Analytic Solution of Barotropic Wind Forced Ocean Gyre Circulation

By

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DECLARATION

This thesis was written in the Department of Mathematics, University of Ghana, Legon from March 2024 to February 2025 in partial fulfilment of the requirements for the award of Master of Philosophy degree in Mathematics under the supervision of Dr. Joseph K. Ansong of the University of Ghana and Dr. Dimitris Menemenlis.

I hereby declare that except where due acknowledgement is made, this work has never been presented wholly or in part for the award of a degree at the University of Ghana or any other University.

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ABSTRACT

In this study, we provide analytic solution of barotropic wind forced ocean gyre circulation. The governing equations are solved analytically using vorticity-stream function together with boundary conditions.

To obtain the analytic solution, we first establish the governing equations of our model by making shallow water assumption on Navier-Stokes equations. Then, the momentum equations of the governing equations are combined to produce vorticity equation by a cross partial derivatives of the momentum equations.

A stream function is defined so as to write the vorticity equation as a fourth-order equation in terms of only the stream function. The solution of the stream function is obtained by the assumptions that streamlines at the boundary are chosen to be zero and that, there are no slippages against the boundary. Finally, the solution of free surface height is established to vary directly with the stream function.

The analytic solution, just like the numerical solution obtained by running the Massachusetts Institute of Technology General Circulation Model(MITgcm), shows western intensification of streamlines, a result that is due to the effect of Coriolis force.

Moreover, different values of the Coriolis force in both the analytic and numerical solution show that the intensification of streamlines at the western boarder is relatively intense with higher Coriolis force.

DEDICATION

To my late father, M'ba Musah, my lovely mom, M'ma Kande, my late grandmother, Hajia Afishaata, and to Alhaji Mohammed Gombila .

ACKNOWLEDGEMENTS

Guided by the verse in Qur'an chapter 14 verse 7, where Almighty Allah says "...if you are grateful, I would certainly give to you more,..", I have at all times appreciated even the little help offered me by anyone.

With humility, I am most grateful to my supervisor: Dr. Joseph K. Ansong for not just his guidance, but his patience and prompt suggestions given to me from the beginning all the way to the finish of this work. Dr. Ansong, but for your invaluable guidance, this work would not have been a reality. Thank you very much. Your supervision of my work has offered me a great learning opportunity. I would like to thank Dr. Demitiris Menemenlis, co-supervisor for this thesis work. I thank you for the partnership with Dr. Ansong in supervising this work.

With gratitude, I praise Almighty Allah for granting me the life, health and time needed to work on this thesis. I say "Al hamdu lil Laah".

Contents

Chapter 1

Introduction And Motivation

The Earth's climatic system is an interesting aspect of nature that needs to be understood by man. In an attempt to better appreciate the Earth's climate system, one can not overlook the interplay between atmospheric and oceanic processes. Of the numerous interactions, wind forced ocean gyre circulation is at the core of energy transfer and global climatic modulation. This study is an exploration of the analytic solution to wind forced barotropic ocean gyre circulation. In addition, this thesis will strive to draw comparison between the analytic solution [7] and observed oceanic behaviors.

A gyre is a system of rotating currents in ocean. [14] defines it as a component of system of circular currents in the ocean, that is due to the effect of global wind patterns and forces created by rotation of the earth. Simply put, it is the movement of water in circular motion. It is the movement of ocean currents in circular path, either clockwise or counter clockwise.

Barotropic fluids are characterized by homogeneous density. Under barotropic conditions, lines of constant density do not meet lines of constant pressure and so both isobaric and isopycnic surfaces are parallel to the sea surface [3].

Stommel's 1945 [10] and Munk's 1950 [8] analytic accounts of the circulation problem provides a foundation of the inquiry this thesis seeks to find. With the mathematical framework provided by Stommel [10], the basic principles guiding ocean circulation patterns was established and as such, laid the foundation for proper appreciation of dynamics of ocean gyres. In 1950, Walter H. Munk improved upon the work of Sverdrup [11],using a baroclinic model. This contribution by Munk [8], enhanced the appreciation of the impact of external forces such as wind on oceanic circulation.

1.1 Motivation

The gyre circulation problem described analytically in 1948 by [10] and by [8] in 1950 and numerically by [2] in 1963 provides the motivation for experiment of the simulation [7] using Massachusetts Institute of Technology General circulation Model(MITgcm).

The Coriolis parameter f, is defined [7] to be linearly dependent on latitude y as

$$f(y) = f_0 + \beta y, \tag{1.1}$$

where y is the distance along the the "north-south" axis of the simulated domain. f_0 set to $10^{-4}s^{-1}$ and β set to $10^{-11}s^{-1}m^{-1}$ [7]. The wind-stress variation is defined as

$$\tau_x(y) = -\tau_0 \cos\left(\frac{\pi y}{L_y}\right) \tag{1.2}$$

where L_y is width of the domain and τ_0 is set to $0.1Nm^{-2}$ [7]. The simulated configuration is seen in figure (1.1). And the set of equations for the configuration as provided in [7] are

$$\frac{Du}{Dt} - fv + g\frac{\partial \eta}{\partial x} - A_h\nabla_h^2 u = \frac{\tau_x}{\rho_c H} \tag{1.3}$$

$$\frac{Dv}{Dt} + fu + g\frac{\partial \eta}{\partial y} - A_h\nabla_h^2 v = 0 \tag{1.4}$$

$$\frac{\partial \eta}{\partial t} + \nabla_h.(H\vec{\mathbf{u}}) = 0 \tag{1.5}$$

where u is the x component of the flow vector $\vec{\mathbf{u}}$ and v is the y component of the flow vector $\vec{\mathbf{u}}$, η is the free surface height, A_h the horizontal Laplacian viscosity, ρ_c is density of the fluid, and g the acceleration due to gravity [7].

The solution of free-surface height η at time $t = 3$ years is as shown in figure (1.2).

A linearized problem is obtained by replacing $\frac{Du}{Dt}$ with $\frac{\partial u}{\partial t}$ in equation (1.3) and $\frac{Dv}{Dt}$

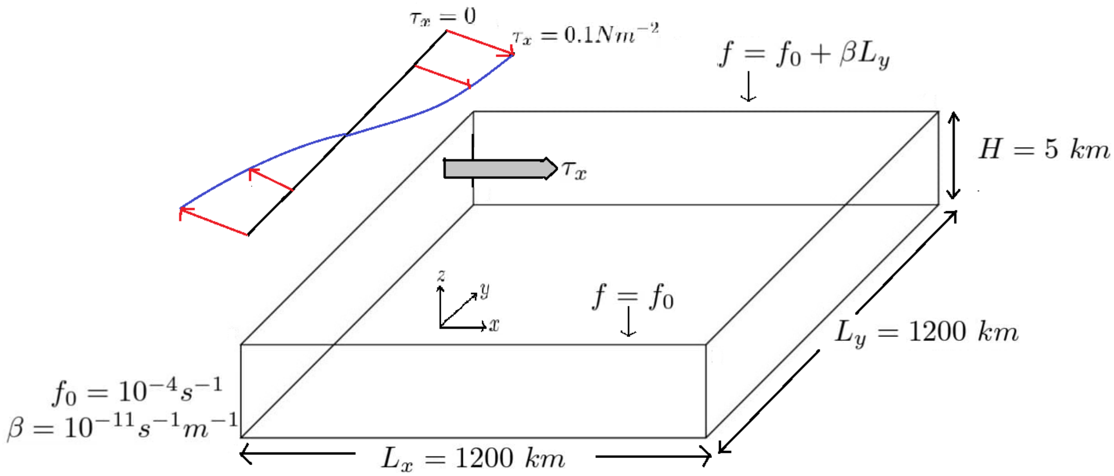

Figure 1.1: Illustration of domain for numerical experiment with effect of wind-stress forcing.

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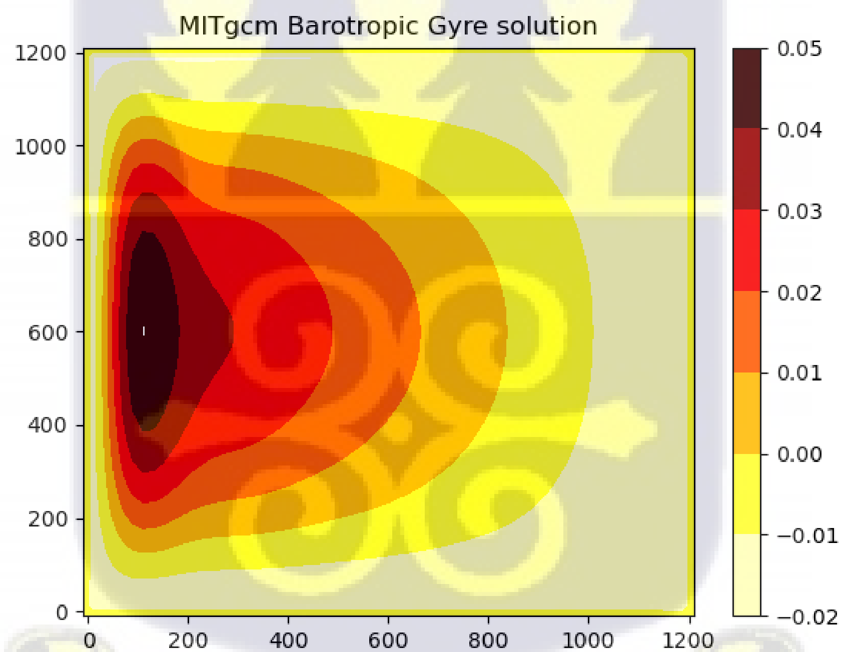

Figure 1.2: Numerical solution of barotropic gyre experiment obtained by using MITgcm .

with $\frac{\partial v}{\partial t}$ in equation (1.4). For this linearized equations, an analytical solution [7] for the free surface height η is

$$\eta(x,y) = \frac{\tau_0}{\rho_c g H}\frac{f}{\beta}\left(1-\frac{x}{L_x}\right)\pi\sin\left(\pi\frac{y}{L_y}\right)\left[1-\exp\left(\frac{-x}{2\delta_m}\right)\left(\cos\frac{\sqrt{3}x}{2\delta_m}+\frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}x}{2\delta_m}\right)\right] \tag{1.6}$$

where $\delta_m = \left(\frac{A_h}{\beta}\right)^{1/3}$

1.2 Goals And Objectives

The main objective of this project is to apply existing theories to understand wind-forced barotropic gyre. The specific objectives include:

1. To derive the set of equations (1.3), (1.4) and (1.5) for the barotropic gyre circulation.
2. To derive the analytic solution (1.6) from equations (1.3), (1.4) and (1.5).
3. To establish similarity between the analytic solution (1.6) and the numerical solution obtained by the MITgcm.

Chapter 2

Literature Review

Generally, in oceanic wind-driven circulation, streamlines along the western boundaries of the ocean are observed to be overly crowded [10]. This occurrence, commonly referred to as Westward Intensification is exemplified by the Kuroshio, the Gulf Stream, and the Agulhas Currents [10]. The explanation behind this crowding of streamlines along the Western Boundaries of ocean basins, has been a concern for scientists. An analytic description of the phenomenon was provided by Henry Stommel in 1948 [10] and by Walter H. Munk in 1950 [8]. A numerical solution of this phenomenon has been obtained by using the Massachusetts Institute of Technology General Circulation Model (MITgcm) [7].

In an attempt to explain the wind-driven circulation of the ocean, Stommel [10] used basic analysis to explain the physical parameter responsible for the western intensification. [10] formulated the problem by modeling the ocean as a rectangular ocean whereby the southwest corner of the Cartesian plain serves as the origin, with the x-axis and y-axis pointing eastward and northward respectively. [10] also considered the ocean to be homogeneous such that, at rest it's depth is constant at $(\mathbf{D})$. The boundaries of the basin [10] are chosen to be

$$x = 0, x = \lambda$$

and

$$y = 0, y = b.$$

With free surface height (h), which is relatively smaller than $(\mathbf{D})$, the total depth of the water column is given as $\mathbf{D} + h$.

[10] considered the effects of Trade winds and westerlies near the equator of the domain and the pole-ward half as the winds over the ocean respectively. The effect of wind stress [10] is represented by

$$-F\cos\left(\frac{\pi y}{b}\right).$$

[10] also introduced a functional dissipative term so as to prevent the ocean from accelerating. Also, components of the frictional force are chosen to be $(-\mathbf{R}u)$ and $(-\mathbf{R}v)$, where $(\mathbf{R})$ is the coefficient of friction, u is the x-component of the velocity vector while u is the y-component of the velocity vector. Representation of Coriolis parameter (f), as a linear function of y is introduced.

By omitting inertial terms, [10] provided the equations of motion for steady state as:

$$f(D+h)v - F\cos\left(\frac{\pi y}{b}\right) - Ru = g\,(D+h)\,\frac{\partial h}{\partial x} \tag{2.1}$$

$$-f(D+h)u - Rv = g\,(D+h)\,\frac{\partial h}{\partial y} \tag{2.2}$$

where $-g\,(D+h)\,\frac{\partial h}{\partial x}$ and $-g\,(D+h)\,\frac{\partial h}{\partial y}$ are respectively the zonal and meridional pressure gradients due to surface height [10].

