RIORDAN ARRAYS, SHEFFER POLYNOMIALS AND THE STIRLING TRANSFORM

KINGSLEY FRIMPONG TANDOH

10247903

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Declaration

This thesis was written in the Department of Mathematics, University of Ghana, Legon from September 2014 to July 2015 in partial fulfilment of the requirement for the award of Master of Philosophy degree in Mathematics under the supervision of Dr. Margaret L. McIntyre and Dr. Douglas Adu - Gyamfi of the University of Ghana.

I hereby declare that except where due acknowledgement is made, this work has never been presented wholly or in part for the award of a degree at the University of Ghana or any other University.

Signature: ....
Kingsley Frimpong Tandoh
(Student)

Signature: ....
Dr. Margaret Lesley McIntyre
(Supervisor)

Signature: ....
Dr. Douglas Adu Gyamfi
(Supervisor)

Abstract

This study brings together a connection among the Sheffer polynomial sequences, the Riordan arrays and the Stirling numbers. Beside the algebraic view, a differential equation that gives rise to Sheffer sequences is discussed; serving as a differential approach to the study of Sheffer polynomials.

Numbers generated by the function $e^{e^t-1} := e^{Gt}$ have already been shown to satisfy the congruence relation $G_{n+p} \equiv G_n + G_{n+1} (mod\ p)$ for any p prime. We derive a congruence relation for the Touchard polynomial sequence and the Stirling numbers of the second kind analogous to the known congruence relation for the G_n.

Also, some Riordan arrays with generating functions related to e^{e^t-1} for polynomial sequences such as the Touchard polynomial, Toscano polynomial, Charlier polynomial and the Poisson - Charlier polynomial are discussed, and their corresponding inverses constructed.

Dedication

This work is wholly dedicated to the loving memory of my cousin Juliana Fosu Tabiri; whom I dearly loved.

Acknowledgements

To God be the glory! That's how I simply want to show my appreciation to my heavenly Father. For I have always been surprised by joy through the interventions He orchestrated in my way. Since, it seems more so to me that my life has been tried almost in every way right from the beginning of my study to the end.

Having settled the scores above, let me venture to add that I am so grateful to Priscilla Manford for releasing her laptop for the work right from start to finish, and Fred Soglohu, who whole heartedly offered to type portions of this work. I pray them well in life.

To my able supervisors, I say thank you very much.

Contents

Introduction and Motivation

There are several ways to describe a polynomial sequence. It may be a description by generating functions, by method of transforms or by a recurrence relation. Still, others may arise as a solution to certain differential equations or by an orthogonality condition. Of particular interest, among such polynomial sequences is the Sheffer class of polynomial sequences. They arise as solutions to certain differential equations, satisfy some recurrence relation (for those known), and possess generating functions which can be used to uniquely determine each.

In recent years, there has been a recapture of interest in the study of this kind of polynomial sequences. Among such key areas of interest, on one hand, is to seek for a differential equation and recursive relations for the Sheffer sequences that will; on the other hand, provide a defining characterization of these sequences, as well as provide a computational means for determining them.

Howbeit, we would like to point out here, first, that there are some things which stand out very sharply from the records of literature done within this research circle which calls for further study. For example, the numbers generated by the function e^{e^t-1}. These numbers are of much importance and have an interesting history. Because, as outlined in [3], the structure of the series $eG_n := \sum_{k=0}^{\infty} \frac{k^n}{k!}, \quad n = 0, 1, 2, \ldots$ hints on the exponential polynomials (also known as single - variable Bell polynomials) should it be approached from the analysis point of view.

Thus, consider the expansion

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

Differentiating and multiplying both sides by x yields;

$$xe^x = \sum_{k=0}^{\infty} \frac{k\, x^k}{k!}.$$

From which we obtain for $x = 1, \quad eG_1 = e \Rightarrow G_1 = 1.$

Repeating this process $n-$ times gives

$$\left(x\,\frac{d}{d\,x}\right)^n e^x = \sum_{k=0}^{\infty} \frac{k^n\, x^k}{k!} = T_n(x)\, e^x, \tag{1}$$

where $T_n(x)$ are polynomials of degree n. We observe that (1) is a defining relation from which we get

$$x\,(T_n(x)\, e^x)' = x\, \left(T_n'(x) + T_n(x)\right)\, e^x = T_{n+1}(x)\, e^x \tag{2}$$

and so it follows that

$$T_{n+1}(x) = x\, \left(T_n'(x) + T_n(x)\right) \tag{3}$$

is a recurrence relation for the polynomial sequence (T_n). Furthermore, it provides a means to explicitly compute T_n for all $n \in \mathbb{N}$.
Note: $T_n'(x) = \frac{d}{d\,x} T_n(x)$.

The Riordan arrays can be described by generating functions. Interesting enough, this function e^{e^t-1} appears as a generating function, (either in the same form or some related yet modified form) in certain Riordan arrays that generate certain Sheffer polynomial sequences. For example Shapiro et al. [19], He et al. [9] have revealed the connection between the Riordan arrays and the Sheffer polynomial sequences.

Thus, the pair consisting of the set of all Riordan arrays $\mathcal{R}$, and the operator $*$ (called the Riordan multiplication operator) forms a group under multiplication. Similarly, the pair $(\mathcal{S}, \otimes)$ forms a group called the Sheffer group. (Where $\mathcal{S}$ is the set of all Sheffer sequences, and $\otimes$ is the multiplicative operator). Algebraically, an isomorphism has been proved by He et al. [9] to exist between $(\mathcal{R}, *)$ and $(\mathcal{S}, \otimes)$. The isomorphism provides for us a way to construct the polynomial inverse of a Sheffer polynomial.

Now here is how the whole work has been organized; the first chapter discusses generating functions and the Riordan arrays. The Riordan group is introduced, and particular attention is paid to the generating function e^{e^t-1}. Chapter two basically deals with the Stirling numbers; both the first and second kind, to the extent we saw applicable to the whole work in general.

In the third chapter, we describe and characterize Sheffer polynomial sequences in terms of recursive relations, operation formulas, generating functions and as rising from certain differential equations. As such, we have approached our discussion on the Sheffer polynomial sequences from a differential equation, determinantal (or matrix) and operators point of view. The delta operators were found to be very instrumental in this study. A connection among the Riordan arrays, the Stirling numbers and the Sheffer polynomials is drawn in this chapter. With that established, we were able to construct inverses of some Sheffer polynomials.

The last chapter (4) introduces some congruence relations among the Touchard polynomial sequences and the Stirling numbers of the second kind. The latter was partly left as a conjecture for further study.

Chapter 1

Generating Functions And The Riordan Arrays

1.1 Generating Series And Functions

A polynomial in a variable z, say is a linear combination of powers of z

$$a_0 + a_1 z + a_2 z^2 + a_3 z^3 + ... + a_n z^n + ...$$

where it is understood, by definition that a linear combination has only finitely many non - zero coefficients.

Suppose we consider the variable z indeterminate, we notice that this linear combination simply generates a sequence (a_n), where (a_n) is in a field $\mathbb{K}$ ($i.e.\mathbb{R}$ or $\mathbb{C}$) for all $n \in \mathbb{N}$. Now, if such a linear combination (function) exists; which expresses the set $\{a_n\}_{n\in\mathbb{N}}$, we shall refer to it as a generating function. Formally, we shall define a generating function (GF) of a sequence $(a_n)_{n\in\mathbb{N}}$ as follows

$$f(z) = \sum_{n\geq 0} a_n \frac{z^n}{c_n}$$

where c_n is a fixed sequence (for all $n \in \mathbb{N}$) of non - zero constants with $c_0 \neq 0$.

Moreover, we say that the function $f(z)$ generates the set $\{a_n\}_{n\in\mathbb{N}}$ and is independent of z. In particular, if $c_n = 1$ for all $n \in \mathbb{N}$, then we have $f(z) = \sum_{n\geq 0} a_n z^n$, and f in this case is called an ordinary generating function (OGF). If $c_n = n!$ for all $n \in \mathbb{N}$, the $f(z)$ is an exponential generating function (EGF). i.e. $f(z) = \sum_{n\geq 0} a_n \frac{z^n}{n!}$.

