involutions, a contradiction. Therefore $G = H_0 \cong X$ as required. □

6.2 Proof of Theorem A

Proof. We prove Theorem A by induction on m . We note that the theorem is true for $m = 4, 5$ by [25]. Thus we may assume $m \geq 6$ and the assertion is true for smaller m , that is Hypothesis B holds. By Theorem 5.0.1, there exist $X \in \{A_{2m}, A_{2m+1}\}$ and for every involution $u \in S$ an isomorphism

$$C_G(u)/O(C_G(u)) \rightarrow C_X(u)/O(C_X(u))$$

with $sO(C_G(u)) \mapsto sO(C_X(u))$ for every $s \in C_S(u)$. By Theorem 6.1.10 we get that $G \cong X$ and thus $G \cong A_{2m}$ or A_{2m+1} . Hence by induction, Theorem A holds. □

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