

ON THE EFFICIENCY OF TREATMENT COMPARISONS  
IN A RANDOMISED BLOCK DESIGN

by

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DECLARATION

I hereby declare that the work presented in this thesis is entirely my own, and that no part thereof has been presented for a degree elsewhere.



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## CHAPTER ONE

### INTRODUCTION

In agricultural field experiments involving treatment comparisons, the statistical tool usually employed is the analysis of variance, and the design most commonly used is the randomised block design. It is also the practice to grow more than one plant of the same kind on each plot, and observations are taken individually on each plant on a plot. Very often, however, the individual observations are not used in the subsequent analysis; the average yields per plot are used instead.

The purpose of this work is to examine, for an agricultural field experiment using a randomised block design with  $n$  plants per plot,

- (i) the various methods of analysis
- (ii) the models underlying these methods
- (iii) the assumptions implicit in these models,

and to determine whether there is a most "efficient" way of comparing treatment effects in such an experiment.

The data used in this study were obtained from an experiment conducted at the Agricultural Research Station, University of Ghana, Kpong, Ghana, in conjunction with the Department of Botany, University of Ghana, Legon, Ghana.

## 2.

It formed part of a study on "Inheritance of Yield Components and their correlation with Yield in Cowpea", conducted by A.N. Aryeetey of the Agricultural Research Station, Kpong, and E. Laing of the Department of Botany, University of Ghana, Legon.

There were 24 varieties of cowpea and these were grown in a randomised block design with three blocks in October, 1971. Each plot consisted of a single ridge 10 metres long with 10 plants spaced 1 metre apart along the ridge. Ridges were spaced 75 cm. apart and only alternative ridges were planted so that the space between two planted ridges was 1.5 metres.

Five relatively healthy plants were individually harvested from all plots which had five or more plants; no probabilistic sampling procedure was used. Some plots had less than 5 plants and were therefore not harvested. The varieties grown on such plots were not considered in this study. Five characters were measured on each plant:

- (i) number of pods per plant (x)
- (ii) average number of seeds per pod (y)
- (iii) average pod length in mm. (h)
- (iv) 100 - grain wt. in grams (z)
- (v) average grain wt. per plant in grams (w).

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The average pod length and average number of seeds per pod for each plant were based on 15 selected pods, the selection of the pods not based on any particular criterion.

From a bowl containing all the seeds of a plant, 50 healthy ones were taken and weighed. The weight obtained was multiplied by two in determining the mean 100 - seed weight ( $z$ ) for a plant as some plants had less than 100 seeds.

The seeds of all the five plants on a plot were weighed to determine the average grain weight per plant ( $w$ ).

In this study, the average yield per plant is defined as  $\theta = xyz/100$  instead of using  $w$ . By this definition,  $\theta$  will be available for each of the five selected plants and this will make it possible to carry out two multivariate analyses of variance.

Essentially,  $d = \frac{\sum_{i=1}^5 \theta_i}{5}$  measures the same characteristic as  $w$ . The data described above will henceforth be referred to as "Kpong data".

The work is broken up into the following parts:

- (i) Discussion of the models for a randomised block design and the assumptions underlying the analysis of variance;
- (ii) Univariate analysis of variance for a randomised block design with  $n$  plants per plot;
- (iii) Multivariate analysis of variance for a randomised

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- block design with  $n$  plants per plot;
- (iv) Multiple comparisons of treatment means;
  - (v) Estimation of variance components;
  - (vi) Pseudo-factorial arrangement;
  - (vii) Comparison of methods of analysis.

#### 1.1 Randomised Block Experiment and the Analysis of Variance.

Suppose that an experiment for comparing  $I$  varieties of maize is to be conducted, and that  $J$  grains of each variety are available. One may acquire a field and divide it into  $IJ$  plots of equal size. The  $I$  varieties may then be randomly allocated to the plots in such a way that  $J$  of the plots receive one kind of variety.

The principal objection to the above design is on grounds of accuracy. In spite of the randomization employed in allocating the varieties to the plots, it is possible that one kind of variety will fall on plots which have high fertility while another kind will fall on plots which have low fertility. In such a situation, there will be differences which are due to the fertility fluctuations of the plots rather than the varieties.

A procedure which will isolate any soil fertility fluctuations from the experimental error is to stratify the field into  $J$  blocks in such a way that within each block there

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is as little variation as possible among the plots. Each block is then divided into  $I$  plots of equal size. The  $I$  varieties are then allocated at random to the plots subject to the constraint that each occurs once in each block. For this design, the soil fertility fluctuations are represented by differences between the blocks. It is possible then to isolate these block differences from differences due to the varieties being compared.

An experimental design of the nature described above is called a randomised block design. It is mostly used in experiments where, though there is considerable variation in the experimental area, it is possible to divide the area into blocks in such a way that within each block there is as little variation as possible.

The mathematical models for a randomised block design will be discussed in Chapter 2.

Analysis of variance, which is used in analysing randomised block experiments, is a technique for estimating how much of the total variation in a set of data can be attributed to one or more sources of variation; the remainder, not attributable to any source, being classed as the residual or error variation. There are related tests of significance by which we can decide whether the sources have probably resulted in real variation or whether the apparent variation ascribed to

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them is only a chance happening.

The related tests of significance, and calculations of fiducial limits for estimates, often require that the errors be normally distributed and have a common variance. Furthermore, with the fixed-effects model (Chapter 2), it is generally necessary to assume that the treatment and environmental effects are additive so as to get exact tests and confidence intervals concerning the main effects.

## 1.2 Univariate Analysis of Variance.

The analysis of variance technique will be used to compare treatment differences in a randomised block design with  $n$  plants per plot in the following cases:

- (i) Using the average yields per plot under the assumption that the yields are normally distributed (normal - theory model).
- (ii) Using yields of the individual plants on a plot under the normal - theory model.
- (iii) Using the average yields per plot under the randomization models.

The F-test will be used to test for treatment differences in (i) and (ii) and as an approximate test in (iii). The tests will be compared to find out which one gives the most sensitive comparison of treatments.

These tests will be based on observations of only one variate the yield - hence the term univariate analysis of

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variance. The tests reduce to the comparison of two independently distributed mean squares. One of the mean squares is an unbiased estimate of the error variance while the other is an unbiased estimate only when the null hypothesis which is being tested is true and may be called the mean square due to deviation from the hypothesis.

A description of the analysis of variance technique and the corresponding analysis - of - variance (ANOVA) tables will be given for each of the three cases in Chapter 3. These results will then be applied to the "Kpong data".

In experiments with more than one plant per plot, sampling of the plots is often resorted to. The loss of information due to sampling will be examined.

### 1.3 Multivariate Analysis of Variance.

In a multivariate analysis of variance, observations on more than one variate are used. This is usually done when the character of primary interest, say yield, is thought to be correlated with other variates such as size of leaf, height of plant, and length of cob. Each observation will then be a random vector consisting of measurements of these correlated variates.

The process of analysing the variances and covariances of multiple correlated variates is termed multivariate analysis of variance or analysis of dispersion. The distribution

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theory of the multivariate analysis of variance is analogous to that of the univariate case and the test criteria are analogous to F-tests.

The multivariate analysis of variance technique will be used to compare treatment differences in a randomised block experiment with  $n$  plants per plot in the following cases:

- (i) Using yields of the individual plants on a plot
- (ii) Using  $p$  correlated variates.

The results will be applied to the "Kpong data" in Chapter 4.

#### 1.4 Multiple Comparisons of Treatment Means.

In experiments involving treatment comparisons, if the treatments have been shown to be significantly different, it is often necessary to investigate further how different they are. Given that treatments A, B, C, and D are significantly different, one may wish to know whether treatment A differs significantly from treatment B, or whether treatments A and C differ significantly from treatments B and D. Investigations of this nature lead to what is termed "multiple comparisons of means" in which the means of the treatments are compared by some special tests.

Various procedures are available for the multiple comparisons of means. These include

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- (i) a "protected" least significant difference (LSD) or Fisher's significant difference (FSD) due to Fisher (18);
- (ii) a multiple range test (MRT) by Duncan (12);
- (iii) the method of Student-Newman-Keuls (SNK) (14);
- (iv) Tukey's significant difference (TSD) due to Tukey (34);
- (v) Scheffé's significant difference (SSD) due to Scheffé (33);
- (vi) Bayes exact test (BET) procedure by Waller (41), which is an improvement upon and extension of a Bayesian procedure developed by Duncan.

Waller and Duncan (41) have discussed the various procedures and the results they give. According to them, a difficulty with these procedures lies in differences in the principles on which they are based and considerable disparities in the results to which they can lead. They considered the multiple comparisons problem as one of many decisions. As such, a full analysis requires the consideration of many different kinds of decision errors and the prior probabilities of their occurrence. Whether it is more appropriate to use one approach other than another depends on which comes closer to minimizing the overall weighted average of the decision errors, its "loss" and its relative prior probability. It was from this point of

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view that the BET was developed. They regarded the BET as not much different from the FSD.

Carmer and Swanson (5) used computer simulation techniques to study the Type I and Type II error rates and the correct decision rates for the procedures listed above. Their results indicate that the SSD, TSD and SNK tests are less appropriate than either the FSD with  $\alpha = 0.05$ , BET or MRT. In fact, they considered FSD with  $\alpha = 0.05$  and BET as the best available procedures so far for use.

The above findings by Carmer, Swanson, Waller, and Duncan will be examined by applying the FSD, BET, MRT, and SSD procedures to the "Kpong data" in Chapter 5.

The error variances in the cases given in Sections 1.2 and 1.3 above differ from one another. Since a multiple comparisons of treatment means is dependent on the error variance, the different error variances will give rise to different results for any given multiple comparisons of means procedure. The different results will be examined in Chapter 5 to find out the extent to which they differ.

### 1.5 Estimation of Variance Components.

Variance components estimation is an important aspect of agricultural experiments. Experimenters normally wish to have an idea of the variances associated with the various effects

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and to test hypotheses about them. The expressions containing the variance components are obtained by taking expectation of the mean squares. These expressions vary with mathematical models. It is therefore expected that estimates of the variance components for cases (i) and (ii) of Section 1.2 may differ from one another. This will be investigated in Chapter 6.

When there are equal number of observations for each treatment (that is, for a "balanced data"), the analysis of variance method (35) is most commonly used to estimate variance components. This choice is motivated by the fact that estimators thus obtained are minimum variance quadratic unbiased when normality is not assumed, and are minimum variance unbiased when normality is assumed. Unfortunately, though variance components are, by definition, positive, the analysis of variance method can lead to negative estimates (28) which cannot be satisfactorily explained.

An alternative method which avoids negative estimates but whose estimators may be biased is the restricted maximum likelihood (39).

In Chapter 6, the two methods given above will be applied to cases (i) and (ii) of Section 1.2.

#### 1.6 The Pseudo-Factorial Type of Arrangement.

With a large number of treatments to be compared in an

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agricultural field experiment, a randomised block design containing the whole of the treatments is unsatisfactory. The blocks are likely to be too large and it may be difficult to eliminate fertility differences efficiently.

To keep the block size small, some authors have suggested the selection of one or more treatments as controls and to divide the rest into sets, each set being arranged with the controls in a number of randomised blocks. Unfortunately, as pointed out by Yates (44), this method has the disadvantage that comparisons between treatments in different sets are of lower accuracy than comparisons between treatments in the same set. There is the further disadvantage that a disproportionate number of plots is devoted to the control treatments, thus further lowering the efficiency.

To cope with the large number of treatments, and at the same time to avoid the use of excessively large blocks and controls, Yates (44) has suggested a new type of arrangement of the treatments. The arrangement is known as pseudo-factorial and is described in Chapter 7.

### 1.7 Comparison of Methods of Analysis.

The F tests used to test for treatment differences in cases (i) and (ii) of Section 1.2 will be compared to determine which one (if any) is the most sensitive for detecting differences between treatments. The comparison will be made by

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examining the powers of the various F tests.

Comparisons will also be made of the following:

- (a) the two multivariate analyses of variance listed in Section 1.3 ;
- (b) the various multiple comparisons of means resulting from the cases listed in Sections 1.2 and 1.3;
- (c) variance components estimation under cases (i) and (ii) of Section 1.2.

The conclusions to be derived from these results, and recommendations, if any, will be given in Chapter 8.

Chapter

CHAPTER TWORANDOMISED BLOCK EXPERIMENTS AND  
THE ANALYSIS OF VARIANCE2.1 Mathematical Models.

Consider the randomised block experiment described in Section 1.1. We shall assume that there are  $n$  plants on a plot instead of one. The symbol  $x_{ijk}$  will be used to denote the  $k^{\text{th}}$  observation on the plot belonging to the  $i^{\text{th}}$  row and the  $j^{\text{th}}$  column.

Let  $\mu$  = general mean ,

$n_{ij}$  = plot mean ,

$A_i$  = mean of the  $i^{\text{th}}$  variety ,

$B_j$  = mean of the  $j^{\text{th}}$  block.

The main effect of the  $i^{\text{th}}$  variety is defined as

$$\alpha_i = A_i - \mu \quad \dots\dots(2.1.1) \cdot$$

Similarly, the main effect of the  $j^{\text{th}}$  block is

$$\beta_j = B_j - \mu \quad \dots\dots(2.1.2) \cdot$$

The main effect of the  $i^{\text{th}}$  variety specific to the  $j^{\text{th}}$  block is  $n_{ij} - B_j$ . The interaction of the  $i^{\text{th}}$  variety with the  $j^{\text{th}}$  block is defined as

$$\begin{aligned} \gamma_{ij} &= (n_{ij} - B_j) - (A_i - \mu) , \\ &= n_{ij} - B_j - A_i + \mu \quad \dots\dots(2.1.3) \end{aligned}$$

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On substituting  $A_i = \mu + \alpha_i$  and  $B_j = \mu + \beta_j$  in (2.1.3), we obtain

$$E(x_{ijk}) = n_{ij} = \mu + \alpha_i + \beta_j + \gamma_{ij} .$$

Thus the mathematical model for the individual observations is given by

$$x_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + e_{ijk} . \quad \dots\dots(2.1.4)$$

The  $e_{ijk}$ 's are errors assumed to be uncorrelated with zero expectations and a common variance  $\sigma_e^2$ . The effects of the violation of this assumption have been studied by Cochran (6) and Scheffé' (34). They found out that there could be substantial biases in the standard errors of estimates, and the t-tests might be impaired, if the errors are correlated. It has however been noted that for randomised experiments the errors may safely be regarded as uncorrelated.

#### I. Fixed-Effects Model.

When interest lies only in the treatments and blocks actually used in the experiment and all inferences made are restricted only to these, then the treatment effects  $\alpha_i$ , block effects  $\beta_j$ , and interaction effects  $\gamma_{ij}$  in the mathematical model (2.1.4) are fixed effects. If the errors  $e_{ijk}$  are assumed to be independently normal with mean zero and a common variance  $\sigma_e^2$ , we obtain what is termed as the fixed-effects model. Under

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this model, and in view of equations (2.1.1), (2.1.2) and (2.1.3), we have the side condition

$$\sum_i \alpha_i = \sum_j \beta_j = \sum_i \gamma_{ij} = \sum_j \gamma_{ij} = 0 . \quad \dots(2.1.5)$$

The data available for this situation are thought of as one of the many possible sets of data (involving these same treatments and blocks) that could be derived in repetitions of the experiment, repetitions in which the  $e_{ijk}$ 's on each occasion would be a random sample from a population of error terms that have zero means and a common variance  $\sigma_e^2$ .

## II. Random-Effects Model.

When interest in the experiment is unlikely to centre only on those treatments and blocks actually used in the experiment, then the  $\alpha_i$ 's,  $\beta_j$ 's and  $\gamma_{ij}$ 's are random effects assumed to be uncorrelated among themselves and with the errors. If, in addition, we assume that the  $e_{ijk}$ 's are independently normal with mean zero and a common variance  $\sigma_e^2$ , we obtain the random-effects model. The  $\alpha_i$ 's,  $\beta_j$ 's and  $\gamma_{ij}$ 's are now random samples from populations of  $\alpha_i$ 's,  $\beta_j$ 's and  $\gamma_{ij}$ 's, respectively, with respective variances  $\sigma_\alpha^2$ ,  $\sigma_\beta^2$ ,  $\sigma_\gamma^2$ . From equations (2.1.1), (2.1.2) and 2.1.3), we observe that the means of the various effects taken over their respective populations are zero. The above conditions can mathematically be

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represented as

$$\begin{array}{l}
 \text{(i)} \quad E(\alpha_i) = E(\beta_j) = E(\gamma_{ij}) = 0 \quad ; \\
 \text{(ii)} \quad \text{Var}(\alpha_i) = \sigma_\alpha^2; \quad \text{Var}(\beta_j) = \sigma_\beta^2; \quad \text{Var}(\gamma_{ij}) = \sigma_\gamma^2 \quad ; \\
 \text{(iii)} \quad \alpha_i, \beta_j, \gamma_{ij} \text{ and } e_{ijk} \text{ are uncorrelated} \quad ; \\
 \text{(iv)} \quad e_{ijk} \sim N(0, \sigma_e^2) .
 \end{array}
 \left. \vphantom{\begin{array}{l} \text{(i)} \\ \text{(ii)} \\ \text{(iii)} \\ \text{(iv)} \end{array}} \right\} \dots(2.1.6)$$

Under these conditions we conceive of repetitions as taking random samples of treatments and blocks on each occasion.

### III. Mixed-Effects Model.

When a model can be conceived of as having both fixed and random effects, we obtain a mixed-effects model. In the mathematical model (2.1.4), if the block effects,  $\beta_j$ , are considered random and the treatment effects,  $\alpha_i$ , fixed, then the interaction effects,  $\gamma_{ij}$ , are random and the model becomes a mixed-effects model. Mathematically, we have

$$\begin{array}{l}
 \text{(i)} \quad \sum \alpha_i = 0 \quad ; \\
 \text{(ii)} \quad E(\beta_j) = E(\gamma_{ij}) = 0 \quad ; \\
 \text{(iii)} \quad \text{Var}(\beta_j) = \sigma_\beta^2; \quad \text{Var}(\gamma_{ij}) = \sigma_\gamma^2 \quad ; \\
 \text{(iv)} \quad \beta_j, \gamma_{ij} \text{ and } e_{ijk} \text{ are uncorrelated} \quad ; \\
 \text{(v)} \quad e_{ijk} \sim N(0, \sigma_e^2) .
 \end{array}
 \left. \vphantom{\begin{array}{l} \text{(i)} \\ \text{(ii)} \\ \text{(iii)} \\ \text{(iv)} \\ \text{(v)} \end{array}} \right\} \dots(2.1.7)$$

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The models above may be called normal-theory models (34) (or infinite models) because of the normality assumption associated with the  $e_{ijk}$ 's.

#### IV. Randomization Models.

Randomization models (4, 34) (or finite models) have the same characteristics as normal-theory models except that there is no normality assumption about the errors. The random errors are regarded as uncorrelated with expectation zero. These models take into account the randomization employed in assigning the treatments to the experimental plots. Detailed discussion of the randomization model for a randomised block experiment is given in Chapter 3.

### 2.2 Assumptions Underlying the Analysis of Variance.

#### I. Normality of Errors.

The general mathematical model for a randomised block design was given in Section 2.1 as

$$x_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + e_{ijk} .$$

If the error  $e_{ijk}$  are normal with mean zero and variance  $\sigma_e^2$ , the observations  $x_{ijk}$  will also be normal with mean

$$\mu + \alpha_i + \beta_j + \gamma_{ij}$$

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and variance  $\sigma_e^2$  if all the effects are fixed, or with mean  $\mu$  and variance

$$\sigma_\alpha^2 + \sigma_\beta^2 + \sigma_\gamma^2 + \sigma_e^2$$

if the effects are random. Thus the assumption of normality of errors implies normality of the observations.

The theoretical distributions of the statistics upon which tests of significance and calculations of fiducial limits are based are derived under the assumption that the observations are normally distributed. The effect of the violation of the normality assumption has been studied by Box (3), Scheffé' (34) and others. In their studies, Scheffé' and Box showed that though the effect of violation of the normality assumption is slight on inferences involving fixed effects (tests of hypotheses about fixed main effects or interactions) it is serious on inferences involving random effects - tests for the equality of variances, and confidence intervals for error variance, variance components or ratio of variance components.

In view of the above, it has been suggested that we verify the normality assumption for any data whose analysis will involve inferences of the type given above. When the normality assumption is violated, the data may be transformed by a suitable transformation (2). Certain transformations are able to change non-normal data to normal.

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Several tests are available for testing normality. Among them are the large sample methods (26), W-test (36), and non-parametric goodness of fit test (26).

The large sample methods use the criteria of Skewness and Kurtosis since these are two important ways by which a distribution may depart from normality. A large sample size is required for this test to be applicable.

Non-parametric goodness of fit test for normality is based on the fact that if  $x_1, x_2, \dots, x_n$  is a random sample from a distribution of the continuous type with cumulative function  $F$  and the transformation

$$Y_i = F(x_i); \quad 0 \leq F(x_i) \leq 1, \quad i=1,2,\dots,n$$

is made, then

$$-2 \sum_{i=1}^n \log Y_i$$

is distributed as  $\chi^2$  with  $2n$  degrees of freedom. As a test of normality, this test statistic tests the hypothesis that the  $x_i$ 's come from a normal population with mean and variance equal to those of the sample against the broad and vague class of alternatives that the  $x_i$ 's come from a population which differs from these specifications. Since the alternatives are undefined, little is known about the power functions of the test in such applications.

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The W-test for normality, proposed by Shapiro and Wilk (36), provides an index or test statistic to evaluate the supposed normality of a complete sample. It is scale and origin invariant and hence supplies a test for the composite null hypothesis of normality. The power functions of the test have been calculated and have shown the statistic to be an effective measure of normality for small samples ( $n \leq 50$ ).

Lack of knowledge about the power functions of the non-parametric goodness of fit test makes the W-test the most plausible for testing normality of small samples ( $n \leq 50$ ). Since the samples provided by the "Kpong data" are small ( $n = 21$ ), the W-test will accordingly be employed in testing for their normality.

#### The W-test for Normality (complete samples).

Let  $m' = (m_1, m_2, \dots, m_n)$  denote the vector of expected values of standard normal order statistics, and let  $V = (v_{ij})$  be the corresponding  $n \times n$  covariance matrix. That is, if

$$x_1 \leq x_2 - \dots \leq x_n$$

denotes an ordered random sample of size  $n$  from a normal distribution with mean zero and variance 1, then

$$\left. \begin{array}{l} \text{(i) } E(x_i) = m_i, \quad i=1,2,\dots,n. \\ \text{(ii) } \text{Cov}(x_i, x_j) = v_{ij}, \quad i,j = 1,2,\dots,n. \end{array} \right\} \dots(2.2.1)$$

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Let  $y' = (y_1, \dots, y_n)$  denote a vector of ordered random observations. The objective is to derive a test for the hypothesis that this is a sample from a normal distribution with unknown mean  $\mu$  and unknown variance  $\sigma^2$ .

If the  $\{y_i\}$  are a normal sample the  $y_i$  may be expressed as

$$y_i = \mu + \sigma x_i, \quad i=1,2,\dots,n.$$

It follows from the generalized least-squares theorem that the best linear unbiased estimates of  $\mu$  and  $\sigma$  are the quantities that minimize the quadratic form

$$Q = (y - \mu I - \sigma m)' V^{-1} (y - \mu I - \sigma m); \quad \dots (2.2.2)$$

where  $I' = (1, 1, \dots, 1)$ .

The estimates are

$$\hat{\mu} = \frac{m' V^{-1} (m I' - I m') V^{-1} y}{I' V^{-1} I m' V^{-1} m - (I' V^{-1} m)^2}; \quad \dots (2.2.3)$$

$$\hat{\sigma} = \frac{I' V^{-1} (I m' - m I') V^{-1} y}{I' V^{-1} I m' V^{-1} m - (I' V^{-1} m)^2}. \quad \dots (2.2.4)$$

For symmetric distributions,  $I' V^{-1} m = 0$ , and hence equations (2.2.3) and (2.2.4) become

$$\hat{\mu} = \frac{1}{n} \sum_{i=1}^n y_i = \bar{y}; \quad \dots (2.2.5)$$

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$$\hat{\sigma} = \frac{m'V^{-1}y}{m'V^{-1}m} \quad \dots\dots(2.2.6)$$

$$\text{Let } R^2 = \frac{n}{\sum_{i=1}^n} (y_i - \bar{y})^2 \quad \dots\dots(2.2.7)$$

denote the usual symmetric unbiased estimate of  $(n-1)\sigma^2$ .

Shapiro and Wilk (36) defined the W-test statistic for normality as

$$W = \frac{L^4 \hat{\sigma}^2}{C^2 R^2} = \frac{b^2}{R^2} = \frac{(a'y)^2}{R^2} = \frac{(\sum_{i=1}^n a_i y_i)^2}{\sum_{i=1}^n (y_i - \bar{y})^2} \quad ; \quad \dots\dots(2.2.8)$$

where (i)  $L^2 = m'V^{-1}m$  ,

(ii)  $C^2 = m'V^{-1}V^{-1}m$  ,

(iii)  $a' = (a_1, a_2, \dots\dots a_n) = \frac{m'V^{-1}}{(m'V^{-1}V^{-1}m)^{\frac{1}{2}}}$  ,

(iv)  $b = \frac{L^2 \hat{\sigma}}{C}$  ;

$b$  is, up to the normalising constant  $C$ , the best linear unbiased estimate of the slope of a linear regression of the ordered observations,  $y_i$ , on the expected values,  $m_i$ , of the standard normal order statistics. The constant  $C$  is so defined that the linear coefficients are normalized.

According to the authors, if one is indeed sampling from a normal population, then the numerator,  $b^2$ , and denominator,  $R^2$ , of  $W$  are both, up to a constant, estimating the same quantity, namely,  $\sigma^2$ .

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The  $\{a_i\}$  used in the W-statistic are defined by

$$a_i = \frac{\sum_{j=1}^n m_j \frac{v_{ij}}{C}}{C}, \quad j=1,2,\dots,n$$

where  $m_j$ ,  $v_{ij}$  and  $C$  have been defined above. To determine the  $a_i$  directly it is necessary to know both the vector of means  $m$  and the covariance matrix  $V$ . However, the elements of  $V$  are known only up to samples of size 20. Various approximations have been suggested but according to the authors, the most plausible one is to define  $b$  as ;

(i) for  $n$  even,  $n = 2k$  ,

$$b = \sum_{i=1}^k a_{n-i+1} (y_{n-i+1} - y_i), \quad \dots\dots(2.2.9)$$

where the values of  $a_{n-i+1}$  are tabulated by Shapiro and Wilk (36).

(ii) for  $n$  odd,  $n = 2k + 1$  ,

$$b = a_n (y_n - y_1) + \dots + a_{k+2} (y_{k+2} - y_k) .$$

In carrying out the test,  $W$  is calculated and its value compared with the tabulated value at the required confidence probability. Small values of calculated  $W$  indicate non-normality.

This test will be applied to the "Kpong data" in Section 1.4.

## II. Additivity of Treatment and Environmental Effects.

The general mathematical model for a randomised block design with one observation per plot is given by

$$x_{ij} = \mu + \alpha_i + \beta_j + \gamma_{ij} + e_{ij} .$$

Since there is only one observation per plot, the error variance cannot be estimated. It is thus not possible to test for variety differences because the standard of comparison provided by the error variance is not available. It has been shown (34), however, that if we assume  $\gamma_{ij} = 0$  we can obtain a valid form of analysis in which the interaction mean square (Chapter 3) is used as the error variance. The assumption of  $\gamma_{ij} = 0$  is equivalent to assuming that the treatment and environmental effects are additive. With the additivity assumption, the mathematical model becomes

$$x_{ij} = \mu + \alpha_i + \beta_j + e_{ij} . \quad \dots\dots(2.2.10)$$

The above discussion shows the importance of testing for additivity of treatment and environmental effects in a randomised block experiment with one observation per plot. When non-additivity is significant the data may be transformed by a suitable transformation (2).

Tukey (40) has devised a test for non-additivity in a randomised block design with one observation per plot. It is based

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on the normal-theory model and his procedure is to obtain a sum of squares with one degree of freedom which will tend to be inflated if there is non-additivity. The mathematical description of the test is presented by Rao (32).

#### Tukey's Test for Additivity.

Let the  $\chi_{ij}$ 's be independently  $N(0, \sigma_e^2)$ , and let them be given by

$$\chi_{ij} = \mu + \alpha_i + \beta_j + \gamma_{ij} + e_{ij} , \quad \dots\dots(2.2.11)$$

where  $\sum_i \alpha_i = \sum_j \beta_j = \sum_i \gamma_{ij} = \sum_j \gamma_{ij} = 0$  .

The hypothesis we would like to test is

$$H_{AB} : \gamma_{ij} = 0, \quad i=1, \dots, I, j=1, \dots, J .$$

As pointed out earlier on, the error variance in model (2.2.11) cannot be estimated. Thus, it is not possible to test for interactions by the F-test.

Tukey (40) managed to obtain a test of  $H_{AB}$  by imposing some restrictions on the  $\{\gamma_{ij}\}$ . These restrictions consist in assuming the  $\{\gamma_{ij}\}$  to be of the form

$$\gamma_{ij} = \lambda \alpha_i \beta_j , \quad \dots\dots(2.2.12)$$

where  $\lambda$  is a constant.

He then asserted that if we could construct a test for the hypothesis  $\lambda = 0$ , it would serve as a test for non-additivity.

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And that is what he showed.