1.1.1 Elementary Relation of Generating Functions

Suppose (a_n), (b_n) and (c_n) for $n \in \mathbb{N}$ are sequences with GFs
$A(z)$, $B(z)$ and $C(z)$ respectively. We have

i.

$$a_n = b_n + c_n \Longleftrightarrow A(z) = B(z) + C(z)$$

ii

$$a_n = b_0 c_n + b_1 c_{n-1} + \ldots + b_n c_0 = \sum_{k=0}^{n} b_k c_{n-k} \Longleftrightarrow A(z) = B(z)C(z).$$

Note that this sum (above) is often times called convolution.We say, $(a_n)_{n\in\mathbb{N}}$ is a convolution of (b_n) and (c_n) whenever $A(z) = B(z)C(z)$. This may also be regarded in Cauchy algebra as a definition of the product of the sequences (c_n) and (b_n) for $n \in \mathbb{N}$.

The second relation expressing the a_n sequence as a product of some other sequences involves the ordinary generating functions. In the case where exponential generating functions are involved, we have

$$a_n = b_0 c_n + n b_1 c_{n-1} + \ldots + \binom{n}{k} b_k c_{n-k} + \ldots + b_n c_0.$$

Note also that both the sum and product of GFs are commutative and associative.

Relating OGFs And EGFs

Let (a_n) be a sequence with OGF $A(z)$. Suppose we can find an EGF for (a_n), say $E(z)$, then these generating functions are formally related by

$$A(z) = \int_0^{\infty} e^{-x} E(xz)\,dx. \tag{1.1}$$

To justify this, we recall the Euler integral for the Gamma function.
i.e.

$$n! = \int_0^\infty e^{-x} x^n dx. \tag{1.2}$$

Then,

$$\begin{aligned} A(z) &= \sum_{n\geq 0} a_n z^n \\ &= \sum a_n \left(\frac{z^n}{n!}\right) \int_0^\infty e^{-x} x^n dx \\ &= \int_0^\infty e^{-x} dx \sum a_n \frac{\left(xz\right)^n}{n!} \\ &= \int_0^\infty e^{-x} E(xz) dx \end{aligned}$$

as required.

We mention that, given any OGF, we can use the Stirling numbers to transform it into an exponential function. Next we fix some notation.

Notation

Remember that for any GF $f(z)$, we are most interested in the number sequence it generates. We shall then make use of this notation $[z^n]$, called the Coefficient Extraction operator whenever we want to extract the coefficients of the GF.

Definition 1.1.2. The notation $[z^n]$ is the *coefficient extraction operator* such that if $f(z) = \sum_{n\in\mathbb{N}} a_n \frac{z^n}{c_n}$, then $c_n[z^n]f(z) = a_n$.

1.1.3 Formal Power Series

A formal power series (f.p.s) in auxiliary variable z is of the form

$$\sum_{n\geq 0} f_n z^n.$$

For a field $\mathbb{K}$ with characteristic zero, let $\mathbb{K}[[z]]$ be the domain of the f.p.s in the indeterminate z with coefficient in the field $\mathbb{K}$. If we let $\mathcal{F}$ be the set of all f.p.s, then for any $f \in \mathcal{F}$, $f(z) = \sum_{n\geq 0} f_n z^n$ with coefficients $f_n \in \mathcal{F}, \quad \forall n \in \mathbb{N}$.

The minimal number $k \in \mathbb{N}$ such that f_k does not vanish ($f_k \neq 0$) is called the order of $f(z) \in \mathcal{F}$. We write $Ord(f) = k$. So we let $\mathcal{F}_k$ be the set of all f.p.s of order k. Hence the unit in $\mathbb{K}[[z]]$ is exactly the power series of order $k = 0$.

Any f.p.s f such that $f(0) \neq 0$ has a multiplicative inverse denoted by $f^{-1} \quad or \quad \frac{1}{f}$. i.e. $f_0 = f(0)$ is invertible in $\mathbb{K}$. Therefore, the series f has a multiplicative inverse if and only if it is invertible in $\mathbb{K}[[z]]$. Let $\mathcal{F}_0$ denote the set of invertible f.p.s.

Further, if $f_0 = f(0) = 0$, then f has a reverse or a compositional inverse series, denoted by $\bar{f}$. Hence, the reverse series $\bar{f}$ of a f.p.s f exists and satisfies $f(\bar{f}(z)) = \bar{f}(f(z)) = z$ if and only if $Ord(f) \geq 1$. Let $\mathcal{F}_1$ denote the set of compositionally invertible f.p.s.

1.1.4 The Generating Function $e^{(e^z-1)}$

This section of the work is heavily influenced G. T. Williams' paper "Numbers generated by the function $e^{(e^z-1)}$". As a matter of fact, most of the ideas that are discussed here shall be the tools we shall employ in studying the Sheffer's polynomial; and to be more precise, the Touchard polynomial, the Toscano polynomial and the Stirling numbers later on in this work. We proceed by making the following definition; stating some algebraic properties as well.

Definition 1.1.5. For any positive $n \in \mathbb{N}$, we define a sequence G_n by $G_n e = \sum_{r=0}^{\infty} \frac{r^n}{r!}$.

The sequence $G_n \quad n \in \mathbb{N}$ is the number of ways in which a product of n distinct primes can be factored. (Silvio Minetola, Principii di Analisi Combinatoria, as cited by [24]).

In a more general sense, we have

$$(aG+b)^n e = \sum_{r=0}^{\infty} \frac{(ar+b)^n}{r!}. \tag{1.3}$$

[1]]

Lemma 1.1.6. ([24]) *For* $n \in \mathbb{N}$, $\quad G_{n+1} = (G+1)^n$.

Proof. From (1.3) , put $a = b = 1$. We have

$$\begin{aligned}(G+1)^n e &= \sum_{r=0}^{\infty} \frac{(r+1)^n}{r!}.\frac{r+1}{r+1} \\ &= \sum_{s=1}^{\infty} \frac{s^{n+1}}{s!} \\ &= G_{n+1}e.\end{aligned}$$

Hence, $(G+1)^n = G_{n+1}$ as required. □

Corollary 1.1.7. ([24]) $\quad G_n = G(G-1)^n$.

Theorem 1.1.8. ([24]) $\quad e^{(e^z-1)} = e^{G_z}$,
where e^{G_z} *is the symbolic representation of* $\sum_{r=0}^{\infty} G_r \frac{z^r}{r!}$.

Proof. We have $e^{(e^z-1)} = e^{G_z} \iff e^{e^z} = ee^{G_z}$. It then follows that;

$$\begin{aligned}e^{e^z} &= \sum_{k=0}^{\infty} \frac{e^{kz}}{k!} \\ &= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{1}{k!}.\frac{(kz)^n}{n!} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{k^n}{k!}.\frac{z^n}{n!} \\ &= \sum_{n=0}^{\infty} eG_n\frac{z^n}{n!} \\ &= e\sum_{n=0}^{\infty} G_n\frac{z^n}{n!} \\ &= e.e^{G_z} \qquad \text{as required.}\end{aligned}$$

[1]We shall adopt Williams' symbolic convention. Thus, whenever we write a product of $G's$, say $G^iG^jG^k...$ we mean the sum of the exponents to be taken as a subscript $G_{i+j+k+...}$

□

From this, the following recurrence relation is obtained for the $G's$.

Theorem 1.1.9. ([24], Theorem 5) $G(G-1)...(G-n+1) = 1$ *for* $n = 0, 1, 2, \ldots$.

Proof. To prove this, we use $\ln(1+z)$ in place of z in Theorem (1.1.8). This gives

$$e^{e^{\ln(1+z)-1}} = e^{G\ln(1+z)}$$

so that

$$e.e^{z} = e.e^{G\ln(1+z)}$$

thus,

$$\begin{aligned} e^{z} &= e^{G\ln(1+z)} \\ &= (1+z)^{G} \\ &= \sum_{n\geq 0} G(G-1)...(G-n+1)\frac{z^{n}}{n!}. \end{aligned}$$

Hence $G(G-1)...(G-n+1) = 1$ as required.

□

The numbers G have the following number - theoretic property.

Lemma 1.1.10. ([24], Lemma 3) *Let p be any prime number, then*

$$G_{p+n} \equiv (G_n + G_{n+1}) \, mod \, p. \tag{1.4}$$

In the case $n = 0$, we obtain

$$G_p \equiv 2 \, mod \, p. \tag{1.5}$$

Take for instance $G_3 = 5$ when $p = 3$, then $G_3 \equiv 2 \, mod \, 3$ as expected.