Cross-differentiating (2.1) and (2.2), and using the continuity equation (2.3),

$$\frac{\partial\left[(D+h)\,u\right]}{\partial x} + \frac{\partial\left[(D+h)\,v\right]}{\partial y} = 0, \tag{2.3}$$

results in the equation

$$v\,(D+h)\left(\frac{\partial f}{\partial y}\right) + \left(F\frac{\pi}{b}\right)\sin\left(\pi\frac{y}{b}\right) + R\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0. \tag{2.4}$$

The solution of the free surface height $h(x,y)$, with details provided in **APPENDIX** (A.15), is:

$$h(x,y) = -\left(\frac{F}{gD}\right)\left(\frac{e^{Ax}p}{A} + \frac{e^{Bx}q}{B}\right) - \left(\frac{b}{\pi}\right)^2\left(\frac{F}{gD}\right)\left(pAe^{Ax} + qBe^{Bx}\right)\left[\cos\left(\frac{\pi y}{b}\right) - 1\right]$$
$$-\left\{\left(\frac{f\gamma}{g}\right)\left(\frac{b}{\pi}\right)^2 \sin\left(\frac{\pi y}{b}\right) - \left(\frac{\partial f}{\partial y}\right)\left(\frac{\gamma}{g}\right)\left(\frac{b}{\pi}\right)^3\left[\cos\left(\frac{\pi y}{b}\right) - 1\right]\right\}\left\{pe^{Ax} + qe^{Bx} - 1\right\} \tag{2.5}$$

[10] considered three cases, a case of non-rotating ocean with no Coriolis parameter, uniformly rotating ocean with constant Coriolis parameter everywhere and the case of rotating ocean where Coriolis force is expressed to be a function of latitude. However, the effects of wind stress, bottom friction and horizontal pressure gradients remained the same in all cases. And the stream function ψ, details of which is provided in **APPENDIX** (A.12) is :

$$\psi = \gamma\left(\frac{b}{\pi}\right)^2 \sin\left(\frac{\pi y}{b}\right)\left[pe^{Ax} + qe^{Bx} - 1\right] \tag{2.6}$$

In a non-rotating ocean, the constants p and q are given [10] as:

$$p = e^{-\frac{\pi\lambda}{b}} \text{ and } q = 1,$$

and the stream function becomes:

$$\psi = \gamma\left(\frac{b}{\pi}\right)^2 \sin\left(\frac{\pi y}{b}\right)\left[e^{\frac{(x-\lambda)\pi}{b}} + e^{-\frac{x\pi}{b}} - 1\right]. \tag{2.7}$$

The instance of zero Coriolis parameter shows a symmetry of the streamlines in the domain. This is due to the fact that, there is no effect of Coriolis force. Also, just as with no Coriolis force, a constant Coriolis parameter of 0.25×10^{-4} shows symmetry of streamlines. The streamline diagram for the two cases are shown in the left side of Figure (2.1).

The instance where Coriolis parameter is a function of latitude shows western intensification of streamlines. The streamline diagram of this case is shown at the right side of Figure (2.1).

Although Stommel's solution correctly explains the western intensification, it was

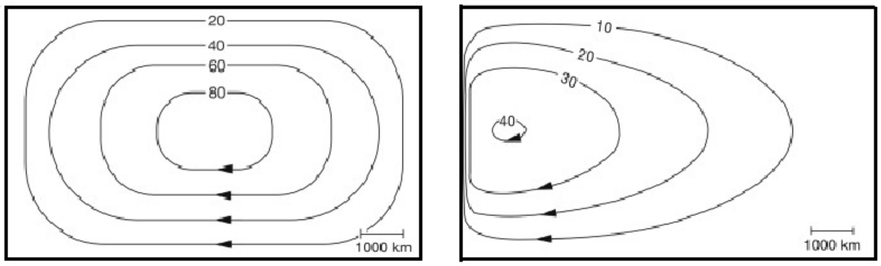

Figure 2.1: Streamlines for three cases of Coriolis parameter. Left plot represents two cases: No Coriolis force($f = 0$) and constant Coriolis force($f = 0.25 \times 10^{-4}$). Right plot represents Coriolis force as a linear function of latitude($f = y \times 10^{-13}$).

very simplistic. Stommel [10] assumed a barotropic ocean with bottom friction. In real ocean however, most currents are in the upper layers, with very little flow at the bottom.

In a 1950 paper [8], Walter H. Munk used lateral friction for the dissipative forces. Even though the rectangular boundaries used in [10] were retained, the basin is extended to both sides of the equator and the full governing equations (1.3), (1.4) and (1.5) were utilized.

With these governing equations, a numerical solution of the free surface height η using the MITgcm model shows crowding of streamlines in the western boundary as in Figure (1.2). Before Stommel [10], Sverdrup in his 1947 paper [11], examined the effects of wind stress on ocean currents. His focus was on baroclinic system [11]. Specifically, he examined the effects of the wind stress on the equatorial currents including counter currents. He used the theory that ocean waters are almost in hydrostatic equilibrium, which says that at any depth (z), the pressure (p) can be determined by the equation

$$dp = g\rho dz,$$

provided the density (ρ) is observed to be a known quantity[11]. Using data he gathered, Sverdrup computed average wind stress by using the relationship

$$\tau = \gamma^2 \rho^* U^2,$$

where (γ^2) is defined to be coefficient of resistance, (ρ^*) is air density and (U^2) is the speed of wind. Sverdrup's study led to the conclusion that average wind stress exerted on the sea surface is responsible for distribution of density and mass transport by the accompanying currents of the eastern pacific[11]. This result of Sverdrup's paper was later improved by Stommel [10] and Munk [8] in 1948 and 1950 respectively.

Chapter 3

Theoretical Background

3.1 Introduction

This chapter seeks to provide understanding of the basic principles, concepts, and equations governing the motion of water in large, rotating oceanic systems. In order to arrive at an analytic solution of barotropic wind-forced ocean gyre circulation we develop a mathematical model based on fundamental principles and solve the governing equations analytically. This will provide an understanding of the physical processes that drive the circulation patterns.

3.1.1 Wind Stress

Energy from the winds blowing over the ocean acts on the surface of the ocean. In some cases, the energy acts in driving the ocean currents. The magnitude of the wind speed determines the frictional force acting on the sea-surface. This sea-surface friction due to wind blowing over the ocean is termed as wind stress [3]. Mathematically, wind stress is expressed to vary directly with the square of wind speed denoted as

$$\tau = cW^2, \tag{3.1}$$

where τ is the wind stress, W is wind speed and c depends on the atmospheric conditions which increases with increasing wind speed [3].

3.1.2 Coriolis Effect

An object moving in a fixed rotating system experiences an inertial or fictitious force proportional to the sine of latitude. This force is known as Coriolis force [3]. For a moving object in a rotating system, the Coriolis force is denoted by

$$F_c = mu2\Omega \sin\theta \tag{3.2}$$

where F_c is the Coriolis force, m is mass of the particle, u is its speed and (Ω) is the angular velocity of the earth while (θ) is the latitude. The term $2\Omega \sin\theta$ is known as the Coriolis parameter and is denoted as f. So equation (3.2) can be written as

$$F_c = mfu \tag{3.3}$$

The Coriolis force causes a moving body to appear to be moving to the right of it's direction in the northern hemisphere and to the left of it's direction in the southern hemisphere [12].

When the Coriolis parameter f varies with latitude, it can be approximated by Taylor series expansion about the central latitude θ_0 as

$$f = f_0 + \beta y$$

where $\beta = \left(\frac{df}{dy}\right)\theta_0 = \left(\frac{df}{d\theta}\frac{d\theta}{dy}\right)\theta_0 = \frac{2\Omega\cos\theta_0}{R}$

with $f = 2\Omega \sin\theta$, $\frac{d\theta}{dy} = \frac{1}{R}$ where R is the earth's radius. A β-plane model is a model that uses the Coriolis parameter in the form $f = f_0 + \beta y$, β being a constant [6].

3.2 Partial Differential Equations

An equation is said to be a Partial Differential Equation (PDE) if it involves an unknown function of more than one variable and some of its partial derivatives [4] .

If $n \geq 1$ is an integer and U is an open subset of $\mathbb{R}^n$, then the expression

$$F(x, u(x), Du(x), D^2u(x), ...D^{n-1}u(x), D^n u(x)) = 0 \tag{3.4}$$

where x $\in$ U, is an n^{th} order partial differential equation. where

$$F : U \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n^2} \times ... \times \mathbb{R}^{n^{d-1}} \times \mathbb{R}^{n^d} \to \mathbb{R}$$

is provided and $u : U \to \mathbb{R}$ is unknown.

The PDE (3.4) is linear if it has the form

$$\sum_{|\alpha| \leq n} a_\alpha(x) D^\alpha u = f(x)$$

for some functions a_α (with $| \alpha | \leq n$) and f. And if $f = 0$, then the linear PDE is also homogeneous.

The PDE (3.4) is semi-linear if it has the form

$$\sum_{|\alpha| \leq n} a_\alpha(x) D^\alpha u + a_0(D^{n-1}u, ..., Du, u, x) = 0$$

(3.4) is a Quasi-linear PDE if it can be written as

$$\sum_{|\alpha| \leq n} a_\alpha(D^{n-1}u, ..., Du, u, x) D^\alpha u + a_0(D^{n-1}u, ..., Du, u, x) = 0$$

Finally, if (3.4) is non-linear in the highest derivative term(s), then it is a fully non-linear PDE.

3.2.1 Second Order linear PDE

An equation is a second order PDE if it can be written in the form

$$a\frac{\partial^2 \phi}{\partial x^2} + b\frac{\partial^2 \phi}{\partial t \partial x} + c\frac{\partial^2 \phi}{\partial t^2} + d\frac{\partial \phi}{\partial x} + e\frac{\partial \phi}{\partial t} + f\phi = g \tag{3.5}$$

where a,b,c,d and e are constants and g may be dependent on x,t and ϕ . The independent variables x and t are space and time coordinates respectively. A solution to (3.5) is a surface in space that consist of different characteristic curves. The discriminant $D = b^2 - 4ac$ is the slope of the characteristic curve and equation (3.5) is hyperbolic, if $D > 0$, parabolic, if $D = 0$ and elliptic, if $D < 0$.

3.2.2 Initial and Boundary Conditions

Generally, PDEs are known to have several solutions. In most instances, auxiliary conditions are set up so as to find single solution to a PDE. These conditions, often motivated by the physics, come in two forms, **initial conditions** and **boundary conditions**.

With an initial condition, the physical state at a particular time t_0 is provided. So, for diffusion equation, the initial condition is

$$u(\mathbf{x}, t_0) = \phi(\mathbf{x}) \tag{3.6}$$

where $\phi(\mathbf{x}) = \phi(x, y, z)$ is a function that represents the initial concentration of dye.

All physical problems have a domain D within which the PDE is valid. So for a diffusing chemical substance, the container within which the liquid is found is the domain, in which case the surface S of the container is the boundary. In specifying the boundary conditions, 3 kinds are considered.

1. When the functional form of the solution $\phi(x)$ is provided at the boundary (∂D) such that

$$u(\mathbf{x}, t) = \phi(\mathbf{x}, t) \tag{3.7}$$

where $\phi(\mathbf{x}, t)$ is a given function and $\mathbf{x} = (x, y, z,) \in \partial D$, the boundary of the domain. This is known as the **Dirichlet condition**.

2. When the derivative of the solution of the PDE normal to the boundary is provided at the boundary of the domain such that

$$\frac{\partial u(\mathbf{x}, t)}{\partial n} = \phi(\mathbf{x}, t) \tag{3.8}$$

where $\phi(\mathbf{x}, t)$ is some function, $\mathbf{x} \in \partial D$ and $\frac{\partial}{\partial n}$ is the directional derivative normal to the boundary ∂D. This is known as the **Neumann boundary condition.**

3. When the function is described as a linear combination of a Dirichlet and a

Neumann condition for the solution of $u(\mathbf{x}, t)$ on the boundary ∂D such that

$$a(\mathbf{x}, t)u(\mathbf{x}, t) + b(\mathbf{x}, t)\frac{\partial u(\mathbf{x}, t)}{\partial n} = c(\mathbf{x}, t) \tag{3.9}$$

where a($\mathbf{x}$,t), b($\mathbf{x}$,t) and c($\mathbf{x}$,t) are some functions. u($\mathbf{x}$,t) $\in \partial D$. This type of boundary condition is known as th **Robin condition** or **Mixed condition.**

When a problem requires both the initial and the boundary conditions, it is referred to as an **Initial Boundary Value Problem (IBVP).** The (IBVPs) are chosen so as to attain a unique solution to the PDE. In order to chose a condition for the PDE (3.5) , we consider the classification of the PDE and the nature of the domain D.