With the above definition of  $\gamma_{ij}$ , the expectation of  $X_{ij}$  is given by

$$E(X_{ij}) = \mu + \alpha_i + \beta_j + \lambda\alpha_i\beta_j \quad \dots\dots(2.2.13)$$

These expectations are not linear when  $\lambda \neq 0$ . The least square theory is therefore not applicable. We note, however, that since

$$E(\bar{X}_{i.} - \bar{X}_{..}) = \alpha_i ,$$

$$E(\bar{X}_{.j} - \bar{X}_{..}) = \beta_j ,$$

$$E(\bar{X}_{..}) = \mu ,$$

where (i)  $\bar{X}_{..} = \frac{1}{IJ} \sum_i \sum_j X_{ij}$  is the overall mean ;

(ii)  $\bar{X}_{i.} = \frac{1}{J} \sum_j X_{ij}$  are the row means ;

(iii)  $\bar{X}_{.j} = \frac{1}{I} \sum_i X_{ij}$  are the column means;

the unbiased estimators of  $\alpha_i$ ,  $\beta_j$  and  $\mu$  are given by

$$\hat{\alpha}_i = \bar{X}_{i.} - \bar{X}_{..} ; \quad \hat{\beta}_j = \bar{X}_{.j} - \bar{X}_{..} ; \quad \hat{\mu} = \bar{X}_{..} .$$

Furthermore,  $E(X_{ij} - \bar{X}_{i.} - \bar{X}_{.j} + \bar{X}_{..}) = \lambda\alpha_i\beta_j$ .

Hence an estimate of  $\lambda$  may be written, when  $\alpha_i$  and  $\beta_j$  are known, by using the least square theory on equation (2.2.13),

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as

$$\lambda^* = \frac{\sum \sum \alpha_i \beta_j (X_{ij} - \alpha_i - \beta_j - \bar{X}_{..})}{\sum \sum \alpha_i^2 \beta_j^2}$$

Substituting estimates  $\hat{\alpha}_i$  and  $\hat{\beta}_j$ , an estimate of  $\lambda$  in terms of known quantities is obtained as

$$\hat{\lambda} = \frac{\sum \sum (\bar{X}_{i.} - \bar{X}_{..}) (\bar{X}_{.j} - \bar{X}_{..}) (X_{ij} - \bar{X}_{i.} - \bar{X}_{.j} + \bar{X}_{..})}{\sum (\bar{X}_{i.} - \bar{X}_{..})^2 \sum (\bar{X}_{.j} - \bar{X}_{..})^2}$$

This simplifies to

$$\hat{\lambda} = \frac{IJ \sum \sum (\bar{X}_{i.} - \bar{X}_{..}) (\bar{X}_{.j} - \bar{X}_{..}) X_{ij}}{(SS_V) (SS_B)} \quad ; \quad \dots (2.2.14)$$

where (i)  $SS_V = J \sum_i (\bar{X}_{i.} - \bar{X}_{..})^2$  ;

(ii)  $SS_B = I \sum_j (\bar{X}_{.j} - \bar{X}_{..})^2$  .

The variance of  $\hat{\lambda}$ , for given  $\hat{\alpha}_i$  and  $\hat{\beta}_j$ , is given by

$$\begin{aligned} \text{Var}(\hat{\lambda}) &= \frac{I^2 J^2 \sigma_e^2 \sum \sum (\bar{X}_{i.} - \bar{X}_{..})^2 (\bar{X}_{.j} - \bar{X}_{..})^2}{(SS_V)^2 (SS_B)^2} , \\ &= \frac{IJ \sigma_e^2}{(SS_V) (SS_B)} \quad \dots (2.2.15) \end{aligned}$$

If  $\lambda = 0$ ,  $E(\hat{\lambda} | \hat{\alpha}_i, \hat{\beta}_j \text{ for all } i, j) = 0$  .

Thus under  $H_{AB}$ ,

$$\hat{\lambda} \sim N(0, \text{Var}(\hat{\lambda}))$$

Therefore

$$\frac{\hat{\lambda}}{\sqrt{\text{Var}(\hat{\lambda})}} \sim N(0, 1)$$

Hence

$$G = \frac{(\hat{\lambda})^2}{\text{Var}(\hat{\lambda})} = \frac{IJ \left[ \sum \sum (\bar{X}_{i.} - \bar{X}_{..}) (\bar{X}_{.j} - \bar{X}_{..}) X_{ij} \right]^2}{(SS_V) (SS_B) \sigma_e^2} \quad \dots (2.2.16)$$

is distributed as chi-square (43) with one degree of freedom ( $\chi_1^2$ ).

The "interaction sum of squares" is given by

$$\begin{aligned} \sum \sum (\gamma_{ij}^*)^2 &= \sum \sum (\lambda^* \alpha_i \beta_j)^2, \\ &= (\lambda^*)^2 \sum \alpha_i^2 \sum \beta_j^2, \\ &= \frac{\left[ \sum \sum \alpha_i \beta_j (X_{ij} - \alpha_i - \beta_j - \bar{X}_{..}) \right]^2}{\sum \alpha_i^2 \sum \beta_j^2}. \end{aligned}$$

Replacing  $\alpha_i$  and  $\beta_j$  by their estimates  $\hat{\alpha}$  and  $\hat{\beta}$ , we obtain

$$SS_N = \frac{IJ \left[ \sum \sum (\bar{X}_{i.} - \bar{X}_{..}) (\bar{X}_{.j} - \bar{X}_{..}) X_{ij} \right]^2}{(SS_V) (SS_B)},$$

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$$= \frac{\left[ \sum \sum (\bar{X}_{i.} - \bar{X}_{..}) (\bar{X}_{.j} - \bar{X}_{..}) X_{ij} \right]^2}{\sum (\bar{X}_{i.} - \bar{X}_{..})^2 \sum (\bar{X}_{.j} - \bar{X}_{..})^2} \quad \dots (2.2.17)$$

Comparing equations (2.2.16) and (2.2.17), we see that

$$G = \frac{SS_N}{\sigma_e^2} \sim \chi_1^2 \quad .$$

Tukey named  $SS_N$  as the "sum of squares due to non-additivity."

$$\text{Let } SS_E = \sum \sum (X_{ij} - \bar{X}_{i.} - \bar{X}_{.j} + \bar{X}_{..})^2 .$$

It has been shown that

$$D = \frac{SS_E}{\sigma_e^2} \sim \chi_{(I-1)(J-1)}^2 \quad .$$

We observe that, since  $X_{ij}$  in (2.2.17) can in fact be written as  $(X_{ij} - \bar{X}_{i.} - \bar{X}_{.j} + \bar{X}_{..})$ ,

$$\frac{SS_E}{\sigma_e^2} - \frac{SS_N}{\sigma_e^2} > 0 \quad ,$$

and is distributed as  $\chi_{(I-1)(J-1)-1}^2$ .

Hence

$$F_0 = \frac{SS_N \left[ (I-1)(J-1) - 1 \right]}{SS_E - SS_N} \quad \dots (2.2.18)$$

has an exact F-distribution (43) with 1 and  $(I-1)(J-1) - 1$

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degrees of freedom  $\left[ F_{1, (I-1)(J-1)-1} \right]$  if  $\lambda = 0$ .

Now  $SS_N$  is the sum of squares due to non-additivity and is part of the error sum of squares  $SS_E$ . Thus

$$SS'_E = SS_E - SS_N$$

is the new error sum of squares devoid of non-additivity. Tukey's test for non-additivity is therefore provided by the test statistic  $F_0$ .

The power of this test is unknown. It is, however, expected to be good against alternatives of the type (2.2.12).

The analysis of variance table for carrying out the test is given in Table 2.2.1.

Table 2.2.1 Analysis of Variance Table for Testing Non-additivity in a Randomised Block Design.

Source	Degrees of Freedom	Sum of Squares	Mean Square	F
Non-additivity	1	$SS_N$	$SS_N$	$SS_N \left[ (I-1)(J-1)-1 \right]$
Residual	$(I-1)(J-1)-1$	$SS'_E$	$\frac{SS'_E}{(I-1)(J-1)-1}$	$SS'_E$

The test will be applied to the "Kpong data" in Section 2.3 (II).

### III. Homogeneity of Error Variances.

The methods for testing equality of means, calculating the true error probabilities of Type I error, and carrying out multiple comparisons of means are derived from assumptions that include the equality of error variances. The effect on these methods when the error variances are not equal has been investigated by Hornsnell (21), Eisenhart (13), and others (34,19). Their studies indicate that the effect is serious when there are unequal number of observations on each plot and the I treatments have unequal sizes. The effect is, however, not very serious for equal treatment and plot sizes.

Scheffé' (34) does not recommend a preliminary test of the assumption of equality of error variances because he is convinced that the effect of the violation of the assumption is not very serious. On the other hand, some authors (6,13,21) feel that even when there are equal numbers of observations on each plot and the treatments have equal sizes, it is a worthwhile practice to test for the equality of the error variances.

When the error variances are unequal, one may use a suitable transformation to transform the data. Sometimes, when the assumptions of normality of errors, additivity of treatment and environmental effects, and homogeneity of error variances are violated simultaneously, one transformation may remedy all the defects. In some cases, however, one defect

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may be remedied at the expense of another. In such a situation, one has to consider, in the light of the experiment, the defect which is more serious.

Several procedures for testing equality of error variances have been proposed. Among them are those developed by Neyman and Pearson (29), Box and Anderson (4), and Bartlett (38).

The original test of this nature was that obtained by Neyman and Pearson (29) who suggested the use of a criterion  $L$  which was the ratio of a weighted geometric to a weighted arithmetic mean of the mean squares from which the variances were estimated. On the assumption that variation followed the normal law, these authors

- (i) gave the sampling moments of  $L$  if the hypothesis of equal sampling variances was true;
- (ii) showed that in the case of large samples,  $-N \log_e L$  was distributed as  $\chi^2$  with  $k - 1$  degrees of freedom (where  $N$  was the total number of observations and  $k$  the number of separate estimates of variance);
- (iii) suggested a method of calculating approximate probability levels for  $L$  in the case of small samples.

A drawback of this test was its non-applicability to small

samples without resorting to approximations.

Approaching from another angle, Bartlett (38) suggested an analogous test in which the sum of squares were weighted with their appropriate degrees of freedom instead of with the number of observations as in Neyman-Pearson criterion. Furthermore, unlike the Neyman-Pearson criterion where approximate probability levels of  $L$  need to be calculated in the case of small samples, Bartlett's test just requires a corrective factor (38) to make it applicable to small samples. Box (3) has shown that Bartlett's test is extremely sensitive to non-normality, and that it is essential to have the original data normally distributed before applying the test.

Making use of the permutation theory, Box and Anderson (4) developed another test for equality of error variances. Essentially, it is a modification of Bartlett's test under the consideration of permutation theory. Unlike Bartlett's test, however, it is insensitive to non-normality, and has less power when the data are normal.

A preliminary test on the "Kpong data" (Section 2.3 (I)) showed them to be normally distributed. Also, the samples furnished by the data have small sizes. From the above discussion therefore, Bartlett's test is the most appropriate (among those discussed in this section) for testing equality of variances of the "Kpong data".

Bartlett's Test for Homogeneity of Error Variances.

Let  $S_i^2$  be the usual unbiased estimate of  $\sigma_i^2$  based on a sum of squares having  $f_i$  degrees of freedom, and let there be  $k$  of these estimates from independent sets of observations.

Bartlett (38) took as his test statistic

$$M = N \operatorname{Log}_e \left[ \frac{\sum_{i=1}^k f_i S_i^2}{N} \right] - \sum_{i=1}^k (f_i \log_e S_i^2), \dots (2.2.19)$$

where  $N = \sum_{i=1}^k f_i$ .

Provided that none of the degrees of freedom  $f_i$  is too small, he found that  $M$  is distributed approximately as  $\chi^2$  with  $k-1$  degrees of freedom if the null hypothesis is true - that is, if the  $\sigma_i^2$  ( $i=1,2,\dots,k$ ) have a common value.

For small samples, Bartlett introduced the corrective factor

$$C = 1 + \frac{1}{3(k-1)} \left[ \sum_{i=1}^k \frac{1}{f_i} - \frac{1}{N} \right] \dots (2.2.20)$$

and showed that the quantity  $M/C$  followed approximately the same  $\chi^2$  distribution law.

If all the samples have equal sizes  $n$ , then

$f_1 = f_2 = \dots = f_k = n-1$ , and  $N = k(n-1)$ . Equations (2.2.19) and (2.2.20) therefore become

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$$M = (n - 1) \left[ k \log_e \frac{\sum_{i=1}^k S_i^2}{k} - \sum_{i=1}^k \log_e S_i^2 \right], \dots\dots(2.2.21)$$

and

$$C = 1 + \frac{k + 1}{3k(n - 1)} \cdot \dots\dots(2.2.22)$$

### 2.3 The "Kpong Data"

The appropriate tests for verifying the assumptions underlying the analysis of variance have been discussed in Section 2.2. We shall now investigate the assumptions for the "Kpong data".

#### I. Testing for Normality.

The "Kpong data" were obtained from a randomised block design with three blocks, each block consisting of 21 plots. The 21 observations from each block form a sample. Thus, there are three samples each of size 21. If there are block effects, the observations from each block may be assumed to have come from a different normal population. In testing for normality therefore, each sample will be considered separately.

It was shown in Section 2.2 (I) that for small samples ( $n \leq 50$ ), the W-test for normality, also discussed in that

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section, is the appropriate one to use. The test statistic was given as

$$W = \frac{b^2}{R^2} \quad , \quad \dots\dots(2.3.1)$$

$$\text{where } R^2 = \sum_{k=1}^n (y_k - \bar{y})^2 = \sum_{k=1}^n y_k^2 - \frac{(\sum y_k)^2}{n} \quad ; \quad \dots\dots(2.3.2)$$

and for  $n = 2k + 1$ ,

$$b = a_n(y_n - y_1) + \dots + a_{k+2}(y_{k+2} - y_k) \cdot \dots\dots(2.3.3)$$

Since the  $y_k$ 's are ordered random observations, the observations in each block will be arranged in ascending order before computing  $W$ . The tables necessary for computing  $W$  are presented below.

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Table 2.3.1: The Observations  $y_k$  and Their Squares,  $y_k^2$ ,  
for the 3 Blocks .

B L O C K 1		B L O C K 2		B L O C K 3		
$y_k$	$y_k^2$	$y_k$	$y_k^2$	$y_k$	$y_k^2$	
5.56	30.914	2.83	8.009	3.62	13.104	
6.37	40.577	5.07	25.705	5.66	32.036	
13.44	180.634	6.42	41.216	6.87	47.197	
13.99	195.720	10.96	120.122	8.76	76.738	
15.23	231.953	12.85	165.123	11.77	138.533	
16.39	268.632	14.52	210.830	12.87	165.637	
19.49	379.860	14.58	212.576	16.15	260.823	
21.30	453.690	14.68	215.502	18.02	324.720	
24.58	604.106	18.44	340.034	18.40	338.560	
24.79	614.544	19.05	362.903	18.41	338.928	
26.39	696.432	22.06	486.644	18.68	348.942	
27.23	741.473	22.39	501.312	21.82	476.112	
27.58	760.656	24.67	608.609	28.93	836.945	
33.21	1102.904	30.62	937.584	29.81	888.636	
34.52	1191.630	34.61	1197.852	34.70	1204.090	
39.47	1557.881	39.84	1587.226	53.67	2880.469	
45.58	2077.536	47.01	2209.940	55.80	3113.640	
47.11	2219.352	52.59	2765.708	60.34	3640.916	
48.71	2372.664	58.20	3387.240	64.20	4121.640	
50.52	2552.270	62.63	3922.517	66.17	4378.469	
53.68	2881.542	70.32	4944.902	71.61	5127.992	
Total	595.14	21155.040	584.34	24251.554	626.26	28754.127

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Table 2.3.2 : The Coefficients  $a_{n-k+1}$  and the Values  $(y_{n-k+1} - y_k)$   
Using the Observations from Block 1 .

k	$a_{n-k+1}$	$(y_{n-k+1} - y_k)$	$a_{n-k+1}(y_{n-k+1} - y_k)$
1	0.4643	48.12	22.342
2	0.3185	44.15	14.062
3	0.2578	35.27	9.093
4	0.2119	33.12	7.018
5	0.1736	30.35	5.269
6	0.1399	23.08	3.229
7	0.1092	15.03	1.641
8	0.0804	11.91	0.958
9	0.0530	3.00	0.159
10	0.0263	2.44	0.064

b =63.835

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Table 2.3.3: The Coefficients  $a_{n-k+1}$  and the Values  $(y_{n-k+1} - y_k)$

Using the Observations from Block 2 .

k	$a_{n-k+1}$	$(y_{n-k+1} - y_k)$	$a_{n-k+1}(y_{n-k+1} - y_k)$
1	0.4643	67.49	31.336
2	0.3185	57.56	18.333
3	0.2578	51.78	13.349
4	0.2119	41.63	8.821
5	0.1736	34.16	5.930
6	0.1399	26.32	3.682
7	0.1092	20.03	2.187
8	0.0804	15.94	1.282
9	0.0530	6.23	0.330
10	0.0263	3.34	0.088

b =85.338

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Table 2.3.4 : The Coefficients  $a_{n-k+1}$  and the Values  $(y_{n-k+1} - y_k)$   
Using the Observations from Block 3 .

k	$a_{n-k+1}$	$(y_{n-k+1} - y_k)$	$a_{n-k+1}(y_{n-k+1} - y_k)$
1	0.4643	67.99	31.568
2	0.3185	60.51	19.272
3	0.2578	57.33	14.780
4	0.2119	51.58	10.930
5	0.1736	44.03	7.644
6	0.1399	40.80	5.708
7	0.1092	38.55	4.210
8	0.0804	11.79	0.948
9	0.0530	10.53	0.558
10	0.0263	3.41	0.090

b =95.708

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Let us consider the observations from Block 1. From Table 2.3.1,  $\Sigma y_k = 595.14$  and  $\Sigma y_k^2 = 21155.04$

Hence

$$\begin{aligned} R^2 &= \Sigma y_k^2 - \frac{(\Sigma y_k)^2}{n} , \\ &= 21155.04 - \frac{(595.14)^2}{21} , \\ &= 4288.773 . \end{aligned}$$

From Table 2.3.2,  $b = 63.835$  .

Therefore

$$\begin{aligned} W &= \frac{b^2}{R^2} , \\ &= \frac{(63.835)^2}{4288.773} , \\ &= 0.950. \end{aligned}$$

For  $n = 21$ , the tabulated 5% point of  $W$  is 0.908. Since the composite null hypothesis of normality is being tested only small values of calculated  $W$  indicate non-normality. Thus, according to the above test, the observations from Block 1 come from a normal population, at the 5% level.

Let us consider the observations from Block 2. Tables 2.3.1 and 2.3.3 provide the following information:

$$\Sigma y_k = 584.34; \quad \Sigma y_k^2 = 24251.554; \quad b = 85.338,$$

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Thus

$$\begin{aligned}
 R^2 &= \Sigma y_k^2 - \frac{(\Sigma y_k)^2}{n} , \\
 &= 24251.554 - \frac{(584.34)^2}{21} , \\
 &= 7991.877 .
 \end{aligned}$$

Therefore

$$W = \frac{b^2}{R^2} = \frac{(85.338)^2}{7991.877} = 0.911 .$$

For  $n = 21$ , the tabulated 5% point of  $W$  is 0.908. Thus there is no evidence of non-normality at the 5% level.

Let us finally consider the observations from Block 3. Tables 2.3.1 and 2.3.4 provide the following information:

$$\Sigma y_k = 626.26; \quad \Sigma y_k^2 = 28754.127; \quad b = 95.708 .$$

Thus

$$\begin{aligned}
 R^2 &= \Sigma y_k^2 - \frac{(\Sigma y_k)^2}{n} , \\
 &= 28754.127 - \frac{(626.26)^2}{21} , \\
 &= 10077.862 .
 \end{aligned}$$



Therefore

$$W = \frac{b^2}{R^2} = \frac{(95.708)^2}{10077.86} = 0.910 .$$

For  $n = 21$ , the tabulated 5% point of  $W$  is 0.908. Thus, there is no evidence of non-normality at the 5% level.

The above results indicate that the observations, and hence the errors, are normally distributed.

## II. Testing for Additivity of Treatment and Environmental Effects.

It was shown in Section 2.2 (II) that the presence of non-additivity in a randomised block experiment could be tested for by using Tukey's test provided the normality assumption was satisfied. The test carried out in Section 2.3 (I) verified the normality assumption. Accordingly, for the "Kpong data", Tukey's test will be employed to test for additivity of treatment and environmental effects. The test was described in Section 2.2 (II) and the analysis of variance table for its performance was given in Table 2.2.1

The sum of squares for non-additivity was given as

$$SS_N = \frac{[\sum \sum (\bar{X}_{i.} - \bar{X}_{..})(\bar{X}_{.j} - \bar{X}_{..})]^2}{\sum (\bar{X}_{i.} - \bar{X}_{..})^2 \sum (\bar{X}_{.j} - \bar{X}_{..})^2} = \frac{p^2 \dots}{(\sum d_i^2) (\sum d_j^2)} \dots (2.3.4)$$

The residual for testing non-additivity was given as

$$SS_E' = SS_E - SS_N ,$$

$$\begin{aligned} \text{where (i) } SS_E &= \sum_{ij} (X_{ij} - \bar{X}_{i.} - \bar{X}_{.j} + \bar{X}_{..})^2 , \\ &= \sum_{ij} X_{ij}^2 - \frac{\sum T_{i.}^2}{iJ} - \frac{\sum T_{.j}^2}{jI} + \frac{T^2}{IJ} ; \dots (2.3.5) \end{aligned}$$

$$\text{(ii) } T_{i.} = \sum_j X_{ij}; \quad T_{.j} = \sum_i X_{ij}; \quad T = \sum_{ij} X_{ij} .$$

The results for carrying out the test are presented below.

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Table 2.3.5 : Tukey's Test for Non-Additivity.  
Yields in grams per Plot for 21 Varieties of Cowpea .

Varieties	B L O C K S			Row Totals	Row Means
	1	2	3	$T_{i\cdot}$	$\bar{X}_{i\cdot}$
V <sub>1</sub>	26.39	22.39	21.82	70.60	23.533
V <sub>2</sub>	50.52	22.06	16.15	88.73	29.577
V <sub>3</sub>	5.56	5.07	5.66	16.29	5.430
V <sub>4</sub>	45.58	58.20	64.20	167.98	55.993
V <sub>5</sub>	39.47	34.61	34.70	108.78	36.260
V <sub>6</sub>	27.58	47.01	60.34	134.93	44.977
V <sub>7</sub>	47.11	24.67	55.80	127.58	42.527
V <sub>8</sub>	34.52	39.84	29.81	104.17	34.723
V <sub>9</sub>	27.23	30.62	18.68	76.53	25.510
V <sub>10</sub>	48.71	62.63	71.61	182.95	60.983
V <sub>11</sub>	53.68	52.59	53.67	159.94	53.313
V <sub>12</sub>	33.21	70.32	66.17	169.70	56.567
V <sub>13</sub>	24.58	12.85	18.41	55.84	18.613
V <sub>14</sub>	19.49	14.52	18.40	52.41	17.470
V <sub>15</sub>	6.37	2.83	3.62	12.82	4.273
V <sub>16</sub>	24.79	18.44	18.02	61.25	20.417
V <sub>17</sub>	13.44	19.05	6.87	39.36	13.120
V <sub>18</sub>	13.99	6.42	8.76	29.17	9.723
V <sub>19</sub>	15.23	10.96	11.77	37.96	12.653
V <sub>20</sub>	16.39	14.58	12.87	43.84	14.613
V <sub>21</sub>	21.30	14.68	28.93	64.91	21.637

46.

Table 2.3.5 (continued)

	B L O C K S			
	1	2	3	
Column Totals $T_{.j}$	595.14	584.14	626.26	
Column Means $\bar{X}_{.j}$	28.340	27.827	29.822	
$d_{.j} = \bar{X}_{.j} - \bar{X}_{..}$	-0.323	-0.836	1.159	$\Sigma d_{.j} = 0$
$d_{.j}^2$	0.104	0.699	1.343	$\Sigma d_{.j}^2 = 2.146$
$T_{.j}^2$	354191.620	341453.236	392201.588	$\Sigma T_{.j}^2 = 1087846.444$

$$\Sigma \Sigma_{i,j} X_{ij}^2 = 74160.721; \quad \bar{X}_{..} = \frac{1805.74}{63} = 28.663$$

47.

Table 2.3.5 (continued)

$d_i = \bar{X}_{i.} - \bar{X}_{..}$	$T_i^2$	$P_i = \sum_j X_{ij} d_j$	$d_i P_i$	$d_i^2$
-5.129	4984.360	-1.953	10.017	26.307
0.914	7873.013	-16.042	-4.662	0.835
-23.233	265.364	0.525	-12.197	539.772
27.331	28217.280	11.031	301.488	746.984
7.597	11833.088	-1.466	-11.137	57.714
16.314	18206.105	21.726	354.438	266.147
13.864	16276.656	28.831	399.713	192.210
6.061	10851.389	-9.906	-60.040	36.736
-3.153	5856.841	-12.743	40.179	9.941
32.321	33470.703	14.904	581.712	1044.647
24.651	25580.804	0.900	22.186	607.672
27.907	28798.090	7.176	200.239	778.633
-10.049	3118.106	2.655	-26.680	100.982
-11.193	2746.808	2.892	-32.370	125.283
-24.389	164.352	-0.228	5.561	594.823
-8.245	3751.563	-2.538	20.926	67.980
-15.543	1549.210	-12.305	191.257	241.585
-18.939	850.889	0.267	-5.057	358.686
16.009	1440.962	-0.441	7.060	256.288
-14.049	1921.946	-2.567	36.064	197.374
-7.026	4213.308	14.378	-101.020	49.365
$\sum d_i = 0$	$\sum T_i^2 = 211970.837$	$\sum P_i = 45.096$	$\sum d_i P_i = P = 1807.677$	$\sum d_i^2 = 6299.964$

48.

From equation (2.3.4)

$$SS_N = \frac{p^2}{(\sum d_i^2)(\sum d_j^2)} = \frac{(1807.677)^2}{(6299.964)(2.146)},$$

$$= \underline{241.698}.$$

From equation (2.3.5)

$$SS_E = \sum_{ij} X_{ij}^2 - \frac{\sum T_i^2}{J} - \frac{\sum T_j^2}{I} + \frac{T^2}{IJ},$$

$$= 74160.721 - \frac{211970.837}{3} - \frac{1087846.444}{21} + \frac{1805.74}{21 \times 3},$$

$$= 3458.657.$$

Hence

$$SS_E' = SS_E - SS_N,$$

$$= 3458.657 - 241.698,$$

$$= \underline{3216.959}.$$

The analysis of variance table is given below.

Table 2.3.6 : Analysis of Variance for Testing  
Non-additivity in a Randomised  
Block Design.

Source	Degrees of Freedom	Sum of Squares	Mean Square
Non-additivity	1	241.698	241.698
Residual	39	3216.959	82.486

49.

From Table 2.3.6,

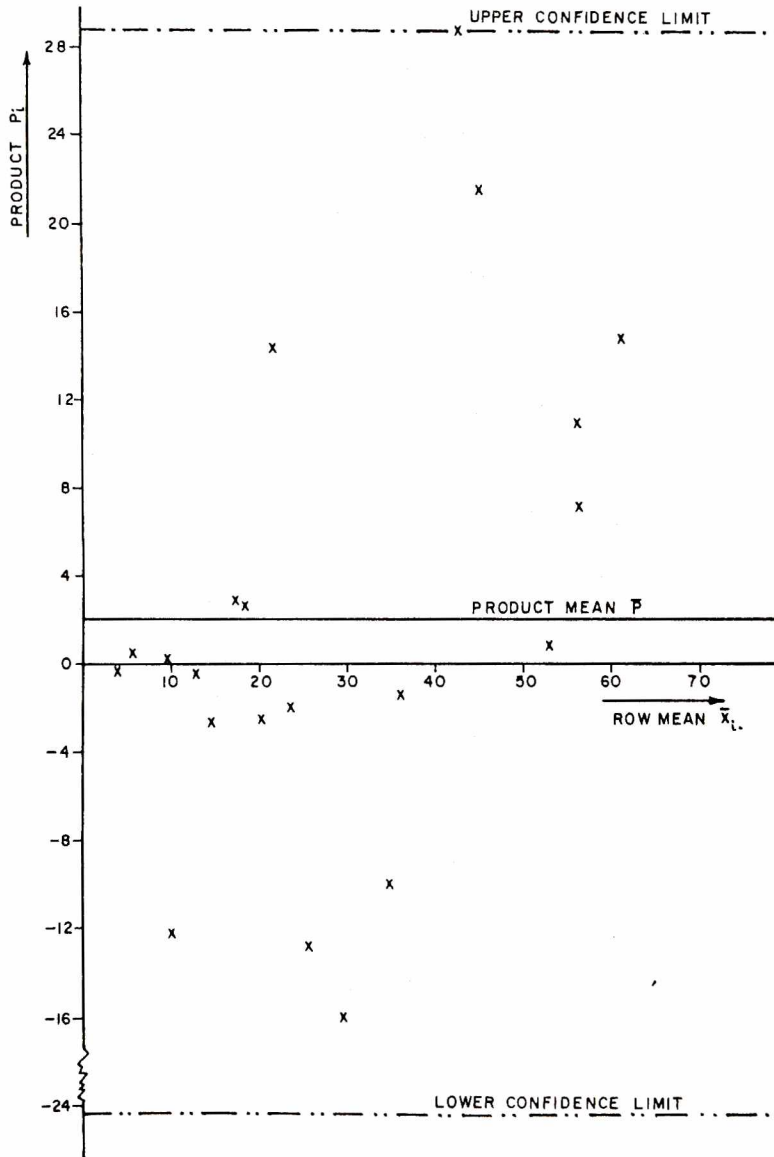
$$F_{1,39} = \frac{241.698}{82.468} = 2.93 .$$

The tabulated value of  $F_{1,39}$  at the 5% level of significance is 4.09 (by linear interpolation). Thus, non-additivity is not significant and hence the treatment and environmental effects may be assumed to be additive.

If the value of  $F$  obtained from the ANOVA table were large, non-additivity would be indicated and a decision would have to be made about procedure. According to Snedecor (37), help in making the decision comes from a graph of  $P_i$  as ordinate against the mean  $\bar{X}_{i.}$ , where  $P_i = \sum_j X_{ij} d_j$  are given in Table 2.3.5. Such a graph has been drawn for the data and is shown in Figure 2.3.1.