1.2 Riordan Array and Group

Introduction

1.2.1 Riordan Array / Matrix

Definition 1.2.2. (Riordan Array/Matrix)
A *Riordan array* with respect to the sequence $c_n, n \in \mathbb{N}$ is an infinite lower triangular matrix $D = \{d_{n,k}\}_{0 \leq k \leq n}$ generated by a pair $(g(z), f(z))$ of power series whose G·F of the k^{th} column is given by

$$g(z)\frac{f(z)^k}{c_k} = \sum_{n \geq 0} d_{n,k} \frac{z^n}{c_n},$$

where $g \in \mathcal{F}_0$ ($g_0 \neq 0$) and $f \in \mathcal{F}_1$ (meaning $f_1 = f^{'}(0) \neq 0$ or $(f) = 1$). Applying the coefficient extraction operator, we obtain $d_{n,k} = c_n[z^n]g(z)\frac{f(z)^k}{c_k}$.

Now, for any pair $(g(z), f(z))$ such that $g \in \mathcal{F}_0$ and$f \in \mathcal{F}_1$, the Riordan array so obtained is referred to as a *Proper Riordan array*. Hence, any proper Riordan array is invertible. This means $D = \{d_{n,k}\}_{0 \leq k \leq n}$ is such that it's diagonals are non-zero (i.e $d_{n,n} \neq 0, \forall n \in \mathbb{N}$) and so the diagonal sums are just the row sums of the vertically stretched array $(g(z), f(z))$. Hence the OGF is $\dfrac{g(z)}{1 - zf(z)}$.

It has been seen in the work of [19] Shapiro et al. how to develop a rule for multiplying Riordan arrays. This development sets a stage to stating the fundamental theorem of Riordan (which follows later) and to defining the Riordan group.
Suppose we have two Riordan arrays $(g(z), f(z))$ and $(d(z), h(z))$, and let $(a_0, a_1, \ldots)^T$ be a column vector with $A(z)$ as the GF for the sequence (a_n).
Suppose further that we obtained the column vector $(b_0, b_1, \ldots)^T$ after multiplying $(g(z), f(z))$ and $A(z)$, i.e.

$$B(z) = g(z)A(f(z)),$$

where $B(z)$ is the GF of the sequence (b_n).

Now, replace $A(z)$ with the second array $(d(z), h(z))$ and consider this matrix one column at a time. Then we will eventually arrive at the following rule for multiplying Riordan arrays,

$$(g(z), f(z)) * (d(z), h(z)) = (g(z) \cdot d(f(z)), h(f(z)))\,.$$

Definition 1.2.3. The operator $*$ such that $(g(z), f(z)) * (d(z), h(z)) =$ $(g(z) \cdot d(f(z)), h(f(z)))$ is called the *Riordan matrix product*, and also satisfies the usual row-by-column product of two Riordan arrays (which is also a Riordan array).

Theorem 1.2.4. *The set of all Riordan arrays $\mathcal{R}$ under the Riordan matrix multiplication $(\mathcal{R}, *)$ forms a group called the Riordan group.*

Proof. $\mathcal{R}$ is non-empty since the matrix $(1(x), Id(x))$ which is everywhere zero (0) except at it's main diagonal where it contains 1 belongs to the set $\mathcal{R}$. Further $I = (1(x), Id(x))$ acts as the identity element for the Riordan group. Thus for any $(g(x), f(x)) \in \mathcal{R}$,

$$(g(x), f(x)) \cdot (1(x), Id(x)) = (g(x), f(x)) = (1(x), Id(x)) \cdot (g(x), f(x)).$$

From these facts, it follows easily by deduction that the inverse $(g(x), f(x))^{-1}$ of the Riordan array $(g(x), f(x))$ is given by

$$(g(x), f(x))^{-1} = \left(\frac{1}{g(\tilde{f}(x))}, \tilde{f}(x)\right).$$

Lastly, $*$ is associative since matrix multiplication is associative. Hence $(\mathcal{R}, *)$ is a group. $\square$

According to [21], Theorem 2.3;

Theorem 1.2.5. *The Fundamental Theorem of Riordan Arrays (FTRA)*

$$(g(z), f(z)) \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \end{bmatrix},$$

where the generating functions of the two column vectors are $A(z)$ and $B(z)$ respectively. Then this identity is true if and only if $g(z)A(f(z)) = B(z)$.

Proof. From LHS

$$\begin{bmatrix} | & | & | & \\ g & gf & gf^2 & \dots \\ | & | & | & \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \end{bmatrix}$$

we get

$$\begin{aligned} & a_0 g(z) + a_1 g(f(z)) + a_2 g(f^2(z)) + \dots \\ = & \; g(a_0 + a_1 f(z) + a_2 f^2(z) + \dots) \\ = & \; g(z)A(f(z)) \\ = & \; B(z) \end{aligned}$$

as required. $\square$

Chapter 2

Stirling Numbers

Introduction

In sums that relate powers of a variable to its factorials, the numbers that appear (as coefficients) are the Stirling numbers. Naturally, these numbers bridge differential calculus and finite difference calculus. Since the factorial in finite difference calculus plays the same role as powers do in differential calculus. We speak of Stirling numbers of the first kind, and Stirling numbers of the second kind.

2.1 Stirling Number of the First Kind

Definition 2.1.1. The *Stirling number of the first kind* $s(n,k)$ is the coefficient of x^k in the falling factorial polynomial x; $(x)_n = x(x-1)...(x-n+1)$ of degree $n \geq 0$, with $s(0,0) := 1$.

Thus, we can explicitly write

$$(x)_n = \sum_{k=0}^{n} s(n,k)x^k,$$

and also the identity $s(n,n) = 1$ for all $n \geq 0$ holds.

Proposition 2.1.2. *The Stirling numbers $s(n,k)$ satisfy the recurrence relation* $s(n,k) = s(n-1,k-1) - (n-1).s(n-1,k)$

Proof. Since $(x)_n = x(x-1)\dots(x-n+1)$ and $s(n,k)$ is the coefficient of x^k for $0 < k < n$ in this polynomial, write $(x)_n = (x)_{n-1}.(x-n+1)$.
Since $(x)_{n-1} = x(x-1)\dots(x-(n-1)+1)$, then $(x)_n = (x)_{n-1}.(x-n+1) = x(x)_{n-1} - (n-1)(x)_{n-1}$ from which we can extract the coefficients to obtain the result $s(n,k) = s(n-1,k-1) - (n-1).s(n-1,k)$ as required. $\square$

Below is a table containing some of the first values of $s(n,k)$.

n \ k	0	1	2	3	4	5	6	7	8
0	1								
1	0	1							
2	0	-1	1						
3	0	2	-3	1					
4	0	-6	11	-6	1				
5	0	24	-50	35	-10	1			
6	0	-120	274	-225	85	-15	1		
7	0	720	-1764	1624	-735	175	-21	1	
8	0	-5040	13068	-13132	6769	-1960	322	-28	1

From the table, we see that the Stirling number of the first kind $s(n,k)$ can be negative (signed). The absolute value of $s(n,k)$ describes the number of permutations on n objects with exactly k cycles.

Definition 2.1.3. The *unsigned Stirling number* of the first kind is $|s(n,k)|$, and describes the number of permutations of an n - set with exactly k - cycles, for all $n \in \mathbb{N}$ with $k = 0, 1, 2, ..., n$.

Proposition 2.1.4. *The Stirling numbers $|s(n,k)|$ satisfy*

i *the recurrence relation* $|s(n,k)| = |s(n-1,k-1)| + (n-1).|s(n-1,k)|$ *and*

ii *the identity* $\sum_{k=0}^{n} |s(n,k)| = n!$

Proof of i. Suppose we have a permutation of $(n-1)$ objects and we want to form a new permutation with n objects by introducing a distinct object. Then, there are exactly two ways of doing this.
First case; we form a singleton cycle, leaving the extra object fixed. This increases the number of the cycles by 1 and so yields the term $|s(n-1,k-1)|$ in the relation.
Second case; we insert the object into one of the existing cycles.
Consider an arbitrary permutation of $(n-1)$ objects with k cycles. To form the new permutation, we insert the new object before any of the $(n-1)$ objects already present. There are $(n-1)|s(n-1,k)|$ ways of doing so. Hence, by taking the sum of the terms in both cases, we exhaust all the possibilities and thus the relation

$$|s(n,k)| = |s(n-1,k-1)| + (n-1).|s(n-1,k)|$$

holds. □

Proof of ii. For $n = 0$, we have by definition that $|s(0,0)| = 1$.
Now, assume for n, the summation $\sum_{k=0}^{n} |s(n,k)| = n!$ holds.
Then, consider