3.3 Navier-Stokes Equations for Fluid Dynamics

3.3.1 Material Derivatives

Consider the flow of an infinitesimally small fluid element. Figure (3.1) shows the motion of fluid element in the Euclidean (Cartesian) space with the unit vectors i, j, and k. In the Cartesian space, the velocity vector field is provided as

$$V = \vec{u}i + \vec{v}j + \vec{w}k.$$

Where $\vec{u} = u(x, y, z, t)$ is the x-component of velocity, $\vec{v} = v(x, y, z, t)$ is the y-component of velocity and $\vec{w} = w(x, y, z, t)$ is the z-component of velocity.

Also, density field is provided as

$$\rho = \rho(x, y, z, t).$$

Figure (3.1) shows that at time t_1, the fluid element is located at point 1 with density $\rho_1 = \rho(x_1, y_1, z_1, t_1)$. But, at time t_2, the same fluid element is located at a new point 2 with new density $\rho_2 = \rho(x_2, y_2, z_2, t_2)$.

An expansion of $\rho = \rho(x, y, z, t)$ in Taylor series about point 1 is given as

$$\rho_2 = \rho_1 + \left(\frac{\partial \rho}{\partial x}\right)_1 (x_2 - x_1) + \left(\frac{\partial \rho}{\partial y}\right)_1 (y_2 - y_1) + \left(\frac{\partial \rho}{\partial z}\right)_1 (z_2 - z_1) + \left(\frac{\partial \rho}{\partial t}\right)_1 (t_2 - t_1) + ... \tag{3.10}$$

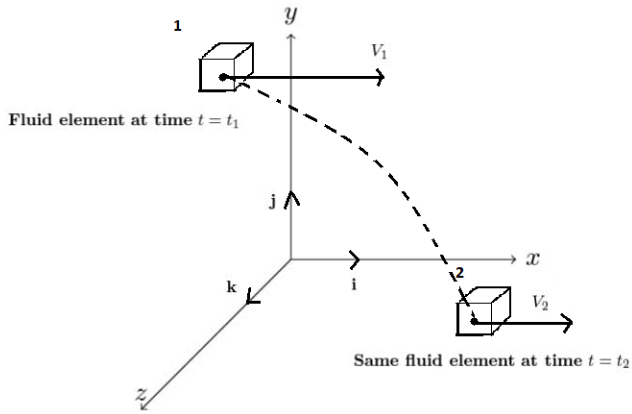

Figure 3.1: Flow of infinitesimally small fluid element.

If we ignore higher-order terms and divide equation (3.10) by time difference $t_2 - t_1$, we obtain

$$\frac{\rho_2 - \rho_1}{t_2 - t_1} = \left(\frac{\partial \rho}{\partial x}\right)_1 \frac{x_2 - x_1}{t_2 - t_1} + \left(\frac{\partial \rho}{\partial y}\right)_1 \frac{y_2 - y_1}{t_2 - t_1} + \left(\frac{\partial \rho}{\partial z}\right)_1 \frac{z_2 - z_1}{t_2 - t_1} + \left(\frac{\partial \rho}{\partial t}\right)_1 \tag{3.11}$$

The left side of equation (3.11) is the change in density of the fluid element with respect to time change from t_1 to t_2 [1]. If we take the limit as t_2 approaches t_1, the left-hand side becomes

$$\lim_{t_2 - t_1} \frac{\rho_2 - \rho_1}{t_2 - t_1} \equiv \frac{D\rho}{Dt},$$

where $\frac{D\rho}{Dt}$ is the derivative of density of the fluid element with respect to time and $\frac{D}{Dt}$ is the Substantial derivative or Material derivative [1].

On the other side of equation(3.11), we note that

$$\lim_{t_2 \to t_1} \frac{x_2 - x_1}{t_2 - t_1} \equiv u$$

$$\lim_{t_2 \to t_1} \frac{y_2 - y_1}{t_2 - t_1} \equiv v$$

$$\lim_{t_2 \to t_1} \frac{z_2 - z_1}{t_2 - t_1} \equiv w$$

Thus, as $t_2 \to t_1$, equation(3.11) becomes

$$\frac{D\rho}{Dt} = \vec{u}\frac{\partial \rho}{\partial x} + \vec{v}\frac{\partial \rho}{\partial y} + \vec{w}\frac{\partial \rho}{\partial z} + \frac{\partial \rho}{\partial t} \tag{3.12}$$

From equation(3.12), the Material derivative in Cartesian coordinates is expressed as

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{u}\frac{\partial}{\partial x} + \vec{v}\frac{\partial}{\partial y} + \vec{w}\frac{\partial}{\partial z}, \tag{3.13}$$

and the vector operator ∇ by

$$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \tag{3.14}$$

Thus, equation (3.13) becomes;

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (\mathbf{V} \cdot \nabla), \tag{3.15}$$

where $\mathbf{V} = (\vec{u}, \vec{v}, \vec{w})$.

Equation (3.15) is the vector notation of substantial(material) derivative operator. Physically, $\frac{D}{Dt}$ is the time rate of change of a fluid element in motion.

The Substantial derivative $\frac{D}{Dt}$ is made up of two components, $\frac{\partial}{\partial t}$ and $(\mathbf{V} \cdot \nabla)$. $\frac{\partial}{\partial t}$ is the **local derivative** which represents the time rate of change at a fixed point, and $(\mathbf{V} \cdot \nabla)$, known as the **Convective derivative**, represents the time rate of change as a result of movement of fluid element from one point to another in the fluid flow field with spatially varied flow properties

3.3.2 Conservation of Mass

Apart from nuclear processes and the effects of relativity, mass cannot be generated or destroyed. Molecules, grains, fluid particles, are elements that can be tracked within a flow field. Thus,they won't vanish or suddenly form new elements.The idea that a particular group of nearby fluid particles has a constant mass is the foundation for the equations illustrating mass conservation in moving fluids.

A material volume V(t) is the volume that a certain group of fluid particles occupy. Within a fluid flow, such a volume flows and deforms, containing the same mass elements at all times so that none of which enter or exit the volume. In view of this, the material surface A(t), which is the surface of the material volume must move at a fluid velocity **u**, so fluid particles inside V(t) will remain inside and those outside V(t) will remain outside [6]. For a flowing fluid, the statement of conservation of mass for a material volume is stated as:

$$\frac{d}{dt}\int_{V(t)} \rho(\mathbf{x},t)dV = 0 \tag{3.16}$$

where ρ is the fluid's density. For a better display of the implication of equation (3.16) for fluid velocity field, we use the following statement of the Reynold Transport Theorem

$$\frac{d}{dt}\int_{V^*(t)} F(\mathbf{x},t)dV = \int_{V^*(t)} \frac{\partial F(\mathbf{x},t)}{\partial t}dV + \int_{A^*(t)} F(\mathbf{x},t)\mathbf{b}.\mathbf{n}dA \tag{3.17}$$

where $V^*(t)$ denotes a moving volume with a closed surface denoted as $A^*(t)$ with outward normal **n** and **b** as the local velocity of $A^*(t)$ [6]. If **F** is taken as ρ and **b** taken as **u**, applying equation (3.17) on the left hand side of equation (3.16) produces

$$\frac{d}{dt}\int_{V(t)} \rho(\mathbf{x},t)dV = \int_{V(t)} \frac{\partial \rho(\mathbf{x},t)}{\partial t}dV + \int_{A(t)} \rho(\mathbf{x},t)\mathbf{u}(\mathbf{x},t)\cdot\mathbf{n}dA \tag{3.18}$$

Using Gauss' divergence theorem, the second term on the right side of equation (3.18) produces

$$\int_{A(t)} \rho(\mathbf{x},t)\mathbf{u}(\mathbf{x},t)\cdot\mathbf{n}dA = \int_{V(t)} \nabla\cdot(\rho(\mathbf{x},t)\mathbf{u}(\mathbf{x},t))dV. \tag{3.19}$$

So the right side of equation (3.18) becomes

$$\int_{V(t)} \frac{\partial \rho(\mathbf{x},t)}{\partial t}dV + \int_{A(t)} \rho(\mathbf{x},t)\mathbf{u}(\mathbf{x},t)\cdot\mathbf{n}dA = \int_{V(t)} \left\{\frac{\partial \rho(\mathbf{x},t)}{\partial t} + \nabla\cdot(\rho(\mathbf{x},t)\mathbf{u}(\mathbf{x},t))\right\}dV = 0. \tag{3.20}$$

It is only when the integrand in equation (3.20) vanishes in space that the equality is possible. Else integrating equation (3.20) in a small volume about a point would produce a nonzero integral, and that would be a contradiction. So, from equation (3.20), we obtain the equation

$$\frac{\partial \rho(\mathbf{x},t)}{\partial t} + \nabla \cdot (\rho(\mathbf{x},t)\mathbf{u}(\mathbf{x},t)) = 0. \tag{3.21}$$

Equation (3.21) may be written in index form as

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u_i)}{\partial x_i} = 0 \tag{3.22}$$

Equations (3.21) and (3.22) referred to as **Continuity equation** [6], shows the principle of conservation of mass in differential form.

3.3.3 Conservation of Momentum

Using Newton's second law of motion, we arrive at the momentum equations. We use the moving fluid element model shown in Figure (3.2) below.

By Newton's second law of motion, we write force as the product of mass and acceleration. However, for a moving fluid element, force equals product of it's mass and acceleration. For simplicity, we consider only one component of force, say the x-component. such that;

$$F_x = ma_x, \tag{3.23}$$

where F_x is the scalar component of the force and a_x is the acceleration [1].

For a start,we consider equation (3.23), with the force on the moving fluid acting in the x-direction. This force is made up of two components- body forces such as gravitational and electric forces which act directly on the volumetric mass of the fluid element and surface forces, which act directly on the surface of the fluid element [1].

If the fluid element is acted upon by the force per unit mass denoted by $\mathbf{f}$ with it's x-component as f_x. Then

$$BF_x = \rho f_x(dxdydz), \tag{3.24}$$

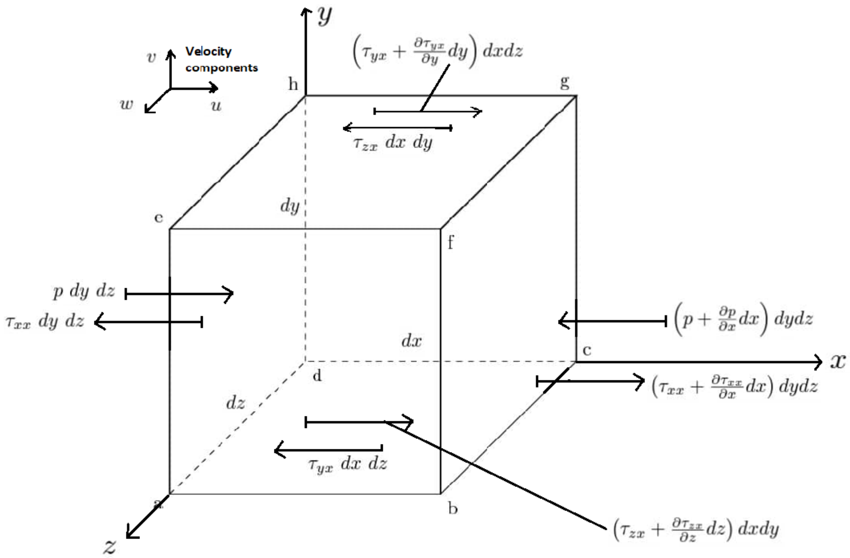

Figure 3.2: Infinitesimally small, moving fluid showing x-direction forces only.

where BF_x is the x-component of the body force acting on the fluid element. ρ is density of the fluid element and $(dxdydz)$ is volume of the fluid element [1]. Also, the net surface force in the x-direction (NSF_x) [1] is given as

$$NSF_x = \left[p - \left(p + \frac{\partial p}{\partial x} dx\right)\right] dydz + \left[\left(\tau_{xx} + \frac{\partial \tau_{xx}}{\partial x} dx\right) - \tau_{xx}\right] dydz + \left[\left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} dy\right) - \tau_{yx}\right] dxdz + \left[\left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z} dz\right) - \tau_{zx}\right] dxdy, \quad (3.25)$$

where ρ is pressure , τ_{xx} is normal stress in the x-direction, τ_{yx} and τ_{zx} are the share stresses that respectively denote the the stress in the y-direction and z-direction exerted on the plane perpendicular to the x-axis [1]. By adding and canceling terms, equation (3.25) is reduced to

$$NSF_x = \left[-\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right] dxdydz \quad (3.26)$$

By summing up equations (3.24) and (3.26), we obtain the force F_x on the left-hand side of equation (3.23) as

$$F_x = \left[-\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right] dxdydz + \rho f_x dxdydz \quad (3.27)$$

For the right-hand side of equation (3.23), we use the fact that fluid element has fixed mass [1] and is given as

$$m = \rho dxdydz \tag{3.28}$$

Also, with acceleration of the fluid element described as the change in its velocity over time, a_x, the x-component of acceleration, is the change in velocity u due to change in time is represented by the material derivative

$$a_x = \frac{Du}{Dt} \tag{3.29}$$

Using equations (3.27),(3.28) and (3.29), equation (3.23) can be written as

$$\rho\frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho f_x \tag{3.30}$$

Equation (3.30) is the x-component of the momentum equation for a viscous flow. The same analysis can be used to obtain the y-component of the momentum equation as

$$\rho\frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \rho f_y \tag{3.31}$$

and z-component of the momentum equation as

$$\rho\frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} + \rho f_z \tag{3.32}$$