50.

FIG.2.3.1 PRODUCT  $P_i$  AS FUNCTION OF ROW MEAN  $\bar{x}_i$ .  
 MEAN OF  $P_i$  IS SHOWN WITH CONFIDENCE LIMITS.



51.

The mean of the  $P_i$ ,  $\bar{P}$ , is plotted as a horizontal line. Similar lines are drawn to show the 95% confidence interval which is given by Snedecor as

$$P \pm 2 \left[ \frac{\text{sum of squares of deviations of column means}}{\quad} \right]^{\frac{1}{2}} \times \left[ \frac{\text{Mean square for Residual}}{\quad} \right]^{\frac{1}{2}} ;$$

$$\text{i.e. } \bar{P} \pm 2 \left( \sum_j d_j^2 \right)^{\frac{1}{2}} \times (\text{Mean square for Residual})^{\frac{1}{2}} ,$$

where the  $d_j$ 's are given in Table 2.3.5.

$$\text{Now } \bar{P} = \frac{\sum_i P_i}{21} = \frac{45.096}{21} = 2.147 .$$

Hence the 95% confidence limits are

$$\bar{P} \pm 2 \left[ (2.146) (82.486) \right]^{\frac{1}{2}} ;$$

$$\text{i.e. } 2.147 \pm 26.610 .$$

These lines are shown in Figure 2.3.1.

Figure 2.3.1 shows that all the points lie within the confidence limits and this corresponds with the non-significance of the Tukey's test for non-additivity. If non-additivity were significant, one or more of the plotted points would be expected to be outside the confidence lines.

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### III. Testing for Homogeneity of Error Variances.

In Section 2.2 (III), Bartlett's test was shown to be appropriate for testing homogeneity of error variances. A theoretical description of the test was also given in that section. The test statistic for small samples was given as  $M/C$ , where

$$\begin{aligned} \text{(i)} \quad M &= (n-1) \left[ k \log_e \frac{\sum S_i^2}{k} - \sum \log_e S_i^2 \right], \\ &= 2.3026(n-1) \left[ k \log_{10} \frac{S_i^2}{k} - \sum \log_{10} S_i^2 \right]; \dots(2.3.6) \end{aligned}$$

(ii)  $C$  is the corrective factor and is given by

$$C = 1 + \frac{k+1}{3k(n-1)} \cdot \dots(2.3.7)$$

For the "Kpong data", the observations from each block are regarded as a sample. Thus, there are three samples each of size 21 and it is the variances of these samples that are to be tested for homogeneity.

Using Table 2.3.1, the three variances are obtained as

$$S_1^2 = 214.439, \quad S_2^2 = 399.594, \quad S_3^2 = 503.893.$$

$$\text{Thus } \log_{10} S_1^2 = 2.3312, \quad \log_{10} S_2^2 = 2.6017, \quad \log_{10} S_3^2 = 2.7024.$$

53.

Hence

$$\sum_{i=1}^3 S_i^2 = 1117.926 \quad ;$$

and

$$\sum_{i=1}^3 \text{Log}_{10} S_i^2 = 7.6353 \quad .$$

From equation (2.3.6),

$$\begin{aligned} M &= (2.3026) (20) \left[ 3 \text{Log}_{10} \frac{1117.926}{3} - 7.6353 \right] , \\ &= 46.052 (7.7136 - 7.6353) , \\ &= \underline{3.606} \quad . \end{aligned}$$

From equation (2.3.7)

$$C = 1 + \frac{4}{9 \times 20} = \underline{1.022} \quad .$$

Therefore

$$\frac{M}{C} = \frac{3.606}{1.022} = \underline{3.53} \quad .$$

The tabulated value of  $\chi^2$  (2 d.f., 5% level) is 5.99. Thus, there is no evidence of heterogeneity of error variances at the 95% confidence level.

The results of the previous section indicate that the three most important assumptions underlying analysis of variance have been satisfied. The analysis of variance methods will therefore be used to analyse the data.

CHAPTER THREEUNIVARIATE ANALYSIS OF VARIANCE3.1 Randomised Block Experiment with n Plants Per Plot Using the Average Yields Per Plot.

Regarding the average yield of a plot as a single observation, we obtain a randomised block experiment with one observation per plot. Such an experiment has been described in Section 1.1. The appropriate mathematical model for this situation is given by

$$X_{ij} = \mu + \alpha_i + \beta_j + \gamma_{ij} + e_{ij} \quad \dots\dots(3.1.1)$$

The application of the analysis of variance technique to the above model has been discussed by Federer (14), Scheffé' (34), and others (23,27).

For model (3.1.1), the total sum of squares is given by

$$\sum_{ij} (X_{ij} - \bar{X}_{..})^2 .$$

Applying the analysis of variance technique, this total sum of squares can be broken up into three different sums of squares, each attributable to a different source. The resulting identity is

$$\begin{aligned} \sum_{ij} (X_{ij} - \bar{X}_{..})^2 &= \sum_i (\bar{X}_{i.} - \bar{X}_{..})^2 + \sum_j (\bar{X}_{.j} - \bar{X}_{..})^2 \\ &\quad + \sum_{ij} (X_{ij} - \bar{X}_{i.} - \bar{X}_{.j} + \bar{X}_{..})^2 ; \end{aligned}$$

55.

where

- (i)  $\sum_{ij} (X_{ij} - \bar{X}_{..})^2 = SS_T = \text{Total sum of squares ;}$
- (ii)  $J \sum_i (\bar{X}_{i.} - \bar{X}_{..})^2 = SS_V = \text{Treatment sum of squares ;}$
- (iii)  $I \sum_j (\bar{X}_{.j} - \bar{X}_{..})^2 = SS_B = \text{Block sum of squares ;}$
- (iv)  $\sum_{ij} (X_{ij} - \bar{X}_{i.} - \bar{X}_{.j} + \bar{X}_{..})^2 = SS_E = \text{"Error" sum of squares .}$

If we let  $\sum_j X_{ij} = T_{i.}$ ,  $\sum_i X_{ij} = T_{.j}$ ,  $\sum_{ij} X_{ij} = T$  ,

we obtain the following convenient computational formulae

- (i)  $J \sum_i (\bar{X}_{i.} - \bar{X}_{..})^2 = \frac{\sum_i T_{i.}^2}{J} - \frac{T^2}{IJ} = SS_V ;$
- (ii)  $I \sum_j (\bar{X}_{.j} - \bar{X}_{..})^2 = \frac{\sum_j T_{.j}^2}{I} - \frac{T^2}{IJ} = SS_B ;$
- (iii)  $\sum_{ij} (X_{ij} - \bar{X}_{..})^2 = \sum_{ij} X_{ij}^2 - \frac{T^2}{IJ} = SS_T .$

The analysis-of-variance (ANOVA) table appropriate to the above situation is given in Table 3.1.1.

**Table 3.1.1:** ANOVA for a Randomised Block Experiment with One Observation Per Plot. Each Observation is an Average Yield.

Source of Variation	Degrees of Freedom (df)	Sum of Squares (SS)	SS for Computation	Mean Square (MS)
Treatments	$I - 1$	$J\sum (\bar{X}_{i.} - \bar{X}_{..})^2 = SS_V$	$\frac{\sum T_i^2}{J} - \frac{T^2}{IJ}$	$\frac{SS_V}{I-1} = MS_V$
Blocks	$J - 1$	$I\sum (\bar{X}_{.j} - \bar{X}_{..})^2 = SS_B$	$\frac{\sum T_{.j}^2}{I} - \frac{T^2}{IJ}$	$\frac{SS_B}{J-1} = MS_B$
"Error"	$(I-1)(J-1)$	$\sum \sum (X_{ij} - \bar{X}_{i.} - \bar{X}_{.j} + \bar{X}_{..})^2 = SS_E$	By subtraction	$\frac{SS_E}{(I-1)(J-1)} = MS_E$
Total	$(IJ - 1)$	$\sum \sum (X_{ij} - \bar{X}_{..})^2 = SS_T$	$\sum \sum X_{ij}^2 - \frac{T^2}{IJ}$	

With the fixed-effects model, we assume that the treatment and environmental effects are additive. That is, the  $\{\gamma_{ij}\}$  in equation (3.1.1) are assumed to be zero.

The expected values of the mean squares in Table 3.1.1 under both the fixed-effects and random-effects models have been calculated by Scheffé' (34), and they are given in Table 3.1.2.

**Table 3.1.2:** Expected Values of the Mean Squares in Table 3.1.1.

	Fixed-Effects ( $\gamma_{ij} = 0$ )	Random-Effects ( $\gamma_{ij} \neq 0$ )
$E(MS_V)$	$\frac{\sigma_e^2}{n} + \frac{J}{I-1} \sum \alpha_i^2$	$\sigma_e^2/n + \sigma_\gamma^2 + J\sigma_\alpha^2$
$E(MS_B)$	$\frac{\sigma_e^2}{n} + \frac{I}{J-1} \sum \beta_j^2$	$\sigma_e^2/n + \sigma_\gamma^2 + I\sigma_\beta^2$
$E(MS_E)$	$\frac{\sigma_e^2}{n}$	$\sigma_e^2/n + \sigma_\gamma^2$

57.

In this experiment, if the fixed-effects model is assumed, the hypotheses that we usually wish to test are

$$H_A: \alpha_i = 0 \quad \text{for } i=1,2,\dots,I \quad ;$$

$$H_B: \beta_j = 0 \quad \text{for } j=1,2,\dots,J \quad .$$

If the random-effects model is assumed, the hypotheses we would like to test are

$$H_A: \sigma_\alpha^2 = 0 \quad ;$$

$$H_B: \sigma_\beta^2 = 0 \quad .$$

Rao (32) has shown that each of the sums of squares (given in Table 3.1.1) divided by the expected value of the corresponding mean square is distributed as central or non-central  $\chi^2$  with the degrees of freedom of the quadratic form in question. In particular, under the fixed-effects model,

$$(i) \quad \frac{SS_E}{\sigma_e^2/n} \sim \chi^2_{(I-1)(J-1)} \quad ;$$

(ii) When  $H_A$  is true,

$$\frac{SS_V}{\sigma_e^2/n} \sim \chi^2_{I-1} \quad ;$$

(iii)  $SS_E$  and  $SS_V$  are independent.

Hence the statistic

$$F_{A1} = \frac{SS_V/(I-1)}{SS_E/((I-1)(J-1))} = \frac{MS_V}{MS_E}$$

....(3.1.2)

58.

has an F-distribution with  $(I - 1)$  and  $(I - 1)(J - 1)$  degrees of freedom when  $H_A$  is true. It may therefore be used to test the hypothesis  $H_A$ .

Similarly, the statistic

$$F_{B1} = \frac{MS_B}{MS_E} \quad \dots\dots(3.1.3)$$

has an F-distribution with  $(J - 1)$  and  $(I - 1)(J - 1)$  degrees of freedom when  $H_B$  is true, and may be used to test the hypothesis  $H_B$ .

Under the random-effects model,

$$(i) \quad \frac{SS_E}{\sigma_e^2/n + \sigma_\gamma^2} \sim \chi^2_{(I-1)(J-1)} ;$$

$$(ii) \quad \text{When } H_A \text{ is true,}$$

$$\frac{SS_V}{\sigma_e^2/n + \sigma_\gamma^2} \sim \chi^2_{I-1} ;$$

(iii)  $SS_E$  and  $SS_V$  are independent.

Hence the statistic

$$F_{A2} = \frac{SS_V/(I-1)}{SS_E/(I-1)(J-1)} = \frac{MS_V}{MS_E} \quad \dots(3.1.4)$$

has an F-distribution with  $(I-1)$  and  $(I-1)(J-1)$  degrees of freedom when  $H_A$  is true. It may therefore be used to test the hypothesis  $H_A$ . A similar test can be constructed to test the hypothesis  $H_B$ .

### 3.2 Randomised Block Experiment with n Plants Per Plot.

In a randomised block experiment with n plants per plot, if the characteristic is observed on each of the n plants on a plot, there will be n observations per plot. A mathematical model appropriate for this situation was given in Section 2.1 as

$$X_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + e_{ijk} .$$

The analysis of variance technique has also been applied to this situation. The procedure is described by Federer (14), Scheffé' (34) and Kempthorne (25).

For the above model, the total sum of squares can be broken up into four different sums of squares attributable to different sources. The various sums of squares and the corresponding sources are given in Table 3.2.1.

Table 3.2.1: ANOVA for a Randomised Block Experiment with n Observations Per Plot.

Source of Variation	Degrees of Freedom	Sum of Squares (SS)	SS for Computation	Mean Square (MS)
Treatment	I - 1	$Jn\sum(\bar{X}_{1..} - \bar{X}...)^2$ = $SS_V$	$\frac{\sum T_{1..}^2}{Jn} - \frac{T^2}{IJn}$	$MS_V = \frac{SS_V}{I-1}$
Blocks	J - 1	$In\sum(\bar{X}_{.j.} - \bar{X}...)^2$ = $SS_B$	$\frac{\sum T_{.j.}^2}{In} - \frac{T^2}{IJn}$	$MS_B = \frac{SS_B}{J-1}$
Interaction	(I-1)(J-1)	$n\sum\sum(\bar{X}_{ij.} - \bar{X}_{i..} - \bar{X}_{.j.} + \bar{X}...)^2$ = $SS_{VB}$	By subtraction	$MS_{VB} = \frac{SS_{VB}}{(I-1)(J-1)}$
Within Plots	IJ(n-1)	$\sum\sum\sum(X_{ijk} - \bar{X}_{ij.})^2$ = $SS_W$	$\sum\sum\sum X_{ijk}^2 - \frac{\sum\sum T_{ij.}^2}{n}$	$MS_W = \frac{SS_W}{IJ(n-1)}$
Total	IJn - 1	$\sum\sum\sum(X_{ijk} - \bar{X}...)^2$ = $SS_T$	$\sum\sum\sum X_{ijk}^2 - \frac{T^2}{IJn}$	

The expected values of the mean squares in Table 3.2.1 under both the fixed-effects and random-effects models have been calculated by Scheffé' (34). They are given below in Table 3.2.2.

Table 3.2.2 : Expected Values of the Mean Squares  
in Table 3.2.1 .

	Fixed-Effects	Random-Effects
$E(MS_V)$	$\sigma_e^2 + \frac{Jn}{I-1} \Sigma \alpha_i^2$	$\sigma_e^2 + n\sigma_\gamma^2 + Jn\sigma_\alpha^2$
$E(MS_B)$	$\sigma_e^2 + \frac{In}{J-1} \Sigma \beta_j^2$	$\sigma_e^2 + n\sigma_\gamma^2 + In\sigma_\beta^2$
$E(MS_{VB})$	$\sigma_e^2 + \frac{n}{(I-1)(J-1)} \Sigma \Sigma \gamma_{ij}^2$	$\sigma_e^2 + n\sigma_\gamma^2$
$E(MS_W)$	$\sigma_e^2$	$\sigma_e^2$

Under the fixed-effects model, the hypotheses of interest are

$$H_A: \alpha_i = 0, \quad i=1,2,\dots,I ;$$

$$H_B: \beta_j = 0, \quad j=1,2,\dots,J ;$$

$$H_{AB}: \gamma_{ij} = 0, \quad i=1,\dots,I; \quad j=1,\dots,J .$$

Under the random-effects model, the hypotheses of interest are

$$H_A: \sigma_\alpha^2 = 0 ;$$

$$H_B: \sigma_\beta^2 = 0 ;$$

$$H_{AB}: \sigma_\gamma^2 = 0 .$$

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It has again been shown by Rao (32) that each of the sums of squares in Table 3.2.1 divided by the expected value of the corresponding mean square is distributed as central or non-central  $\chi^2$  with the degrees of freedom of the quadratic form in question.

In particular, under the fixed-effects model,

$$(i) \quad \frac{SS_W}{\sigma_e^2} \sim \chi_{IJ(n-1)}^2 ;$$

(ii) When  $H_A$  is true,

$$\frac{SS_V}{\sigma_e^2} \sim \chi_{I-1}^2 ;$$

(iii)  $SS_W$  and  $SS_V$  are independent .

Hence the statistic

$$F_{A3} = \frac{SS_V/(I-1)}{SS_W/IJ(n-1)} = \frac{MS_V}{MS_W} \dots\dots(3.2.1)$$

has an F-distribution with  $(I-1)$  and  $IJ(n-1)$  degrees of freedom when  $H_A$  is true, and may be used to test the hypothesis  $H_A$ . Similar test statistics can be obtained to test for the hypotheses  $H_B$  and  $H_{AB}$ .

Under the random-effects model, we have

63.

$$(i) \quad \frac{SS_{VB} -}{\sigma_e^2 + n\sigma_\gamma^2} \sim \chi_{(I-1)(J-1)}^2 \quad ;$$

(ii) When  $H_A$  is true,

$$\frac{SS_V}{\sigma_e^2 + n\sigma_\gamma^2} \sim \chi_{I-1}^2 \quad \circ$$

Hence the statistic

$$F_{A4} = \frac{SS_V/(I-1)}{SS_{VB}/(I-1)(J-1)} = \frac{MS_V}{MS_{VB}} \quad \dots\dots(3.2.2)$$

has an F-distribution with  $I-1$  and  $(I-1)(J-1)$  degrees of freedom when  $H_A$  is true.  $F_{A4}$  may therefore be used to test for the hypothesis  $H_A$ . Similarly, the test statistics  $MS_B/MS_{VB}$  and  $MS_{VB}/MS_W$  may be used to test for the hypotheses  $H_B$  and  $H_{AB}$ , respectively.

When the hypothesis  $H_{AB}$  is not true, we may have treatment, say,  $V_1$  to be better than treatment  $V_2$  when averaged over all the  $J$  blocks but for some of the blocks  $V_2$  may be better than  $V_1$ . Under this condition, any treatment differences detected should be viewed as being averaged over all the  $J$  blocks in the experiment. For a true picture of the situation, however, it is necessary to test for differences in treatments for every block. The sum of squares with  $I-1$  degrees of freedom due to treatments for the  $j^{\text{th}}$  block is given by Rao (31) as

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$$SS'_V = \frac{1}{n} \sum_i T_{ij}^2 - \frac{T_{.j}^2}{In} \quad \dots\dots(3.2.3)$$

The mean square corresponding to this is tested against  $MS_{VB}$  or  $MS_W$  depending on whether we are using the random-effects or fixed-effects model.

### 3.3 Comparisons of the Various Tests for Treatment Effects.

We noted in Section 3.1 that when the average yields per plot are used, the hypothesis of no treatment effects ( $H_A$ ) under both the fixed-effects and random-effects models could respectively be tested by the F statistics  $F_{A1}$  and  $F_{A2}$ ;

where

$$(i) \quad F_{A1} = \frac{MS_V}{MS_E} \quad (\text{Fixed-Effects Model}) \ ;$$

$$(ii) \quad F_{A2} = \frac{MS_V}{MS_E} \quad (\text{Random-Effects Model}) \ .$$

We also noted in Section 3.2 that when yields of the individual plants on a plot are used, the same hypothesis  $H_A$  under both the fixed-effects and random-effects models could respectively be tested by the F statistics  $F_{A3}$  and  $F_{A4}$ ;

where

$$(iii) \quad F_{A3} = \frac{MS'_V}{MS_W} \quad (\text{Fixed-Effects Model}) \ ;$$

where  $MS'_V$  is the treatments mean square given in Table 3.2.1 ;

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$$(iv) \quad F_{A4} = \frac{MS_V'}{MS_{VB}} \quad (\text{Random-Effects Model}) .$$

Thus, under the fixed-effects model, the hypothesis  $H_A$  can be tested by using either  $F_{A1}$  or  $F_{A3}$ . Let these two statistics be generally represented by

$$F_f = \frac{M_1}{M_2} ,$$

where  $M_1$  and  $M_2$  are treatments and error mean squares with  $f_1$  and  $f_2$  degrees of freedom, respectively. As pointed out in Sections 3.1 and 3.2,  $F_f$  has an F-distribution with  $f_1$  and  $f_2$  degrees of freedom when  $H_A$  is true. When  $H_A$  is not true,  $F_f$  has been found to have a non-central  $F(F_{f_1, f_2; \delta})$  with non-centrality parameter  $\delta$ .

Using  $F_f$  to test  $H_A$  at the  $\alpha$  level of significance consists in rejecting  $H_A$  if and only if

$$F_f \geq F_{\alpha; f_1, f_2} ,$$

where  $F_{\alpha; f_1, f_2}$  is the tabular value of the F-distribution with  $f_1$  and  $f_2$  degrees of freedom at the  $\alpha$  level of significance. Thus the power of the test - the probability of rejecting the hypothesis  $H_A$  - is

$$\beta = P_r (F_f > F_{\alpha; f_1, f_2}) ,$$

$$\text{or} \quad \beta = P_r (F_{f_1, f_2; \delta} > F_{\alpha; f_1, f_2}) . \quad \dots\dots(3.3.1)$$

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The non-centrality parameter  $\delta$  is calculated by

$$\delta^2 = \frac{SS_V(m_{ij})}{\sigma_e^2}, \quad \dots\dots(3.3.2)$$

where (i)  $m_{ij}$  = expected value of  $X_{ij}$  ;

(ii)  $SS_V(m_{ij})$  = treatment sum of squares with the  $X_{ij}$ 's replaced by  $m_{ij}$  .

Equation (3.3.1) has been tabulated by Tang (25) for  $\alpha = 0.01$  and  $0.05$ . He, however, chose to work in terms of  $\phi^*$ , where

$$\phi^* = \delta(f_1 + 1)^{-\frac{1}{2}} \cdot \quad \dots\dots(3.3.3)$$

Pearson and Hartley (34) have prepared charts from Tang's table for  $\alpha = 0.01$  and  $0.05$ . The charts show that the power  $\beta$  increases with  $f_2$  and  $\phi^*$  for a given  $f_1$ . From equations (3.3.2) and (3.3.3),  $\phi^*$  will be large when  $\sigma_e^2$  is small. Thus the F-test based on  $F_f$  will be powerful, and hence sensitive, if the error variance is small or has more degrees of freedom.

When  $F_f = F_{A1}$ , we have

- (i)  $f_1 = I-1$  ;
- (ii)  $f_2 = (I-1)(J-1)$  ;
- (iii)  $\hat{\sigma}_e^2 = n(MS_E)$  .

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On the other hand, when  $F_f = F_{A3}$ , we have

- (i)  $f_1 = I-1$  ;
- (ii)  $f_2 = IJ(n-1)$  ;
- (iii)  $\hat{\sigma}_e^2 = MS_W$  .

$$\text{Now } n(MS_E) = \frac{n(SS_E)}{(I-1)(J-1)} \sim \frac{\sigma_e^2}{(I-1)(J-1)} \chi_{(I-1)(J-1)}^2 ,$$

$$\text{and } MS_W = \frac{SS_{VB}}{IJ(n-1)} \sim \frac{\sigma_e^2}{IJ(n-1)} \chi_{IJ(n-1)}^2 .$$

Since the variance of any  $\chi^2$  variable is 2 times its degrees of freedom, we have

- (i)  $\text{Var} [n(MS_E)] = \frac{2\sigma_e^4}{(I-1)(J-1)} ;$
- (ii)  $\text{Var}(MS_W) = \frac{2\sigma_e^4}{IJ(n-1)} .$

$$\text{Thus } \text{Var}(MS_W) < \text{Var} [n(MS_E)]$$

since  $IJ(n-1) > (I-1)(J-1)$  .

That is,  $MS_W$  is a more efficient estimator of  $\sigma_e^2$  than  $n(MS_E)$ .

From the above results, we may conclude that the F-test based on  $F_{A3}$  will be more sensitive than the one based on  $F_{A1}$ . That is, if the fixed-effects model is assumed, the F-test for treatment differences using yields of the individual plants on

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a plot is more sensitive than a corresponding F-test using the average yields per plot.

Under the random-effects model, the hypothesis  $H_A$  can be tested by using either  $F_{A2}$  or  $F_{A4}$ . Let these two statistics be generally represented by

$$F_r = \frac{M_3}{M_4} ,$$

where  $M_3$  and  $M_4$  are treatments and error mean squares with  $f_3$  and  $f_4$  degrees of freedom, respectively. When  $H_A$  is true,  $F_r$  has an F-distribution with  $f_3$  and  $f_4$  degrees of freedom. Under this model, however,  $F_r$  is distributed as  $bF_{f_3, f_4}$  when  $H_A$  is not true, where  $F_{f_3, f_4}$  is a central F-variable with  $f_3$  and  $f_4$  degrees of freedom, and  $b$  is a constant. The test consists in rejecting  $H_A$  at the  $\alpha$  level of significance if

$$F_r \geq F_{\alpha; f_3, f_4} .$$

The power of the test is therefore given by

$$\beta = P_r(F_r \geq F_{\alpha; f_3, f_4}) ,$$

$$\text{or } \beta = P_r(F_{f_3, f_4} \geq \frac{1}{b} F_{\alpha; f_3, f_4}) . \quad \dots (3.3.4)$$

When  $F_r = F_{A2}$ , we have

$$(i) \quad f_3 = I-1 ;$$

$$(ii) \quad f_4 = (I-1)(J-1) ;$$

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$$\begin{aligned}
 \text{(iii)} \quad \frac{MS_V}{MS_E} &= \frac{(\sigma_e^2/n + \sigma_\gamma^2 + J\sigma_\alpha^2)}{\sigma_e^2/n + \sigma_\gamma^2} \cdot \frac{(I-1)(J-1)}{(I-1)} \cdot \frac{x_{I-1}^2}{x_{(I-1)(J-1)}^2} , \\
 &= \left(1 + \frac{Jn\sigma_\alpha^2}{\sigma_e^2 + n\sigma_\gamma^2}\right) F_{(I-1), (I-1)(J-1)} .
 \end{aligned}$$

$$\text{Thus} \quad b = \left(1 + \frac{Jn\sigma_\alpha^2}{\sigma_e^2 + n\sigma_\gamma^2}\right) .$$

From equation (3.3.4), we obtain

$$\beta_1 = P_r \left[ F_{(I-1), (I-1)(J-1)} \geq \frac{1}{\left(1 + \frac{Jn\sigma_\alpha^2}{\sigma_e^2 + n\sigma_\gamma^2}\right)} F_{\alpha; (I-1), (I-1)(J-1)} \right]$$

.....(3.3.5)

When  $F_r = F_{A4}$ , we have

$$\text{(i)} \quad f_3 = I-1 \quad ;$$

$$\text{(ii)} \quad f_4 = (I-1)(J-1) \quad ;$$

$$\text{(iii)} \quad \frac{MS'_V}{MS_{VB}} = \frac{\sigma_e^2 + n\sigma_\gamma^2 + Jn\sigma_\alpha^2}{\sigma_e^2 + n\sigma_\gamma^2} \cdot \frac{(I-1)(J-1)}{(I-1)} \cdot \frac{x_{I-1}^2}{x_{(I-1)(J-1)}^2} ,$$

$$= \left(1 + \frac{Jn\sigma_\alpha^2}{\sigma_e^2 + n\sigma_\gamma^2}\right) F_{(I-1), (I-1)(J-1)} .$$

$$\text{Thus} \quad b = 1 + \frac{Jn\sigma_\alpha^2}{\sigma_e^2 + n\sigma_\gamma^2} .$$

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From equation (3.3.4) therefore,

$$\beta_2 = P_r \left[ F_{I-1, (I-1)(J-1)} \geq \frac{1}{\left(1 + \frac{Jn\sigma_\alpha^2}{\sigma_e^2 + n\sigma_\gamma^2}\right)} F_{\alpha; I-1, (I-1)(J-1)} \right] \circ$$

.....(3.3.6)

Comparing equations (3.3.6) and (3.3.5), we observe that

$$\beta_1 = \beta_2 \circ$$

When there are no interactions, however, we will expect

$$\beta_2 > \beta_1$$

since the error variance,  $\sigma_e^2$ , is estimated more efficiently in equation (3.3.6) than in equation (3.3.5) -  $MS_W$  is a more efficient estimator of  $\sigma_e^2$  than  $n(MS_E)$ .

From the above results, we may conclude that the F-test based on  $F_{A4}$  will be more sensitive than a corresponding one based on  $F_{A2}$  if there are no interaction effects. That is, if the random-effects model is assumed and there are no interaction effects, the F-test for treatment differences using yields of the individual plants on a plot is more sensitive than a corresponding F-test using the average yields per plot. When there are interaction effects, the two tests may have the same sensitivity.

### 3.4 Randomised Block Experiment with one Observation Per Plot Under Randomization Models.

As pointed out in Section 2.1 (IV), the basic difference between normal-theory models and randomization models is the normality assumption associated with the errors in the former models. The errors in randomization models are assumed only to be uncorrelated with expectation zero.

Under randomization models, exact tests of certain hypotheses are possible. These exact tests are called permutation tests (or randomization tests) (4,17,34).

#### I. Permutation Tests.

The rudiments of permutation tests were given by Fisher (17). Major studies which stimulated interest in this field were however started by Box and Anderson (4) as well as Wilk and Kempthorne (42). Permutation tests for a hypothesis were found to exist whenever the joint distribution of the observations under the hypothesis had a certain kind of symmetry (a requirement which may often be guaranteed by randomization). That is, when the distribution was invariant under a group of permutations. The tests were found to be exact for a very broad kind of hypotheses in the sense that the validity of the calculated significance level depended only on the

symmetry of the distribution and not on any further assumptions such as normality of errors, homogeneity of error variances, or independence as in normal-theory tests. Box and Anderson also found permutation tests to be robust (4).

The attractiveness of permutation tests to statisticians lies in the fact that they are exact tests under less restrictive model, and can often be approximated by the F-test (or a slight modification of it) derived for the corresponding hypothesis under the fixed-effects normal-theory test. For this reason, the tests are often based on the same statistics that would be used in normal-theory tests.