$$\begin{aligned}
\sum_{k=0}^{n+1} |s(n+1,k)| &= \sum_{k=0}^{n+1} |(s(n,k-1)| + n|s(n,k)|) \qquad by \quad (2.1.4) \\
&= \sum_{k=0}^{n+1} |s(n,k-1)| + n\sum_{k=0}^{n+1} |s(n,k)| \\
&= \sum_{k=1}^{n+1} |s(n,k-1)| + n\sum_{k=0}^{n+1} |s(n,k)| \qquad \text{since} \quad |s(n,-1)| = 0 \\
&= \sum_{k=0}^{n} |s(n,k)| + n\sum_{k=0}^{n+1} |s(n,k)|
\end{aligned}$$

$$\begin{aligned}\sum_{k=0}^{n+1} |s(n+1,k)| &= n! + n(n!) \qquad \text{since} \quad |s(n,n+1)| = 0 \\ &= (n+1)!\end{aligned}$$

as required.
Hence $\sum_{k=0}^{n} |s(n,k)| = n!$ for all n. $\square$

2.1.5 Representing $s(n,k)$ by graphs

We take the permutation $\sigma \in S(n)$ as a directed graph with $i \longrightarrow j$ if $j = \sigma(i)$. In figure (2.1) and (2.2) , we give the graphs for the Stirling numbers $s(n,k)$ where $n = 5$ and $k = 0, 1, ..., n$.

Notation

It should be noted that there are different notations used by different authors to represent the Stirling numbers. Thus, there is no standardized notation for the Stirling numbers yet. Owing to this, the following have been adopted in this work to denote the Stirling numbers.
$s(n,k) = \left[\begin{array}{c} n \\ k \end{array} \right] (-1)^{n-k}$ shall denote the ordinary signed Stirling numbers of the first kind, while;

$|s(n,k)| = \left[\begin{array}{c} n \\ k \end{array} \right]$ shall be used to denote the unsigned Stirling numbers of the first kind.
The Stirling number of the second kind shall be denoted by $\left\{ \begin{array}{c} n \\ k \end{array} \right\}$.
It shall be observed that the Stirling number of the second kind can never be negative. That's why only the above notation is mentioned.

Figure 2.1:

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Figure 2.2:

This will replace page 18

Historically, these notations $\left[\begin{matrix} n \\ k \end{matrix} \right]$ and $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$ were first introduced by a Serbian mathematician Jovan Karamata.

2.2 Stirling Number of the Second kind

Definition 2.2.1. The *Stirling number of the second kind* $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$ is the number of partitions of a set of size n into exactly k non - empty sets.

In other words, the number of equivalence relations on a set of size n. We denote by $[n]$ a set of size n. We can also obtain $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$ explicitly as

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \frac{1}{k!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n \tag{2.1}$$

and from the recurrence that follows:

Lemma 2.2.2. $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$ *satisfies the recurrence* $\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \left\{ \begin{matrix} n-1 \\ k-1 \end{matrix} \right\} + k \left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\}$, $\left\{ \begin{matrix} n \\ n \end{matrix} \right\} = 1$, *for all* $n \geq 0$ *and* $\left\{ \begin{matrix} n \\ 0 \end{matrix} \right\} = 0$, *for all* $n \geq 1$.

Proof. Given a partition of an $[n-1]$ set into k non - empty subsets, there are two ways of constructing a partition of $[n]$ into k non - empty subsets. In one case, we add the n^{th} element to an existing non - empty subset. There are $k \left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\}$ ways of doing this.
In the other case, we form the singleton set $\{n\}$ and add this to a partition of $[n-1]$ with $k-1$ subsets. There are $\left\{ \begin{matrix} n-1 \\ k-1 \end{matrix} \right\}$ ways to do this. Hence, taking the sum concludes the proof. □

If $f : [n] \to [k]$ is a surjection, then the pre-images $f^{-1}(1), \ldots, f^{-1}(k)$ partition $[n]$. There are $k!$ bijective mappings from parts of a partition with k parts to $[k]$. Thus,

$$k!\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$$

is the number of surjective functions $f : [n] \longrightarrow [k]$.
Now, we use this to deduce (2.1).
i.e. $k!\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n$.

Proof. For $j \in [k]$, let $A_j = \{f : [n] \mapsto [k] | \forall x \in [n], f(x) \neq j\}$.
Then, $\cup_{j=1}^{k} A_j$ is the set of non - surjective functions $f : [n] \mapsto [k]$. For $B \subseteq [k]$ with cardinality $|B| = i \quad (0 \geq i \geq k)$, the set $\cap_{j\in B} A_j$; where $\cap_{j\in B} A_j = \{f : [n] \mapsto [k] \mid \forall x \in [n],\ f(x) \in [k] - B\}$ has the cardinality $|\cap_{j\in B} A_j| = (k-i)^n$. Also,

$$\begin{aligned} |\cup_{j=1}^{k} A_j| &= \sum_{i=1}^{k}(-1)^{i+1} \sum_{B\subseteq[k],\, |B|=i} (k-i)^n \\ &= \sum_{i=1}^{k}(-1)^{i+1}\binom{k}{i}(k-i)^n. \end{aligned}$$

The number of surjective functions is the total number of functions $f : [n] \mapsto [k]$ less the number of non - surjective functions. i.e.

$$\begin{aligned} k^n - |\cup_{j=1}^{k} A_j| &= k^n - \sum_{i=1}^{k}(-1)^{i+1}\binom{k}{i}(k-i)^n \\ &= k^n + \sum_{i=1}^{k}(-1)^{i}\binom{k}{i}(k-i)^n \\ &= \sum_{i=0}^{k}(-1)^{i}\binom{k}{i}(k-i)^n. \end{aligned}$$

Notice the term on the LHS: $k^n - |\cup_{j=1}^k A_j| = k!\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$. Hence the result (2.1) follows. $\square$

Lemma 2.2.3. $\sum_{k=1}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\}(x)_k = x^n, \forall x \in \mathbb{N}.$

Proof. Let $\#A$ denote the cardinality of the set A. Then

$$\begin{aligned}
x^n &= \#\{\text{all maps } f : [n] \longrightarrow [x]\} \\
&= \sum_{k=1}^n \#\{\text{all maps } f : [n] \longrightarrow [x] \text{ such that } |im(f)| = k\} \\
&= \sum_{k=1}^n \binom{x}{k} \#\{\text{surjective maps } f : [n] \longrightarrow [k]\} \\
&= \sum_{k=1}^n \binom{x}{k} k! \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \\
&= \sum_{k=1}^n x(x-1), \ldots, (x-k+1) \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \\
&= \sum_{k=1}^n (x)_k \left\{ \begin{matrix} n \\ k \end{matrix} \right\}
\end{aligned}$$

as required. $\square$

.

Stirling Transform

We have

$$x^n = \sum_{k=0}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\}(x)_k \quad \text{and} \tag{2.2}$$

$$(x)_n = \sum_{k=0}^n s(n,k)x^k, \tag{2.3}$$

hence, the resulting matrices of the Stirling numbers $s(n,k)$ and $\left\{ \begin{array}{c} n \\ k \end{array} \right\}$ are inverses of each other.

Theorem 2.2.4. *Let* $\delta_{m,n} = \begin{cases} 1, & if \;\; m = n \\ 0, & otherwise \end{cases}$ *and* $s(n,k) = (-1)^{n-k}|s(n,k)|$ *, then* $\sum_{k=n}^{m} \left\{ \begin{array}{c} m \\ k \end{array} \right\} s(k,n) = \delta_{m,n}$ *and* $\sum_{k=n}^{m} s(m,k) \left\{ \begin{array}{c} k \\ n \end{array} \right\} = \delta_{m,n}$.

Proof. Let M and N be $n \times n$ matrices of the Stirling numbers of first and second kind respectively. Notice for $n < k$, $s(n,k) = 0 = \left\{ \begin{array}{c} n \\ k \end{array} \right\}$.

Let V_n be a vector space of polynomials p_n in x with zero constant term and $deg \;\; p_n \leq n$ with bases $E = (e_i | i = 1, \ldots, n)$ and $F = (f_i | i = 1, \ldots, n)$, where $e_i = x^i$ and $f_i = (x)_i$.

From (2.3), put $x = -x$ and multiply the result by $(-1)^n$ on both sides to get $f_i = \sum_{k=1}^{i} e_k s(i,k)$ for all i. This implies that, $F = EM$. Thus, M is a change - of - basis matrix from E to F.