The partial differential equations from (3.30) to (3.32) obtained by applying fundamental physical principles to an infinitesimal fluid element moving with a flow are Navier-Stokes equations in non-conservation form. To obtain the conservation form of the Navier-Stokes equations, we consider equation (3.30), and expressed the right-hand side as

$$\rho\frac{Du}{Dt} = \rho\frac{\partial u}{\partial t} + \rho\mathbf{V}\cdot\nabla u. \tag{3.33}$$

By applying product rule on (ρu) and rearranging terms, we have

$$\rho\frac{\partial u}{\partial t} = \frac{\partial(\rho u)}{\partial t} - u\frac{\partial \rho}{\partial t}. \tag{3.34}$$

Using the vector identity for the divergence of product of a scalar and vector [1], we have

$$\nabla \cdot (\rho u \mathbf{V}) = u\nabla \cdot (\rho \mathbf{V}) + (\rho \mathbf{V}) \cdot \nabla u. \tag{3.35}$$

By rearranging terms, we express equation (3.35) as

$$(\rho \mathbf{V}) \cdot \nabla u = \nabla \cdot (\rho u \mathbf{V}) - u\nabla \cdot (\rho \mathbf{V}). \tag{3.36}$$

Equations (3.34) and (3.36) in equation (3.33) produces

$$\begin{aligned}\rho\frac{Du}{Dt} &= \frac{\partial(\rho u)}{\partial t} - u\frac{\partial \rho}{\partial t} - u\nabla \cdot (\rho \mathbf{V}) + \nabla \cdot (\rho u \mathbf{V}) \\ &= \frac{\partial(\rho u)}{\partial t} - u\left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V})\right] + \nabla \cdot (\rho u \mathbf{V})\end{aligned} \tag{3.37}$$

We observe that the term in brackets in equation (3.37) is exactly the right side of the continuity equation (3.21), which equals to zero. So equation (3.37) is reduced to

$$\rho\frac{Du}{Dt} = \frac{\partial(\rho u)}{\partial t} + \nabla \cdot (\rho u \mathbf{V}) \tag{3.38}$$

Equation (3.38) into equation (3.30) yields

$$\frac{\partial(\rho u)}{\partial t} + \nabla \cdot (\rho u \mathbf{V}) = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho f_x \tag{3.39}$$

In a similar fashion, equation (3.31) becomes

$$\frac{\partial(\rho v)}{\partial t} + \nabla \cdot (\rho v \mathbf{V}) = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \rho f_y \tag{3.40}$$

and equation (3.32) as

$$\frac{\partial(\rho w)}{\partial t} + \nabla \cdot (\rho w \mathbf{V}) = -\frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} + \rho f_z \tag{3.41}$$

Equations (3.39),(3.40) and (3.41) together represent the conservation form of the Navier-Stokes equations.

In Newtonian fluids where the shear stress is noted to be proportional to the time rate of strain (velocity gradient), the following expressions [1] are obtained

$$\begin{aligned}
\tau_{xx} &= \lambda\left(\nabla \cdot \mathbf{V}\right) + 2\mu\frac{\partial u}{\partial x} \\
\tau_{yy} &= \lambda\left(\nabla \cdot \mathbf{V}\right) + 2\mu\frac{\partial v}{\partial y} \\
\tau_{zz} &= \lambda\left(\nabla \cdot \mathbf{V}\right) + 2\mu\frac{\partial w}{\partial z} \\
\tau_{xy} = \tau_{yx} &= \mu\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right] \\
\tau_{xz} = \tau_{zx} &= \mu\left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right] \\
\tau_{yz} = \tau_{zy} &= \mu\left[\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right]
\end{aligned} \tag{3.42}$$

where μ and λ are respectively molecular viscosity coefficient and viscosity coefficient with

$$\lambda = -\frac{2}{3}\mu$$

To obtain the complete Navier-Stokes equation in conservation form, we perform the necessary substitutions of equations from (3.42) into equations (3.39),(3.40) and (3.41) to arrive at the following equations

$$\begin{aligned}
\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u^2)}{\partial x} + \frac{\partial(\rho uv)}{\partial y} + \frac{\partial(\rho uw)}{\partial z} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left[\lambda\nabla \cdot \mathbf{V} + 2\mu\frac{\partial u}{\partial x}\right] + \\
\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right] + \rho f_x
\end{aligned} \tag{3.43}$$

$$\frac{\partial(\rho v)}{\partial t} + \frac{\partial(\rho uv)}{\partial x} + \frac{\partial(\rho v^2)}{\partial y} + \frac{\partial(\rho vw)}{\partial z} = -\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] + \frac{\partial}{\partial y}\left[\lambda\nabla\cdot\mathbf{V} + 2\mu\frac{\partial v}{\partial y}\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)\right] + \rho f_y \quad (3.44)$$

$$\frac{\partial(\rho w)}{\partial t} + \frac{\partial(\rho uw)}{\partial x} + \frac{\partial(\rho vw)}{\partial y} + \frac{\partial(\rho w^2)}{\partial z} = -\frac{\partial p}{\partial z} + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)\right] + \frac{\partial}{\partial y}\left[\lambda\nabla\cdot\mathbf{V} + 2\mu\frac{\partial w}{\partial z}\right] + \rho f_z \quad (3.45)$$

3.4 Barotropic fluid flow

A flow in ocean waters such that density is a function of only pressure is said to be barotropic. Surfaces of constant density (Isopycnals) are parallel to surfaces of equal pressure(Isobars) [3]. In a barotropic flow, density increases with depth as it does with pressure.

In this section, we provide the set of equations for the configuration that stimulate the barotropic, wind-forced, ocean gyre circulation.

We begin with Navier-Stokes equations for a fluid

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2 u \quad (3.46)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\nabla^2 v \quad (3.47)$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\nabla^2 w \quad (3.48)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (3.49)$$

For simplicity, we consider the N-S equations without Coriolis and viscous terms so that:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{3.50}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} \tag{3.51}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + g = -\frac{1}{\rho}\frac{\partial p}{\partial z} \tag{3.52}$$

Considering shallow water condition, with negligible vertical velocity component (w) [6]. By the shallow water assumption, (u) and (v) are independent of (z). This produces:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{3.53}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} \tag{3.54}$$

$$\frac{\partial p}{\partial z} = -\rho g \tag{3.55}$$

To express the pressure gradients in terms of free surface, we consider a layer of fluid with depth(H) over flat bottom as shown in Figure (3.3). Let $z = 0$ on the bottom surface and let η be the displacement of free surface [6]. The surface height of the fluid at any point[6] is $H + \eta$.

Integrating the vertical momentum equation over depth z, from 0 to $H + \eta$ gives:

$$\int_0^{H+\eta} \frac{\partial p}{\partial z} dz = -\int_0^{H+\eta} \rho g dz$$

this yields:

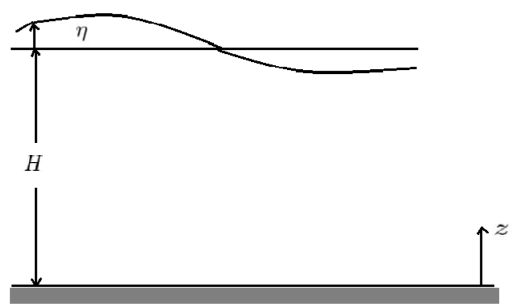

Figure 3.3: Fluid layer with average thickness (H) above bottom $z = 0$.

$$p_s - p_b = -\rho g(H + \eta).$$

Where p_s and p_b are the surface and bottom fluid pressure respectively.

For a zero fluid surface pressure, pressure at any height (z) from the bottom is given as:

$$p = \rho g(H + \eta - z)$$

This yields the pressure gradients:

$$\frac{\partial p}{\partial x} = \rho g \frac{\partial \eta}{\partial x} \quad and \quad \frac{\partial p}{\partial y} = \rho g \frac{\partial \eta}{\partial y} \tag{3.56}$$

By substituting the horizontal pressure gradients, into the respective momentum equations (3.53) and (3.54) we obtain :

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + g\frac{\partial \eta}{\partial x} = 0 \tag{3.57}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + g\frac{\partial \eta}{\partial y} = 0, \tag{3.58}$$

so that

$$\frac{Du}{Dt} + g\frac{\partial \eta}{\partial x} = 0 \tag{3.59}$$

$$\frac{Dv}{Dt} + g\frac{\partial \eta}{\partial y} = 0 \tag{3.60}$$

We next integrate the continuity equation (3.49) vertically over depth z, from 0 to $H + \eta$. Thus,

$$\int_0^{H+\eta} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) dz = 0$$

This gives:

$$(H + \eta)\frac{\partial u}{\partial x} + (H + \eta)\frac{\partial v}{\partial y} + w(\eta) - w(0) = 0$$

Where $w(\eta)$ and $w(0)$ are the vertical velocity at the surface and bottom respectively [6].

However, velocity at the bottom is zero [6] and velocity at the surface is such that:

$$w(\eta) = \frac{D\eta}{Dt} = \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y}$$

The integrated continuity equation becomes:

$$(H + \eta)\frac{\partial u}{\partial x} + (H + \eta)\frac{\partial v}{\partial y} + \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y} = 0.$$

So that:

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}\left[u(H + \eta)\right] + \frac{\partial}{\partial y}\left[v(H + \eta)\right] = 0$$

However, by neglecting the non linear terms due to small amplitude waves [6], the integrated continuity equation reduces to:

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}(uH) + \frac{\partial}{\partial y}(vH) = 0.$$

Which is expressed as:

$$\frac{\partial \eta}{\partial t} + \nabla_h.(H\vec{\mathbf{u}}) = 0 \tag{3.61}$$

Thus, the free surface is depressed by the divergence of the horizontal fluid transport [6].

We next assume a rotating system and introduce the Coriolis and viscous terms in the momentum equations (3.59) and (3.60)

$$\frac{Du}{Dt} - fv + g\frac{\partial \eta}{\partial x} - A_h\nabla_h^2 u = 0, \tag{3.62}$$

$$\frac{Dv}{Dt} + fu + g\frac{\partial \eta}{\partial y} - A_h\nabla_h^2 v = 0, \tag{3.63}$$

where $(A_h\nabla_h^2 u)$ and $(A_h\nabla_h^2 v)$ represent the viscous terms with (A_h) been the horizontal Laplacian viscosity whiles (fv) and (fu) are the Coriolis terms [7].

We also introduce wind stress terms τ_x and τ_y for wind forcing and normalize each by dividing the wind stress by the total mass of the fluid column $(\rho_c H)$ to obtain:

$$\frac{Du}{Dt} - fv - A_h\nabla_h^2 u + g\frac{\partial \eta}{\partial x} = \frac{\tau_x}{\rho_c H}, \tag{3.64}$$

$$\frac{Dv}{Dt} + fu - A_h\nabla_h^2 v + g\frac{\partial \eta}{\partial y} = \frac{\tau_y}{\rho_c H}, \tag{3.65}$$

where (ρ_c) is the fluid density.

The equations (3.64) and (3.65) together with the equation (3.61) produces the set of equations:

$$\frac{Du}{Dt} - fv - A_h\nabla_h^2 u + g\frac{\partial \eta}{\partial x} = \frac{\tau_x}{\rho_c H} \tag{3.66}$$

$$\frac{Dv}{Dt} + fu - A_h \nabla_h^2 v + g\frac{\partial \eta}{\partial y} = \frac{\tau_y}{\rho_c H} \tag{3.67}$$

$$\frac{\partial \eta}{\partial t} + \nabla_h \cdot (H\vec{\mathbf{u}}) = 0 \tag{3.68}$$

However, our model assumes that the fluid is forced by zonal wind stress (τ_x) with negligible meridional wind stress (that is $\tau_y = 0$) and so, the set of equations become:

$$\frac{Du}{Dt} - fv - A_h \nabla_h^2 u + g\frac{\partial \eta}{\partial x} = \frac{\tau_x}{\rho_c H} \tag{3.69}$$

$$\frac{Dv}{Dt} + fu - A_h \nabla_h^2 v + g\frac{\partial \eta}{\partial y} = 0 \tag{3.70}$$

$$\frac{\partial \eta}{\partial t} + \nabla_h \cdot (H\vec{\mathbf{u}}) = 0 \tag{3.71}$$

The set of equations (3.69,3.70 and 3.71) are the set of equations for the modeled barotropic gyre circulation. And the analytic solution to these system of equations is the subject of the next chapter.

Chapter 4

Analytic and Numerical Solutions

In this chapter, we present analytic solution to the governing equations representing the barotropic wind-forced ocean gyre circulation. We subject the analysis to various initial and boundary conditions to arrive at the solution.