In spite of its attractiveness, some statisticians have given concern about this new procedure. Yates and Barnard (4), for example, give caution of the possibility of finding "ourselves in the position where two statistical tests, while applying with apparently equal appropriateness to answering apparently the same question on the basis of the same data, give different answers". Welch (4) observed that: "the more complicated the experimental situation becomes, the less heavily one can lean on permutation theory to validate the usual normal-theory tests and the more heavily one must lean on the mathematical assumptions made".

On the power of the permutation test, Box and Anderson (4) found out that if the observations were really normally distributed the permutation test would necessarily be less powerful

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than the normal-theory test since the latter is uniformly most powerful for the normal distribution. For a randomised block design, Hoeffding (20) showed that the permutation test has asymptotically the same power as the usual F-test against alternatives of the normal-theory model. To the latter claim, Scheffé' (34) comments that what is of interest is the power of F-tests against alternatives in the randomization models. According to him, it is from such a study that we shall see whether the usual practice of using the normal-theory test as approximation to the exact permutation test is justified.

## II. Permutation Test for a Randomised Block Design.

Consider the randomised block experiment described in Section 1.1. There are  $(I!)^J$  possible assignments of treatments to the plots. Box and Anderson (4) and Scheffé' (34) have developed a permutation test for such an experiment. The following treatment is mainly due to Scheffé'.

Let the plots in each block be numbered with  $t=1,2,\dots,I$ . Also let  $\mu_{ij t}$  be the 'true' response under the  $i^{\text{th}}$  treatment on the  $(j,t)$  plot. Scheffé' defined the 'true' response as

$$\mu_{ij t} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \epsilon_{ij t}, \dots (3.4.1)$$

where

- (i)  $\mu = \mu_{\dots}$  ;
- (ii)  $\alpha_i = \mu_{i \dots} - \mu_{\dots}$  ;

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$$(iii) \beta_j = \mu_{\cdot j \cdot} - \mu_{\dots} \ ;$$

$$(iv) \gamma_{ij} = \mu_{ij \cdot} - \mu_{i \cdot \cdot} - \mu_{\cdot j \cdot} + \mu_{\dots} \ ;$$

$$(v) \epsilon_{ij t} = \mu_{ij t} - \mu_{ij \cdot} \ .$$

Thus  $\sum_t \epsilon_{ij t} = 0$  for all  $i, j$ .

He adopted the terminology of Wilk and Kempthorne (42) by calling the quantity  $\epsilon_{ij t}$  a plot error. It is a constant and it arises from the inequality of the true responses of different plots in the same block  $j$  to the same treatment  $i$ .

In a conceptual experiment involving a sequence of repetitions under the same conditions, Scheffé noted that the observed response  $X_{ij t}$  of the  $(j, t)$  plot to the  $i^{\text{th}}$  treatment would differ from the conceptual 'true' response  $\mu_{ij t}$  of the  $(j, t)$  plot to the  $i^{\text{th}}$  treatment on any particular trial by a technical error,  $e_{ij t}$ , (42). Thus

$$X_{ij t} = \mu_{ij t} + e_{ij t} \ ;$$

$e_{ij t}$  is regarded as a random variable with  $E(e_{ij t}) = 0$  by definition of  $\mu_{ij t}$  as the expectation of  $X_{ij t}$ .

The technical error,  $e_{ij t}$ , is a measurement error, the difference between an observed value and the corresponding true value, an error caused by the measuring instrument or the observer. The randomization by which the treatments are assigned

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ned to the plots is performed in such a way that the plot errors are independent of the technical errors  $\{e_{ijt}\}$ .

Denoting the observation on the  $i^{\text{th}}$  treatment in the  $j^{\text{th}}$  block by  $X_{ij}$ , Scheffé gave the mathematical model as

$$X_{ij} = \mu + \alpha_i + \beta_j + \gamma_{ij} + n_{ij} + e_{ij} , \dots(3.4.2)$$

where

- (i)  $n_{ij} = \sum_t d_{ijt} e_{ijt} \equiv$  plot error ;
- (ii)  $e_{ij} = \sum_t d_{ijt} e_{ijt} \equiv$  technical error ;
- (iii)  $\{d_{ijt}\}$  are  $I^2J$  random variables taking on the values 0 and 1.

He noted that, under the above model, the joint distribution of the observations does not have sufficient symmetry under the hypothesis  $H_A$ : all  $\alpha_i = 0$  to permit a permutation test. However, by assuming that the treatment-block interactions and the treatment-plot interactions within a block are zero, and that the technical errors are additive, he obtained a model for which the joint distribution of the observations is symmetric under the hypothesis  $H_A$ . Under the hypothesis  $H_A$ , he gave the model as

$$X_{ij} = \mu + \beta_j + \sum_t d_{ijt} (C_{jt} + Z_{jt}) , \dots(3.4.3)$$

where

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- (i)  $C_{jt}$  = plot main effect within the  $j^{\text{th}}$  block ;
- (ii)  $Z_{jt}$  = technical error associated with the  $(j,t)$  plot, formerly written  $e_{ijt}$ , but now assumed not to depend on the treatment applied to the plot.

Permutation tests will be based on the set  $G$  of permutations within blocks. The set  $G$  consists of  $(I!)^J$  permutations. Let  $X_0$  be the observed sample of  $\{X_{ij}\}$  and  $S(X_0)$  the set of  $(I!)^J$  samples obtained from  $X_0$  by applying the set  $G$  of permutations. Scheffé has shown that, given that  $X_0$  is in  $S(X_0)$ , all  $X_0$  in  $S(X_0)$  have the same conditional probability under the hypothesis  $H_A$ . Thus, symmetry is established.

The exact permutation test of the hypothesis  $H_A$  will be based on the statistic

$$F_0 = C \frac{SS_V}{SS_E} ,$$

where the SS's are those defined in Section 3.1. We note that  $SS_V + SS_E$  has a constant value in the set  $S(X_0)$  since

$$SS_V + SS_E = SS_T - SS_B .$$

It follows that  $F_0$  is a strictly increasing function of  $SS_V$ , and hence of the sum of squares of the treatment totals  $\sum_i T_i^2$ .

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The permutation test based on  $\sum_i T_i^2$ . would thus be equivalent to the permutation test based on  $F_0$ . It turns out, however, that the calculations involved in using this statistic is impracticable because of the large number of permutations involved.

Box and Anderson (4), as well as Scheffé approximated the exact calculation by basing the permutation test on the statistic

$$U = \frac{SS_V}{SS_V + SS_E} \cdot$$

This was found to be equivalent to basing it on  $F_0$ . They chose  $U$  instead of  $SS_V$  because it offered them a way both for approximating the exact calculation and for comparing the permutation test with the normal-theory test. By assuming that there were no technical errors, Box and Anderson obtained the expected value and variance of  $U$  as

$$\begin{aligned} \text{(i)} \quad E(U) &= \frac{1}{J} \quad ; \\ \text{(ii)} \quad \text{Var}(U) &= \frac{2(J-1)}{J^3(I-1)} \left(1 - \frac{W}{J}\right), \end{aligned} \quad \left. \vphantom{\begin{aligned} \text{(i)} \quad E(U) &= \frac{1}{J} \quad ; \\ \text{(ii)} \quad \text{Var}(U) &= \frac{2(J-1)}{J^3(I-1)} \left(1 - \frac{W}{J}\right), \end{aligned}} \right] \dots\dots(3.4.4)$$

where  $W$  is the square of the coefficient of variation (ratio of the variance to the square of the mean) of the block variances.

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Under the various assumptions given in this section,  $U$  was found to have a discrete distribution with range  $0 \leq U \leq 1$ . Box and Anderson approximated this distribution by a (continuous)  $\beta$ -distribution (43) with the same mean and variance. Now if a random variable  $X$  has the  $\beta$ -distribution with  $f_1$  and  $f_2$  degrees of freedom, then

$$\begin{aligned} \text{(i)} \quad E(X) &= \frac{f_1}{f_1 + f_2} \quad ; \\ \text{(ii)} \quad \text{Var}(X) &= \frac{2f_1 f_2}{(f_1 + f_2)^2 (f_1 + f_2 + 2)} \quad \cdot \end{aligned} \quad \left. \vphantom{\begin{aligned} \text{(i)} \quad E(X) &= \frac{f_1}{f_1 + f_2} \quad ; \\ \text{(ii)} \quad \text{Var}(X) &= \frac{2f_1 f_2}{(f_1 + f_2)^2 (f_1 + f_2 + 2)} \quad \cdot \end{aligned}} \right] \dots\dots(3.4.5)$$

Equating the expected values given in equations (3.4.4) and (3.4.5), the variances given in equations (3.4.4) and (3.4.5) and solving for  $f_1$  and  $f_2$ , we obtain

$$\begin{aligned} \text{(i)} \quad f_1 &= \phi(I-1) \quad ; \\ \text{(ii)} \quad f_2 &= \phi(I-1)(J-1) \quad , \end{aligned} \quad \left. \vphantom{\begin{aligned} \text{(i)} \quad f_1 &= \phi(I-1) \quad ; \\ \text{(ii)} \quad f_2 &= \phi(I-1)(J-1) \quad , \end{aligned}} \right] \dots\dots(3.4.6)$$

where

$$\phi = \frac{1}{1 - \frac{W}{J}} - \frac{2}{J(I-1)} \quad \cdot \dots\dots(3.4.7)$$

Scheffé's approximation to the permutation test of  $H_A$  based on the statistic  $U$  consists in rejecting  $H_A$  if the value of  $U$  for the observed sample  $\geq$  the upper  $\alpha$  point of the  $\beta$ -distribution with the degrees of freedom  $f_1$  and  $f_2$  given above.

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Let  $X$  be a  $\beta$ - variable with  $f_1$  and  $f_2$  degrees of freedom, It has been shown that the random variable

$$\frac{f_2 X}{f_1(1-X)}$$

is a strictly increasing function of  $X$ , and has the  $F$  distribution with  $f_1$  and  $f_2$  degrees of freedom. Hence, according to Scheffé, it is equivalent to reject  $H_A$  if  $U \geq$  the upper  $\alpha$  point of the  $\beta$ -distribution with  $f_1$  and  $f_2$  degrees of freedom and to reject  $H_A$  if  $F_0 \geq F_{f_1, f_2}(\alpha\% \text{ level})$ . He therefore regarded his approximation to the permutation test of  $H_A$  based on the statistic  $F_0$  as being equivalent to modifying the normal-theory test by multiplying the numbers of degrees of freedom of the  $F$ -distribution by the factor  $\phi$ .

Scheffé observed that if  $\phi > 1$ , the adjusted normal-theory test will be more sensitive than the unadjusted normal-theory test. This is seen from inspection of the  $F$ -tables. For  $\alpha \leq 0.1$  and  $f_2 > 2$ ,  $F_{f_1, f_2}(\alpha\% \text{ level})$  is a decreasing function of  $f_1$  and  $f_2$ ; thus values of the statistic  $F_0$  which miss significance with the unadjusted numbers of degrees of freedom may achieve it with the adjusted numbers.

Scheffé further justified the above approximation to the permutation test in the case where the technical errors  $\{\mu_{jt}\}$  are not zero but have an arbitrary distribution subject only to

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$$E(\mu_{jt}) = 0.$$

In carrying out the permutation test, the factor  $\phi$  may be calculated from the observations  $\{X_{ij}\}$  by using equation (3.4.7) with

$$\frac{W}{J} = \frac{1}{J-1} \left[ \frac{J \sum_j S_j^2}{(\sum_j S_j)^2} - 1 \right], \dots (3.4.8)$$

$$\text{where } S_j = \sum_i X_{ij}^2 - \frac{(\sum_i X_{ij})^2}{I}.$$

Equation (3.4.8) is due to Box and Anderson (4).

### 3.5 Loss of Information due to Sampling.

In some agricultural field experiments using the randomized block design, there may be  $N$  plants per plot from which a sample of  $n$  plants is taken. The characteristic is then measured on the  $n$  plants. In such experiments, it is worth considering the loss of information due to sampling. The theory has been discussed by Yates and Zaccopani (45). Other comments have been given by Immer (22) and Finney (15,16).

Whenever a plot is sampled, a loss in information relative to the information obtained for complete recording results. The fractional loss in information due to the sampling and the efficiency of sampling relative to complete recording are dis-

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cussed below.

The variance of a variety mean is

$$\frac{MS_{VB}}{Jn} \cdot$$

From Table 3.2.2,  $E(MS_{VB}) = \sigma_e^2 + n\sigma_\gamma^2$ . Hence the information on each treatment mean is

$$\frac{Jn}{\sigma_e^2 + n\sigma_\gamma^2} \cdot$$

If the whole plot is harvested, the information on each treatment is

$$\frac{JN}{\sigma_e^2 + N\sigma_\gamma^2} \cdot$$

Thus the efficiency of sampling relative to complete recording is

$$\begin{aligned} \epsilon &= \frac{Jn}{\sigma_e^2 + n\sigma_\gamma^2} / \frac{JN}{\sigma_e^2 + N\sigma_\gamma^2} \quad , \\ &= \frac{\sigma_\gamma^2 + \sigma_e^2 / N}{\sigma_\gamma^2 + \sigma_e^2 / n} \quad \dots\dots(3.5.1) \end{aligned}$$

The loss of information, L, due to sampling is estimated by

$$L = 1 - \epsilon \quad \dots\dots(3.5.2)$$

### 3.6 Analysis of the "Kpong Data"

The "Kpong data" come from a randomised block experiment with 10 plants per plot from which a sample of 5 plants was taken. The different ways of analysing such data have been given in the preceding sections. These different procedures will be applied to the data to illustrate some of the points given in those sections.

#### I. The Case of Using the Average Yields Per Plot Under the Normal-Theory Models.

The analysis of a randomised block experiment with  $n$  plants per plot using the average yields per plot has been discussed in Section 3.1. The calculations necessary for carrying out the analysis are given in Tables 3.6.1 and 3.6.2.

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Table 3.6.1: Yields in grams per Plot for 21 Varieties of Cowpea .

Varieties	B L O C K S			$T_{i\cdot}$	$T_{i\cdot}^2$
	1	2	3		
V <sub>1</sub>	26.39	22.39	21.82	70.60	4984.360
V <sub>2</sub>	50.52	22.06	16.15	88.73	7873.013
V <sub>3</sub>	5.56	5.07	5.66	16.29	265.364
V <sub>4</sub>	45.58	58.20	64.20	167.98	28217.280
V <sub>5</sub>	39.47	34.61	34.70	108.78	11833.088
V <sub>6</sub>	27.58	47.01	60.34	134.93	18206.105
V <sub>7</sub>	47.11	24.67	55.08	127.58	16276.656
V <sub>8</sub>	34.52	39.84	29.81	104.17	10851.389
V <sub>9</sub>	27.23	30.62	18.68	76.53	5856.841
V <sub>10</sub>	48.71	62.63	71.61	182.95	33470.703
V <sub>11</sub>	53.68	52.59	53.67	159.94	25580.804
V <sub>12</sub>	33.21	70.32	66.17	169.70	28798.090
V <sub>13</sub>	24.58	12.85	18.41	55.84	3118.106
V <sub>14</sub>	19.49	14.52	18.40	52.41	2746.808
V <sub>15</sub>	6.37	2.83	3.62	12.82	164.352
V <sub>16</sub>	24.79	18.44	18.02	61.25	3751.563
V <sub>17</sub>	13.44	19.05	6.87	39.36	1549.210
V <sub>18</sub>	13.99	6.42	8.76	29.17	850.889
V <sub>19</sub>	15.23	10.96	11.77	37.96	1440.962
V <sub>20</sub>	16.39	14.58	12.87	43.84	1921.946
V <sub>21</sub>	21.30	14.68	28.93	64.91	4213.308
T <sub>·j</sub>	595.14	584.34	626.26	1805.74 = T	211970.837 = $\sum T_{i\cdot}^2$
$T_{\cdot j}^2$	354191.620	341453.236	392201.588	$\sum T_{\cdot j}^2 = 1087846.444$	

$$\sum \sum X_{ij}^2 = 74160.721$$

84.

$$\frac{T^2}{IJ} = \frac{(1805.74)^2}{21 \times 3} = 51757.094 \quad .$$

$$SS_V = \frac{\sum_i T_{i\cdot}^2}{J} - \frac{T^2}{IJ} = \frac{211970.837}{3} - 51757.094 \quad ,$$

$$= 18899.852 \quad .$$

$$SS_B = \frac{\sum_j T_{\cdot j}^2}{I} - \frac{T^2}{IJ} = \frac{1087846.444}{21} - 51757.094 \quad ,$$

$$= 45.118 \quad .$$

$$SS_T = \sum_{ij} X_{ij}^2 - \frac{T^2}{IJ} = 74160.721 - 51757.094 \quad ,$$

$$= 22403.627 \quad .$$

$$SS_E = SS_T - SS_V - SS_B = 3458.657.$$

Table 3.6.2: Analysis of Variance for Table 3.6.1.

Source	df	SS	MS	F
Varieties	20	18899.852	944.993	10.929
Blocks	2	45.118	22.559	
Error	40	3458.657	86.466	
Total	62	22403.627		

85.

From the ANOVA table,

$$F_{20,40} = 10.93 .$$

From standard tables,

$$F_{20,40} \text{ (5\% level of significance)} = 1.84 .$$

Thus, the varieties are significantly different at the 5% level.

## II. The Case of Using Yields of the Individual Plants On a Plot Under the Normal-Theory Models.

The analysis of a randomised block experiment with  $n$  plants per plot using yields of the individual plants on a plot has been discussed in Section 3.2. Given below in Tables 3.6.3 and 3.6.4 are the calculations necessary for carrying out the analysis of the "Kpong data".

Table 3.6.3 : Totals of Plot Yields (in grams) for  
21 Varieties of Cowpea.

Varieties	B L O C K S			$T_{i..}$	$T_{i..}^2$
	1	2	3		
V <sub>1</sub>	131.97	111.92	109.13	353.02	124623.121
V <sub>2</sub>	252.61	110.31	80.74	443.66	196834.196
V <sub>3</sub>	27.78	25.37	28.28	81.43	6630.845
V <sub>4</sub>	227.92	291.01	321.02	839.95	705516.003
V <sub>5</sub>	197.35	173.03	173.51	543.89	295816.332
V <sub>6</sub>	137.92	235.06	301.72	674.70	455220.090
V <sub>7</sub>	235.56	123.35	279.02	637.93	406954.685
V <sub>8</sub>	172.58	199.21	149.06	520.85	271284.723
V <sub>9</sub>	136.15	153.09	93.43	382.67	146436.329
V <sub>10</sub>	243.56	313.13	358.06	914.75	842265.063
V <sub>11</sub>	263.41	262.94	268.33	799.68	639488.103
V <sub>12</sub>	166.03	351.59	330.84	848.46	719884.372
V <sub>13</sub>	122.90	64.23	92.06	279.19	77947.056
V <sub>14</sub>	97.44	72.59	92.01	262.04	68664.962
V <sub>15</sub>	31.87	14.14	18.12	64.13	4112.657
V <sub>16</sub>	123.96	92.20	90.12	306.28	93807.439
V <sub>17</sub>	67.21	95.24	34.34	196.79	38726.304
V <sub>18</sub>	69.94	32.10	43.77	145.81	21260.556
V <sub>19</sub>	76.16	54.78	58.86	189.80	36024.040
V <sub>20</sub>	81.95	72.89	64.35	219.19	48044.256
V <sub>21</sub>	106.48	73.43	144.64	324.55	105332.703
T <sub>.j.</sub>	2975.75	2921.61	3131.41	9028.55	5304393.835
T <sub>.j.</sub> <sup>2</sup>	8855088.063	8535804.992	9805728.588		

$$\sum_j T_{.j}^2 = 27196621.643$$

$$\sum_{i=1}^21 \sum_{k=1}^3 X_{ik}^2 = 403876.001$$

$$\sum_{ij} T_{ij}^2 = 1854057.80$$

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$$CF = \frac{T^2}{IJn} = \frac{(9028.77)^2}{21 \times 3 \times 5} = 258789.484 \quad .$$

$$SS_B = \frac{\sum_j T_{.j}^2}{In} - CF = \frac{27196621.643}{21 \times 5} - CF = 225.96 \quad .$$

$$SS_V = \frac{\sum_i T_{i..}^2}{Jn} - CF = \frac{5304393.835}{3 \times 5} - CF = 94836.772 \quad .$$

$$SS_T = \sum_{ijk} X_{ijk}^2 - CF = 403876.001 - CF = 145086.517 \quad .$$

$$SS_W = \sum_{ijk} X_{ijk}^2 - \frac{\sum_{ij} T_{ij.}^2}{n} = 403876.001 - 370811.561 \quad ,$$

$$= 33064.440 \quad .$$

$$SS_{VB} = SS_T - SS_B - SS_V - SS_W = 16959.345.$$

Table 3.6.4 : Analysis of Variance for Table 3.6.3 .

Source	df	SS	MS	F
Varieties	20	94836.772	4741.839	
Blocks	2	225.960	112.980	
Interaction	40	16959.345	423.984	3.232
Within Plots	252	33064.440	131.208	
Total	314	151824.762		

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The F ratio for testing the interaction is

$$F_{40,252} = \frac{423.984}{131.208} = 3.23 .$$

From standard tables,

$$F_{40,252} (5\% \text{ level}) = 1.45 .$$

The interaction effects are therefore significant at the 5% level. In view of the reasons given in Section 3.2, we have to test for variety effects for each block.

The variety sum of squares for each block is given by equation (3.2.3). Under the random-effects model, the variety mean square is tested against  $MS_{VB}$  while under the fixed-effects model it is tested against  $MS_W$ . The various mean squares and the corresponding F values for the three blocks are given in Table 3.6.5.  $F_r$  and  $F_f$  denote the F ratios under the random-effects and fixed-effects models, respectively.

Table 3.6.5: Variety Mean Squares and the Corresponding F Values for Blocks 1, 2 and 3 .

BLOCKS	$d_f$	$SS_V$	$MS_V$	$F_f$	$F_r$
1	20	21445.88	1072.294	8.17	2.53
2	20	39958.52	1997.926	15.24	4.71
3	20	50391.67	2519.584	19.21	5.94

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From standard tables,

$$(i) \quad F_{20,40} \quad (5\% \text{ level}) = 1.84 \quad ;$$

$$(ii) \quad F_{20,252} \quad (5\% \text{ level}) = 1.52 \quad .$$

Thus, for each block, and under both the fixed-effects and random-effects models, the varieties are significantly different at the 5% level.

Using yields of the individual plants on a plot, the interaction effects are found to be significant. Using the average yields per plot, however, the interaction effects are not significant. The mathematical models for the two situations are, respectively,

$$(i) \quad X_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + e_{ijk} \quad ;$$

$$(ii) \quad \bar{X}_{ij\cdot} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \bar{e}_{ij\cdot}$$



Thus, the various effects are the same in both models. In case (ii) we assume that  $\gamma_{ij} = 0$  (explained in Section 2.2 (II)). We then use Tukey's test to justify this assumption. As we noted in Section 2.2 (II), Tukey's test imposes restrictions on the  $\{\gamma_{ij}\}$  and, furthermore, the power of the test is unknown. Since the test for interactions in case (i) does not impose any restrictions on the  $\{\gamma_{ij}\}$ , it is possible for this

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test to detect interaction effects otherwise not detectable by Tukey's test. It is also possible that Tukey's test is not powerful enough to detect small interaction effects. These arguments may explain why Tukey's test failed to detect interaction effects in the "Kpong data".

From ANOVA Tables 3.6.2 and 3.6.4, we have  $MS_W = 131.208$ ,  $MS_{VB} = 423.984$ ,  $n(MS_E) = 432.330$ . Theoretically, whether there are interaction effects or not,  $MS_W$  and  $n(MS_E)$  estimate the same thing, namely  $\sigma_e^2$ . However, as established on page 67,  $MS_W$  is a more efficient estimator of  $\sigma_e^2$  than  $n(MS_E)$ . If  $\sigma_e^2$  is estimated by  $n(MS_E)$ , the standard error of the estimate is

$$\begin{aligned} \sqrt{\left(\frac{2 \hat{\sigma}_e^4}{(I-1)(J-1)}\right)} &= \frac{\hat{\sigma}_e^2}{\sqrt{\left(\frac{2}{(I-1)(J-1)}\right)}} \quad , \\ &= n(MS_E) \sqrt{\left(\frac{2}{(I-1)(J-1)}\right)} \quad , \\ &= 432.830 \sqrt{\left(\frac{2}{20 \times 2}\right)} \quad , \\ &= \underline{96.672} \quad . \end{aligned}$$

On the other hand, if  $\sigma_e^2$  is estimated by  $MS_W$ , the standard error of the estimate is

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$$\begin{aligned}
 \sqrt{\left(\frac{2 \hat{\sigma}_e^4}{IJ(n-1)}\right)} &= \hat{\sigma}_e^2 \sqrt{\left(\frac{2}{IJ(n-1)}\right)}, \\
 &= (MS_W) \sqrt{\left(\frac{2}{IJ(n-1)}\right)}, \\
 &= 131.208 \sqrt{\left(\frac{2}{21 \times 3 \times 4}\right)}, \\
 &= \underline{11.689}.
 \end{aligned}$$

Comparing these two standard errors, we observe that  $n(MS_E)$  has grossly overestimated  $\sigma_e^2$  in this case.

Under both the random-effects and fixed-effects models, the F-tests using

- (i) the average yields per plot, and
- (ii) yields of the individual plants on a plot

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showed the varieties to be significantly different at the 5% level. The test based on case (ii) is, however, more sensitive than the one based on case (i) in view of the discussions given in Section 3.3 and the fact that  $MS_W$  is smaller and have more degrees of freedom than  $n(MS_E)$ .

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### III. The Case of Using the Average Yields Per Plot Under the Randomization Model.

The analysis of a randomised block experiment with one observation per plot under randomization models has been discussed in Section 3.4. The application of this technique to the "Kpong data" is presented below. The observations in this case are average yields. The calculations necessary to carry out the test are given in Tables 3.6.5 and 3.6.6.

Tables 3.6.5: Values of  $S_j$  and their Squares.

Block	$S_j$	$S_j^2$
1	4288.767	18393522.380
2	7991.867	63869938.146
3	10077.851	101563080.778
Total	22358.485	183826541.304

$$(\sum S_j)^2 = 499901851.495 .$$

From equation (3.4.8)

$$\begin{aligned} \frac{W}{J} &= \frac{1}{J-1} \left[ \frac{J \sum S_j^2}{(\sum S_j)^2} - 1 \right] , \\ &= \frac{1}{2} \left( 3 \times \frac{18382654.304}{499901851.495} - 1 \right) , \\ &= 0.05 . \end{aligned}$$

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Hence from equation (3.4.9)

$$\begin{aligned}\phi &= \frac{1}{1 - 0.05} - \frac{2}{3(21 - 2)} , \\ &= \underline{1.02} .\end{aligned}$$

Table 3.6.6 : Analysis of Variance for Table 3.6.1  
Using the Permutation Test.

Source	df	$\phi \times df$	SS	MS	F
Varieties	20	20.40	18899.852	926.463	10.929
Blocks	2	2.04	45.118	22.117	
Error	40	40.80	3458.657	84.771	
Total	62	63.24	22403.627		

From the ANOVA table,

$$F_{20.4,40.8} = 10.93 .$$

From standard tables,

$$F_{20.4,40.8} (5\% \text{ level}) = 1.81 .$$

Thus the varieties are significantly different at the 5% level.

Let  $F_N$  and  $F_P$  be the F statistics for the normal-theory and permutation tests, respectively, when the average yields per plot are used. Let  $F'_N$  and  $F'_P$  be the corresponding tabular values. From Tables 3.6.2 and 3.6.6,  $F_N = 10.93$  and  $F_P = 10.93$ .

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The corresponding tabular values at the 5% level are  $F'_N = 1.84$  and  $F'_P = 1.81$ . Thus, the F-tests based on  $F_N$  and  $F_P$  give the same verdict. Though  $F'_P$  is smaller than  $F'_N$  (this is expected in view of the deductions given in Section 3.4 (II) ), the large value of  $F_N (= F_P)$  does not make the reduction in the tabular value any important. For the "Kpong data" therefore, the permutation test for variety differences gives the same verdict as the normal-theory test.

#### IV. Loss of Information due to Sampling.

The importance of examining the loss of information due to sampling for experiments involved with sampling has been discussed in Section 3.5. The theoretical treatment given in that section gave the efficiency of sampling relative to complete recording as

$$\epsilon = \frac{\sigma_\gamma^2 + \sigma_e^2 / N}{\sigma_\gamma^2 + \sigma_e^2 / n} .$$

For the "Kpong data", the restricted maximum likelihood estimates of  $\sigma_e^2$  and  $\sigma_\gamma^2$  have been calculated in Chapter 6 and are given as

$$\hat{\sigma}_e^2 = 131.208, \quad \hat{\sigma}_\gamma^2 = 55.593 .$$

Substituting these values in the above equation, we obtain

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$$\begin{aligned}\epsilon &= \frac{55.593 + 131.208/10}{55.593 + 131.208/5} , \\ &= \frac{68.7138}{81.8346} , \\ &= \underline{0.840} .\end{aligned}$$

From equation (3.5.2), the loss of information,  $L$ , due to sampling is

$$\begin{aligned}L &= 1 - \epsilon , \\ &= 1 - 0.84 , \\ &= \underline{0.16} .\end{aligned}$$

Thus, the loss of information is 16% which is relatively small.

In experiments where sampling is involved, it is possible to choose a sampling size and a number of replications that will minimize the error variance by making use of the results of a previous similar experiment or a pilot experiment. The theory leading to such a sample size and number of replications - often referred to as optimum conditions of sampling - has been discussed by Kempthorne (25).

CHAPTER FOURMULTIVARIATE ANALYSIS OF VARIANCE4.1 The Multivariate Analysis of Variance Test.

In the univariate analysis of variance described in Chapter 3, the test statistics for testing the hypothesis of no treatment effects,  $H_A$ , were of the form

$$F = \frac{SS_V/q}{SS_E/n_e} ,$$

where (i)  $SS_V$  = Treatment sum of squares with  $q$  degrees of freedom ;

(ii)  $SS_E$  = Error sum of squares with  $n_e$  degrees of freedom.