Similarly, from (2.2) we have $e_i = \sum_{k=1}^{i} f_k \left\{ \begin{array}{c} n \\ k \end{array} \right\}$ for all i. Also implying that, $E = FN$. In other words, N is the change - of - basis matrix from F to E. Hence the proof of the theorem.

□

Figure 2.3: A table showing some values of Stirling numbers of the second kind.

n \ k	0	1	2	3	4	5	6	7	8	9	10	11
0	1											
1	0	1										
2	0	1	1									
3	0	1	3	1								
4	0	1	7	6	1							
5	0	1	15	25	10	1						
6	0	1	31	90	65	15	1					
7	0	1	63	301	350	140	21	1				
8	0	1	127	966	1701	1050	266	28	1			
9	0	1	255	3025	7770	6951	2646	462	36	1		
10	0	1	511	9330	34105	42525	22827	5880	750	45	1	
11	0	1	1023	28501	145750	246730	179487	63987	11880	1155	55	1
12	0	1	2047	86526	611501	1379400	1323652	627396	159027	22275	1705	66
13	0	1	4095	261625	2532530	7508501	9321312	5715424	1899612	359502	39325	2431
14	0	1	8191	788970	10391745	40075035	63436373	49329280	20912320	5135130	752752	66066
15	0	1	16383	2375101	42355950	210766920	420693273	408741333	216627840	67128490	12662650	1479478

Chapter 3

Sheffer Polynomial Sequences

Introduction

Generally speaking, the Sheffer sequence is a class of polynomial sequence which arises as the solution to certain differential equations. One aspect of the study of Sheffer polynomial sequences which has recently been of research interest is finding a differential equation and recursive formulas for these sequences. See for example [9] and [25]. We give, in this section, some of the differential equations that give rise to some known Sheffer polynomial sequences like the Touchard, Toscano and the Poisson - Charlier polynomial sequences. In the sequel we shall always assume that a set of polynomials $\{p_n(x)\}_{n=0}^{\infty}$ is such that each $p_n(x)$ has degree exactly n.

3.1 Definitions

Definition 3.1.1. Let $g(z)$ be an invertible analytic function, with $g(0) \neq 0$. Let $f(z)$ be an analytic function with $f(0) = 0$ and $\mathrm{f}'(0) \neq 0$ and suppose f admits a compositional inverse $\bar{f}(z)$. Then, $(s_n(x))$ is the *Sheffer polynomial sequence for* $(g(z), f(z))$ if and only if

$$\frac{1}{g(\bar{f}(z))} e^{x\bar{f}(z)} = \sum_{k=0}^{\infty} \frac{s_k(x)}{k!} z^k. \tag{3.1}$$

By Taylor's theorem, we have

$$s_k(x) = \left(\frac{d}{dz}\right)^{(k)} \frac{1}{g(\bar{f}(z))} e^{x\bar{f}(z)}\Big|_{z=0} \tag{3.2}$$

since $\frac{1}{g(\bar{f}(z))} e^{x\bar{f}(z)}$ is analytic.
Note that by the use of $f^{(k)}$, we mean the k^{th} order derivative of f. f^k shall represent the k^{th} power of f.

Definition 3.1.2. ([9]) Let $g(z)$ and $f(z)$ be any given f.p.s over the field $\mathbb{K}$ with $g(0) = 1, f(0) = 0$ and $f^{'}(0) \neq 0$. Then the polynomial $s_n(x), (n = 0, 1, 2, 3, \ldots)$ defined by the GF

$$g(z)e^{xf(z)} = \sum_{n\geq 0} s_n(x)\frac{z^n}{n!} \tag{3.3}$$

is the *Sheffer-type polynomial* with $s_0 = 1$. Accordingly, $s_n(D)$ with $D = \frac{d}{dt}$ is called the Sheffer-type differential operator of degree n associated with $g(z)$ and $f(z)$. In particular, $s_0(D) = I$ is the identity operator.

3.2 Differential Equation Of Sheffer Polynomial Sequences

A differential equation which give rise to a Sheffer sequence has been established in [25] as follows;

Theorem 3.2.1. ([25]) *Let $s_n(x)$ be the Sheffer polynomial sequence for $(g(z), f(z))$. Then, it satisfies the following differential equation:*

$$\sum_{k=1}^{n} (\beta_k x + \alpha_k)\frac{s_n^{(k)}(x)}{k!} - ns_n(x) = 0, \tag{3.4}$$

where $\beta_k = \left(\frac{f(z)}{\mathrm{f}^{(1)}(z)}\right)^{(k)}\Big|_{z=0}$ *and* $\alpha_k = \left(\frac{-\mathrm{g}^{(1)}(z)f(z)}{g(z)\mathrm{f}^{(1)}(z)}\right)^{(k)}\Big|_{z=0}$.

Corollary 3.2.2. ([25]) *Let $(q_n(x))$ be the associated polynomial sequence for $f(z)$. Then, it satisfies the following differential equation:*

$$\sum_{k=1}^{n} \beta_k x \frac{q_n^{(k)}(x)}{k} - nq_n(x) = 0, \tag{3.5}$$

where $\beta_k = \left(\frac{f(z)}{\mathrm{f}^{(1)}(z)}\right)^{(k)}\Big|_{z=0}$.

Proof. Since $(q(x))$ is an associated polynomial sequence, $g(z) = 1$, and so $\alpha_k = 0$ for all $k > 0$. Hence, equation (3.4) reduces to (3.5). $\square$

Corollary 3.2.3. ([25]) *Let $(a_n(x))$ be the Appell polynomial sequence for $g(z)$. Then, it satisfies the following differential equation:*

$$\sum_{k=2}^{n} \frac{\bar{\alpha}_{k-1}}{k-1} a_n^{(k)}(x) + (x + \bar{\alpha}_0)\mathrm{a}^{(1)}(x) - na_n(x) = 0, \tag{3.6}$$

where $\bar{\alpha}_k = (-\frac{\mathrm{g}^{(1)}(z)}{g(z)})^{(k)}\Big|_{z=0}$.

Proof. Since $f(z) = z, \beta_1 = 1$ and $\beta_k = 0$ for all $k > 1$. We compute

$$\begin{aligned}
\alpha_0 &= \left(-\frac{z\, g^{(1)}(z)}{g(z)}\right)^{(0)}\Big|_{z=0} = 0 \\
\alpha_1 &= \left(-\frac{z\, g^{(1)}(z)}{g(z)}\right)^{(1)}\Big|_{z=0} \\
&= -\frac{g^{(1)}(z)}{g(z)} - z\left(\frac{g^{(2)}(z)}{g(z)} - \left(\frac{g^{(1)}(z)}{g(z)}\right)^2\right)\Big|_{z=0} \\
&= -\frac{g^{(1)}(z)}{g(z)} = \bar{\alpha}_0 \\
\alpha_2 &= \left(-\frac{z\, g^{(1)}(z)}{g(z)}\right)^{(2)}\Big|_{z=0}
\end{aligned}$$

$$\alpha_2 = \frac{d\left(-\frac{g^{(1)}(z)}{g(z)} - z\left(\frac{g^{(2)}(z)}{g(z)} - \left(\frac{g^{(1)}(z)}{g(z)}\right)^2\right)\right)}{d\,z}\Bigg|_{z=0}$$

$$= -\left(\frac{g^{(2)}(z)}{g(z)} - \left(\frac{g^{(1)}(z)}{g(z)}\right)^{(2)}\right) - \left(\frac{g^{(2)}(z)}{g(z)} - \left(\frac{g^{(1)}(z)}{g(z)}\right)^{(2)}\right) +$$

$$z\,\frac{d\left(\frac{g^{(2)}(z)}{g(z)} - \left(\frac{g^{(1)}(z)}{g(z)}\right)^2\right)}{d\,z}\Bigg|_{z=0}$$

$$= 2\left(-\frac{g^{(1)}(z)}{g(z)}\right)^{(1)}\Bigg|_{z=0} = 2\,\bar{\alpha}_1.$$