4.1 Vorticity and stream function equation

Vorticity is a concept in fluid dynamics that is used to measure the rotation of fluids. It describes the direction and how fast a fluid particle rotates. It is instrumental in understanding ocean circulation patterns. For any given fluid with velocity field $\mathbf{V} = \vec{u}i + \vec{v}j + \vec{w}k$ the vorticity of the fluid, denoted by ζ is defined as the curl of the velocity field $\mathbf{V}$ [15]. Mathematically, vorticity is represented as

$$\zeta = \nabla \times \mathbf{V} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

$$\zeta = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) i - \left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right) j + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) k$$

For a 2-dimensional flow, where w is assumed negligible [13] and $\frac{\partial u}{\partial z} = 0 = \frac{\partial v}{\partial z}$. The vorticity of the fluid becomes

$$\zeta = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \tag{4.1}$$

To obtain an analytic solution to our problem, we begin by forming a vorticity equation by differentiate the x-momentum equation (3.69) with respect to y and differentiate the y-momentum equation(3.70) with respect to x. Thus:

$$\frac{\partial}{\partial y}\left[\frac{Du}{Dt} - fv\right] = \frac{\partial}{\partial y}\left[-g\frac{\partial \eta}{\partial x} + A_h \nabla_h^2 u + \frac{\tau_x}{\rho_c H}\right] \tag{4.2}$$

$$\frac{\partial}{\partial x}\left[\frac{Dv}{Dt} + fu\right] = \frac{\partial}{\partial x}\left[-g\frac{\partial \eta}{\partial y} + A_h \nabla_h^2 u\right] \tag{4.3}$$

equation (4.2) minus equation (4.3) yields;

$$\frac{\partial}{\partial y}\left(\frac{Du}{Dt}\right) - \frac{\partial}{\partial x}\left(\frac{Dv}{Dt}\right) - \frac{\partial}{\partial y}(fv) - \frac{\partial}{\partial x}(fu) = -\frac{\partial}{\partial y}\left(g\frac{\partial \eta}{\partial x}\right) + \frac{\partial}{\partial x}\left(g\frac{\partial \eta}{\partial y}\right) +$$
$$A_h\left[\frac{\partial}{\partial y}\left(\nabla_h^2 u\right) - \frac{\partial}{\partial x}\left(\nabla_h^2 v\right)\right] + \frac{\partial}{\partial y}\left(\frac{\tau_x}{\rho_c H}\right)$$

Simplifying further, we have

$$\frac{\partial}{\partial y}\left(\frac{Du}{Dt}\right) - \frac{\partial}{\partial x}\left(\frac{Dv}{Dt}\right) - f\left[\frac{\partial v}{\partial y} + \frac{\partial u}{\partial x}\right] - v\frac{\partial f}{\partial y} - u\frac{\partial f}{\partial x} = -g\left[\frac{\partial^2 \eta}{\partial y \partial x} - \frac{\partial^2 \eta}{\partial x \partial y}\right] +$$
$$A_h\left\{\frac{\partial^2}{\partial x^2}\left[\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right] + \frac{\partial^2}{\partial y^2}\left[\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right]\right\} + \frac{1}{\rho_c H}\left(\frac{\partial \tau_x}{\partial y}\right)$$

By considering geostrophic balance, where the pressure gradients and the Coriolis parameter are the dominant terms in the momentum equation [6], so that equations (3.69) and (3.70) respectively becomes:

$$fu = -g\tfrac{\partial \eta}{\partial y} \quad \text{and} \quad fv = g\tfrac{\partial \eta}{\partial x}.$$

For a steady-state, where the continuity equation is such that;

$$\tfrac{\partial u}{\partial x} + \tfrac{\partial v}{\partial y} = 0.$$

Taking cross partial derivatives of the resulting momentum equations, assuming negligible advective terms and simplifying the reduced momentum equations will yield;

$$\frac{\partial^2\eta}{\partial y\partial x} - \frac{\partial^2\eta}{\partial x\partial y} = 0.$$

Also, with the Coriolis force defined as $f = f_0 + \beta y$, implies

$$\frac{\partial f}{\partial t} = 0 = \frac{\partial f}{\partial x} \text{ and } \frac{\partial f}{\partial y} = \beta.$$

Making the necessary substitutions and using (4.1) yields:

$$\frac{\partial}{\partial y}\left(\frac{Du}{Dt}\right) - \frac{\partial}{\partial x}\left(\frac{Dv}{Dt}\right) - v\beta - f\left[\frac{\partial v}{\partial y} + \frac{\partial u}{\partial x}\right] + A_h\frac{\partial^2}{\partial x^2}\zeta + A_h\frac{\partial^2}{\partial y^2}\zeta = \frac{1}{\rho_c H}\left(\frac{\partial \tau_x}{\partial y}\right) \tag{4.4}$$

Now, we consider the first two terms of equation (4.4)

$$\frac{\partial}{\partial y}\left(\frac{Du}{Dt}\right) - \frac{\partial}{\partial x}\left(\frac{Dv}{Dt}\right) = \frac{\partial}{\partial y}\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right] - \frac{\partial}{\partial x}\left[\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right]$$

$$= \frac{\partial}{\partial y}\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right] - \frac{\partial}{\partial x}\left[\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right]$$

$$= \frac{\partial}{\partial t}\left[\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right] + \left[\frac{\partial u}{\partial y}\frac{\partial u}{\partial x} + u\frac{\partial^2 u}{\partial y\partial x}\right] + \left[\frac{\partial v}{\partial y}\frac{\partial u}{\partial y} + v\frac{\partial^2 u}{\partial y^2}\right] - \left[\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + u\frac{\partial^2 v}{\partial x^2}\right] - \left[\frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + v\frac{\partial^2 v}{\partial x\partial y}\right]$$

Re-arranging like terms, we have

$$\frac{\partial}{\partial y}\left(\frac{Du}{Dt}\right) - \frac{\partial}{\partial x}\left(\frac{Dv}{Dt}\right) = \frac{\partial}{\partial t}\left[\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right] + \frac{\partial v}{\partial y}\left[\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right] + \frac{\partial u}{\partial x}\left[\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right] +$$

$$u\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right] + v\frac{\partial}{\partial y}\left[\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right]$$

Now, using equation (4.1) we have

$$\frac{\partial}{\partial y}\left(\frac{Du}{Dt}\right) - \frac{\partial}{\partial x}\left(\frac{Dv}{Dt}\right) = -\frac{\partial\zeta}{\partial t} - \zeta\frac{\partial v}{\partial y} - \zeta\frac{\partial u}{\partial x} - u\frac{\partial\zeta}{\partial x} - v\frac{\partial\zeta}{\partial y}$$

Now by considering a steady state solution and assuming negligible advective terms, we have that the time derivative of ζ as well as the non-linear terms are set to zero and so,

$$\frac{\partial}{\partial y}\left(\frac{Du}{Dt}\right) - \frac{\partial}{\partial x}\left(\frac{Dv}{Dt}\right) = -\zeta\frac{\partial v}{\partial y} - \zeta\frac{\partial u}{\partial x}$$

Equation (4.4) thus becomes:

$$-\zeta\frac{\partial v}{\partial y} - \zeta\frac{\partial u}{\partial x} - v\beta - f\left[\frac{\partial v}{\partial y} + \frac{\partial u}{\partial x}\right] + A_h\frac{\partial^2}{\partial x^2}\zeta + A_h\frac{\partial^2}{\partial y^2}\zeta = \frac{1}{\rho_c H}\left(\frac{\partial \tau_x}{\partial y}\right)$$

$$\Rightarrow -v\beta - \zeta\left[\frac{\partial v}{\partial y} + \frac{\partial u}{\partial x}\right] - f\left[\frac{\partial v}{\partial y} + \frac{\partial u}{\partial x}\right] + A_h\frac{\partial^2}{\partial x^2}\zeta + A_h\frac{\partial^2}{\partial y^2}\zeta = \frac{1}{\rho_c H}\left(\frac{\partial \tau_x}{\partial y}\right)$$

$$\Rightarrow -v\beta - (f+\zeta)\left[\frac{\partial v}{\partial y} + \frac{\partial u}{\partial x}\right] + A_h\frac{\partial^2}{\partial x^2}\zeta + A_h\frac{\partial^2}{\partial y^2}\zeta = \frac{1}{\rho_c H}\left(\frac{\partial \tau_x}{\partial y}\right)$$

For incompressible flow, we note that, $\left[\frac{\partial v}{\partial y} + \frac{\partial u}{\partial x}\right] = 0$, and so, equation (4.4) becomes:

$$-v\beta + A_h\left[\frac{\partial^2\zeta}{\partial x^2} + \frac{\partial^2\zeta}{\partial y^2}\right] = \frac{1}{\rho_c H}\frac{\partial \tau_x}{\partial y},$$

which may be written as:

$$-v\beta + A_h\nabla^2\zeta = \frac{1}{\rho_c H}\frac{\partial \tau_x}{\partial y}. \tag{4.5}$$

To proceed, we define a stream function ψ, such that for each of the velocity components u and v, we have that:

$$u = -\frac{\partial \psi}{\partial y} \qquad \text{and} \qquad v = \frac{\partial \psi}{\partial x}.$$

Thus,

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial y}\right) = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = \nabla^2\psi.$$

Making the necessary substitutions and with the sinusoidal wind stress defined as $\tau_x = -\tau_0\cos\left(\pi\frac{y}{L_y}\right)$, equation (4.5) becomes the fourth-order equation:

$$A_h\nabla^4\psi - \beta\frac{\partial \psi}{\partial x} = \frac{\tau_0}{\rho_c H}\frac{\pi}{L_y}\sin\left(\frac{\pi y}{L_y}\right). \tag{4.6}$$

For a solution, we notice that, equation (4.6) is a vorticity-stream function equation in terms of ψ for the interior and boundaries of the ocean basin. And that, the boundary is a streamline whose value is chosen to be zero [8] and, there are no slippages against the boundary [8]. This yields the boundary conditions:

$$\psi_{boundary} = 0, \left(\frac{\partial \psi}{\partial x}\right)_{boundary} = \left(\frac{\partial \psi}{\partial y}\right)_{boundary} = 0 \tag{4.7}$$

For a solution of ψ, we consider the interior and boundary solutions denoted as ψ_I and ψ_B respectively such that:

$$\psi = \psi_I + \psi_B \tag{4.8}$$

For the solution of ψ_I, we notice that, equation (4.6) without friction represented by the lateral stress curl satisfies the Sverdrup balance [9] such that:

$$-\beta \frac{\partial \psi}{\partial x} = \frac{\tau_0}{\rho_c H} \frac{\pi}{L_y} \sin\left(\frac{\pi y}{L_y}\right) \tag{4.9}$$

Integrating equation (4.9) with respect to x along the lateral domain, we obtain:

$$\psi_I(x, y) = -\int_{L_y}^{0} \tfrac{\tau_0}{\rho_c H \beta} \tfrac{\pi}{L_y} \sin\left(\tfrac{\pi y}{L_y}\right) dx$$

Thus,

$$\psi_I(x, y) = \frac{\tau_0}{\rho_c H \beta} \pi \sin\left(\frac{\pi y}{L_y}\right) \tag{4.10}$$

For the boundary stream function ψ_B, we define a stretched boundary layer α as [9]:

$$\alpha = \tfrac{(x - X_W)}{\delta_M}. \qquad \alpha \in (0, \infty),$$

where X_W denote the x value at the western boundary and $\delta_M = \left(\frac{A_h}{\beta}\right)^{\frac{1}{3}}$.

For the purposes of analysis, we define the boundary layer stream function for the flow $\psi_B(\alpha, y)$ [9] as

$$\psi_B(\alpha, y) = \psi_I(x, y) + \phi_B(\alpha, y), \tag{4.11}$$

where $\psi_I(x, y)$ is the interior stream function and $\phi_B(\alpha, y)$ is a correction function [9] and chosen such that:

$$\lim_{\alpha \to \infty} \phi_B(\alpha, y) = 0 \tag{4.12}$$

So that, for large values of α we have,

$$\psi_B(\alpha, y) = \psi_I(x, y)$$

Now, at the boundary, the effect of wind stress curl is assumed negligible. So at the boundary, equation (4.6) becomes the biharmonic homogeneous equation:

$$A_h \nabla^4 \psi - \beta \frac{\partial \psi}{\partial x} = 0, \tag{4.13}$$

where we note that $\nabla^4 \psi = \frac{\partial \psi^4}{\partial x^4} + 2\frac{\partial \psi^4}{\partial x^2 \partial y^2} + \frac{\partial \psi^4}{\partial y^4}$.