We noted that since  $SS_E$  is distributed as  $\sigma_e^2 \chi^2$  with  $n_e$  degrees of freedom, and, under the null hypothesis  $H_A$ ,  $SS_V$  is distributed as  $\sigma_e^2 \chi^2$  with  $q$  degrees of freedom, the statistic  $F$  has an F-distribution with  $q$  and  $n_e$  degrees of freedom.

It has been shown (43) that the likelihood ratio criterion for the same hypothesis  $H_A$  is

$$\lambda = \left[ \frac{SS_E}{SS_E + SS_V} \right]^{N/2} , \quad \dots\dots(4.1.1)$$

where  $N$  is the total number of observations.

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In the multivariate analysis of variance, the observations  $X_{ij}$  are column vector variables with  $P$  components. They are assumed to be normally and independently distributed with common covariance matrix  $\Phi$ . The expected value of  $X_{ij}$  is assumed to be

$$E(X_{ij}) = \mu + \alpha_i + \beta_j, \dots (4.1.2)$$

where  $\mu$ ,  $\alpha_i$ 's and  $\beta_j$ 's are column vectors.

Under the above situation, the error sum of squares is shown (1) to be

$$SS'_E = \sum_{ij} (X_{ij} - \bar{X}_{i\cdot} - \bar{X}_{\cdot j} + \bar{X}_{\cdot\cdot}) (X_{ij} - \bar{X}_{i\cdot} - \bar{X}_{\cdot j} + \bar{X}_{\cdot\cdot})' \bullet \dots (4.1.3)$$

Similarly, the treatment sum of squares is

$$SS'_V = J \sum_i (\bar{X}_{i\cdot} - \bar{X}_{\cdot\cdot}) (\bar{X}_{i\cdot} - \bar{X}_{\cdot\cdot})' \bullet \dots (4.1.4)$$

Anderson (1) and Wilks (43) have shown that  $SS'_E$  has the Wishart distribution,  $W(p, n_e, \Phi)$ , and, under the null hypothesis  $H_A$ ,  $SS'_V$  has the Wishart distribution,  $W(p, q, \Phi)$ . Furthermore,  $SS'_E$  and  $SS'_V$  are independent. They then showed that the statistic

$$\Lambda = \frac{|SS'_E|}{|SS'_E + SS'_V|} \dots (4.1.5)$$

has the  $U_{p, q, n_e}$  distribution (1) under the null hypothesis  $H_A$ .

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The multivariate test statistic analogous to (4.1.1) for the same hypothesis  $H_A$  is shown (1) to be (4.1.5).

It has been shown (1,31) that for

- (i)  $p \leq 2$ , any  $q$ , and
- (ii)  $q \leq 2$ , any  $p$  ,

the statistic  $\Lambda$  is exact for testing the null hypothesis  $H_A$ .

On the other hand, if  $p > 2$  and  $q > 2$ , the appropriate test statistic to use is

$$Q = -m \text{Log}_e \Lambda \quad , \quad \dots\dots(4.1.6)$$

where (i)  $m = n - \frac{p+q+1}{2}$  ;

(ii)  $n =$  total degrees of freedom.

If  $m$  is large ( $m > 30$ ), the first approximation consists in using  $Q$  as  $\chi^2$  with  $pq$  degrees of freedom.

If  $m$  is not large, one may use the first approximation suggested by Bartlett (31). The approximation gives the test statistic as

$$Q' = \left( \frac{1 - \Lambda^{1/s}}{\Lambda^{1/s}} \right) \left( \frac{ms + 2\lambda}{2r} \right) \quad ,$$

where

$$(i) \quad s = \frac{(p^2 q^2 - 4)^{1/2}}{p^2 + q^2 - 5} \quad ;$$

$$(ii) \quad \lambda = - \frac{(pq - 2)}{4} \quad ;$$

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$$(iii) \quad r = \frac{pq}{2}.$$

Q's is then a variance ratio with  $2r$  and  $(ms + 2\lambda)$  degrees of freedom, where  $(ms + 2\lambda)$  need not be integral.

#### 4.2. Analysis of the "Kpong Data".

The multivariate analysis of variance test described in Section 4.1 will be applied to the "Kpong data".

Normality of the observations will be assumed. We shall further assume that the observation vectors have a common dispersion matrix for, as pointed out by Keith Hope (24), the multivariate analysis of variance can survive a certain amount of heterogeneity among the dispersions of the groups.

Each observation  $X_{ij}$  is a column vector of dimension 5, consisting of the 5 correlated variates  $X_1, X_2, \dots, X_5$ . Thus  $X'_{ij} = (X_1, X_2, \dots, X_5)$ . The square of  $X_{ij}$  is defined as

$$X_{ij}X'_{ij} = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_5 \end{pmatrix} (X_1, X_2, \dots, X_5) ,$$

$$= \begin{pmatrix} X_1^2 & X_1X_2 & \dots & X_1X_5 \\ X_2X_1 & X_2^2 & \dots & X_2X_5 \\ \vdots & \vdots & \ddots & \vdots \\ X_5X_1 & X_5X_2 & \dots & X_5^2 \end{pmatrix}$$

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I. The Case of Using Yields of the Individual Plants on a Plot.

Under this condition, the five correlated variates  $X_1, X_2, \dots, X_5$  are the yields of the five plants on a plot.

The values of the row totals ( $X_{i.}$ ), column totals ( $X_{.j}$ ) and overall total ( $X_{..}$ ) are given in Table 4.2.1.

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Table 4.2.1 : Values of  $X'_{i.}$ ,  $X'_{.j}$ , and  $X'_{..}$  .

Varieties	$X'_{i.}$				
$V_1$	83.12	63.46	70.57	62.20	73.67
$V_2$	96.41	88.34	69.85	55.12	103.94
$V_3$	11.93	31.44	16.21	12.95	18.90
$V_4$	173.57	154.06	237.73	157.76	116.83
$V_5$	122.03	82.77	154.78	108.37	75.94
$V_6$	137.27	105.35	142.25	166.99	122.84
$V_7$	152.88	129.07	161.80	128.85	65.33
$V_8$	128.72	108.59	100.06	88.40	95.08
$V_9$	91.56	89.60	89.90	42.99	68.62
$V_{10}$	238.85	178.16	140.21	147.39	210.14
$V_{11}$	174.02	194.44	125.04	131.67	174.51
$V_{12}$	196.52	141.14	194.91	133.14	182.75
$V_{13}$	51.26	75.24	63.73	56.04	32.92
$V_{14}$	53.52	42.34	59.77	41.99	65.22
$V_{15}$	9.94	10.91	18.46	15.57	9.25
$V_{16}$	56.37	82.73	63.33	57.15	46.72
$V_{17}$	31.51	34.91	34.87	30.48	65.02
$V_{18}$	26.99	26.72	28.61	39.58	23.91
$V_{19}$	38.04	34.80	43.13	37.97	35.86
$V_{20}$	44.38	37.38	49.37	39.40	48.66
$V_{21}$	63.48	75.69	85.46	51.95	47.97

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Blocks	$X'_{.j}$				
1	579.55	675.79	552.40	544.40	678.80
2	629.66	669.03	640.28	565.11	580.39
3	822.18	614.59	691.79	561.84	598.57

$$X'_{.j} = (2031.39, 1959.41, 1884.47, 1671.21, 1857.76)$$

$$\sum_{ij} X_{ij} X'_{ij} = \begin{pmatrix} 100472.73 & 80828.21 & 86656.28 & 75583.81 & 79792.64 \\ 80828.21 & 75790.35 & 71752.14 & 63296.61 & 66440.59 \\ 86656.28 & 71752.14 & 91070.21 & 69634.35 & 69581.10 \\ 75583.81 & 63296.61 & 69634.35 & 64070.64 & 61522.69 \\ 79792.64 & 66440.59 & 69581.10 & 61522.69 & 72470.62 \end{pmatrix}$$

$$\frac{\sum_i X_i X'_i}{J} = \begin{pmatrix} 90806.51 & 77297.37 & 84405.46 & 71123.84 & 74975.23 \\ 77297.37 & 68202.96 & 72195.76 & 61028.35 & 64121.91 \\ 84405.46 & 72195.76 & 84669.01 & 68473.45 & 67682.95 \\ 71123.84 & 61028.35 & 68473.45 & 58484.14 & 58332.55 \\ 74975.23 & 64121.91 & 67682.95 & 58332.55 & 65159.55 \end{pmatrix}$$

$$\frac{\sum_j X_j X'_j}{I} = \begin{pmatrix} 67063.31 & 62772.31 & 61527.50 & 53961.25 & 59570.42 \\ 62772.31 & 61048.28 & 58420.94 & 51961.03 & 57582.35 \\ 61527.50 & 58420.94 & 56841.76 & 50054.89 & 55269.78 \\ 53961.25 & 51961.03 & 50054.89 & 44344.39 & 49225.15 \\ 59570.42 & 57852.35 & 55269.78 & 49225.15 & 55043.21 \end{pmatrix}$$

$$\frac{X_{..} X'_{..}}{IJ} = \begin{pmatrix} 65500.71 & 63179.76 & 60763.38 & 53886.96 & 59902.13 \\ 63179.76 & 60941.07 & 58610.30 & 51977.54 & 57779.57 \\ 60763.38 & 58610.30 & 56368.68 & 49989.60 & 55569.71 \\ 53886.96 & 51977.54 & 49989.60 & 44332.42 & 49281.06 \\ 59902.13 & 57779.57 & 55569.71 & 49281.06 & 54782.08 \end{pmatrix}$$

103.

From equation (4.1.3)

$$SS'_E = \sum_{i,j} X_{ij} X'_{ij} - \frac{\sum_i X_{i.} X'_{i.}}{J} - \frac{\sum_j X_{.j} X'_{.j}}{I} + \frac{X_{..} X'_{..}}{IJ} ,$$

$$= \begin{pmatrix} 8103.62 & 3937.93 & 1486.68 & 4385.68 & 5149.11 \\ 3937.93 & 7480.18 & -254.26 & 2284.76 & 2245.89 \\ 1486.68 & -254.26 & 5928.12 & 1095.60 & 2198.08 \\ 4385.68 & 2284.76 & 1095.60 & 5574.54 & 3246.04 \\ 5149.11 & 2245.89 & 2198.08 & 3246.04 & 7049.93 \end{pmatrix} .$$

From equation (4.1.4)

$$SS'_V = \sum_i X_{i.} X'_{i.} - \frac{X_{..} X'_{..}}{IJ} ,$$

$$= \begin{pmatrix} 25305.80 & 14117.96 & 23642.08 & 17236.87 & 15073.09 \\ 14117.96 & 7261.89 & 13585.45 & 9050.81 & 6342.34 \\ 23642.08 & 13585.45 & 28300.32 & 18483.84 & 12113.22 \\ 17236.87 & 9050.81 & 18483.84 & 14151.72 & 9051.49 \\ 15073.09 & 6342.34 & 12113.22 & 9051.49 & 10377.47 \end{pmatrix} .$$

Hence

$$SS'_E + SS'_V = \begin{pmatrix} 33409.42 & 18055.90 & 25128.76 & 21622.55 & 20222.21 \\ 18055.90 & 14742.07 & 13331.18 & 11335.58 & 8588.23 \\ 25128.76 & 13331.18 & 34228.45 & 19579.45 & 14311.31 \\ 21622.55 & 11335.58 & 19579.45 & 19726.26 & 12297.54 \\ 20222.21 & 8588.23 & 14311.31 & 12297.54 & 17427.40 \end{pmatrix} .$$

104.

$$\text{Now } |SS'_E| = 0.27 \times 10^{19} ,$$

$$\text{and } |SS'_E + SS'_V| = 0.49 \times 10^{20} .$$

$$\begin{aligned} \text{Therefore } \Lambda &= \frac{|SS'_E|}{|SS'_E + SS'_V|} , \\ &= \frac{0.27 \times 10^{19}}{0.49 \times 10^{20}} , \\ &= 5.51 \times 10^{-2} . \end{aligned}$$

$$\begin{aligned} \text{Hence } \text{Log}_e \Lambda &= \text{Log}_e 5.51 + \text{Log}_e 10^{-2} , \\ &= 1.7066 + \bar{5}.3945 , \\ &= \bar{3}.1014 , \\ &= -2.8986 . \end{aligned}$$

From equation (4.1.6),

$$\begin{aligned} Q &= -m \text{Log}_e \Lambda , \\ &= m \times 2.8986 . \end{aligned}$$

$$\text{Now } m = n - \frac{p+q+1}{2} = 62 - \frac{5+20+1}{2} = 49 ,$$

$$\begin{aligned} \text{Therefore } Q &= 49 \times 2.8986 , \\ &= \underline{142.03} . \end{aligned}$$

105.

Since  $m = 49$  is large, the  $\chi^2$  approximation is appropriate for the test.

From standard tables,  $\chi_{100}^2$  (5% level) = 124.3.

Thus,  $Q$  is significant, and hence the varieties are significantly different at the 5% level.

II. The Case of Using the Five Correlated  
Variates  $x, y, h, z, \theta$ .

Under this situation, the five correlated variates are  $x, y, h, z, \theta$ .

The row totals ( $X_{i.}$ ), column totals ( $X_{.j}$ ), and overall total ( $X_{..}$ ) are given in Table 4.2.2.

Table 4.2.2 : Values of  $X'_{i.}$ ,  $X'_{.j}$ , and  $X..$ 

Varieties	$X'_{i.}$				
V <sub>1</sub>	478.00	42.75	45.00	33.00	70.60
V <sub>2</sub>	567.65	45.12	46.40	37.60	88.73
V <sub>3</sub>	359.36	19.88	12.00	60.20	16.29
V <sub>4</sub>	608.76	45.14	54.80	60.60	167.98
V <sub>5</sub>	527.77	44.90	57.60	37.40	108.78
V <sub>6</sub>	390.97	39.07	88.80	33.80	134.93
V <sub>7</sub>	527.28	39.35	65.80	42.90	127.58
V <sub>8</sub>	509.92	37.06	51.20	49.30	104.17
V <sub>9</sub>	465.80	29.31	38.60	56.50	76.53
V <sub>10</sub>	469.93	43.74	85.60	43.50	182.95
V <sub>11</sub>	446.16	45.38	78.80	40.20	159.94
V <sub>12</sub>	414.27	41.12	86.60	42.10	169.70
V <sub>13</sub>	447.33	34.45	40.60	35.50	55.84
V <sub>14</sub>	355.19	27.47	29.80	58.50	52.41
V <sub>15</sub>	376.47	23.79	13.80	36.50	12.82
V <sub>16</sub>	502.58	37.66	33.60	44.00	61.25
V <sub>17</sub>	438.37	20.65	25.80	62.80	39.36
V <sub>18</sub>	348.03	22.01	25.20	46.90	29.17
V <sub>19</sub>	394.75	32.57	34.80	30.30	37.96
V <sub>20</sub>	486.44	29.77	23.00	57.80	43.84
V <sub>21</sub>	466.70	33.44	37.40	46.60	64.91

Blocks	$X'_{.j}$				
1	3236.01	249.39	333.00	310.40	595.14
2	3207.68	244.70	311.80	315.30	584.34
3	3138.12	240.54	330.40	330.30	626.26

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$$X'_{..} = (9581.81, \quad 734.63, \quad 975.20, \quad 956.00, \quad 1805.74) \cdot$$

$$\Sigma_{i,j} X_{ij} X'_{ij} = \begin{Bmatrix} 1492786.75 & 114639.57 & 151793.96 & 145539.65 & 287600.18 \\ 114639.57 & 9085.10 & 12497.05 & 10871.36 & 23720.98 \\ 151793.96 & 12497.05 & 19465.37 & 14210.61 & 37225.89 \\ 145539.65 & 10871.36 & 14210.61 & 15252.36 & 26992.21 \\ 287600.18 & 23720.98 & 37225.89 & 26992.21 & 74160.42 \end{Bmatrix} \cdot$$

$$\frac{\Sigma X_{i.} X'_{i.}}{J} = \begin{Bmatrix} 1490855.50 & 114495.90 & 151324.03 & 145562.00 & 286427.25 \\ 114495.90 & 9061.59 & 12417.16 & 10872.97 & 23529.39 \\ 151324.03 & 12417.16 & 18766.94 & 14196.38 & 35718.58 \\ 145562.00 & 10872.97 & 14196.38 & 15202.95 & 26893.41 \\ 286427.25 & 23529.39 & 35718.58 & 26893.41 & 70656.85 \end{Bmatrix} \cdot$$

$$\frac{\Sigma X_{.j} X'_{.j}}{I} = \begin{Bmatrix} 1457560.00 & 111751.93 & 148313.34 & 145350.43 & 274549.12 \\ 111751.93 & 8568.23 & 11372.31 & 11143.56 & 21050.01 \\ 148313.34 & 11372.31 & 15108.20 & 14800.23 & 27966.42 \\ 145344.43 & 11143.56 & 14800.23 & 14517.15 & 27420.35 \\ 274549.12 & 21050.01 & 27966.42 & 27420.35 & 51802.19 \end{Bmatrix} \cdot$$

$$\frac{X_{..} X'_{..}}{IJ} = \begin{Bmatrix} 1457318.25 & 111731.46 & 148320.31 & 145400.15 & 274638.93 \\ 111731.46 & 8566.36 & 11371.60 & 11147.71 & 21056.35 \\ 148320.31 & 11371.60 & 15095.47 & 14798.27 & 27951.70 \\ 145400.15 & 11147.71 & 14798.27 & 14506.92 & 27401.38 \\ 274638.93 & 21056.35 & 27951.70 & 27401.38 & 51757.09 \end{Bmatrix} \cdot$$

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From equation (4.1.3),

$$SS'_E = \begin{pmatrix} 1689.50 & 123.20 & 476.90 & 27.37 & 1262.75 \\ 123.20 & 21.64 & 79.17 & 2.53 & 197.92 \\ 476.90 & 79.17 & 685.69 & 12.27 & 1492.59 \\ 27.37 & 2.53 & 12.27 & 39.17 & 79.83 \\ 1262.75 & 147.92 & 1492.59 & 79.83 & 3458.46 \end{pmatrix} .$$

From equation (4.1.4)

$$SS'_V = \frac{\sum_i X_{i.} \cdot X'_{i.}}{J} - \frac{X_{..} \cdot X'_{..}}{IJ} ,$$

$$= \begin{pmatrix} 33537.25 & 2764.43 & 3003.71 & 161.84 & 11788.31 \\ 2764.43 & 495.22 & 1045.55 & -274.74 & 2473.03 \\ 3003.71 & 1045.55 & 3671.46 & -601.88 & 7766.87 \\ 161.84 & -274.74 & -601.88 & 696.03 & -507.96 \\ 11788.31 & 2473.03 & 7766.87 & -507.96 & 18899.76 \end{pmatrix} .$$

Hence

$$SS'_E + SS'_V = \begin{pmatrix} 35226.75 & 2887.64 & 3480.62 & 189.21 & 13051.06 \\ 2887.64 & 516.86 & 1124.73 & -272.20 & 2670.96 \\ 3480.62 & 1124.73 & 4357.16 & -589.61 & 9259.47 \\ 189.21 & -272.00 & -589.61 & 735.21 & -428.13 \\ 13051.06 & 2670.96 & 9259.47 & -428.13 & 22358.22 \end{pmatrix} .$$

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$$\text{Now } |SS'_E| = 0.28 \times 10^{11} \text{ ,}$$

$$\text{and } |SS'_E + SS'_V| = 0.41 \times 10^{16} \text{ .}$$

$$\begin{aligned} \text{Therefore } \Lambda &= \frac{|SS'_E|}{|SS'_E + SS'_V|} \text{ ,} \\ &= \frac{0.28 \times 10^{11}}{0.41 \times 10^{16}} \text{ ,} \\ &= 6.83 \times 10^{-6} \text{ .} \end{aligned}$$

Hence

$$\begin{aligned} \text{Log}_e \Lambda &= \text{Log}_e 6.83 + \text{Log}_e 10^{-6} \text{ ,} \\ &= 1.9213 + \bar{1}4.1845 \text{ ,} \\ &= \bar{1}2.1058 \text{ ,} \\ &= -11.8942 \text{ .} \end{aligned}$$

From equation (4.1.6),

$$\begin{aligned} Q &= -m \text{Log}_e \Lambda \text{ ,} \\ &= m \times 11.8942 \text{ ,} \end{aligned}$$

$$\text{where } m = n - \frac{p+q+1}{2} = 62 - \frac{5+20+1}{2} = 49 \text{ .}$$

$$\begin{aligned} \text{Therefore } Q &= 49 \times 11.8942 \text{ ,} \\ &= \underline{582.82} \text{ .} \end{aligned}$$

110.

Since  $m = 49$  is large, the  $\chi^2$  approximation is approximate for the test.

From standard tables,  $\chi_{100}^2$  (5% level) = 124.3.

Thus, the varieties are significantly different at the 5% level.

The two tests given in Sections 4.2(I) and 4.2(II) show the varieties to be significantly different. We note, however, that the  $\chi^2$  value for the case of the correlated variates  $x, y, z, h,$  and  $\theta$  is more than 4 times the corresponding value for the case of the yields of the individual plants on a plot. This is an indication that the correlated variates  $x, y, z, h,$  and  $\theta$  do discriminate between the varieties better and may therefore be a better criterion for comparing the varieties.

CHAPTER FIVEPAIRWISE MULTIPLE COMPARISONS OF MEANS5.1 Multiple Comparisons of Means Procedures.

Six multiple comparisons of means procedures - FSD, MRT, SNK, TSD, SSD, and BET - have been listed in Section 1.4. We shall in this section define these various procedures.

I. The Studentized Range.

Let  $\mu_i$ ,  $i=1, \dots, I$ , be  $I$  uncorrelated equally replicated treatment means. Let  $\hat{\mu}_i$ ,  $i=1, \dots, I$ , be unbiased estimates of the means, each distributed as  $N(\mu_i, b^2 \sigma^2)$  where  $b$  is a known positive constant. Furthermore, let  $S^2$  be an independent mean square estimate of  $\sigma^2$  with  $f$  degrees of freedom. Suppose the means to be ordered, and denote them by

$$\hat{\mu}_1 \leq \hat{\mu}_2 \leq \dots \leq \hat{\mu}_I \cdot$$

To any pair  $\hat{\mu}_i$  and  $\hat{\mu}_r$ ,  $i < r$ , there corresponds a difference  $(\hat{\mu}_r - \hat{\mu}_i)$  which is the range of a subset of  $(r - i + 1)$  adjacent ordered means. Under the above assumptions, we have

- (i)  $(\hat{\mu}_r - \hat{\mu}_i) \sim N(0, b^2 \sigma^2)$  ;
- (ii)  $\frac{fS^2}{\sigma^2} \sim \chi_f^2$  ;
- (iii)  $(\hat{\mu}_r - \hat{\mu}_i)$  and  $\frac{fS^2}{\sigma^2}$  are independent (43).

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It has been shown (34) that the random variable

$$q = \frac{(\hat{\mu}_r - \hat{\mu}_i)}{b_\sigma} / \left( \frac{fs^2}{\sigma^2} \cdot \frac{1}{f} \right)^{\frac{1}{2}} = \frac{\hat{\mu}_r - \hat{\mu}_i}{bs}$$

has the distribution of the studentized range (43) denoted by  $q(\alpha, I, f)$ , where  $\alpha$  is the upper percentage point and  $I$  is the number of means included in the range.

For the randomised block experiment described in Section 1.1, estimates of the treatment means are given by  $\bar{X}_{i.}$ , and each is distributed as  $N(\mu_i, \sigma^2/J)$ , where  $\sigma^2$  is the variance of  $X_{ij}$  with  $S^2$  as its estimate. They are equally replicated and assumed to be uncorrelated. Hence from the above discussion

$$q = \frac{\bar{X}_{r.} - \bar{X}_{i.}}{S/J^{\frac{1}{2}}} \quad \dots (5.1.1)$$

has the distribution of the studentized range,  $q(\alpha, I, f)$ .

## II. Fisher's Significant Difference (FSD),

The earliest multiple comparisons of means procedure was that of calculating an estimated standard error of the difference between two means and comparing the observed difference with it, using the appropriate number of degrees of freedom in the t-distribution (43). We thus set up a least significant difference (LSD) and compare each of the  $\frac{1}{2}I(I-1)$  observed differences with it. The critical value is given (14) by

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$$\text{LSD} = t(\alpha, f) S_d, \quad \dots\dots(5.1.2)$$

where  $t(\alpha, f)$  is the tabular value of student's  $t$  at the chosen Type I error rate  $\alpha$  for the number of degrees of freedom  $f$  associated with the standard error,  $S_d$ .

A modification of the LSD by Fisher (18) led to what is termed Fisher's Significant Difference (FSD). The modification consists in performing the LSD only after obtaining a significant observed  $F$  ratio of the treatments mean square to the error mean square. He computed the critical value of the FSD as

$$\text{FSD} = \text{LSD} = t(\alpha, f) S_d \quad \dots\dots(5.1.3)$$

if the observed  $F$  value is significant

or  $\text{FSD} = \infty$

if the observed  $F$  value is not significant.

### III. Tukey's Significant Difference (TSD).

Utilizing the distribution of the studentized range - equation (5.1.1) - Tukey (34) showed that, of the  $\frac{1}{2}I(I-1)$  differences of the type  $(\bar{X}_r. - \bar{X}_i.)$ , the probability is  $1 - \alpha$  that all these differences simultaneously satisfy

$$(\bar{X}_r. - \bar{X}_i.) - TS \leq \mu_r - \mu_i \leq (\bar{X}_r. - \bar{X}_i.) + TS, \quad ,$$

where the constant  $T$  is

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$$T = J^{-\frac{1}{2}} q(\alpha, I, f) .$$

As a multiple comparisons of means procedure, Tukey regarded the estimated differences  $(\bar{X}_{r.} - \bar{X}_{i.})$  as being significantly different from zero at the  $\alpha$  level if

$$\bar{X}_{r.} - \bar{X}_{i.} \geq TS = SJ^{-\frac{1}{2}} q(\alpha, I, f) .$$

Let  $Sd$  be the standard error of the difference between any two means. Then

$$Sd = \left(\frac{2S^2}{J}\right)^{\frac{1}{2}} ,$$

$$\text{or } \frac{Sd}{\sqrt{2}} = \frac{S}{J^{\frac{1}{2}}} .$$

Hence the probability is  $\alpha$  that

$$(\bar{X}_{r.} - \bar{X}_{i.}) \geq q(\alpha, I, f) \frac{Sd}{\sqrt{2}} . \quad \dots(5.1.4)$$

Tukey adjudged the difference  $(\bar{X}_{r.} - \bar{X}_{i.})$  significantly different from zero (and the extreme means  $\bar{X}_{r.}$  and  $\bar{X}_{i.}$  therefore regarded as from different populations) if equation (5.1.4) holds. His critical value, denoted by  $TSD$ , is therefore given by

$$TSD = q(\alpha, I, f) \frac{Sd}{\sqrt{2}} . \quad \dots(5.1.5)$$

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In practice,  $\bar{X}_I.$  is successively compared with  $\bar{X}_1.$ ,  $\bar{X}_2.$ , etc. until a non-significant difference is obtained. If  $(\bar{X}_I. - \bar{X}_1.)$  is non-significant the test ends there; otherwise all means  $\bar{X}_1., \bar{X}_2., \dots$  found to be significant from  $\bar{X}_I.$  are tested against  $\bar{X}_{I-1}.$ , again in succession starting from  $\bar{X}_1.$ . The procedure is continued until no further differences are significant.

The TSD is equivalent to FSD if  $\alpha$  is the same for both and  $I=2$ , since

$$t(\alpha, f)Sd = q(\alpha, 2, f) \frac{Sd}{\sqrt{2}} \quad \bullet$$

If  $I > 2$ , the TSD value will be larger than the FSD value.

The TSD utilizes the upper percentage points of the studentized range as found in (30).

#### IV. Student-Newman-Keuls (SNK).

Instead of using the fixed critical value of the studentized range based on the  $I$  means, Newman and Keuls (26) suggested testing the difference  $(\bar{X}_r. - \bar{X}_i.)$  against  $Sd/\sqrt{2}$  times the studentized range based on  $(r - i + 1)$  means. Their procedure adjudges a pair of treatment means significantly different only if every subset of adjacent treatment means containing that pair is significantly different by the studentized range test suggested by them. This test takes account of

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the number of means in the subset. From equation (5.1.4), we obtain the critical value as

$$SNK_v = q(\alpha, v, f) \frac{Sd}{\sqrt{2}} , \quad \dots (5.1.6)$$

for  $v = 2, 3, \dots, I$  .

Thus,  $SNK_2$  equals the FSD value,  $SNK_I$  equals the TSD value, and, for intermediate values of  $v$ ,  $SNK_v$  is intermediate to the FSD and TSD.

The computational procedure is the same as that for the TSD, except that the critical value now varies ( $I-1$  critical values in all) in the component tests instead of being fixed as previously.

The SNK also utilizes the upper percentage points of the studentized range as found in (30).

#### V. Duncan's Multiple Range Test (MRT).

Duncan's MRT (12) is essentially a modification of the SNK procedure in which each difference ( $\bar{X}_r - \bar{X}_i$ ) is tested against the  $100(1 - \alpha_{r-i+1})$  per cent point of the studentized range based on  $(r-i+1)$  means.  $\alpha_{r-i+1}$  depends on  $(r-i)$  through the relation (26)

$$\alpha_{r-i+1} = 1 - (1 - \alpha_2)^{r-i} .$$

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With the above modification, Duncan showed that the probabilities of error are redistributed among the components of the test, falling as the means compared get closer together in the ordering.

He computed the critical values as

$$\text{MRT}_v = q(\alpha_v, v, f) \frac{S_d}{\sqrt{2}}, \quad \dots (5.1.7)$$

for  $v = 2, 3, \dots, I$ .

Except for  $v=2$ , the MRT values are all larger than the FSD, but smaller than the TSD, and smaller than the corresponding  $\text{SNK}_v$  values.