Finally,

$$\begin{aligned}
\alpha_k = \left(\frac{-z\mathrm{g}^{(1)}(z)}{g(z)}\right)^{(k)}\Bigg|_{z=0} &= \frac{d\left(-\frac{z\,g^{(1)}(z)}{g(z)}\right)^{(k-1)}}{d\,z}\Bigg|_{z=0} \\
&= \frac{d\,(k-1)\,\left(-\frac{g^{(1)}(z)}{g(z)}\right)^{(k-1)}}{d\,z}\Bigg|_{z=0} + \frac{d\,z\left(-\frac{g^{(1)}(z)}{g(z)}\right)^{(k-1)}}{d\,z}\Bigg|_{z=0} \\
&= (k-1)\bar{\alpha}_{k-1} + \left(-\frac{g^{(1)}(z)}{g(z)}\right)^{(k-1)}\Bigg|_{z=0} + z\,\left(-\frac{g^{(1)}(z)}{g(z)}\right)^{(k)}\Bigg|_{z=0} \\
&= k\,\bar{\alpha}_{k-1},
\end{aligned}$$

substituting into equation (3.4) we obtain (3.6). □

Example 3.2.4. ([25]) The exponential polynomial, denoted by $\phi_n(x)$ is the associated sequence for $f(z) = \ln(1+z)$, and $\phi_n(x) = \sum_{k=0}^{n} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} x^k$. We apply Theorem (3.2.1) to obtain the differential equation

$$x\sum_{k=2}^{n} \frac{(-1)^k \phi_n^{(k)}(x)}{k(k-1)} + x\phi^{'}(x) - n\phi_n(x) = 0, \quad \text{for } n \geq 2.$$

Example 3.2.5. ([25]) The falling factorial polynomial $\left(\frac{x}{a}\right)_n = \left(\frac{x}{a}\right)\left(\frac{x}{a}-1\right)\ldots \left(\frac{x}{a}-n+1\right)$ is an associated polynomial sequence generated by $(1, e^{az}-1)$. It satisfies the differential equation

$$x\sum_{k=1}^{n}\frac{(-a)^{k-1}}{k!}\left(\frac{x}{a}\right)_n^{(k)}(x) - n\left(\frac{x}{a}\right)_n(x) = 0.$$

We will later on see that these two polynomial sequences describe the Touchard polynomial sequence, and thus are related.

Example 3.2.6. ([25]) Let $L_n^{(\lambda)}(x)$ denote the Laguerre polynomial of order λ, which is a Sheffer polynomial for $(g(z), f(z)) = \left((1-z)^{-(\lambda+1)}, \frac{z}{z-1}\right)$ Then, $L_n^{(\lambda)}(x) = \sum_{k=0}^{n}\binom{n+\lambda}{n-k}\left(\frac{n!}{k!}\right)(-x)^k$ satisfies the differential equation

$$x\left(L_n^{(\lambda)}(x)\right)^{''} - (x-\lambda-1)\left(L_n^{(\lambda)}(x)\right)^{'} + nL_n^{(\lambda)}(x) = 0.$$

In the case $\lambda = 0$, we obtain the generally known Laguerre polynomial $L_n^{(0)}(x) = L_n(x)$ and the differential equation

$$xL_n^{''}(x) - (x-1)L_n^{'}(x) + nL_n(x) = 0.$$

Example 3.2.7. ([25]) The Hermite polynomial of order λ, which we denote by $H_n^{(\lambda)}(x)$ is the Appell polynomial sequence for $g(z) = e^{\lambda z^2/2}$.
Thus, $H_n^{(\lambda)}(x) = \sum_{k=o}^{\frac{n}{2}}\left(\frac{-\lambda}{2}\right)^k\left(\frac{(n)_{2k}}{k!}\right)x^{n-2k}$ and arises from the differential equation

$$\lambda\left(H_n^{(\lambda)}(x)\right)^{''} - x\left(H_n^{(\lambda)}(x)\right)^{'} + nH_n^{(\lambda)}(x) = 0.$$

3.3 Polynomial Sequence of Binomial Type

3.3.1 Notations and Definitions

Let $\mathcal{L}$ denote the space of all linear operators, $T : \Pi \to \Pi$, where we've set

$$\Pi := \cup_{n\geq 0}\Pi_n,$$

with Π_n denoting the linear space of polynomials of degree at most n. Furthermore, we let Π_n^* denote the set of all polynomials of degree n.

Definition 3.3.2. We define the *shift operator, E^y* by

$$\Big(E^y p\Big)(x) = p(x+y) \tag{3.7}$$

for every $x, y \in \mathbb{K}$, and $p \in \Pi$.

E^y is an element of $\mathcal{L}$.

Definition 3.3.3. If $p_0(x), p_1(x), \ldots (n = 0, 1, 2, \ldots)$ is a sequence of polynomials satisfying;

i. $p_0(x) = 1$

ii. $p_n(0) = 0$ for $n > 0$, and the identity;

iii. $\sum_{n\geq 0} p_n(x)\frac{x^n}{n!} = \Big(\sum_{n\geq 0} p_n(1)\frac{x^n}{n!}\Big)^x$,
then we call $(p_n(x))_{n\geq 0}$ a *sequence of polynomial of binomial - type,* (or simply *a polynomial of binomial - type*).

We give the following characterization of a sequence of binomial type.

Theorem 3.3.4. [15] *A sequence $\Big(p_n(x)\Big)_{n\geq 0}$ of polynomials such that $p_0(x) = 1$ is of binomial - type if and only if*

$$p_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} p_k(x)p_{n-k}(y),\ n \geq 0. \tag{3.8}$$

3.3.5 Examples of Polynomials of Binomial - Type

Example 3.3.6. For $p_k(a) = a^k$, we have

$$\sum_{k\geq 0} a^k \frac{x^k}{k!} = \Big(\sum_{k\geq 0} \frac{x^k}{k!}\Big)^a = e^{ax}.$$

Example 3.3.7. For $p_k(a) = a(a-1)\dots(a-k+1) = (a)_k$ the falling (or lower) factorial, we have

$$\sum_{k\geq 0} (a)_k \frac{x^k}{k!} = \sum_{k\geq 0} \binom{n}{k} x^k = (1+x)^n.$$

Example 3.3.8. For $p_k(a) = a(a+1)\dots(a+k-1) = a^{(k)}$ the upper factorial, we have

$$\sum_{k\geq 0} a^{(k)} \frac{x^k}{k!} = (1-x)^{-n}.$$

Example 3.3.9. For $p_k(a) = a(a-\alpha k)^{k-1}, a \in \mathbb{K}$, we have the Abel polynomial.

$$\sum_{k\geq 0} a(a-\alpha k)^{k-1} \frac{x^k}{k!} = \Big(\sum_{k\geq 0} (1-\alpha k)^{k-1} \frac{x^k}{k!}\Big)^a.$$

Note: The binomial- type is equivalent to Abel's identity

$$i.e. \quad (x+y)^k = \sum_{i=0}^{k} \binom{k}{i} x(x-iz)^{i-1}(y+iz)^{k-i}.$$

For $z = 0$, this yields the binomial theorem.

Example 3.3.10. For $p_k(a) = \sum_{i=1}^{k} \left\{ \begin{matrix} k \\ i \end{matrix} \right\} a^i$, we have the exponential (or Touchard) polynomial.

$$\sum_{k\geq 0} \Big(\sum_{i=1}^{k} \left\{ \begin{matrix} k \\ i \end{matrix} \right\} a^i\Big) \frac{x^k}{k!} = \Big(\sum_{k\geq 0} B(k) \frac{x^k}{k!}\Big)^a,$$

where $B(k)$ is the Bell number.

In a recent study on Bell polynomials, [1] proposed the following identity

$$B_{n,k}(P_0(1), 2P_1(1), 3P_2(1), \ldots) = \binom{n}{k} q_{n-k}(k),$$

where $q_n(x)$ is a sequence of binomial type which satisfies (3.8).

According to [15], a sequence is a binomial type if and only if it is an associated sequence.
Since an associated sequence is a special Sheffer sequence, we characterize Sheffer sequences this way also. Before that, we define what we mean by an associated sequence.

Definition 3.3.11. The Sheffer sequence for $(1, f(t))$ is the *associated sequence for* $f(t)$. If $p_n(x)$ is associated to $f(t)$, then

$$\sum_{n=0}^{\infty} p_n(x)\frac{t^n}{n!} = exp(x\bar{f}(t)).$$

The Sheffer sequence for $(g(t), t)$ is the *Appell sequence for* $g(t)$ and if $P_n(x)$ is Appell for $g(t)$, then

$$\sum_{n=0}^{\infty} P_n(x)\frac{t^n}{n!} = \frac{exp(xt)}{g(t)}.$$

Theorem 3.3.12. *A sequence $s_n(x)$ is Sheffer for $(g(z), f(z))$ for some invertible $g(z)$, if and only if*

$$s_n(x+y) = \sum_{k=0}^{n} s_{n-k}(x)q_k(y),$$

where $q_n(y)$ is the associated sequence for $f(z)$.