Now, using equation (4.11) in equation (4.13), we have:

$$\frac{A_h}{\beta}\left[\frac{\partial^4 \psi_I}{\partial \alpha^4} + \frac{\partial^4 \phi_B}{\partial \alpha^4} + 2\frac{\partial^4 \psi_I}{\partial \alpha^2 \partial y^2} + 2\frac{\partial^4 \phi_B}{\partial \alpha^2 \partial y^2} + \frac{\partial^4 \psi_I}{\partial y^4} + \frac{\partial^4 \phi_B}{\partial y^4}\right] - \frac{\partial \psi_I}{\partial \alpha} - \frac{\partial \phi_B}{\partial \alpha} = 0$$

By considering the definition of ψ_I and ϕ_B, if we denote $\delta_M = \left(\frac{A_h}{\beta}\right)^{\frac{1}{3}}$, we obtain:

$$(\delta_M)^3 \frac{\partial^4 \phi_B}{\partial \alpha^4} - \frac{\partial \phi_B}{\partial \alpha} = 0. \tag{4.14}$$

To solve for the correction boundary function in equation (4.14), we assume a solution;

$$\phi_B = e^{\lambda \alpha},$$

where λ is yet to be determined. So equation (4.14) becomes:

$$(\delta_M)^3 \lambda^4 e^{\lambda \alpha} - \lambda e^{\lambda \alpha} = 0$$

So that,

$$\lambda \left[(\delta_M)^3 \lambda^3 - 1\right] = 0 \tag{4.15}$$

Solving equation (4.15), we obtain four solutions for λ as :

$$\lambda = 0, \lambda = \frac{1}{\delta_M}, \lambda = \frac{-1 \pm i\sqrt{3}}{2\delta_M}$$

With the solutions of λ, the general solution of $\phi_B(\alpha, y)$ is given as:

$$\phi_B(\alpha, y) = C_1 + C_2 e^{\frac{\alpha}{\delta_M}} + C_3 e^{-\frac{\alpha}{2\delta_M}} \cos\left(\tfrac{\sqrt{3}}{2\delta_M}\alpha\right) + C_4 e^{-\frac{\alpha}{2\delta_M}} \sin\left(\tfrac{\sqrt{3}}{2\delta_M}\alpha\right),$$

where C_1, C_2, C_3 and C_4 are functions of y to be determined. For the condition in equation (4.12) to be satisfied, C_1 and C_2 must be zero and with the order $O(\frac{1}{\delta_M})$, we have;

$$\phi_B(\alpha, y) = C_3 e^{-\frac{\alpha}{2}} \cos\left(\tfrac{\sqrt{3}}{2}\alpha\right) + C_4 e^{-\frac{\alpha}{2}} \sin\left(\tfrac{\sqrt{3}}{2}\alpha\right)$$

So that equation (4.11) becomes;

$$\psi_B(\alpha, y) = \psi_I(x, y) + C_3 e^{-\frac{\alpha}{2}} \cos\left(\frac{\sqrt{3}}{2}\alpha\right) + C_4 e^{-\frac{\alpha}{2}} \sin\left(\frac{\sqrt{3}}{2}\alpha\right) \tag{4.16}$$

Now using the no-slip condition of equation (4.7), we have ;

$$\frac{\partial \psi_B(\alpha, y)}{\partial x} = 0 = \frac{\partial \psi_B(\alpha, y)}{\partial y}.$$

We differentiate equation (4.16) with respect to y to obtain;

$$\frac{\partial \psi_I(X_W, y)}{\partial y} + \frac{\partial C_3}{\partial y} e^{-\frac{\alpha}{2}} \cos\left(\frac{\sqrt{3}}{2}\alpha\right) + \frac{C_4}{\partial y} e^{-\frac{\alpha}{2}} \sin\left(\frac{\sqrt{3}}{2}\alpha\right) = 0. \tag{4.17}$$

Also, differentiating equation (4.16) with respect to x, we have

$$\frac{\partial \psi_I(X_W, y)}{\partial x} + C_3 \left[-\frac{1}{2} e^{-\frac{\alpha}{2}} \cos\left(\frac{\sqrt{3}}{2}\alpha\right) + \frac{\sqrt{3}}{2} \sin\left(\frac{\sqrt{3}}{2}\alpha\right)\right] + C_4 \left[-\frac{1}{2} e^{-\frac{\alpha}{2}} \sin\left(\frac{\sqrt{3}}{2}\alpha\right) + \frac{\sqrt{3}}{2} \cos\left(\frac{\sqrt{3}}{2}\alpha\right)\right] = 0 \tag{4.18}$$

At the boundary where $x = X_W$, by definition, $\alpha = 0$ and so, with the solution of $\psi_I(x,y)$ in equation(4.10), equation (4.17) becomes;

$$\frac{\partial \psi_I}{\partial y} + \frac{\partial C_3}{\partial y} = 0 \tag{4.19}$$

And equation (4.18) becomes;

$$-\tfrac{C_3}{2} + \tfrac{\sqrt{3}}{2} C_4 = 0$$

such that,

$$C_4 = \frac{C_3}{\sqrt{3}} \tag{4.20}$$

Integrating equation (4.19) with respect to y, we have:

$$-\int \frac{\partial \psi_I}{\partial y} dy = \int \frac{\partial C_3}{\partial y} dy$$

$$-\psi_I(X_W, y) + k = C_3. \tag{4.21}$$

where k is a constant, and so C_4 in equation (4.20) is:

$$C_4 = \frac{\psi_I(X_W, y)}{\sqrt{3}} + k \tag{4.22}$$

Now, using equations (4.21) and (4.22) in equation (4.16), the boundary stream function ψ_B becomes:

$$\psi_B(\alpha, y) = \psi_I(x,y) + [-\psi_I(x,y) + k]\, e^{-\frac{\alpha}{2}} \cos\left(\tfrac{\sqrt{3}}{2}\alpha\right) + \left[-\tfrac{\psi_I(x,y)}{\sqrt{3}} + k\right] e^{-\frac{\alpha}{2}} \sin\left(\tfrac{\sqrt{3}}{2}\alpha\right)$$

$$= \psi_I(x,y) - \psi_I e^{-\frac{\alpha}{2}} \cos\left(\tfrac{\sqrt{3}}{2}\alpha\right) - \psi_I e^{-\frac{\alpha}{2}} \sin\left(\tfrac{\sqrt{3}}{2}\alpha\right) + k e^{-\frac{\alpha}{2}} \left[\cos\left(\tfrac{\sqrt{3}}{2}\alpha\right) + \sin\left(\tfrac{\sqrt{3}}{2}\alpha\right)\right]$$

$$\Rightarrow \psi_B(\alpha, y) = \psi_I(x, y)\left[1 - e^{-\frac{\alpha}{2}}\left[\cos\left(\frac{\sqrt{3}}{2}\alpha\right) + \frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}\alpha\right)\right]\right] + ke^{-\frac{\alpha}{2}}\left[\cos\left(\frac{\sqrt{3}}{2}\alpha\right) + \frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}\alpha\right)\right] \quad (4.23)$$

Now, at the boundary where $x = X_W$, by definition, $\alpha = 0$ and so, for the boundary condition (4.7) to be satisfied, the constant k must be zero. And so, equation (4.23) reduces to:

$$\psi_B(\alpha, y) = \psi_I(x, y)\left[1 - e^{-\frac{\alpha}{2}}\left[\cos\left(\frac{\sqrt{3}}{2}\alpha\right) + \frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}\alpha\right)\right]\right] \quad (4.24)$$

For large values of x, $(x - X_W) \to x$. And by definition, $\alpha = \frac{x}{\delta_M}$. So from equation (4.24) we obtain the stream function as :

$$\psi(x, y) = \psi_I(x, y)\left[1 - e^{-\frac{x}{2\delta_M}}\left[\cos\left(\frac{\sqrt{3}}{2\delta_M}x\right) + \frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2\delta_M}x\right)\right]\right]$$

Using the interior stream function obtained in equation (4.10), we obtain :

$$\psi(x, y) = \frac{\tau_0}{\rho_c H\beta}\pi\sin\left(\frac{\pi y}{L_y}\right)\left[1 - e^{-\frac{x}{2\delta_M}}\left[\cos\left(\frac{\sqrt{3}}{2\delta_M}x\right) + \frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2\delta_M}x\right)\right]\right] \quad (4.25)$$

Now, for the free surface height η, assuming a constant Coriolis force f, the free surface height η is proportional to the stream function ψ [5]. Such that for any interval (x, L_x) in the zonal direction, we have that:

$$\eta = \frac{f}{g}\left[\frac{L_x - x}{L_x}\right]\psi = \frac{f}{g}\left[1 - \frac{x}{L_x}\right]\psi$$

So that by using equation (4.25), we obtain η as

$$\eta(x, y) = \frac{f}{g}\left(1 - \frac{x}{L_x}\right)\frac{\tau_0}{\rho_c H\beta}\pi\sin\left(\frac{\pi y}{L_y}\right)\left[1 - e^{\frac{-x}{2\delta_M}}\left(\cos\left(\frac{\sqrt{3}x}{2\delta_M}\right) + \frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}x}{2\delta_M}\right)\right)\right] \quad (4.26)$$

4.2 Results

For the contour plot of the free surface height (η), shown in equation (4.26), as displayed in figure (4.1), we use the parameters [7]

$$f = 10^{-4}s^{-1}, \quad \beta = 10^{-11}s^{-1}m^{-1}, \quad \rho_c = 10^3, \quad g = 9.81, \quad H = 5000,$$

$$L_x = 12 \times 10^5, \quad L_y = 12 \times 10^5, \quad A_h = 400, \quad \tau_0 = 0.1, \quad \delta_m = \left(\frac{A_h}{\beta}\right)^{1/3}.$$

A look at the plot in Figure (4.1) shows that streamlines in the western boarder travel through a relatively narrow area than in the eastern boarder. The currents in the western boundary are faster and narrower than the eastern boundary currents . Also, the center of gyre is close to the western boundary.

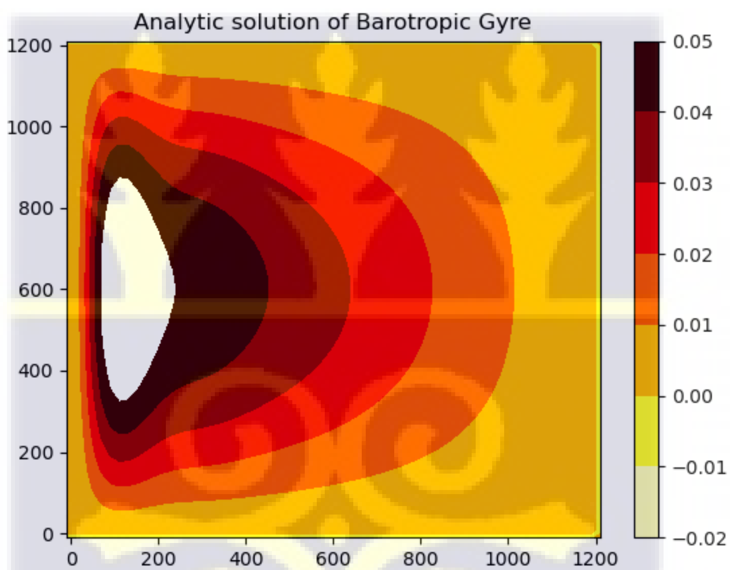

Figure 4.1: Analytic solution of Barotropic Wind Forced Ocean Gyre Circulation.

The contour plot of both the numerical solution obtained through the MITgcm [7] and the analytic solution of Barotropic wind forced Ocean gyre circulation show intensification of streamlines along the western boundary of the ocean. Albeit there is a slight difference at the center of each gyre, where the center of the gyre in the MITgcm solution is relatively smaller than that in the Analytic solution, both the solution from the MITgcm simulation and that of the analytic solution show the western intensification of ocean currents as shown in Figure (4.2), the intensification of streamlines are stronger at higher surface heights in both solutions.

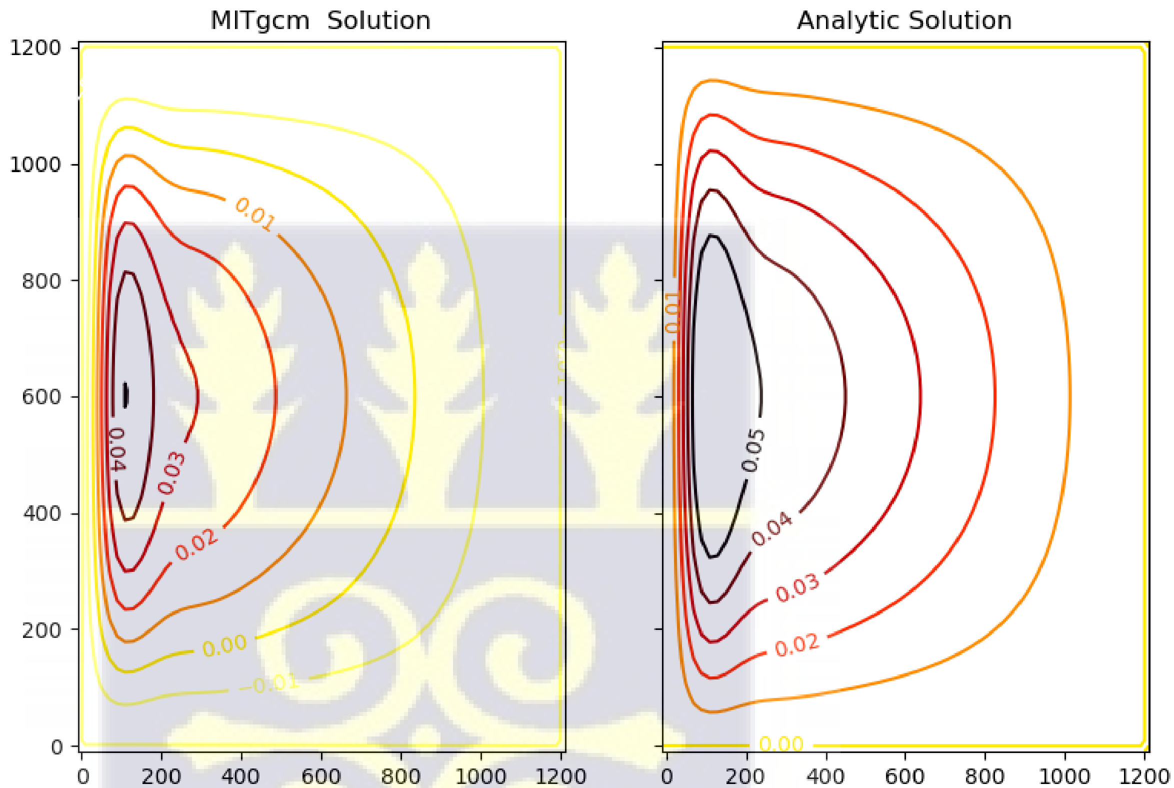

Figure 4.2: MITgcm Solution and Analytic Solution of Barotropic Wind Forced Ocean Gyre Circulation.