Modified studentized ranges in line with the modification in the Type I error rate  $\alpha$ , and calculated by Duncan (12), are used for the MRT.

#### VI. Scheffé's Significant Difference (SSD).

Suppose that the F-test of the hypothesis  $H_A: \text{all } \alpha_i = 0$  at the  $\alpha$  level of significance has rejected  $H_A$ . Suppose further that the F-test has  $I-1$  and  $f_e$  degrees of freedom for treatments and error, respectively. Let  $\mu_i, i=1, \dots, I$ , be the treatment means with estimates  $\hat{\mu}_i$ . Scheffé' (33) has devised a method, based on the F distribution, of judging all contrasts among the means  $\mu_i$ .

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By defining

$$C = \sum_i \lambda_i \mu_i, \quad \sum \lambda_i = 0,$$

as any contrast with  $\hat{C}$  as its estimate and  $S_C^2$  its estimated variance, he showed (33) that for the totality of contrasts, no matter what the true values of the C's, the probability is  $1 - \alpha$  that they all satisfy

$$\hat{C} - F_S S_C \leq C \leq \hat{C} + F_S S_C,$$

where (i)  $F_S^2 = (I-1)F(\alpha, I-1, f_e)$  ;

(ii)  $F(\alpha, I-1, f_e)$  is the  $\alpha$  upper point of the F distribution with  $I-1$  and  $f_e$  degrees of freedom.

As a pairwise multiple comparisons of means procedure, Scheffé' regarded the estimated contrast  $\hat{C}$  as being significantly different from zero at the  $\alpha$  level if

$$|\hat{C}| \geq F_S S_C.$$

If the contrasts are the  $\frac{1}{2}I(I-1)$  differences of the type  $(\bar{X}_r. - \bar{X}_i.)$ , the treatment means  $\bar{X}_i.$  and  $\bar{X}_r.$  will be adjudged significantly different at the  $\alpha$  level if

$$\bar{X}_r. - \bar{X}_i. \geq F_S S_d,$$

where in this case  $S_C = S_d$ .

The critical value of Scheffé's method is therefore given by

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$$SSD = \left[ (I-1) F(\alpha, I-1, f_e) \right]^{\frac{1}{2}} S_d \cdot \dots\dots(5.1.8)$$

According to Scheffé, the SSD, unlike the methods so far discussed, is insensitive to violations of the assumptions of normality of errors and equality of error variances, and does not require that the treatments be equally replicated. He, however, gave a warning that if all the  $\bar{X}_i$  have the same variance, and all pairs  $\bar{X}_i$ ,  $\bar{X}_r$  have the same covariance, and the only contrasts of interest are the  $\frac{1}{2}I(I-1)$  differences  $(\bar{X}_r - \bar{X}_i)$ , the method of Tukey (TSD) should be used in preference to the SSD, because the confidence intervals will then be shorter.

#### VII. Bayes Exact Test (BET).

Duncan (11) and others have investigated Bayesian approaches to the multiple comparisons problem from which they have developed procedures which directly incorporate the observed F value into the critical value. In lieu of choosing a significant level  $\alpha$ , a measure of the relative seriousness of a Type I error to a Type II error is selected.

Waller and Duncan (41) improved upon the earlier work of Duncan and the result was Bayes exact test (BET). An outline of their work is presented below.

As usual, we consider the  $\frac{1}{2}I(I-1)$  contrasts of the form

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$(\bar{X}_r. - \bar{X}_i.)$ . Waller and Duncan (41) defined a three-decision problem  $Q(i,r)$  where for each pair of estimated means  $(\bar{X}_r., \bar{X}_i.)$  we choose one of the three decisions

- (i)  $d_{ir}^1$  :  $\bar{X}_r.$  is significantly larger than  $\bar{X}_i$ ;
- (ii)  $d_{ir}^2$  :  $\bar{X}_r.$  is not significantly different from  $\bar{X}_i$ ;
- (iii)  $d_{ir}^3$  :  $\bar{X}_r.$  is significantly smaller than  $\bar{X}_i$ .

The multiple comparisons problem was then defined as that of solving the three-decision problem  $Q(i,r)$  for all the  $\frac{1}{2}I(I-1)$  combinations of  $(\bar{X}_r., \bar{X}_i.)$  considered simultaneously.

They formulated a linear additive loss model and defined  $k_1$  and  $k_2$  as positive constants measuring the seriousness of the Type I and Type II errors, respectively. They then obtained the loss of any multiple comparisons decision. This loss was found to be the sum of the linear losses for the component decisions.

Noting that the multiple comparisons problem was invariant with respect to changes in location, they concerned themselves only with solutions depending on the treatment means through the  $(I-1)$  Helmert comparisons (41)

$$\begin{aligned} C_1 &= (\bar{X}_1. - \bar{X}_2.) / \sqrt{2} \quad , \\ C_2 &= (\bar{X}_1. + \bar{X}_2.) / \sqrt{6} \quad , \\ &\vdots \\ C_{I-1} &= \left[ \bar{X}_1. + \dots + \bar{X}_{I-1} - (I-1)\bar{X}_I. \right] / \left[ I(I-1) \right]^{\frac{1}{2}} . \end{aligned}$$

121.

In the theoretical discussions, they considered a two-decision problem  $P(i,r)$  and showed that the derivation of the overall solution of the decision problem was equivalent to solving the two-decision problems  $P(i,r)$ . From this angle they obtained a Bayes solution  $t(k,F,I-1,f_e)$  for the critical equation they obtained.  $t(k,F,I-1,f_e)$  is the minimum average risk  $t$  value for the chosen Type I to Type II error weight ratio  $k$ , the observed  $F$  value, and the degrees of freedom  $(I-1)$  and  $f_e$  for treatments and error, respectively.

Applying their results to the three-decision problem  $Q(i,r)$ , they obtained the following three-decision component rules:

- (i)  $\bar{X}_r.$  is significantly larger than  $\bar{X}_i.$  if
- $$\bar{X}_r. - \bar{X}_i. > \text{BET} ;$$
- (ii)  $\bar{X}_r.$  is not significantly different from  $\bar{X}_i.$  if
- $$\bar{X}_r. - \bar{X}_i. \leq \text{BET} ;$$
- (iii)  $\bar{X}_r.$  is significantly smaller than  $\bar{X}_i.$  if
- $$\bar{X}_r. - \bar{X}_i. < -\text{BET} ;$$

where BET denotes the critical value (the  $k$ -ratio least significant difference), and is given by

$$\text{BET} = t(k,F,I-1,f_e)S_d \cdot \dots\dots(5.1.9)$$

Fairly extensive tables of minimum risk  $t$  values have been

published by the authors (41).

Examining the properties of the BET procedure, Carmer and Swanson (5) observed that large observed F ratios result in small critical values similar in magnitude to the FSD, while small F values ( $F < 2.5$ ) result in considerable larger critical values. According to them therefore, BET should avoid Type I errors when the observed F value indicates little or no variation among the means, and should avoid Type II errors when the F value indicates considerable variation among the means.

## 5.2 The Effect of the Standard Error on Multiple Comparisons of Treatment Means.

From Section 5.1, we observe that each of the multiple comparisons of means procedures utilizes the standard error of the difference between any two means,  $S_d$ . We shall in this section investigate how the different standard errors from the various analyses given in Chapters 3 and 4 affect multiple comparisons of treatment means.

We shall concern ourselves only with the case where the treatments are fixed. If the treatments are random, no meaningful multiple comparisons of their means can be made, for the treatments used in the experiment are then a random sample from a population of treatments. On the other hand, if the treatments are random but we wish to compare the treatments for a given sample, then the treatments under this condition are fixed

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We can therefore perform a multiple comparisons of the treatment means, where any inferences made are strictly confined to the treatments in the sample.

Let  $\bar{X}_S.$  and  $\bar{X}_R.$  be any two treatment means when the average yields per plot are used. Then, under the fixed-effects model,

$$(i) \quad \text{Var}(\bar{X}_S.) = \text{Var}(\bar{X}_R.) = \frac{\sigma_e^2}{Jn} \quad ;$$

$$(ii) \quad \text{Cov}(\bar{X}_S., \bar{X}_R.) = 0 \quad .$$

Therefore

$$\text{Var}(\bar{X}_S. - \bar{X}_R.) = S_{d1}^2 = \frac{2\sigma_e^2}{Jn} \quad \dots\dots(5.2.1)$$

Let  $\bar{X}_{S..}$  and  $\bar{X}_{R..}$  be any two treatment means when yields of the individual plants on a plot are used. Then, under the fixed-effects model,

$$(i) \quad \text{Var}(\bar{X}_{S..}) = \text{Var}(\bar{X}_{R..}) = \frac{\sigma_e^2}{Jn} \quad ;$$

$$(ii) \quad \text{Cov}(\bar{X}_{S..}, \bar{X}_{R..}) = 0 \quad .$$

Therefore

$$\text{Var}(\bar{X}_{S..} - \bar{X}_{R..}) = S_{d2}^2 = \frac{2\sigma_e^2}{Jn} \quad \dots\dots(5.2.2)$$

From equations (5.2.1) and (5.2.2),

$$S_{d1}^2 = S_{d2}^2 \quad .$$

124.

$Sd_1^2$  is estimated by  $2n(MS_E)/Jn$  and  $Sd_2^2$  is estimated by  $2(MS_W)/Jn$ . We found in Chapter 3 that  $MS_W$  is a more efficient estimator of  $\sigma_e^2$  than  $n(MS_E)$ . Hence, it is expected that

$$Sd_1 \geq Sd_2 .$$

Thus, a multiple comparisons procedure based on  $Sd_2$  will have shorter confidence intervals for the differences than if it is based on  $Sd_1$ .

An approximation to the permutation test requires the degrees of freedom of the F-test to be multiplied by  $\phi$  (defined in Section 3.4). If  $\phi > 1$ , the error mean square, and hence the standard error of the difference between any two treatment means, will be smaller than the corresponding one under the normal-theory test. A multiple comparisons of treatment means under the permutation test may therefore have shorter confidence intervals for the differences than a corresponding one under the normal-theory test.

### 5.3. Application of the FSD, SSD, MRT, and BET Procedures to the "Kpong Data"

The FSD, MRT and BET procedures have theoretically been shown (5,41) to be more sensitive than the TSD, SSD and SNK; the sensitivity being related to the lengths of the intervals for contrasts. We shall, in this section, apply the FSD, MRT,

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BET, and SSD procedures to the "Kpong data" (using the average yields per plot) to illustrate the theoretical findings.

The varietal means in descending order are:

$V_{10}$	$V_{12}$	$V_4$	$V_{11}$	$V_6$	$V_7$	$V_5$
60.98	56.57	55.99	53.31	44.98	42.53	36.26
$V_8$	$V_2$	$V_9$	$V_1$	$V_{21}$	$V_{16}$	$V_{13}$
34.72	29.58	25.51	23.53	21.64	20.42	18.61
$V_{14}$	$V_{20}$	$V_{17}$	$V_{19}$	$V_{18}$	$V_3$	$V_{15}$
17.47	14.61	13.12	12.65	9.72	5.43	4.27 .

There are 21 varieties and hence 210 varietal mean differences.

#### I. Fisher's Significant Difference (FSD).

From equation (5.1.3), the critical value of the FSD test is given by

$$\text{FSD} = t(\alpha, f) S_d ,$$

$$\text{where } S_d = \left( 2 \times \frac{MS_E}{J} \right)^{\frac{1}{2}} .$$

Making use of ANOVA Table 3.6.2 and choosing  $\alpha = 0.05$ , we obtain

$$\begin{aligned} \text{FSD} &= t(0.05, 40) \left( 2 \times \frac{86.466}{3} \right)^{\frac{1}{2}} , \\ &= 2.02 \left( \frac{172.932}{3} \right)^{\frac{1}{2}} , \\ &= \underline{15.34} . \end{aligned}$$

126.

With this critical value there are 117 significant differences and the varietal means may be partitioned into 11 groups; the members of each group are not significantly different. A graphical array of the means with the groups indicated by brackets is shown in Figure 5.3.1 (page 130).

### II. Scheffé's Significant Difference (SSD).

From equation (5.1.8), the critical value of the SSD is given by

$$SSD = \left[ (I-1) F(\alpha, I-1, f_e) \right]^{\frac{1}{2}} S_d \cdot$$

From ANOVA Table 3.6.2 and choosing  $\alpha = 0.05$ , we obtain

$$\begin{aligned} SSD &= \left[ 20 F(0.05, 20, 40) \right]^{\frac{1}{2}} \left( 2 \times \frac{86.466}{3} \right)^{\frac{1}{2}}, \\ &= (20 \times 1.84)^{\frac{1}{2}} \left( \frac{172.932}{3} \right)^{\frac{1}{2}}, \\ &= \underline{46.06} \cdot \end{aligned}$$

With this critical value, there are 14 significant differences and the varietal means may be partitioned into 4 groups. A graphical array of the means with the groups indicated by brackets is shown in Figure 5.3.2 (page 131).

### III. Duncan's Multiple Range Test (MRT).

From equation (5.1.5), the I-1 critical values of the MRT are given by

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$$MRT_v = q(\alpha_v, v, f) S_d / \sqrt{2} ; \quad v = 2, 3, \dots, I \ .$$

Choosing  $\alpha = 0.05$  and making use of ANOVA Table 3.6.2, we obtain

$$MRT_v = q(\alpha_v, v, 40) \left( \frac{86.466}{3} \right)^{\frac{1}{2}} \ .$$

The resulting 20 critical values are given below.

$$\begin{array}{lllll} MRT_{21} = 18.63 & MRT_{17} = 18.52 & MRT_{13} = 18.28 & MRT_9 = 17.88 & MRT_5 = 17.02 \\ MRT_{20} = 18.63 & MRT_{16} = 18.47 & MRT_{12} = 18.20 & MRT_8 = 17.72 & MRT_4 = 16.64 \\ MRT_{19} = 18.60 & MRT_{15} = 18.41 & MRT_{11} = 18.09 & MRT_7 = 17.56 & MRT_3 = 16.16 \\ MRT_{18} = 18.58 & MRT_{14} = 18.36 & MRT_{10} = 17.98 & MRT_6 = 17.29 & MRT_2 = 15.35 \ . \end{array}$$

With these critical values there are 112 significant differences giving rise to 10 subsets of the means. A graphical array of the means with the subsets indicated by brackets is shown in Figure 5.3.3 (page 132).

#### IV. Bayes Exact Test (BET).

From equation (5.1.9), the critical value of the BET is given by

$$BET = t(k, F, I-1, f_e) S_d \ .$$

Using ANOVA Table 3.6.2 and Table A2 (41), we obtain

$$BET = t(100, 10.929, 20, 40) \left( 2 \times \frac{86.466}{3} \right)^{\frac{1}{2}} \ .$$

By linearly interpolating with respect to F, the critical value is given by

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$$\begin{aligned} \text{BET} &= 1.859 \left( 2 \times \frac{86.466}{3} \right)^{\frac{1}{2}} , \\ &= \underline{14.11} . \end{aligned}$$

With this critical value there are 112 significant differences. The resulting 11 groups of means are shown in Figure 5.3.4 (p.133)

Theoretically, of all the four procedures discussed above, it is the BET which avoids Type I errors when there is little or no variation among the means, and avoids Type II errors when the F value indicates considerable variation among the means. It may therefore be used as a criterion for comparing the other tests.

The observed F value of 10.93, as compared with the 5% level tabular value of 1.84, indicates considerable variation among the means. The results of the BET, shown in Figure 5.3.4, reflect this fact. The FSD results, shown in Figure 5.3.1, are almost the same as those of BET. In each case, there are eleven groups with almost the same composition. In effect, four of the groups have the same elements while the rest differ only by about one or two elements. These results are in agreement with the findings of Carmer, Swanson, Waller, and Duncan mentioned in Section 1.4.

Duncan's MRT results, shown in Figure 5.3.3, are somewhat different, though not markedly so, from those of BET and FSD. The heterogeneous nature of the means is reflected but not to

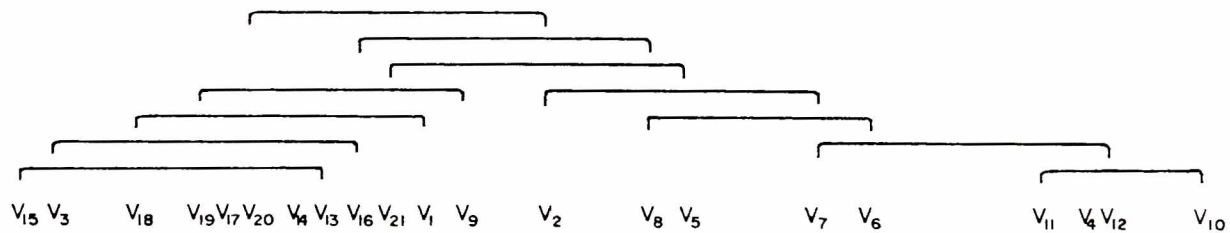
129.

the same degree as found with BET. In this case, there are 10 groups with sizes larger than those of BET and FSD.

The results of SSD, shown in Figure 5.3.2, are markedly different from any of the tests discussed above. There are only four groups. This gives an impression that the means are not different contrary to the verdict of the F-test that the varieties are significantly different. The SSD is therefore less appropriate than either the BET, FSD, or MRT. This is in agreement with the findings of Carmer and Swanson (2) stated in Section 1.4.

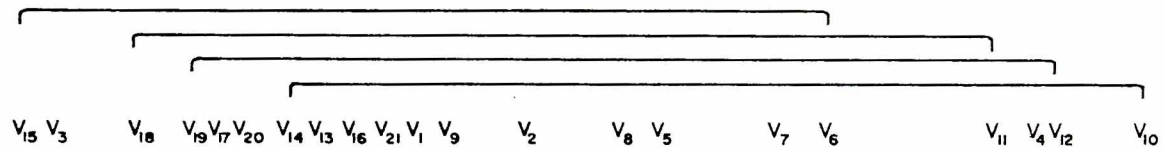
130.

FIG. 5.3.1 : GRAPHICAL ARRAY OF 21 VARIETAL MEANS INDICATED BY THE VARIETAL SYMBOLS. THE BRACKETING IS POSSIBLE AT THE END OF FSD TEST THE MEANS WITHIN A BRACKET ARE SAID NOT TO DIFFER, BUT MEANS NOT BRACKETED TOGETHER ARE ASSERTED TO BE DIFFERENT.



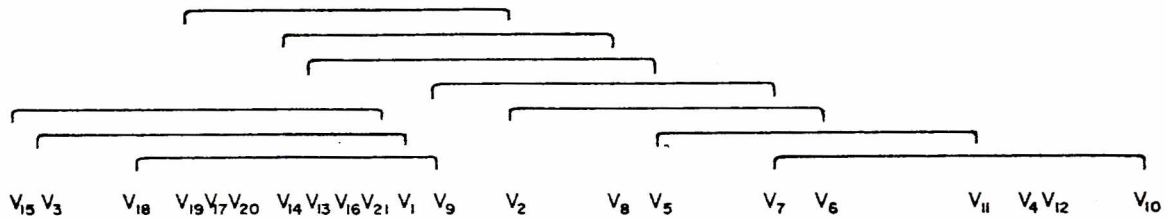
131.

FIG.5.3.2: GRAPHICAL ARRAY OF 21 VARIETAL MEANS. THE BRACKETING IS POSSIBLE AT THE END OF SSD TEST. MEANS BRACKETED TOGETHER ARE NOT DIFFERENT.



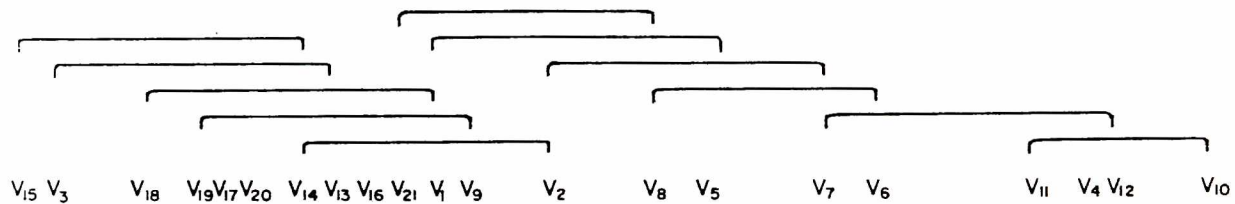
132.

FIG. 5.3.3. GRAPHICAL ARRAY OF 21 VARIETAL MEANS. THE BRACKETING IS POSSIBLE AT THE END OF THE DUNCAN'S MULTIPLE RANGE TEST (MRT). MEANS BRACKETED TOGETHER ARE NOT DIFFERENT



133.

FIG 5.3.4 : GRAPHICAL ARRAY OF 21 VARIETAL MEANS. THE BRACKETING IS POSSIBLE AT END OF THE BAYE'S EXACT TEST (BET). MEANS BRACKETED TOGETHER ARE NOT DIFFERENT.



#### 5.4 Multiple Comparisons of Treatment Means for the Various Analyses Using Duncan's MRT.

We saw from Section 5.3 that the FSD and BET are the best available procedures, among those discussed in Section 5.1, for use in multiple comparisons of treatment means. Unfortunately, neither of them is applicable to all the various analyses discussed in Chapters 3 and 4. Since Duncan's MRT is not markedly different from the BET and FSD, and is also applicable to the various analyses, it will be used in this section to compare treatment means for those analyses considered in Chapters 3 and 4.

##### I. MRT Using the Average Yields Per Plot.

Using the average yields per plot, under the fixed-effects model, the standard error of the difference between any two treatment means is obtained from equation (5.2.1). Its estimated value is

$$S_d = \left( 2 \cdot \frac{MS_E}{J} \right)^{\frac{1}{2}} .$$

Making use of equation (5.1.7) and ANOVA Table 3.6.2, the critical values of Duncan's MRT are

$$MRT_v = q(\alpha_v, v, 40) \left( \frac{86.466}{3} \right)^{\frac{1}{2}} .$$

The resulting critical values are the same as those given in Section 5.3 (III). There will therefore be 10 groupings of the means with the graphical array of Figure 5.3.3 (p.132).

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## II. MRT Using Yields of the Individual Plants on a Plot.

Using yields of the individual plants on a plot, under the fixed-effects model, the standard error of the difference between any two treatment means is obtained from equation (5.2.2). Its estimated value is

$$S_d = \left( 2 \cdot \frac{MS_W}{Jn} \right)^{\frac{1}{2}} .$$

From equation (5.1.7) and ANOVA Table 3.2.4, the critical values are given by

$$MRT_v = q(\alpha_v, v, 252) \left( \frac{131.208}{3 \times 5} \right)^{\frac{1}{2}} .$$

The 20 critical values are

$$\begin{array}{llll} MRT_{21} = 10.28 & MRT_{16} = 10.09 & MRT_{11} = 9.81 & MRT_6 = 9.32 \\ MRT_{20} = 10.26 & MRT_{15} = 10.04 & MRT_{10} = 9.73 & MRT_5 = 9.14 \\ MRT_{19} = 10.22 & MRT_{14} = 10.00 & MRT_9 = 9.64 & MRT_4 = 8.93 \\ MRT_{18} = 10.18 & MRT_{13} = 9.94 & MRT_8 = 9.55 & MRT_3 = 8.64 \\ MRT_{17} = 10.13 & MRT_{12} = 9.88 & MRT_7 = 9.44 & MRT_2 = 8.19 . \end{array}$$

With these critical values there are 148 significant differences resulting in 12 groupings of the means. These are shown in Figure 5.4.1, (page 140).

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### III. MRT Using Results of the Permutation Test.

Making use of equation (5.1.7) and ANOVA Table 3.6.6, the critical values of Duncan's MRT are

$$\text{MRT}_v = q(\alpha_v, v, 40.8) \left( \frac{84.771}{3} \right)^{\frac{1}{2}} .$$

The individual critical values are

$$\begin{array}{llll} \text{MRT}_{21} = 18.45 & \text{MRT}_{16} = 18.29 & \text{MRT}_{11} = 17.91 & \text{MRT}_6 = 17.12 \\ \text{MRT}_{20} = 18.45 & \text{MRT}_{15} = 18.23 & \text{MRT}_{10} = 17.81 & \text{MRT}_5 = 16.50 \\ \text{MRT}_{19} = 18.42 & \text{MRT}_{14} = 18.18 & \text{MRT}_9 = 17.70 & \text{MRT}_4 = 16.48 \\ \text{MRT}_{18} = 18.39 & \text{MRT}_{13} = 18.10 & \text{MRT}_8 = 17.54 & \text{MRT}_3 = 16.00 \\ \text{MRT}_{17} = 18.34 & \text{MRT}_{12} = 18.02 & \text{MRT}_7 = 17.38 & \text{MRT}_2 = 15.20 . \end{array}$$

These critical values are not much different from those of Section 5.3 (III) - MRT using the average yields per plot.

### IV. MRT Using Results of the Multivariate Analyses of Variance.

The varietal means are the same as those given in Section 5.3. From Section 4.2 (I), the error sum of squares is given by

$$\text{SS}'_E = \begin{array}{ccccc} 8103.62 & 3987.93 & 1486.68 & 4385.68 & 5149.11 \\ 3937.93 & 7480.18 & -254.26 & 2284.76 & 2245.89 \\ 1486.68 & -254.26 & 5928.12 & 1095.60 & 2198.08 \\ 4385.68 & 2284.76 & 1095.60 & 5574.54 & 3246.04 \\ 5149.11 & 2245.89 & 2198.08 & 3246.04 & 7049.93 \end{array} .$$

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Dividing  $SS'_E$  by the degrees of freedom 40 gives a variance-covariance matrix whose diagonal elements are the variances of the five observations. The average of the five variances will be taken as the mean square for use in the multiple comparisons. Thus

$$S^2_E = \frac{8103.62 + 7480.18 + 5928.12 + 5574.54 + 7049.93}{40 \times 5}$$

$$= 170.682.$$

The critical values of Duncan's MRT are therefore given by

$$MRT_v = q(\alpha_v, v, 40) \left( \frac{170.682}{3} \right)^{\frac{1}{2}}.$$

The individual critical values are given below.

$$\begin{array}{llll} MRT_{21} = 26.17 & MRT_{16} = 25.95 & MRT_{11} = 25.42 & MRT_6 = 24.29 \\ MRT_{20} = 26.17 & MRT_{15} = 25.87 & MRT_{10} = 25.27 & MRT_5 = 23.91 \\ MRT_{19} = 26.14 & MRT_{14} = 25.80 & MRT_9 = 25.12 & MRT_4 = 23.38 \\ MRT_{18} = 26.10 & MRT_{13} = 25.68 & MRT_8 = 24.89 & MRT_3 = 22.70 \\ MRT_{17} = 26.02 & MRT_{12} = 25.75 & MRT_7 = 24.66 & MRT_2 = 21.57 \end{array}.$$

With these critical values there are 71 significant differences. The resulting grouping of the means is shown in Figure 5.4.2, (page 141).

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(ii) The Case of Using the Correlated Variates x,y,h,z and  $\theta$ .

In this case, the varietal means are defined by the average of the five correlated variates x,y,h,z and  $\theta$ . As a result, the means are different from those given in Section 5.3. They are given below in descending order.

$V_4$	$V_{10}$	$V_7$	$V_2$	$V_5$	$V_{11}$	$V_{12}$
187.46	165.14	160.58	157.10	155.29	154.10	150.76
$V_8$	$V_6$	$V_{16}$	$V_1$	$V_9$	$V_{21}$	$V_{20}$
150.13	137.51	135.82	133.87	133.36	129.81	128.17
$V_{13}$	$V_{17}$	$V_{19}$	$V_{14}$	$V_{18}$	$V_3$	$V_{15}$
122.74	117.40	106.08	104.67	94.26	93.55	92.68

From Section 4.2 (II), the error sum of squares is given by

$$SS'_E = \begin{bmatrix} 1689.50 & 123.20 & 476.90 & 27.37 & 1262.75 \\ 123.20 & 21.64 & 79.17 & 2.53 & 197.92 \\ 476.90 & 79.17 & 685.69 & 12.27 & 1492.59 \\ 27.37 & 2.53 & 12.27 & 39.17 & 79.83 \\ 1262.75 & 197.92 & 1492.59 & 79.83 & 3458.46 \end{bmatrix}$$

By the same argument given in Section 5.4 (IV) (i), the error mean square is given by

$$\begin{aligned} S^2_E &= \frac{1689.50 + 21.64 + 685.69 + 39.17 + 3458.46}{40 \times 5} \\ &= 29.472 \end{aligned}$$

139.

The critical values of Duncan's MRT are given by

$$\text{MRT}_v = q(\alpha_v, v, 40) \left( \frac{29.472}{3} \right)^{\frac{1}{2}} .$$

The individual critical values are given below.

$\text{MRT}_{21} = 10.88$	$\text{MRT}_{16} = 10.78$	$\text{MRT}_{11} = 10.56$	$\text{MRT}_6 = 10.09$
$\text{MRT}_{20} = 10.88$	$\text{MRT}_{15} = 10.75$	$\text{MRT}_{10} = 10.50$	$\text{MRT}_5 = 9.94$
$\text{MRT}_{19} = 10.86$	$\text{MRT}_{14} = 10.72$	$\text{MRT}_9 = 10.44$	$\text{MRT}_4 = 9.72$
$\text{MRT}_{18} = 10.84$	$\text{MRT}_{13} = 10.67$	$\text{MRT}_8 = 10.34$	$\text{MRT}_3 = 9.43$
$\text{MRT}_{17} = 10.81$	$\text{MRT}_{12} = 10.63$	$\text{MRT}_7 = 10.25$	$\text{MRT}_2 = 8.96$ .

With these critical values there are 170 significant differences.

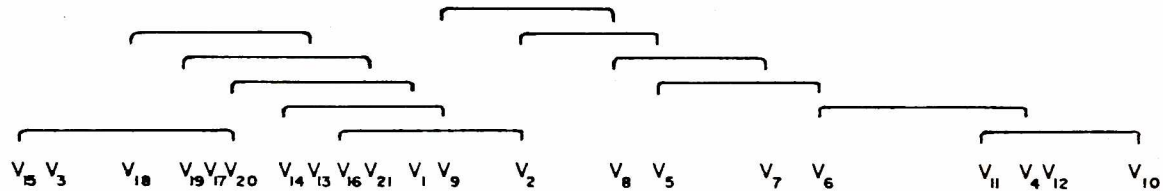
The resulting grouping of the means is shown in Figure 5.4.3,

(page 142).