Particularly, if $s_n(x)$ is itself an associated sequence, then the theorem reduces to the identity in (3.8).

3.4 Delta Operators

Suppose $T \in \mathcal{L}$ is an operator which commutes with all shift operators. i.e.

$$TE^y = E^y T \quad \text{for all} \quad y \in \mathbb{K}.$$

We shall call such an operator a *shift - invariant operator*, and denote by $\mathcal{L}_s$ the set of all shift - invariant operators.

Definition 3.4.1. ([2], Def. 2.1) An operator Q is called a *delta operator* if $Q \in \mathcal{L}_s$ and Qe_1 is a non - zero constant.

We denote by $\mathcal{L}_\delta$ the set of all delta operators, and $(e_n)_{n\geq 0}, e_n(x) = x^n$ (the monomials). Note; every delta operator commutes with shift - invariant operators.

3.4.2 Example Of Delta Operators

Example 3.4.3. The derivative operator, denoted by D.
Note that every $Q \in \mathcal{L}_\delta$ also belongs to $\mathcal{L}_s$. i.e. $\mathcal{L}_\delta \subset \mathcal{L}_s$. Take for instance $Q \in \mathcal{L}_\delta$, then for any constant h, $Qh = 0$.
Since $Q \in \mathcal{L}_\delta$, $Qe_1 = c \neq 0$. Hence,

$$\begin{aligned} (QE^h)(e_1) &= (E^h Q)(e_1) \\ \Leftrightarrow Q(e_1 + h) &= E^h c \\ \Leftrightarrow c + Qh &= c \\ \Leftrightarrow Qh &= 0. \end{aligned}$$

We generalize this, as follows [18];
For every $Q \in \mathcal{L}_\delta$, we have

$$Q(\Pi_n^*) \subset \Pi_{n-1}^* \quad n \in \mathbb{N}. \tag{3.9}$$

Example 3.4.4. The Laguerre operator,

$$L := \frac{D}{D - I} = -D(I - D)^{-1},$$

where I is the identity.
For any polynomial $p \in \Pi$ that we apply the operator L to, the power series (infinite) reduces to a finite sum. Also

$$(Lp)(x) = -\int_0^\infty \frac{dp}{dt}(x + t)dt.$$

Example 3.4.5. The Touchard operator, $T := \ln(I + D)$.
Another way to represent this operator is

$$(Tp)(x) = \int_0^\infty e^{-t}[x, x - t; p]dt.$$

Example 3.4.6. The Abel operator, $A_y := DE^y$.
For any $p \in \Pi$, we have

$$(A_y p)(x) = \frac{dp}{dx}(x + y).$$

Symbolically, we can write the Taylor's series

$$E^y = \sum_{n=0}^{\infty} \frac{y^n D^n}{n!} = e^{yD}$$

and so we obtain $A_y = D(e^{yD})$.

Example 3.4.7. The difference operators. Let $h \in \mathbb{K}$ be fixed. We have the following;

i. The forward difference operator; $\triangle_h := E^h - I$.

ii. The backward difference operator; $\nabla_h := I - E^{-h}$.

iii. The central difference operator; $\delta_h := E^{\frac{h}{2}} - E^{-\frac{h}{2}}$.

iv. The mean value operator; $M_h := \frac{1}{2}(I + E^h)$.

Example 3.4.8. The Gould operator, $G_{a,b}$

$$G_{a,b} := \triangle_b E^a = E^{a+b} - E^a, \quad ab \neq 0.$$

3.4.9 Connecting Delta Operators and Binomial - Type Sequences

Definition 3.4.10. Let Q be a delta operator. A polynomial sequence $p \in (p_n)_{n\geq 0}$ is called the *sequence of basic polynomials associated to Q* if

i. $p_0(x) = 1$ for any $x \in \mathbb{K}$,

ii. $p_n(0) = 0$ for any $n \in \mathbb{N}$, and

iii. $(Qp_n)(x) = np_{n-1}(x)$ for any $n \in \mathbb{N}$ and $x \in \mathbb{K}$.

If $p = (p_n)_{n\geq 0}$ is a sequence of basic polynomials associated to Q, then $\{p_0, p_1, \ldots, p_{n-1}, e_n\}$ is a basis of the vectorial space Π_n. Hence the proposition;

Proposition 3.4.11. ([18]) *Every delta operator has a unique sequence of basic polynomials.*

Theorem 3.4.12. ([2])

(a) If $p = (p_n)_{n\geq 0}$ is a basic sequence for some delta operator Q, then it is a sequence of binomial - type.

(b) If $p = (p_n)_{n\geq 0}$ is a sequence of binomial - type, then it is a basic sequence for some delta operator.

Theorem 3.4.13. ([2]) *Let T be a shift - invariant operator and let Q be a delta operator with its basic sequence $(p_n)_{n\geq 0}$. Then the identity $T = \sum_{k\geq 0} \frac{(Tp_k)(0)}{k!} Q^k$ holds.*

We began this chapter using GFs to define the Sheffer polynomials, then we described it via delta operators. Now, we take an algebraic view of these two via an isomorphism. Here, product will mean the Cauchy product between two series.
Let $(\mathcal{L}_s, +, .)$ be the ring of shift - invariant operators, and $(\mathcal{F}, +, .)$ the ring of formal power series in the variable z over the field $\mathbb{K}$.
Define, for any $Q_1, Q_2 \in \mathcal{L}_s$ the product $Q_1Q_2 : \Pi \to \Pi$ by

$$(Q_1Q_2)(p) = Q_1(Q_2(p))$$

for any $p \in \Pi$. Then, there is an isomorphism [2] $\Psi : \mathcal{F} \to \mathcal{L}_s$ such that

$$\Psi\Big(\sum_{k\geq 0} \frac{a_k}{k!} z^k\Big) = \sum_{k\geq 0} \frac{a_k}{k!} Q^k$$

from which we draw the following conclusions;

i. A shift - invariant operator T is invertible if and only if $Te_0 \neq 0$ since for every $Q \in \mathcal{L}_s$ we have $Qe_0 = 0$. This, also, implies that there is no delta operator which is invertible.

ii. Since $(\mathcal{F}, +, .)$ is a commutative ring, any two shift - invariant operators commute. i.e. $TS = ST \quad \forall S, T \in \mathcal{L}_s$.

3.4.14 The Sheffer Sequences and Delta Operators

In [8], the Sheffer sequence is defined as a polynomial sequence of type - zero. Here, we state conditions for a set of type - zero, and then connect them with the delta operators. Basically, a Sheffer sequence for $Q \in \mathcal{L}_\delta$ is related to the set of basic polynomials associated to Q (or a sequence of binomial - type).

Definition 3.4.15. A set $s = \{s_n\}_{n\geq 0}$ is of *type - zero* if $s_0 \neq 0$ and $T(s_n) = s_{n-1}, n \in \mathbb{N}$, where for every $p \in \Pi, T : \Pi \to \Pi$ is given by; $T(p) = \sum_{k\geq 1} a_k p^{(k)}, \quad a_1 \neq 0$ and T is a delta operator.

Definition 3.4.16. *Let Q be a delta operator. The polynomial sequence $s = (s_n)_{n\geq 0}$ is of type - zero (or a Sheffer sequence) for the operator Q if $s_0 \neq 0$ and*

$$Qs_n = ns_{n-1}, n \in \mathbb{N}. \tag{3.10}$$

Theorem 3.4.17. [2] *Let $Q \in \mathcal{L}_s$ with its basic sequence polynomial $(p_n)_{n\geq 0}$. Then $s = (s_n)_{n\geq 0}$ is a Sheffer sequence relative to Q if and only if there exists an invertible shift - invariant operator S such that*

$$s_n = S^{-1}(p_n),\ n \in \mathbb{N}. \tag{3.11}$$

Theorem 3.4.18. ([2], [6]) *Let $Q \in \mathcal{L}_s$ with its basic sequence $(p_n)_{n\geq 0}$. For a Sheffer sequence $s = (s_n)_{n\geq 0}$ relative to Q the following identity*

$$s_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} s_k(x)p_{n-k}(y) = \sum_{k=0}^{n} \binom{n}{k} p_k(y)s_{n-k}(x) \tag{3.12}$$

holds for every $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $(x, y) \in \mathbb{K} \times \mathbb{K}$.