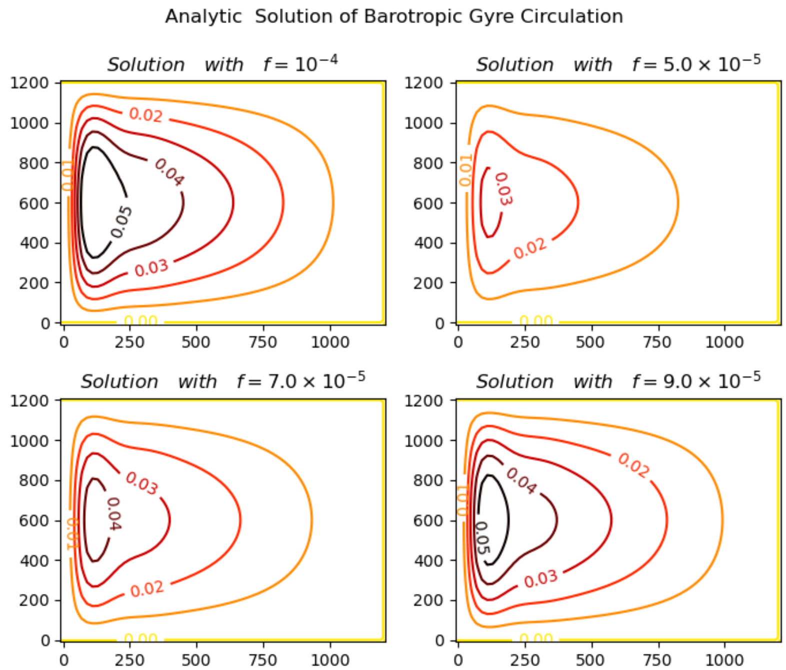

Figure 4.3: Comparison of different parameter values of the analytic solution of barotropic wind forced ocean gyre circulation.

Figure (4.3) shows how the Coriolis force influences the intensification of streamlines at the western boundary. A higher Coriolis force results in a more crowding of stream lines at the western border, whiles a low Coriolis force results in less crowding of stream lines at the western boundary. A Coriolis force set at $f = 10^{-4}s^{-1}$ produces a solution with more crowding of streamlines in the western boundary than when the Coriolis force is set relatively lower at $f = 5 \times 10^{-5}s^{-1}$. In between these two forces, a Coriolis force set at $f = 7 \times 10^{-5}s^{-1}$ produces streamlines at western boundary that are more crowded relative to that of the force set at $f = 5 \times 10^{-5}s^{-1}$ but less crowded as compared to that of the force $f = 10^{-4}s^{-1}$. Moreover, a force of $f = 9 \times 10^{-5}s^{-1}$ results in crowding of streamlines at the western boundary that are more intense relative to that of the force set at $f = 7 \times 10^{-5}s^{-1}$ and less intense relative to that of force set at $f = 10^{-4}s^{-1}$.

The effect of the Coriolis force is as a result of the fact that it is a linear function of latitude [3]. In the Northern hemisphere, a parcel of water in motion is deflected to the right side of motion due to Coriolis force as the earth rotates [3]. Near the equator, where latitude is zero, Coriolis force is negligible. In contrast the Coriolis force is strong at the poles, where latitude in 90 degrees[3]. A parcel of water near the poles and flowing towards the east is deflected towards the equator due to Coriolis force, and since Coriolis force is strong at the poles, the deflection is stronger, resulting in widely spaced streamlines at the eastern boundary. However, at the equator, a parcel of water flowing towards the west is deflected northward due to Coriolis force, and since Coriolis force is week at he equator, the deflection is week resulting in crowded streamlines along the western boundary.

In Figure (4.4), a numerical solution by the MITgcm [7] using four different values of Coriolis force shows that a high Coriolis force $f = 10^{-4}s^{-1}$ produces more intensification of steam lines in the western boundary than a small Coriolis force $f = 5 \times 10^{-5}s^{-1}$. Also, intensification produced by the force $f = 7 \times 10^{-5}s^{-1}$ is stronger than that produced by the force $f = 5 \times 10^{-5}s^{-1}$ but less than the effect of the force $f = 9 \times 10^{-5}s^{-1}$ which in turn is weaker that of the force $f = 10^{-4}s^{-1}$.

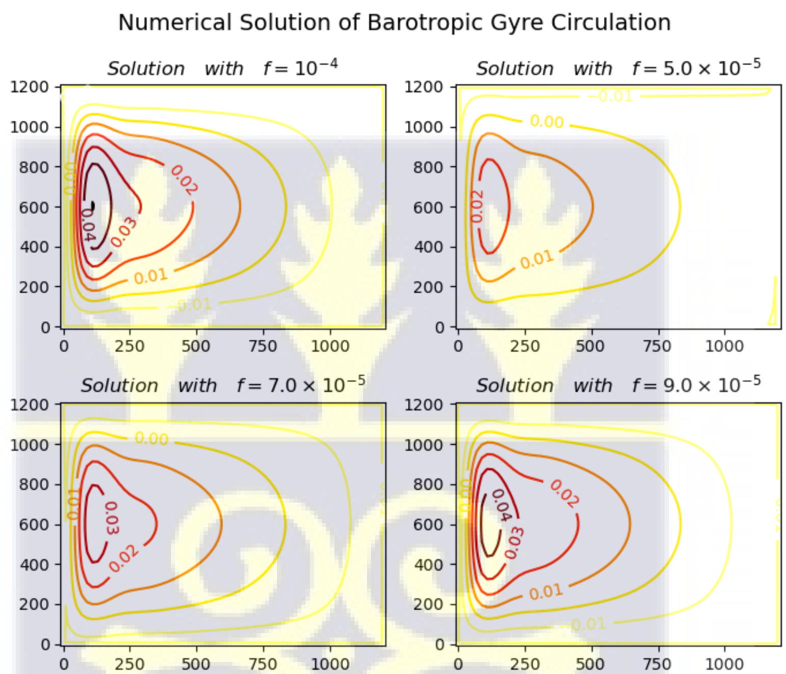

Figure 4.4: Comparison of different parameter values of the numerical solution of barotropic wind forced ocean gyre circulation by MITGcm.

The results as demonstrated in Figure (4.3) and Figure (4.4) show that:

1. The analytic solution shows western intensification of stream lines just as the numerical solution obtained by the MITgcm.
2. Western intensification of stream lines is due to the effect of Coriolis force.
3. Change in Coriolis force in both analytic and numerical solution affects the intensity of the crowding of streamlines at the western boundary.

Chapter 5

Conclusion

Western intensification is an oceanic phenomenon that has been of interest to scientists and oceanographers alike. A proper appreciation of it helps in understanding ocean circulation and its effects such as upwelling. This project explores the concept of western intensification by providing an analytic solution to barotropic wind forced ocean circulation.

Chapter 1 introduced some concepts relevant to the study, the motivation, and objectives of the study. In chapter 2, some literature related to this project was reviewed. Chapter 3 provided a comprehensive study of the theory underpinning this study. The Navier-Stokes equation for fluid dynamics which serves as a foundation in deriving the governing equations for our model was derived.

In chapter 4, we provided an analytic solution to the derived equations by first obtaining a Vorticity-stream function equation. A solution to the stream function is then obtained subject to boundary conditions. A relation between the stream function and the free surface height is used to find the solution of the free surface height. The contour plot of the analytic solution was compared to numerical solution obtained from the MITgcm simulation. Different parameter values were used to observe the impact of those parameters on both the analytic and numerical solution.

A comparison of the analytic solution with the numerical solution shows intensification of streamlines at the western border of the domain in both cases. This occurrence, known as western intensification, is due to the effect of the Coriolis force. By using different values of Coriolis force in both numerical and analytic solutions, it revealed that intensification of streamlines in both solutions is stronger with higher Coriolis force and weaker with lower Coriolis force.

Appendix A

Analytical solution of steady state equations of motion by Henry Stommel [10] .

Consider the equation:

$$v\left(D+h\right)\left(\frac{\partial f}{\partial y}\right)+\left(F\frac{\pi}{b}\right)\sin\left(\pi\frac{y}{b}\right)+R\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)=0 \tag{A.1}$$

Taking notice that (h) is smaller than $(\mathbf{D})$ and defining $\alpha=\left(\frac{D}{R}\right)\left(\frac{\partial f}{\partial y}\right)$ and $\gamma=\frac{F\pi}{Rb}$, equation (A.1) is approximated to first degree as:

$$\alpha v+\gamma\sin\left(\pi\frac{y}{b}\right)+\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=0 \tag{A.2}$$

Similar approximation of the continuity equation (2.3) $\frac{\partial[(D+h)u]}{\partial x}+\frac{\partial[(D+h)v]}{\partial y}=0$ yields:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0.$$

We introduce a stream function ψ such that:

$$v = -\frac{\partial \psi}{\partial x} \; and \; u = \frac{\partial \psi}{\partial y}. \tag{A.3}$$

Equation (A.2) then becomes :

$$\nabla^2 \psi + \alpha \frac{\partial \psi}{\partial x} = \gamma \sin\left(\frac{\pi y}{b}\right) \tag{A.4}$$

To obtain a solution, equation (A.4) is subjected to the boundary conditions:

$$\psi(0, y) = \psi(\lambda, y) = \psi(x, 0) = \psi(x, b) = 0 \tag{A.5}$$

For a general solution, we first obtain the homogeneous equation by dropping the right hand side of equation (A.4), which is solved by separation of variables. By adding the homogeneous solution to a particular integral of equation (A.4), we obtain the general solution. The particular integral [10] of (A.4) is provided as:

$$-\gamma \left(\frac{b}{\pi}\right)^2 \sin\left(\frac{\pi y}{b}\right) \tag{A.6}$$

In solving the homogeneous part of equation (A.4), the functions X and Y are defined such that, function X is dependent only on the variable x and the function Y is dependent only on the variable y and ψ is a function of X and Y such that:

$$\psi = XY$$

Equation (A.4) can then be expressed as a system of equations as:

$$Y'' + n^2 Y = 0, \tag{A.7}$$
$$X'' + \alpha X' + n^2 X = 0, \tag{A.8}$$

where n^2 is determined by condition (A.5). The solution of the system of equations is written in the form:

$$Y = \sum \left[c_j \sin(n_j y) + d_j \cos(n_j y) \right], \tag{A.9}$$

$$X = \sum \left(p_j e^{A_j x} + q_j e^{B_j x} \right), \tag{A.10}$$

where A_j and B_j are constants such that:

$A_j = -\frac{\alpha}{2} + \sqrt{\frac{\alpha^2}{4} + n_j^2}$ and $B_j = -\frac{\alpha}{2} - \sqrt{\frac{\alpha^2}{4} + n_j^2}$.

The quantities c_j, d_j, p_j and q_j are constants yet to be determined. The general solution of equation (A.4) is therefore:

$$\psi = XY - \gamma \left(\frac{b}{\pi}\right)^2 \sin\left(\frac{\pi y}{b}\right). \tag{A.11}$$

Using the boundary condition (A.5) and noting that all the $c_j s$ and $d_j s$ except c_1 which corresponds with $n_1 = \frac{\pi}{b}$ vanishes, the general solution (A.11) can be reduced to a simple closed form. Also, p_1 and q_1 absorb c_1 and so, with the subscripts dropped, ψ has the form:

$$\psi = \gamma \left(\frac{b}{\pi}\right)^2 \sin\left(\frac{\pi y}{b}\right) \left[p e^{Ax} + p e^{Bx} - 1 \right], \tag{A.12}$$

where $p = \frac{(1 - e^{B\lambda})}{(e^{A\lambda} - e^{B\lambda})}$ and $q = 1 - p$.