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FIG. 5.4.1: GRAPHICAL ARRAY OF 21 VARIETY MEANS. THE BRACKETING IS THE RESULT OF DUNCAN'S MULTIPLE RANGE TEST (MRT). MEANS BRACKETED TOGETHER ARE NOT DIFFERENT

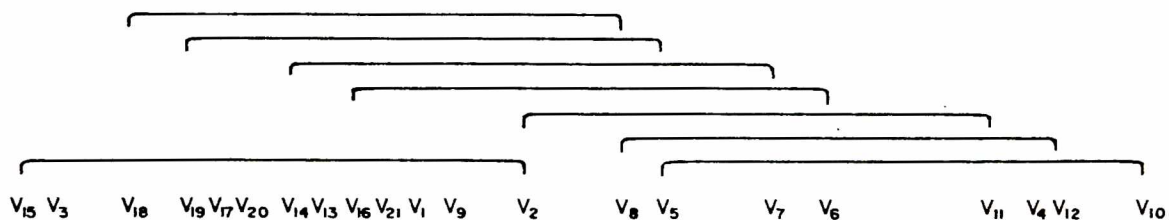
(The case of using Yields of the individual plants on a plot with within plots mean square as error)



141.

FIG. 5.4.2? GRAPHICAL ARRAY OF 21 VARIETAL MEANS. THE BRACKETING IS POSSIBLE AT THE END OF DUNCAN'S MULTIPLE RANGE TEST (MRT). MEANS BRACKETED TOGETHER ARE NOT DIFFERENT

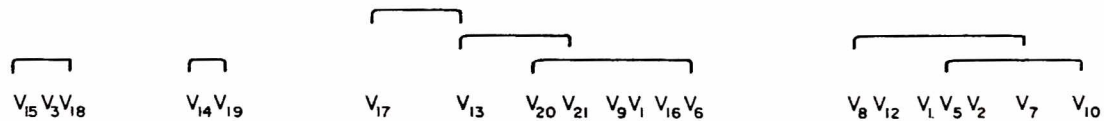
(The case of Multivariate Analysis of Variance using Yields of the individual plants on a plot)



142.

FIG. 5.4.3: GRAPHICAL ARRAY OF 21 VARIETAL MEANS OBTAINED FROM THE CORRELATED VARIABLES. THE BRACKETING IS THE RESULT OF DUNCAN'S MULTIPLE RANGE TEST (MRT). MEANS BRACKETED TOGETHER ARE NOT DIFFERENT.

(The case of multivariate Analysis of Variance using the correlated Variates  $x, y, h, z$  and  $\theta$ )



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There is considerable variation in the results shown in Figures 5.4.1, 5.4.2 and 5.4.3 even though the same test was employed. The variation is due to the different standard errors used in each case.

Duncan's MRT produced 10 groups when the average yields per plot were used. Using yields of the individual plants on a plot, the MRT produced 12 groups. Both cases show evidence of heterogeneous means but the latter one is more pronounced. This is to be expected in view of the deductions given in Section 5.2.

The MRT applied to the multivariate analysis of variance using yields of the individual plants on a plot produced 8 groups. This gives an impression of homogeneous means contrary to the verdicts of the  $\chi^2$  and F tests.

For the multivariate analysis of variance using the correlated variates  $x, y, h, z$ , and  $\theta$ , the MRT also produced 8 groups (Figure 5.4.3). The nature of these 8 groups is, however, different from all the other cases considered earlier on. They do not overlap to the extent as found in the other cases; in fact, three of the groups are disjoint. Thus, the heterogeneous nature of the means is reflected.

The MRT applied to the permutation test produced the same results as the case of the average yields per plot under the fixed-effects model. This is to be expected since, because

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only three blocks were used,  $\phi$  is not much different from unity.

## CHAPTER SIX

### ESTIMATION OF VARIANCE COMPONENTS

#### 6.1 Standard Errors and Confidence Limits.

A point estimate of a parameter is not very meaningful without some measure of the possible error in the estimate. A measure of the possible error could be in the form of a standard error attached to the estimate or an interval about the estimate with some measure of assurance that the true parameter lies within the interval. The interval in the latter approach is called confidence interval; the limits are correspondingly called confidence limits.

Crump (8,9) and Davies (10) have provided a method for finding standard errors of estimates of variance components. They have also given an approximate method for finding confidence limits for variance components. The discussion below is a summary of the methods.

#### I. Standard Errors.

Let us assume each of the random elements in the mathematical model (2.1.4) follows a normal distribution. Then each of the analysis of variance mean squares is distributed as

$$\chi^2 \frac{\sigma_o^2}{f} ,$$

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where  $\sigma_0^2$  is the expectation of the mean square in question and  $f$  the corresponding degrees of freedom. Hence, since the variance of  $\chi^2$  is  $2f$ , the variance of any mean square is

$$2\sigma_0^4 / f \cdot$$

Further, the mean squares are independently distributed, and thus we may write the variance of any linear function of the mean squares.

According to Crump (8), the variance  $2\sigma_0^4 / f$  may unbiasedly be estimated by

$$\frac{2\hat{\sigma}_0^4}{2+f}, \quad \dots\dots(6.1.1)$$

where  $\hat{\sigma}_0^2$  is the appropriate mean square value in the ANOVA table.

## II. Confidence Limits.

(a) Let  $M_1$  and  $M_2$  be estimates of the population variances  $\sigma_1^2$  and  $\sigma_2^2$  with  $f_1$  and  $f_2$  degrees of freedom respectively. Then

$$F = \frac{M_1}{M_2} \cdot \frac{1}{\lambda}, \quad \lambda = \frac{\sigma_1^2}{\sigma_2^2},$$

has the F-distribution with  $f_1$  and  $f_2$  degrees of freedom.

Davies (10) method consists in finding approximate confidence limits for  $\sigma_1^2$

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In the F tables, the values of  $f_1$  or  $f_2$  equal to infinity correspond to the case where one variance is known exactly. The confidence limits for  $\sigma_1^2$  are obtained by multiplying those for the ratio by the known value of  $\sigma_2^2$ . If  $\sigma_2^2 = 1$ , the confidence limits of the ratio are those of  $\sigma_1^2$ . To find the limits for a single variance  $\sigma_1^2$  therefore, we find the percentage points corresponding to  $f_2 = \infty$  and  $f_1 =$  number of degrees of freedom on which the estimate of  $\sigma_1^2$  is based.

- Let (i)  $L_1(\alpha)$  = the multiplier for the lower confidence limit of the ratio of two variances based on  $f_1$  and  $f_2$  degrees of freedom for probability  $\alpha$ ;
- (ii)  $L_2(\alpha)$  = the corresponding multiplier for the upper confidence limit.

Then, according to Davies, the  $100(1-2\alpha)\%$  confidence limits for  $\sigma_1^2/\sigma_2^2$  are

$$\frac{M_1}{M_2} L_1(\alpha) \quad \text{and} \quad \frac{M_1}{M_2} L_2(\alpha) \cdot$$

Hence, approximate confidence limits for  $\sigma_1^2$  are

$$M_1 L_1(\alpha) \quad \text{and} \quad M_1 L_2(\alpha) \cdot \quad \dots\dots(6.1.2)$$

(b) Consider the case where

$$\hat{\sigma}_1^2 = M_1 \cdot$$

$$\text{and} \quad \hat{\sigma}_2^2 = \frac{M_2 - M_1}{r} \cdot$$

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Davies method consists in finding approximate confidence limits for  $\sigma_2^2$ .

From above, we have

$$\frac{\hat{\sigma}_2^2}{\hat{\sigma}_1^2} = \frac{M_2 - M_1}{rM_1}.$$

Hence

$$1 + r \frac{\hat{\sigma}_2^2}{\hat{\sigma}_1^2} = \frac{M_2}{M_1}.$$



Now  $M_2/M_1$  is the ratio of two independent mean squares and hence the  $100(1-2\alpha)\%$  confidence limits for the ratio of the true variances are

$$\frac{M_2}{M_1} L_1(\alpha) \quad \text{and} \quad \frac{M_2}{M_1} L_2(\alpha).$$

The confidence limits for  $(1 + r \hat{\sigma}_2^2 / \hat{\sigma}_1^2)$  are therefore given by

$$\frac{M_2}{M_1} L_1(\alpha) \quad \text{and} \quad \frac{M_2}{M_1} L_2(\alpha).$$

Accordingly, the confidence limits for  $\hat{\sigma}_2^2 / \hat{\sigma}_1^2$  are

$$\frac{1}{r} \left[ \frac{M_2}{M_1} L_1(\alpha) - 1 \right] \quad \text{and} \quad \frac{1}{r} \left[ \frac{M_2}{M_1} L_2(\alpha) - 1 \right].$$

Davies then chose the following as approximate confidence limits for  $\sigma_2^2$ :

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$$\frac{1}{r} \left[ M_2 L_1(\alpha) - M_1 \right] \quad \text{and} \quad \frac{1}{r} \left[ M_2 L_2(\alpha) - M_1 \right] \quad \dots (6.1.3)$$

## 6.2. Estimation of Variance Components by the Analysis of Variance Method.

The analysis of variance method (35) of estimating variance components involves equating mean squares to their expected values and solving the resulting equations. The method applied to the cases of using the average yields per plot and yields of the individual plants on a plot is discussed below.

### I. The Case of Using Average Yields Per Plot.

The ANOVA table for a randomised block experiment using the average yields per plot is given by Table 3.1.1. The expected values of the mean squares under the random-effects model are given in Table 3.1.2. Equating the mean squares to their expected values, we obtain the following estimators:

$$\left. \begin{aligned} \text{(i)} \quad \hat{\sigma}_\alpha^2 &= \frac{MS_V - MS_E}{J} ; \\ \text{(ii)} \quad \hat{\sigma}_\beta^2 &= \frac{MS_B - MS_E}{I} . \end{aligned} \right] \quad \dots (6.2.1)$$

The error variance,  $\sigma_e^2$ , and the interactions component of variance,  $\sigma_\alpha^2$ , cannot be separately estimated. If the treat-

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ment and environmental effects are additive, we shall have  $\sigma_{\gamma}^2 = 0$ . The error variance can then be estimated and is given by

$$\sigma_e^2 / n = MS_E \quad \dots\dots(6.2.2)$$

## II. The Case of Using Yields of the Individual Plants on a Plot .

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The ANOVA table for a randomised block experiment using yields of the individual plants on a plot is given by Table 3.2.1. The expected values of the mean squares under the random-effects model are given in Table 3.2.2.

Equating the mean squares to their expected values, we obtain the following estimators:

$$\begin{aligned} \text{(i)} \quad \hat{\sigma}_e^2 &= MS_W ; \\ \text{(ii)} \quad \hat{\sigma}_\alpha^2 &= \frac{MS_V - MS_{VB}}{Jn} ; \\ \text{(iii)} \quad \hat{\sigma}_\beta^2 &= \frac{MS_B - MS_{VB}}{In} ; \\ \text{(iv)} \quad \hat{\sigma}_\gamma^2 &= \frac{MS_{VB} - MS_W}{n} , \end{aligned} \quad \dots\dots(6.2.3)$$

We observe that the nature of the estimators permit negative values. Such negative values are, however, not acceptable since variance components are by definition positive.

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This unfortunate situation led to the search for other procedures. One of the methods discovered will be discussed in Section 6.3.

III. Comparison of Estimators.

We shall compare the estimators given in equations (6.2.1), (6.2.2) and (6.2.3) by examining their variances under additivity and non-additivity of treatments and environmental effects.

(i) Treatments and Environmental Effects are Additive ( $\sigma_{\gamma}^2 = 0$ ).

When treatments and environmental effects are additive, we obtain the estimators given in equations (6.2.1), (6.2.2) and (6.2.3). We shall consider  $\hat{\sigma}_{\alpha}^2$  and  $\hat{\sigma}_e^2$  since  $\hat{\sigma}_{\beta}^2$  is similar to  $\hat{\sigma}_{\alpha}^2$ .

The variances of  $MS_E$  and  $MS_W$  have been calculated in Section 3.3. By the same approach, those of  $MS_V$  and  $MS_{VB}$  under the random-effects model can be calculated. The mean squares occurring in a particular estimator are independent. Thus, the variances of  $\hat{\sigma}_e^2$  and  $\hat{\sigma}_{\alpha}^2$  can easily be calculated. They are given in Table 6.2.1.

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Table 6.2.1: Variances of  $\hat{\sigma}_e^2$  and  $\hat{\sigma}_\alpha^2$  (Analysis of Variance Method).  
Treatments and Environmental Effects are assumed to be Additive.

	$\text{Var}(\hat{\sigma}_e^2)$	$\text{Var}(\hat{\sigma}_\alpha^2)$
Average yields per plot	$D_1 = \frac{2\sigma_e^4}{(I-1)(J-1)}$	$D_2 = \frac{2}{J^2n^2(I-1)} \left[ (\sigma_e^2 + Jn\sigma_\alpha^2)^2 + \frac{\sigma_e^4}{J-1} \right]$
Yields of Individual Plants on a Plot	$D_3 = \frac{2\sigma_e^4}{IJ(n-1)}$	$D_4 = \frac{2}{J^2n^2(I-1)} \left[ (\sigma_e^2 + Jn\sigma_\alpha^2)^2 + \frac{\sigma_e^4}{J-1} \right]$

From Table 6.2.1,

$$D_3 < D_1 \cdot$$

Hence, the error variance is more efficiently estimated if yields of the individual plants on a plot are used. We also note from Table 6.2.1 that

$$D_4 = D_2 \cdot$$

Thus,  $\sigma_\alpha^2$  is estimated with the same efficiency in both cases.

(ii) Treatments and Environmental Effects are Non-additive ( $\sigma_\gamma^2 \neq 0$ ).

When treatments and environmental effects are non-additive,  $\sigma_e^2$  and  $\sigma_\gamma^2$  cannot be separately estimated for the case of average yields per plot. We are therefore left with  $\hat{\sigma}_\alpha^2$  and  $\hat{\sigma}_\beta^2$  to do the comparison. Since  $\hat{\sigma}_\alpha^2$  and  $\hat{\sigma}_\beta^2$  are similar, we will only consider  $\hat{\sigma}_\beta^2$ .

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From equation (6.2.1),

$$\begin{aligned} \text{Var}(\hat{\sigma}_\alpha^2) &= D_5 = \frac{1}{J^2} \left[ \text{Var}(\text{MS}_V) + \text{Var}(\text{MS}_E) \right], \\ &= \frac{2}{J^2 n^2 (I-1)} \left[ (\sigma_e^2 + n\sigma_\gamma^2 + Jn\sigma_\alpha^2)^2 + \frac{(\sigma_e^2 + n\sigma_\gamma^2)^2}{J-1} \right]. \end{aligned}$$

From equation (6.2.3),

$$\begin{aligned} \text{Var}(\hat{\sigma}_\alpha^2) &= D_6 = \frac{1}{J^2 n^2} \left[ \text{Var}(\text{MS}_V) + \text{Var}(\text{MS}_{VB}) \right], \\ &= \frac{2}{J^2 n^2 (I-1)} \left[ (\sigma_e^2 + n\sigma_\gamma^2 + Jn\sigma_\alpha^2)^2 + \frac{(\sigma_e^2 + n\sigma_\gamma^2)^2}{J-1} \right]. \end{aligned}$$

Thus  $D_5 = D_6$ . That is, the efficiency with which  $\sigma_\alpha^2$  is estimated is the same whether one uses the average yields or yields of the individual plants on a plot.

### 6.3 Estimation of Variance Components by the Restricted Maximum Likelihood Method.

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We remarked in Section 6.2 that the analysis of variance method for estimating variance components could lead to negative estimates, and that this objectionable situation led to the search for other methods. One of the methods which evolved, and which is appropriate for the randomised block experiment described in Section 1.1, is based on a restricted maximum likelihood princi-

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ple. It is called the restricted maximum likelihood method and is due to Thompson (39). Generally, it consists of maximizing the joint likelihood of that portion of the set of sufficient statistics which is location invariant.

#### I. The Restricted Maximum Likelihood Method.

To begin with, Thompson (39) considered a maximization problem. He defined the following:

- (i)  $w$  and  $\hat{w}$  are  $p$  dimensional column vectors where  $\hat{w}$  is the restricted maximum likelihood estimate of  $w$ ;
- (ii)  $A$  is a non-singular  $p \times p$  matrix with  $A'$  as its transpose;
- (iii)  $g(w)$  is a function of  $p$  variables;
- (iv)  $G_i(w) = \frac{\partial g(w)}{\partial w_i}$ ,  $i=1,2,\dots,p$ ;
- (v)  $G(w)$  is the column vector with elements  $G_1(w), G_2(w), \dots, G_p(w)$ ;
- (vi)  $b_i$  and  $a_i$  are the  $i^{\text{th}}$  rows of  $A^{-1}$  and  $A'$  respectively.

He then set about the maximization problem by considering the following theorem from the theory of non-linear programming.

Theorem I: If  $g(w)$  is a differentiable function at  $\hat{w}$ , then a necessary set of conditions that it has a relative maximum at  $\hat{w}$  subject to the constraints  $A^{-1}w \geq 0$  is

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$$\begin{array}{l}
 \text{(i)} \quad A^{-1}\hat{w} \geq 0 \quad ; \\
 \text{(ii)} \quad A'G(\hat{w}) \leq 0 \quad ; \\
 \text{(iii)} \quad \hat{w}'G(\hat{w}) = 0 \quad \cdot
 \end{array}
 \left. \vphantom{\begin{array}{l} \text{(i)} \\ \text{(ii)} \\ \text{(iii)} \end{array}} \right\} \dots\dots(6.3.1)$$

An alternative form of the theorem is: for each  $i$ ,  $i=1,2,\dots,p$ , either

$$\begin{array}{l}
 \text{(i)} \quad b_i\hat{w} = 0 \quad \text{and} \quad a_iG(\hat{w}) \leq 0 \\
 \text{or} \quad \text{(ii)} \quad b_i(\hat{w}) > 0 \quad \text{and} \quad a_iG(\hat{w}) = 0 \quad \cdot
 \end{array}
 \left. \vphantom{\begin{array}{l} \text{(i)} \\ \text{(ii)} \end{array}} \right\} \dots\dots(6.3.2)$$

He referred to the set of all points  $w$  satisfying  $A^{-1}w \geq 0$  as the admissible region.

Thompson studied the conditions under which there will be a unique solution to the relations (6.3.1) or (6.3.2) by considering a property of strictly concave functions and their tangent planes. In effect, he came out with the following theorem and its corollary.

Theorem 2: If  $g(w)$  is a strictly concave function having gradient vector  $G(w)$  and if  $\hat{w}$  satisfies (6.3.1), then  $g(\hat{w}) > g(w)$  whenever  $A^{-1}w \geq 0$  and  $w \neq \hat{w}$ .

Corollary: If  $g(w)$  is a strictly concave function having gradient vector  $G(w)$ , then the solution to (6.3.1) or (6.3.2) is unique.

He next considered a general theory for multiple classifica-

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- (i)  $M_1, \dots, M_p$  be  $p$  mean squares whose expected values are the  $p$  independently distributed variables  $w_1, \dots, w_p$  with degrees of freedom  $f_1, \dots, f_p$  respectively;
- (ii)  $M = (M_1, \dots, M_p)'$  ;
- (iii)  $\sigma^2 = (\sigma_1^2, \dots, \sigma_p^2)'$  ;
- (iv)  $w = (w_1, \dots, w_p)'$  .

Then  $E(M) = w = A\sigma^2$  .

Thus,  $\sigma^2 = A^{-1}w$

if the inverse exists.

He noted that each  $M_i$  has a gamma distribution with parameters  $f_i/2$  and  $2w_i/f_i$ . He obtained the logarithm of the density function of  $M_1, \dots, M_p$  as

$$g(w) = \text{constant} - \sum_{i=1}^p f_i \left( \log w_i + \frac{M_i}{w_i} \right) \text{ . } \dots (6.3.3)$$

In order to find  $\hat{\sigma}^2$  (the vector of restricted maximum likelihood estimates of the components of  $\sigma^2$ ) he maximized equation (6.3.3) twice with respect to  $w_1, \dots, w_p$  and subject to the set of linear constraints

$$A^{-1}w \geq 0 \text{ .}$$

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For the function  $g(w)$  - equation (6.3.3),

$$G_i(w) = \frac{\partial g(w)}{\partial w_i} = f_i \frac{(M_i - w_i)}{w_i^2} \cdot \dots (6.3.4)$$

Hence condition (6.3.1) (iii) becomes

$$\sum_i f_i = \sum_i \frac{f_i M_i}{w_i} \cdot \dots (6.3.5)$$

Thompson applied the above results to two special cases - randomised block experiment with one observation per plot and  $n$  observations per plot. Discussed below are the two special cases with observations defined as average yields per plot in one case and yields of individual plants on a plot in the other case.

## II. The Case of Using Average Yields Per Plot.

The ANOVA table for this situation is given by Table 3.1.1. The expected values of the mean squares are given in Table 3.1.2 and is reproduced below in Table 6.3.1,

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Table 6.3.1 : Expected Values of Mean Squares of Table 3.1.1 Under the Random-Effects Model. Treatments and Environmental Effects are assumed to be Additive.

Source	Mean Square	Expected Mean Square
Treatments	$MS_V = M_3$	$E(MS_V) = w_3 = \frac{\sigma_e^2}{n} + J\sigma_\alpha^2$
Blocks	$MS_B = M_2$	$E(MS_B) = w_2 = \frac{\sigma_e^2}{n} + I\sigma_\beta^2$
Error	$MS_E = M_1$	$E(MS_E) = w_1 = \frac{\sigma_e^2}{n}$

Thus,  $E(M) = w = A\sigma^2$  ,

where (i)  $M = \begin{pmatrix} M_1 \\ M_2 \\ M_3 \end{pmatrix}$  ; (ii)  $w = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$  ; (iii)  $\sigma^2 = \begin{pmatrix} \sigma_e^2/n \\ I\sigma_\beta^2 \\ J\sigma_\alpha^2 \end{pmatrix}$  ;

(iv)  $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$  ; (v)  $A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$  .

The conditions (6.3.1) then become

$$\begin{aligned}
 & \text{(i)} \quad \hat{w}_1 \geq 0 \ ; \\
 & \text{(i)'} \quad 0 \leq \frac{f_1(\hat{w}_1 - M_1)}{\hat{w}_1^2} + \frac{f_2(\hat{w}_2 - M_2)}{\hat{w}_2^2} + \frac{f_3(\hat{w}_3 - M_3)}{\hat{w}_3^2} \ ; \\
 & \text{(ii)} \quad \hat{w}_1 \leq \hat{w}_2 \ ; \quad \text{(ii)'} \quad \hat{w}_2 \geq M_2 \ ; \\
 & \text{(iii)} \quad \hat{w}_1 \leq \hat{w}_3 \ ; \quad \text{(iii)'} \quad \hat{w}_3 \geq M_3 \ .
 \end{aligned}$$

....(6.3.6)

According to Thompson, the necessary conditions for maximum likelihood estimates are that, for the conditions given in (6.3.6), equality must hold in at least one equation of each pair; further, if  $M_1 > 0$ , a maximum likelihood estimate cannot occur at  $w_1 = 0$ . From these considerations, he obtained four possibilities.

The solutions under the various conditions are given in Table 6.3.2 below. These solutions are found to be mutually exclusive and hence the maximum likelihood estimates are unique.

He denoted  $M_{r \dots k}$  to mean the mean square obtained by pooling  $M_r, \dots, M_k$ . That is

$$M_{r \dots k} = \frac{(f_r M_r + \dots + f_k M_k)}{(f_r + \dots + f_k)} \quad \dots (6.3.7)$$

Table 6.3.2 : Estimates of Variance Components of Table 6.3.1 Under Four Possible Conditions.

CONDITIONS	$\hat{\sigma}_e^2/n$	$I\hat{\sigma}_\beta^2$	$J\hat{\sigma}_\alpha^2$
(i) $M_1 < M_2, M_1 < M_3$	$M_1$	$M_2 - M_1$	$M_3 - M_1$
(ii) $M_1 \geq M_2, M_2 \leq M_{12} < M_3$	$M_{12}$	0	$M_3 - M_{12}$
(iii) $M_1 \geq M_3, M_3 \leq M_{13} < M_2$	$M_{13}$	$M_2 - M_{13}$	0
(iv) $M_{13} \geq M_2, M_{12} \geq M_3$ and either $M_1 \geq M_2$ or $M_1 \geq M_3$	$M_{123}$	0	0

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III. The Case of Using Yields of the Individual Plants on a Plot.

The ANOVA table for this situation is given by Table 3.2.1. The expected values of the mean squares (under the random-effects model) are given in Table 3.2.2 and is reproduced below in Table 6.3.3.

Table 6.3.3: Expected Values of Mean Squares of Table 3.2.1 Under the Random-Effects Model.

Source	Mean Square	Expected Mean Square
Treatments	$MS_V = M_4$	$E(MS_V) = w_4 = \sigma_e^2 + n\sigma_\gamma^2 + Jn\sigma_\alpha^2$
Blocks	$MS_B = M_3$	$E(MS_B) = w_3 = \sigma_e^2 + n\sigma_\gamma^2 + In\sigma_\beta^2$
Interaction	$MS_{VB} = M_2$	$E(MS_{VB}) = w_2 = \sigma_e^2 + n\sigma_\gamma^2$
Within Plots	$MS_W = M_1$	$E(MS_W) = w_1 = \sigma_e^2$

Thus,  $E(M) = w = A\sigma^2$ ,

where (i)  $M = \begin{pmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \end{pmatrix}$ ; (ii)  $w = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix}$ ; (iii)  $\sigma^2 = \begin{pmatrix} \sigma_e^2 \\ n\sigma_\gamma^2 \\ In\sigma_\beta^2 \\ Jn\sigma_\alpha^2 \end{pmatrix}$ ;

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$$(iv) \quad A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \quad ; \quad (v) \quad A^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}.$$

The conditions given by equation (6.3.1) now become

$$(i) \quad \hat{w}_1 > 0 \quad ;$$

$$(i)' \quad 0 = \frac{f_1(\hat{w}_1 - M_1)}{\hat{w}_1^2} + \frac{f_2(\hat{w}_2 - M_2)}{\hat{w}_2^2} + \frac{f_3(\hat{w}_3 - M_3)}{\hat{w}_3^2} + \frac{f_4(\hat{w}_4 - M_4)}{\hat{w}_4^2}$$

$$(ii) \quad \hat{w}_1 \leq \hat{w}_2 \quad ; \quad (ii)' \quad \hat{w}_1 \leq M_1 \quad ;$$

$$(iii) \quad \hat{w}_2 \leq \hat{w}_3 \quad ; \quad (iii)' \quad \hat{w}_3 \geq M_3 \quad ;$$

$$(iv) \quad \hat{w}_2 \leq \hat{w}_4 \quad ; \quad (iv)' \quad \hat{w}_4 \geq M_4 \quad . \quad \dots(6.3.8)$$

In this case, Thompson obtained eight possibilities depending on whether equality does or does not hold in (ii), (iii) and (iv) of equation (6.3.8). Table 6.3.4 gives the various solutions and again they are found to be mutually exclusive showing that the maximum likelihood estimates are unique. The notation is that of equation (6.3.7).

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Table 6.3.4: Estimates of the Variance Components of Table 6.3.3 Under Eight Possible Conditions.

Conditions	$\hat{\sigma}_e^2$	$n\hat{\sigma}_\alpha^2$	$In\hat{\sigma}_\beta^2$	$Jn\hat{\sigma}_\alpha^2$
(i) $M_1 < M_2 < M_3, M_2 < M_4$	$M_1$	$M_2 - M_1$	$M_3 - M_2$	$M_4 - M_2$
(ii) $M_1 < M_{23} < M_4, M_2 \geq M_3$	$M_1$	$M_{23} - M_1$	0	$M_4 - M_{23}$
(iii) $M_1 < M_{24} < M_3, M_2 \geq M_4$	$M_1$	$M_{24} - M_1$	$M_3 - M_{24}$	0
(iv) $M_1 \geq M_2, M_{12} < M_3, M_{12} < M_4$	$M_{12}$	0	$M_3 - M_{12}$	$M_4 - M_{12}$
(v) $M_{123} < M_4, M_3 \leq M_{12}, M_{23} \leq M_1$	$M_{123}$	0	0	$M_4 - M_{123}$
(vi) $M_{124} < M_3, M_4 \leq M_{12}, M_{24} \leq M_1$	$M_{124}$	0	$M_3 - M_{124}$	0
(vii) $M_1 < M_{234}, M_3 \leq M_{24}, M_4 \leq M_{23}$	$M_1$	$M_{234} - M_1$	0	0
(viii) $M_3 \leq M_{124}, M_4 \leq M_{123}, M_1 \geq M_{234}$	$M_{1234}$	0	0	0

#### IV. Comparison of Estimators.

The four possible conditions of Table 6.3.2 - (i), (ii), (iii), and (iv) - correspond respectively with conditions (i), (ii), (iii), and (vii) of Table 6.3.4. We shall consider

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the estimators  $\hat{\sigma}_e^2$  and  $\hat{\sigma}_\alpha^2$  obtained under condition (ii) of both cases.

From Table 6.3.2,

$$\hat{\sigma}_e^2 = n(M_{12}) = n \frac{(f_1 M_1 + f_2 M_2)}{f_1 + f_2} ;$$

$$\hat{\sigma}_\alpha^2 = \frac{M_3 - M_{12}}{J} .$$

From Table 6.3.4,

$$\hat{\sigma}_e^2 = M_1 ;$$

$$\hat{\sigma}_\alpha^2 = \frac{M_4 - M_{23}}{Jn} ,$$

where  $M_{23} = \frac{f_2 M_2 + f_3 M_3}{f_2 + f_3} .$

The variances of these estimators are given in Table 6.3.5.

Table 6.3.5 : Variances of  $\hat{\sigma}_e^2$  and  $\hat{\sigma}_\alpha^2$  (Restricted Maximum Likelihood Method). Treatments and Environmental Effects are assumed to be Additive in Both Cases .

	$\text{Var}(\hat{\sigma}_e^2)$	$\text{Var}(\hat{\sigma}_\alpha^2)$
Average Yields Per Plot	$D_5 = \frac{2\sigma_e^4}{I(J-1)}$	$D_6 = \frac{2}{J^2 n^2} \left[ \frac{(\sigma_e^2 + Jn\sigma_\alpha^2)^2}{I-1} + \frac{\sigma_e^4}{I(J-1)} \right]$
Yields of Individual Plants on a Plot	$D_7 = \frac{2\sigma_e^4}{IJ(n-1)}$	$D_8 = \frac{2}{J^2 n^2} \left[ \frac{(\sigma_e^2 + Jn\sigma_\alpha^2)^2}{I-1} + \frac{\sigma_e^4}{I(J-1)} \right]$

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From Table 6.3.5,

$$D_7 < D_5$$

and

$$D_8 = D_6 \bullet$$

This situation is the same as that obtained in Section 6.2 (III). Accordingly, the inferences made in that section hold good for this section.

#### V. Comparison of the Analysis of Variance and Restricted Maximum Likelihood Methods

Examination of the estimates provided by the analysis of variance and restricted maximum likelihood methods indicates that the two methods are identical when the former gives positive estimates. For example, under condition (i) of Tables 6.3.2 and 6.3.4, the analysis of variance method gives positive estimates which are the same as those obtained by the restricted maximum likelihood method. The two methods are, however, different when the analysis of variance method gives negative estimates. What the restricted maximum likelihood method does in this situation is to assign the value zero to the otherwise negative estimates and to simultaneously adjust the values of the other estimates by pooling some of the mean squares.

We shall examine one of the conditions under which the

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analysis of variance method gives negative estimates. One such condition is (ii) of Tables 6.3.2 and 6.3.4.

(a) The Case of Using Average Yields Per Plot.

We shall assume that the treatments and environmental effects are additive. Let  $\hat{\sigma}_{e1a}^2$ ,  $\hat{\sigma}_{\alpha1a}^2$  and  $\hat{\sigma}_{\beta1a}^2$  be the analysis of variance estimators of  $\sigma_e^2/n$ ,  $\sigma_\alpha^2$ , and  $\sigma_\beta^2$ , respectively, when the average yields per plot are used.

Then under condition (ii),

$$\begin{aligned}\hat{\sigma}_{e1a}^2 &= MS_E \quad ; \\ \hat{\sigma}_{\alpha1a}^2 &= \frac{MS_V - MS_E}{J} \quad ; \\ \hat{\sigma}_{\beta1a}^2 &= \frac{MS_B - MS_E}{I} \quad .\end{aligned}$$

Their respective variances are

$$\begin{aligned}\text{(i)} \quad \text{Var}(\hat{\sigma}_{e1a}^2) &= \frac{2\sigma_e^4}{n^2(I-1)(J-1)} \quad ; \\ \text{(ii)} \quad \text{Var}(\hat{\sigma}_{\alpha1a}^2) &= \frac{2}{J^2n^2} \left[ \frac{(\sigma_e^2 + Jn\sigma_\alpha^2)^2}{I-1} + \frac{\sigma_e^4}{(I-1)(J-1)} \right] .\end{aligned}$$

Let  $\hat{\sigma}_{e1r}^2$ ,  $\hat{\sigma}_{\alpha1r}^2$  and  $\hat{\sigma}_{\beta1r}^2$  be the restricted maximum likelihood estimators of  $\sigma_e^2/n$ ,  $\sigma_\alpha^2$  and  $\sigma_\beta^2$  when the average yields per plot are used.

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Then under condition (ii),

$$\hat{\sigma}_{e1r}^2 = M_{12} ;$$

$$\hat{\sigma}_{\beta1r}^2 = 0 ;$$

$$\hat{\sigma}_{\alpha1r}^2 = \frac{M_3 - M_{12}}{J} .$$

Their respective variances are

$$(i) \quad \text{Var}(\hat{\sigma}_{e1r}^2) = \frac{2\sigma_e^4}{n^2 I(J-1)} ;$$

$$(ii) \quad \text{Var}(\hat{\sigma}_{\alpha1r}^2) = \frac{2}{J^2 n^2} \left[ \frac{(\sigma_e^2 + Jn\sigma_\alpha^2)^2}{I-1} + \frac{\sigma_e^4}{I(J-1)} \right] .$$

Comparing the four variances given above, we observe that

$$\left. \begin{array}{l} (i) \quad \text{Var}(\hat{\sigma}_{e1r}^2) < \text{Var}(\hat{\sigma}_{e1a}^2) ; \\ (ii) \quad \text{Var}(\hat{\sigma}_{\alpha1r}^2) < \text{Var}(\hat{\sigma}_{\alpha1a}^2) . \end{array} \right] \quad \dots\dots(6.3.9)$$

(b) The Case of Using Yields of the Individual Plants on a Plot.

Considering yields of the individual plants on a plot, and using the notation in (a), the analysis of variance method provides the following estimators under condition (ii):

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$$(i) \quad \hat{\sigma}_{e2a}^2 = MS_W \quad ;$$

$$(ii) \quad \hat{\sigma}_{\alpha 2a}^2 = \frac{MS_V - MS_{VB}}{Jn} \quad ;$$

$$(iii) \quad \hat{\sigma}_{\gamma 2a}^2 = \frac{MS_{VB} - MS_W}{n} \quad .$$

Their respective variances are

$$(i) \quad \text{Var}(\hat{\sigma}_{e2a}^2) = \frac{2\sigma_e^4}{IJ(n-1)} \quad ;$$

$$(ii) \quad \text{Var}(\hat{\sigma}_{\alpha 2a}^2) = \frac{2}{J^2 n^2} \left[ \frac{(\sigma_e^2 + n\sigma_\alpha^2 + Jn\sigma_\alpha^2)^2}{I-1} + \frac{(\sigma_e^2 + n\sigma_\gamma^2)^2}{(I-1)(J-1)} \right] ;$$

$$(iii) \quad \text{Var}(\hat{\sigma}_{\gamma 2a}^2) = \frac{2}{n^2} \left[ \frac{(\sigma_e^2 + n\sigma_\gamma^2)^2}{(I-1)(J-1)} + \frac{\sigma_e^4}{IJ(n-1)} \right] .$$

For the same situation, the restricted maximum likelihood method provides the following estimators under condition (ii):

$$(i) \quad \hat{\sigma}_{e2r}^2 = M_1 \quad ;$$

$$(ii) \quad \hat{\sigma}_{\alpha 2r}^2 = \frac{M_4 - M_{23}}{Jn} \quad ;$$

$$(iii) \quad \hat{\sigma}_{\gamma 2r}^2 = \frac{M_{23} - M_1}{n} \quad .$$

Their respective variances are

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$$\begin{aligned}
 \text{(i)} \quad \text{Var}(\hat{\sigma}_{e2r}^2) &= \frac{2\sigma_e^4}{IJ(n-1)} ; \\
 \text{(ii)} \quad \text{Var}(\hat{\sigma}_{\alpha 2r}^2) &= \frac{2}{J^2 n^2} \left[ \frac{(\sigma_e^2 + n\sigma_\gamma^2 + Jn\sigma_\alpha^2)^2}{I-1} + \frac{(\sigma_e^2 + n\sigma_\gamma^2)^2}{I(J-1)} \right] ; \\
 \text{(iii)} \quad \text{Var}(\hat{\sigma}_{\gamma 2r}^2) &= \frac{2}{n^2} \left[ \frac{(\sigma_e^2 + n\sigma_\gamma^2)^2}{I(J-1)} + \frac{\sigma_e^4}{IJ(n-1)} \right] .
 \end{aligned}$$

Comparing the six variances given above, we see that

$$\begin{aligned}
 \text{(i)} \quad \text{Var}(\hat{\sigma}_{e2r}^2) &= \text{Var}(\hat{\sigma}_{e2a}^2) ; \\
 \text{(ii)} \quad \text{Var}(\hat{\sigma}_{\alpha 2r}^2) &< \text{Var}(\hat{\sigma}_{\alpha 2a}^2) ; \\
 \text{(iii)} \quad \text{Var}(\hat{\sigma}_{\gamma 2r}^2) &< \text{Var}(\hat{\sigma}_{\gamma 2a}^2) .
 \end{aligned} \quad \dots (6.3.10)$$

The results shown in equations (6.3.9) and (6.3.10) indicate that the restricted maximum likelihood estimators are more efficient than those of the analysis of variance. Thompson has, however, shown that the estimators may be biased due to the pooling of the mean squares. Nevertheless, the restricted maximum likelihood method is more plausible than the analysis of variance method when the latter gives negative estimates. Biased estimates are more tolerable than negative

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estimates of essentially positive parameters since no suitable explanation can be offered for the negative estimates.

#### 6.4. Estimation of Variance Components for the "Kpong Data"

Assuming the random-effects model, the variance components and their estimates for the cases of using the average yields per plot and yields of the individual plants on a plot, employing the analysis of variance and restricted maximum likelihood methods, are those given in Sections 6.2 and 6.3.

##### I. The Analysis of Variance Method.

Employing the analysis of variance method, the estimators using the average yields per plot are those given by equations (6.2.1) and (6.2.2). From Table 3.6.2 therefore, we obtain

$$\begin{aligned} \text{(i)} \quad \hat{\sigma}_e^2 / 5 &= \underline{86.466} \quad ; \\ \text{(ii)} \quad \hat{\sigma}_\beta^2 &= \frac{22.559 - 86.466}{21} = \underline{-3.043} \quad ; \\ \text{(iii)} \quad \hat{\sigma}_\alpha^2 &= \frac{944.993 - 86.466}{3} = \underline{286.176} \quad . \end{aligned}$$

Making use of equations (6.1.1), (6.1.2), and (6.1.3), the standard errors of the estimates and the 95% confidence intervals for the variance components are:

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- (i)  $S.E.(\hat{\sigma}_e^2/5) = 18.868$  ,  
 $58.105 \leq \sigma_e^2/5 \leq 141.631$  ;
- (ii)  $S.E.(\hat{\sigma}_\alpha^2) = 95.183$  ,  
 $125.527 \leq \sigma_\alpha^2 \leq 689.373$  ;
- (iii)  $S.E.(\hat{\sigma}_\beta^2) = 1.117$  ,  
 $-3.849 \leq \sigma_\beta^2 \leq 38.248$  .

When yields of the individual plants on a plot are used, the estimators provided by the analysis of variance method are those given by equations (6.2.3) and (6.2.4). From Table 3.6.4 therefore, we obtain

- (i)  $\hat{\sigma}_e^2 = 131.208$  ;
- (ii)  $\hat{\sigma}_\beta^2 = \frac{112.980 - 423.984}{21 \times 5} = \underline{-2.962}$  ;
- (iii)  $\hat{\sigma}_\alpha^2 = \frac{4741.839 - 423.984}{3 \times 5} = \underline{287.857}$  ;
- (iv)  $\hat{\sigma}_\gamma^2 = \frac{423.984 - 131.208}{5} = \underline{58.555}$  .

From equations (6.1.1), (6.1.2), and (6.1.3), the standard errors of the estimates and the 95% confidence intervals for the true parameters are:

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$$(i) \quad \text{S.E.}(\hat{\sigma}_e^2) = 11.643 ;$$

Confidence limits coincide with its value

$$(ii) \quad \text{S.E.}(\hat{\sigma}_\beta^2) = 1.355 ,$$

$$-3.772 \leq \sigma_\beta^2 \leq 38.397 ;$$

$$(iii) \quad \text{S.E.}(\hat{\sigma}_\alpha^2) = 95.514 ,$$

$$126.634 \leq \sigma_\alpha^2 \leq 692.494 ;$$

$$(iv) \quad \text{S.E.}(\hat{\sigma}_\gamma^2) = 18.650 ,$$

$$30.742 \leq \sigma_\gamma^2 \leq 112.654 .$$

We observe from the above estimates that estimates of  $\sigma_\beta^2$  are negative. A remark on the possibility of such a situation occurring was made in Section 6.2 (II).

Estimates of  $\sigma_\alpha^2$  and  $\sigma_\beta^2$  using yields of the individual plants on a plot are slightly bigger than the corresponding ones when the average yields per plot are used. The same is not true for  $\sigma_e^2$  - the value of  $\hat{\sigma}_e^2$  for the latter case is more than three times the corresponding value for the former case. This has been explained in Section 3.6. (II).

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## II. The Restricted Maximum Likelihood Method.

The estimators provided by the restricted maximum likelihood method using the average yields per plot are those given in Table 6.3.2. From Table 3.6.2,

$$M_1 = 86.466; \quad M_2 = 22.559; \quad M_3 = 944.993 .$$

This is condition (ii) of Table 6.3.2. Hence, the estimates are:

$$(i) \quad \hat{\sigma}_e^2/n = M_{12} = \frac{f_1 M_1 + f_2 M_2}{f_1 + f_2} ;$$

$$\text{Hence } \hat{\sigma}_e^2/5 = \frac{40 \times 86.466 + 2 \times 22.559}{40 + 2} = \underline{83.375} ;$$

$$(ii) \quad I\hat{\sigma}_\beta^2 = 0. \quad \text{Hence } \hat{\sigma}_\beta^2 = 0 ;$$

$$(iii) \quad J\hat{\sigma}_\alpha^2 = M_3 - M_{12} ;$$

$$\text{Hence } \hat{\sigma}_\alpha^2 = \frac{M_3 - M_{12}}{J} = \frac{944.993 - 83.375}{3} = \underline{287.206} .$$

The standard errors and 95% confidence intervals are:

$$(i) \quad \text{S.E.}(\hat{\sigma}_e^2/5) = 17.776 ,$$

$$56.445 \leq \sigma_e^2/5 \leq 135.318 ;$$

$$(ii) \quad \text{S.E.}(\hat{\sigma}_\alpha^2) = 95.169 ,$$

$$127.502 \leq \sigma_\alpha^2 \leq 688.533 ;$$

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When yields of the individual plants on a plot are used, the estimators provided by the restricted maximum likelihood method are those given in Table 6.3.4. From Table 3.6.4,

$$M_1 = 131.208; \quad M_2 = 423.984; \quad M_3 = 112.980; \quad M_4 = 4741.839.$$

This is condition (ii) of Table 6.3.4. Hence the estimates are:

$$(i) \quad \hat{\sigma}_e^2 = M_1 = \underline{131.208} \quad ;$$

$$(ii) \quad n\hat{\sigma}_\gamma^2 = M_{23} - M_1 \quad ,$$

$$\text{where } M_{23} = \frac{f_2 M_2 + f_3 M_3}{f_2 + f_3} = \frac{16959.345 + 225.960}{40 + 2} = 409.174 \quad ;$$

$$\text{Hence } \hat{\sigma}_\gamma^2 = \frac{409.174 - 131.208}{5} = \underline{55.593} \quad .$$

$$(iii) \quad \ln\hat{\sigma}_\beta^2 = 0. \quad \text{Hence } \hat{\sigma}_\beta^2 = 0 \quad ;$$

$$(iv) \quad Jn\hat{\sigma}_\alpha^2 = M_4 - M_{23} \quad ;$$

$$\text{Hence } \hat{\sigma}_\alpha^2 = \frac{4741.839 - 409.174}{3 \times 5} = \underline{288.844} \quad .$$

The standard errors and 95% confidence intervals are:

$$(i) \quad \text{S.E.}(\hat{\sigma}_e^2) = 11.643 \quad ;$$

The confidence limits coincide with its value.

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$$(ii) \quad S.E.(\hat{\sigma}_\gamma^2) = 17.602,$$

$$29.161 \leq \sigma_\gamma^2 \leq 106.576 ;$$

$$(iii) \quad S.E.(\hat{\sigma}_\alpha^2) = 95.492,$$

$$128.570 \leq \sigma_\alpha^2 \leq 691.584 .$$

Estimates of  $\sigma_\alpha^2$  and  $\sigma_e^2$  for the cases of average yields per plot and yields of individual plants on a plot follow the same trend as obtained in Section 6.4 (I).

With the notation of Section 5.4, we have

$$(i) \quad \hat{\sigma}_{e1a}^2 = 86.466, \quad \hat{\sigma}_{\alpha1a}^2 = 286.176, \quad \hat{\sigma}_{\beta1a}^2 = -3.043 ;$$

$$(ii) \quad \hat{\sigma}_{e1r}^2 = 83.375, \quad \hat{\sigma}_{\alpha1r}^2 = 287.206, \quad \hat{\sigma}_{\beta1r}^2 = 0 ;$$

$$(iii) \quad \hat{\sigma}_{e2a}^2 = 131.208, \quad \hat{\sigma}_{\alpha2a}^2 = 287.857, \quad \hat{\sigma}_{\beta2a}^2 = -2.962, \quad \hat{\sigma}_{\gamma2a}^2 = 58.555 ;$$

$$(iv) \quad \hat{\sigma}_{e2r}^2 = 131.208, \quad \hat{\sigma}_{\alpha2r}^2 = 288.844, \quad \hat{\sigma}_{\beta2r}^2 = 0, \quad \hat{\sigma}_{\gamma2r}^2 = 55.593 .$$

Thus the restricted maximum likelihood method has eliminated the objectionable negative estimates while not causing any major changes in the other estimates as compared with those obtained by the analysis of variance method.

## CHAPTER SEVEN

### PSEUDO-FACTORIAL ARRANGEMENT

The pseudo-factorial arrangement, proposed by Yates (44), is a way of ensuring increased efficiency in an agricultural field experiment with considerable soil fertility fluctuations. It increases the efficiency by avoiding the use of excessively large blocks while at the same time avoiding the use of some of the treatments as controls.

In this type of arrangement, the treatments are divided into sets for comparison in more than one way, the sets of each division being so arranged that they cut across those of all the other divisions. Thus 64 treatments, numbered 00 - 63, may be divided into sets of 8 in two ways, the first group of 8 sets consisting of treatments 00 - 07, 08 - 15, 16 - 23, etc., and the second group of 8 sets consisting of treatments 00, 08, 16, .....48; 01, 09, 17, .....49, 57; 02, 10, 18, .....50, 58; etc. Each set of 8 can be arranged in the field in the form of one or more randomised blocks of 8 plots each, or in the form of a 8 x 8 Latin square, according to the number of replications that are feasible.

#### Pseudo-Factorial Arrangement in Two Equal Groups of Sets.

According to Yates, two equal groups of sets are possible if the number of treatments is a perfect square, say  $p^2$ . He

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defined the following:

$X_{ij}$  = mean yield of the treatment in the  $i^{\text{th}}$  set of the first group and the  $j^{\text{th}}$  set of the second group over the replicates of the first group of sets;

$Y_{ij}$  = mean yield of the treatment in the  $i^{\text{th}}$  set of the first group and the  $j^{\text{th}}$  set of the second group over the replicates of the second group of sets.

With the above notation, the two groups may be set up as in Table 7.1.

Table 7.1: Mean Yields and Marginal Means of Each Group of Sets.

FIRST GROUP				SECOND GROUP							
Set	1	2	.....	p	Mean	Set	1	2	.....	p	Mean
	$X_{11}$	$X_{21}$	.....	$X_{p1}$	$\bar{X}_{.1}$	1	$Y_{11}$	$Y_{21}$	.....	$Y_{p1}$	$\bar{Y}_{.1}$
	$X_{12}$	$X_{22}$	.....	$X_{p2}$	$\bar{X}_{.2}$	2	$Y_{12}$	$Y_{22}$	.....	$Y_{p2}$	$\bar{Y}_{.2}$
	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
	$X_{1p}$	$X_{2p}$	.....	$X_{pp}$	$\bar{X}_{.p}$	p	$Y_{1p}$	$Y_{2p}$	.....	$Y_{pp}$	$\bar{Y}_{.p}$
Mean	$\bar{X}_{1.}$	$\bar{X}_{2.}$	...	$\bar{X}_{p.}$	$\bar{X}_{..}$	Mean	$\bar{Y}_{1.}$	$\bar{Y}_{2.}$	...	$\bar{Y}_{p.}$	$\bar{Y}_{..}$

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According to Yates, such an experimental arrangement in two equal groups of sets is analogous to a factorial experiment (7,14) involving two factors (each with  $p$  values) in which the main effects of one factor are confounded in one half (the first group) of the replications and the main effects of the other factor are confounded in the other half (the second group).

He obtained the adjusted treatment means as

$$t_{ij} = \frac{1}{2}(X_{ij} + Y_{ij} - \bar{X}_{i\cdot} + \bar{X}_{\cdot j} + \bar{Y}_{i\cdot} - \bar{Y}_{\cdot j})$$

Differences between the adjusted values  $t_{ij}$  will give efficient estimates of the treatment differences freed from block effects. He obtained the variance of the difference of two such adjusted means.

The analysis of variance table for such an experiment has been given by him (Yates). For the case of  $r$  randomised blocks, he partitioned the degrees of freedom in the analysis of variance in the way shown in Table 7.2.

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Table 7.2 : Partition of Degrees of Freedom for a Pseudo-Factorial Arrangement in Two Groups of Sets in  $r$  Randomised Blocks.

Source	df													
BLOCKS	<table border="0"> <tr> <td>Between Groups</td> <td>1</td> <td rowspan="5">Pr - 1</td> </tr> <tr> <td rowspan="2">Between Sets</td> <td>Group I</td> <td>p-1</td> </tr> <tr> <td>Group II</td> <td>p-1</td> </tr> <tr> <td rowspan="2">Within Sets</td> <td>Group I</td> <td><math>p(\frac{1}{2}r - 1)</math></td> </tr> <tr> <td>Group II</td> <td><math>p(\frac{1}{2}r - 1)</math></td> </tr> </table>	Between Groups	1	Pr - 1	Between Sets	Group I	p-1	Group II	p-1	Within Sets	Group I	$p(\frac{1}{2}r - 1)$	Group II	$p(\frac{1}{2}r - 1)$
Between Groups	1	Pr - 1												
Between Sets	Group I		p-1											
	Group II		p-1											
Within Sets	Group I		$p(\frac{1}{2}r - 1)$											
	Group II		$p(\frac{1}{2}r - 1)$											
TREATMENTS	<table border="0"> <tr> <td>First factor</td> <td>p-1</td> <td rowspan="3"><math>p^2 - 1</math></td> </tr> <tr> <td>Second factor</td> <td>p-1</td> </tr> <tr> <td>Interactions</td> <td><math>(p-1)^2</math></td> </tr> </table>	First factor	p-1	$p^2 - 1$	Second factor	p-1	Interactions	$(p-1)^2$						
First factor	p-1	$p^2 - 1$												
Second factor	p-1													
Interactions	$(p-1)^2$													
ERROR	<table border="0"> <tr> <td>Between Groups</td> <td><math>(p-1)^2</math></td> <td rowspan="3"><math>(p-1)(p_r - p - 1)</math></td> </tr> <tr> <td>Within Group I</td> <td><math>p(p-1)(\frac{1}{2}r - 1)</math></td> </tr> <tr> <td>Within Group II</td> <td><math>p(p-1)(\frac{1}{2}r - 1)</math></td> </tr> </table>	Between Groups	$(p-1)^2$	$(p-1)(p_r - p - 1)$	Within Group I	$p(p-1)(\frac{1}{2}r - 1)$	Within Group II	$p(p-1)(\frac{1}{2}r - 1)$						
Between Groups	$(p-1)^2$	$(p-1)(p_r - p - 1)$												
Within Group I	$p(p-1)(\frac{1}{2}r - 1)$													
Within Group II	$p(p-1)(\frac{1}{2}r - 1)$													
TOTAL	$p_r^2 - 1$													

He obtained the various sums of squares as follows:

$$\text{First factor: } M = \frac{1}{2}r(p\sum \bar{Y}_{i.}^2 - p^2\sum \bar{Y}_{..}^2) ;$$

$$\text{Second factor; } N = \frac{1}{2}r(p\sum \bar{X}_{.j}^2 - p^2\sum \bar{X}_{..}^2) ;$$

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$$\text{Interactions: } S = \frac{1}{4}r \left[ \sum_{ij} (X_{ij} + Y_{ij})^2 - p \sum (\bar{X}_{i.} + \bar{Y}_{i.})^2 - p \sum (\bar{X}_{.j} + \bar{Y}_{.j})^2 + p^2 (\bar{X}_{..} + \bar{Y}_{..})^2 \right] ;$$

$$\text{Sum of Squares for sets and Groups : } R = \frac{1}{2}r \left[ p \sum \bar{X}_{i.}^2 + \sum \bar{Y}_{.j}^2 - \frac{1}{2}p^2 (\bar{X}_{..} + \bar{Y}_{..})^2 \right] ;$$

$$\text{Put } H = M + N + S + R = \frac{1}{4}r \left[ \sum_{ij} (X_{ij} + Y_{ij})^2 + p \sum_i (\bar{X}_{i.} - \bar{Y}_{i.})^2 + p \sum_j (\bar{X}_{.j} - \bar{Y}_{.j})^2 - p^2 (\bar{X}_{..} - \bar{Y}_{..})^2 - p^2 (\bar{X}_{..} + \bar{Y}_{..})^2 \right]$$

$$\text{Total Sum of Squares between all X and Y : } K = \frac{1}{2}r \left[ \sum_{ij} X_{ij}^2 + \sum_{ij} Y_{ij}^2 - p^2 (\bar{X}_{..} + \bar{Y}_{..})^2 \right]$$

Hence, Treatment Sum of Squares:  $SS_V = H - R$  ;

Sum of Squares for Error:  $SS_E = K - H$  .

To test for treatment differences, the mean squares for treatments and error are compared by means of the Z-test.

The cases of two-dimensional pseudo-factorial arrangements in three equal groups of sets forming a Latin square, three-dimensional pseudo-factorial arrangements in two unequal groups

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of sets, and three-dimensional pseudo-factorial arrangements in three unequal groups of sets have also been discussed by Yates.

From the efficiencies of the various arrangements relative to that of ordinary randomised blocks, he found out that pseudo-factorial arrangements are the most efficient when there are considerable soil fertility fluctuations. Furthermore, three-dimensional arrangements are likely to be most advantageous with a larger number of treatments and in cases in which there are sufficient replications for the randomised blocks to be replaced by Latin squares.

## CHAPTER EIGHT

### CONCLUSION

This thesis has shown that, for an agricultural field experiment involving treatment comparisons, if one uses a randomised block design with  $n$  plants per plot the following will be the results:

I. Under both the fixed-effects and random-effects models, analysing the experiment using yields of the individual plants on a plot is more efficient than a corresponding analysis using the average yields per plot. In particular, because the error variance is efficiently estimated,

- (i) the F test for treatment differences is more sensitive;
- (ii) the standard error of the difference between any two treatment means is smaller and thus a corresponding multiple comparisons of the treatment means is more efficient.

II. When there are three or more blocks, an F test for treatment differences under randomization models is more sensitive than the corresponding F test under normal-theory models.

III. As far as testing of treatment differences and multiple comparisons of means are concerned, a multivariate analysis of variance using measurements of some correlated variates may produce better results than a corresponding analysis using yields of the individual plants on a plot. In fact, of all the multiple comparisons of means treated in this thesis, the one resulting from the correlated variates was the least ambiguous.

This thesis has also confirmed the findings that of all the multiple comparisons of means procedures considered, Bayes exact test (BET) and Fisher's significant difference (FSD) are the best. A reasonable alternative to any of these two is Duncan's multiple range test (MRT).

We saw further that, in estimating variance components, the restricted maximum likelihood estimators are more efficient than those of the analysis of variance. They may, however, produce biased estimates. In spite of this, we noted that they are preferable to those of the analysis of variance since they avoid negative estimates.

We also identified a type of arrangement, called pseudo-factorial, which could be used in conjunction with the randomised block and Latin square designs to increase efficiency when there are many treatments and considerable soil fertility fluctuations.

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With regard to the "Kpong data", if no variance components estimation is required, the most efficient method of comparing treatments is to use measurements of the correlated variates to perform a multivariate analysis of variance. In the subsequent multiple comparisons of means, Duncan's multiple range test (MRT) is the appropriate procedure to use. But if variance components estimation is required, then one may use yields of the individual plants on a plot. In this case, Bayes exact test (BET) and Fisher's significant difference (FSD) are the best procedures to use in the multiple comparisons of means.

Generally, for an agricultural field experiment involving treatment comparisons,

(a) one may use a randomised block design with  $N$  plants per plot if the treatments are not many. Of the  $N$  plants on a plot, if it is necessary to take a sample of  $n$  plants ( $n < N$ ), the selection must be random. Efforts should be made to use the optimum sampling size and the optimum number of replications. In this situation, the interaction mean square is the appropriate error mean square to use in the analysis.

In analysing the experiment,

(i) one may take measurements of some correlated variates to perform a multivariate analysis of variance. This approach is advantageous to those experimenters who will go on further to do discriminant and canonical analysis. It is not necessary

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to measure the actual yield as long as the yield is a function of the correlated variates.

(ii) If it is not possible to take measurements of some correlated variates, one may use yields of the individual plants on a plot to perform a univariate analysis of variance.

(iii) If in using yields of the individual plants on a plot, the F test for treatment differences just misses significance, one may use the average yields per plot to perform a permutation test provided there are enough blocks.

(b) If there are many treatments and there is evidence of high soil fertility fluctuations, the pseudo-factorial type of arrangement may be employed in conjunction with the randomised block design. The average yields per plot may then be used with the formulae given by Yates (44). In such situations, the blocking should efficiently be done.

(c) A multiple comparisons of means may be performed by using either Bayes exact test (BET) or Fisher's significant difference (FSD) with  $\alpha = 0.05$ . Where any of these is not applicable, Duncan's multiple range test (MRT) will be a suitable substitute.

(d) If variance components estimation is required, the restricted maximum likelihood method is quite appropriate.

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