From equation (A.3), we obtain the velocity components u and v by differentiating ψ as:

$$u = \gamma \left(\frac{b}{\pi}\right)^2 \cos\left(\frac{\pi y}{b}\right) \left[p e^{Ax} + p e^{Bx} - 1 \right], \tag{A.13}$$

$$v = -\gamma \left(\frac{b}{\pi}\right)^2 \sin\left(\frac{\pi y}{b}\right) \left[pA e^{Ax} + pB e^{Bx} \right]. \tag{A.14}$$

Now, by integrating equations (2.1) and (2.2) the value of h at any point is obtained as :

$$h(x,y) = -\left(\frac{F}{gD}\right)\left(\frac{e^{Ax}p}{A} + \frac{e^{Bx}q}{B}\right) - \left(\frac{b}{\pi}\right)^2\left(\frac{F}{gD}\right)\left(pAe^{Ax} + qBe^{Bx}\right)\left[\cos\left(\frac{\pi y}{b}\right) - 1\right]$$
$$-\left\{\left(\frac{f\gamma}{g}\right)\left(\frac{b}{\pi}\right)^2 \sin\left(\frac{\pi y}{b}\right) - \left(\frac{\partial f}{\partial y}\right)\left(\frac{\gamma}{g}\right)\left(\frac{b}{\pi}\right)^3\left[\cos\left(\frac{\pi y}{b}\right) - 1\right]\right\}\left\{pe^{Ax} + qe^{Bx} - 1\right\} \tag{A.15}$$

For the purposes of numerical computations, the dimensions of the ocean are taken as:

$$\lambda = 10^9 cm = 10,000 km$$
$$b = 2\pi \times 10^8 cm = 6249 km$$
$$D = 2 \times 10^4 cm = 200 m$$

The maximum wind stress F is assumed one $dyne/cm^2$ and the coefficient of friction R is assumed to be 0.02. In a non-rotating ocean, the constants p and q are simple to within one percent, and given as:

$$p = e^{-\frac{\pi\lambda}{b}} \text{ and } q = 1.$$

Thus the stream function becomes:

$$\psi = \gamma\left(\frac{b}{\pi}\right)^2 \sin\left(\frac{\pi y}{b}\right)\left[e^{\frac{(x-\lambda)\pi}{b}} + e^{-\frac{x\pi}{b}} - 1\right]. \tag{A.16}$$

Appendix B

Python Code for the Analytic Solution.

A python code for the plot of the analytical solution:

$$\eta(x,y) = \frac{\tau_0}{\rho_c g H}\frac{f}{\beta}\left(1-\frac{x}{L_x}\right)\pi\sin\left(\pi\frac{y}{L_y}\right)\left[1-\exp\left(\frac{-x}{2\delta_m}\right)\left(\cos\frac{\sqrt{3}x}{2\delta_m}+\frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}x}{2\delta_m}\right)\right],$$

where:

$$\delta_m = \left(\frac{A_h}{\beta}\right)^{1/3}, f = 10^{-4}, \quad \beta = 10^{-11}, \quad \rho_c = 10^3, \quad g = 9.81, \quad H = 5000$$

$$L_x = 12\times 10^5, \quad L_y = 12\times 10^5, \quad A_h = 400, \quad \tau_0 = 0.1$$

is shown below:

```
import numpy as np
import matplotlib.pyplot as plt
#directory of data
dir  = 'C:/Users/Kasutaja/model/MITgcm/
   verification/tutorial_barotropic_gyre/run/'
      #for f=1.0*10**(-4)
dir1 = 'C:/Users/Kasutaja/model/MITgcm/
   verification/tutorial_barotropic_gyre1/run/'
```

```
      #for f=5.0*10**(-5)
dir2 = 'C:/Users/Kasutaja/model/MITgcm/
   verification/tutorial_barotropic_gyre2/run/'
      #for f=7.0*10**(-5)
dir3 = 'C:/Users/Kasutaja/model/MITgcm/
   verification/tutorial_barotropic_gyre3/run/'
      #for f=9.0*10**(-5)
dirsave = 'C:/Users/Kasutaja/Desktop/THESIS/
   Images/'

nx = 62
ny = 62
# Get the grid data for f=1.0*10**(-4)
XC  = np.fromfile(dir  + 'XC.001.001.data',
   dtype='>f4').reshape((ny, nx))
YC  = np.fromfile(dir  + 'YC.001.001.data',
   dtype='>f4').reshape((ny, nx))
Eta = np.fromfile(dir  + 'Eta
   .0000064800.001.001.data', dtype='>f4').
   reshape((ny, nx))
# Get the grid data for f=5.0*10**(-5)
XC1  = np.fromfile(dir1  + 'XC.001.001.data',
   dtype='>f4').reshape((ny, nx))
YC1  = np.fromfile(dir1  + 'YC.001.001.data',
   dtype='>f4').reshape((ny, nx))
Eta1 = np.fromfile(dir1  + 'Eta
   .0000064800.001.001.data', dtype='>f4').
   reshape((ny, nx))
# Get the grid data for f=7.0*10**(-5)
XC2  = np.fromfile(dir2  + 'XC.001.001.data',
   dtype='>f4').reshape((ny, nx))
YC2  = np.fromfile(dir2  + 'YC.001.001.data',
   dtype='>f4').reshape((ny, nx))
Eta2 = np.fromfile(dir2  + 'Eta
   .0000064800.001.001.data', dtype='>f4').
   reshape((ny, nx))
# Get the grid data for f=9.0*10**(-5)
XC3  = np.fromfile(dir3  + 'XC.001.001.data',
   dtype='>f4').reshape((ny, nx))
```

```
YC3  = np.fromfile(dir3  +  'YC.001.001.data',
   dtype='>f4').reshape((ny, nx))
Eta3 = np.fromfile(dir3  +  'Eta
   .0000064800.001.001.data', dtype='>f4').
   reshape((ny, nx))
#Code for numerical solution
plt.contourf(XC/1000, YC/1000, Eta, np.linspace
   (-0.02, 0.05,8), cmap='hot_r')
plt.title('MITgcm Barotropic Gyre solution')
plt.colorbar()
plt.savefig(dirsave+'MITGCM.png')
#Code for Analytic solution
f = 10**(-4)
beta=10**(-11)
rhoc=1000.0
g =9.81
H=5000.0
Lx, Ly=12*10**(5), 12*10**(5)
Ah = 400.0
tau0= 0.1
delm = (Ah/beta)**(1.0/3.0)
p1 = (tau0/(rhoc*g*H))*(f/beta)
p2 = np.sqrt(3.0)/(2.0*delm)
p3 = 1.0/np.sqrt(3.0)
pi = np.pi
x=XC[1,:]
y=YC[:,1]
Lenx = np.size(x)
Leny = np.size(y)

Etaa = np.zeros((Leny,Lenx))
for i in range(0,Lenx):
for j in range(0,Leny):
Etaa[j,i] = p1*(1-(x[i]/Lx))*pi*np.sin(pi*(y[j
   ]/Ly))*\
(1- np.exp(-x[i]/(2*delm))*( np.cos(p2*x[i]) +
   p3*np.sin(p2*x[i]))
plt.contourf(XC/1000, YC/1000, Etaa, np.
   linspace(-0.02,0.05,8), cmap='hot_r')
```

```
plt.colorbar()
plt.title('Analytic solution of Barotropic Gyre
    ')
plt.savefig(dirsave+'Analytic.png')
plt.show()

#Code for comparing numericalsolution with
    analytic solution
fig, axes = plt.subplots(1,2,figsize=(9,6),
    sharex=True, sharey=False)
img1 = axes[0].contour(XC/1000,YC/1000,Eta,np.
    linspace(-0.02,0.05,8),cmap='hot_r')
axes[0].set_ylabel('')
axes[0].set_title('MITgcm  Solution')
img2 = axes[1].contour(XC/1000,YC/1000,Etaa,np.
    linspace(-0.02,0.05,8),cmap='hot_r')
axes[1].set_yticklabels('')
axes[1].set_title('Analytic Solution')
ax=axes[1]
plt.clabel(img1, inline= True,)
plt.clabel(img2, inline= True, )
plt.savefig(dirsave+'Compare.png')
plt.show()

# Code for comparing numerical solutions with
    different coriolis force
fig, axes = plt.subplots(2,2,)
plt.title('Numerical solution of Barotropic
    Gyre')
axes[0,0].contour(XC/1000,YC/1000,Eta,np.
    linspace(-0.02,0.05,8),cmap='hot_r')
axes[0,0].set_title(r'$Solution \quad with \
    quad f = 10^{-4}$')
axes[0,1].contour(XC1/1000,YC1/1000,Eta1,np.
    linspace(-0.02,0.05,8),cmap='hot_r')
axes[0,1].set_title(r'$Solution \quad with \
    quad f = 5.0 \times 10^{-5}$')
fig.tight_layout()
fig.tight_layout()
```

```
plt.suptitle('Numerical Solution of Barotropic
   Gyre Circulation', fontsize=14)
axes[1,0].contour(XC2/1000,YC2/1000,Eta2,np.
   linspace(-0.02,0.05,8),cmap='hot_r')
axes[1,0].set_title(r'$Solution \quad with \
   quad f = 7.0 \times 10^{-5}$')
axes[1,1].contour(XC3/1000,YC3/1000,Eta3,np.
   linspace(-0.02,0.05,8),cmap='hot_r')
axes[1,1].set_title(r'$Solution \quad with \
   quad f = 9.0 \times 10^{-5}$')
fig.tight_layout()
fig.tight_layout()
plt.savefig(dirsave+'parameter1.png')
plt.show()

# Code for comparing analytic solutions with
   different coriolis force
# Coriolis force
f = 10**(-4)
f1 = 5.0*10**(-5)
f2 = 7.0*10**(-5)
f3 = 9.0*10**(-5)

beta=10**(-11)
rhoc=1000.0
g =9.81
H=5000.0
Lx, Ly=12*10**(5), 12*10**(5)
Ah = 400.0
tau0= 0.1
delm = (Ah/beta)**(1.0/3.0)
p1 = (tau0/(rhoc*g*H))*(f/beta)
p2 = np.sqrt(3.0)/(2.0*delm)
p3 = 1.0/np.sqrt(3.0)

q1 = (tau0/(rhoc*g*H))*(f1/beta)
q2 = p2
q3 = p3
r1 = (tau0/(rhoc*g*H))*(f2/beta)
```

```
r2 = p2
r3 = p3
s1 = (tau0/(rhoc*g*H))*(f3/beta3)
s2 = p2
s3 = p3
pi = np.pi
x=XC[1,:]
y=YC[:,1]
Lenx = np.size(x)
Leny = np.size(y)
Etaa = np.zeros((Leny,Lenx))
for i in range(0,Lenx):
for j in range(0,Leny):
Etaa[j,i] = p1*(1-(x[i]/Lx))*pi*np.sin(pi*(y[j
   ]/Ly))*\
(1- np.exp(-x[i]/(2*delm))*( np.cos(p2*x[i]) +
   p3*np.sin(p2*x[i])) )

Etaa1 = np.zeros((Leny,Lenx))
for i in range(0,Lenx):
for j in range(0,Leny):
Etaa1[j,i] = q1*(1-(x[i]/Lx))*pi*np.sin(pi*(y[j
   ]/Ly))*\
(1- np.exp(-x[i]/(2*delm1))*( np.cos(q2*x[i]) +
    q3*np.sin(q2*x[i])) )

Etaa2 = np.zeros((Leny,Lenx))
for i in range(0,Lenx):
for j in range(0,Leny):
Etaa2[j,i] = r1*(1-(x[i]/Lx))*pi*np.sin(pi*(y[j
   ]/Ly))*\
(1- np.exp(-x[i]/(2*delm2))*( np.cos(r2*x[i]) +
    r3*np.sin(r2*x[i])) )

Etaa3 = np.zeros((Leny,Lenx))
for i in range(0,Lenx):
for j in range(0,Leny):
Etaa3[j,i] = s1*(1-(x[i]/Lx))*pi*np.sin(pi*(y[j
   ]/Ly))*\
```

```
(1- np.exp(-x[i]/(2*delm3))*( np.cos(s2*x[i]) +
     s3*np.sin(s2*x[i])) )

fig, axes = plt.subplots(2,2,)
plt.title('Analytic solution of Barotropic Gyre
    ')
axes[0,0].contour(XC/1000,YC/1000,Etaa,np.
    linspace(-0.02,0.05,8),cmap='hot_r')
axes[0,0].set_title(r'$Solution \quad with \
    quad f = 10^{-4}$')
axes[0,1].contour(XC/1000,YC/1000,Etaa1,np.
    linspace(-0.02,0.05,8),cmap='hot_r')
axes[0,1].set_title(r'$Solution \quad with \
    quad f = 5.0 \times 10^{-5}$')

fig.tight_layout()
fig.tight_layout()
plt.suptitle('Analytic Solution of Barotropic
    Gyre', fontsize=14)
axes[1,0].contour(XC/1000,YC/1000,Etaa2,np.
    linspace(-0.02,0.05,8),cmap='hot_r')
axes[1,0].set_title(r'$Solution \quad with \
    quad f = 7.0 \times 10^{-5}$')
axes[1,1].contour(XC/1000,YC/1000,Etaa3,np.
    linspace(-0.02,0.05,8),cmap='hot_r')
axes[1,1].set_title(r'$Solution \quad with \
    quad f = 9.0 \times 10^{-5}$')

fig.tight_layout()
fig.tight_layout()
fig.suptitle('Analytic  Solution of Barotropic
    Gyre Circulation')
plt.savefig(dirsave+'parameter.png')
plt.show()
```

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