Proof. Since $(p_n)_{n\geq 0}$ is of binomial - type, we have

$$p_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} p_k(x)p_{n-k}(y), \quad n \in \mathbb{N}_0$$

by Theorem (3.3.4). Applying the shift - invariant operator S^{-1} to both sides we obtain;

$$\begin{aligned}(S^{-1}p_n)(x+y) &= \sum_{k=0}^{n} \binom{n}{k} (S^{-1}p_k)(x)p_{n-k}(y) \\ &= \sum_{k=0}^{n} \binom{n}{k} s_k(x)p_{n-k}(y) \quad \text{by} \quad \text{eqn.(3.11)}.\end{aligned}$$

It is easy to see that the LHS;

$$\begin{aligned}(S^{-1}p_n)(x+y) &= \Big(S^{-1}(E^y p_n)\Big)(x)\\ &= \Big(E^y(S^{-1}p_n)\Big)(x)\\ &= (E^y s_n)(x)\\ &= s_n(x+y)\end{aligned}$$

as expected, and that establishes the theorem. $\square$

Corollary 3.4.19. *Let $Q \in \mathcal{L}_s$ with its basic sequence $(p_n)_{n\geq 0}$. If $s = (s_n)_{n\geq 0}$ is the Sheffer sequence relative to Q, then*

$$s_n(x+y) = \sum_{k\geq 0} \frac{1}{(n-k)!} Q^k s_n(x) p_k(y) \quad \text{for } n \text{ fixed.}$$

Proof. From definition (3.4.16), we have that $(Qs_n)(x) = n\, s_{n-1}(x)$, and iterating $k-$ times we obtain $s_{n-k}(x) = \frac{Q^k s_n(x)}{n(n-1)\ldots(n-k+1)}$.
Alternatively, start with $s_n(x) = (S^{-1}p_n)(x)$ and apply Q^k, to both sides for $Q \in \mathcal{L}_\delta$ then

$$Q^k s_n(x) = ((Q^k \circ S^{-1})p_n)(x).$$

Notice that $S^{-1} \in \mathcal{L}_s$ is invertible and commutes with all shift - invariant operators. Hence, Q commutes with S^{-1}.
Then,

$$\begin{aligned}Q^k s_n(x) = ((Q^k \circ S^{-1})p_n)(x) &= ((S^{-1} \circ Q^k)p_n)(x)\\ &= (S^{-1}(n(n-1)\ldots(n-k+1))p_{n-k})(x)\\ &= n(n-1)\ldots(n-k+1)(S^{-1}p_{n-k})(x)\\ &= n(n-1)\ldots(n-k+1)s_{n-k}(x).\end{aligned}$$

Therefore,

$$s_{n-k}(x) = \frac{Q^k s_n(x)}{n(n-1)\ldots(n-k+1)}$$

and putting it in equation (3.12) yields

$$s_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} \frac{Q^k s_n(x)}{n(n-1)\dots(n-k+1)} p_k(y)$$

and so we get

$$s_n(x+y) = \sum_{k\geq 0} \frac{1}{(n-k)!} Q^k s_n(x) p_k(y).$$

□

Theorem 3.4.20. ([2]) *$Q \in \mathcal{L}_s$ is a delta operator if and only if $Q = DS$ for some shift - invariant operator S, where the inverse operator S^{-1} exists and D is the derivative operator.*

Proof. We know from Theorem (3.4.13), T can be written in terms of D as $T = \sum_{k\geq 0} \frac{a_k}{k!} D^k$, where $a_k = (Te_k)(0)$.
Put $T = Q \in \mathcal{L}_\delta$, then we have $a_0 = Q(e_0) = 0$ and $a_1 = Q(e_1) = c \neq 0$. It follows that $Q = \sum_{k\geq 1} \frac{a_k}{k!} D^k$.
Denoting $\sum_{k\geq 1} \frac{a_k}{k!} D^{k-1}$ by S, we have $S \in \mathcal{L}_s$ and $S(e_0) = a_1 \neq 0$, then S is invertible. So we can write $Q = DS$.
Conversely, for every $S \in \mathcal{L}_s$ such that S is invertible, DS is a shift - invariant operator, i.e. $E^y(DS) = (DS)E^y \quad \forall y \in \mathbb{K}$ and
$(DS)(e_1) = S(D(e_1)) = S(e_0) = c \neq 0$. Thus, $DS \in \mathcal{L}_\delta$. □

Proposition 3.4.21. *Let Q be a delta operator with $p = (p_n)_{n\geq 0}$, its sequence of basic polynomials, and $s_n(x)$ the Sheffer sequence for the operator Q, then $(Qs_n)(x) = (Dp_n)(x)$, and more generally $(Q^k s_n)(x) = ((D^k \circ S^{k-1})p_n)(x)$, where $S \in \mathcal{L}_s$ is invertible and D is the derivative operator.*

Proof. Since $Q = DS$ by Theorem (3.4.20), then $Q^k = (DS)^k$ for $k \in \mathbb{N}$ and it follows that

$$\begin{aligned} D^k = (Q \circ S^{-1})^k &= (Q^k \circ S^{-1})S^{-(k-1)} \\ D^k \circ S^{k-1} &= Q^k \circ S^{-1}. \end{aligned}$$

Also, $(Q^k s_n)(x) = (Q^k \circ S^{-1})p_n(x)$ according to equation (3.11),
hence,

$$(Q^k s_n)(x) = ((D^k \circ S^{k-1})p_n)(x).$$

For $k = 1$, we obtain $(Qs_n)(x) = (Dp_n)(x)$ as required.

□

3.4.22 An Approach to Sheffer Polynomials Using Determinants

So far, we have seen the Sheffer polynomials approached from the differential and algebraic point of view. In this last subsection, we give a determinantal form of the Sheffer polynomial sequences as functions of polynomial sequences of binomial - type and then conclude by obtaining a Wronskian matrix for the Sheffer vector.
Let $p_n(x)$ be a polynomial sequence, we can decompose its coefficients $a_{n,k}$ over power monomials x^k. According to [15] Roman et al.

Proposition 3.4.23. *A polynomial $p_n(x) = \sum_{k=0}^{n} a_{n,k}x^k$ is of binomial - type if and only if $\binom{i+j}{i}a_{n,i+j} = \sum_{k=0}^{n} \binom{n}{k} a_{i,k}a_{n-k,j}$ for all $n \geq 0$ and for all i, $j \geq 0$.*

As observed in [[6] ,pp. 296], $a_{n,0} = \delta_{n,0}$ and all the coefficients $a_{n,k}$, $k = 2, \ldots, n$, are completely determined by the sequence $a_{n,1}$, $n = 0, 1, \ldots,$ as follows ;

$$a_{n,k} = \frac{1}{k} \sum_{i=1}^{n-k+1} \binom{n}{i} a_{i,1}a_{n-1,k-1}, k = 2, \ldots, n. \tag{3.13}$$

Let β_n, $n \in \mathbb{N}$ be a sequence with $\beta_0 \neq 0$. We define Sheffer polynomial sequences in terms associated with $(\beta_n, p_n(x))$ as follows;

Definition 3.4.24. ([6]) The polynomial sequence $s_n(x), n = 0, 1, \ldots,$ defined by $s_0 = \frac{1}{\beta_0}$

$$s_n = \frac{(-1)^n}{(\beta_0)^{n+1}} \begin{bmatrix} p_0 & p_1(x) & p_2(x) & \dots & \dots & p_{n-1}(x) & p_n(x) \\ \beta_0 & \beta 1 & \beta_2 & \dots & \dots & \beta_{n-1} & \beta_n \\ 0 & \beta_0 & \binom{2}{1}\beta_1 & \dots & \dots & \binom{n-1}{1}\beta_{n-2} & \binom{n}{1}\beta_{n-1} \\ 0 & 0 & \beta_0 & \dots & \dots & \binom{n-1}{2}\beta_{n-3} & \binom{n}{2}\beta_{n-2} \\ \vdots & & & \ddots & & \vdots & \vdots \\ \vdots & & & & \ddots & \vdots & \vdots \\ 0 & \dots & \dots & \dots & 0 & \beta_0 & \binom{n}{n-1}\beta_1 \end{bmatrix}, n = 0, 1, \dots$$

is called the *Sheffer polynomial sequence* for $(\beta_n, p_n(x))$.

Theorem 3.4.25. ([6]) *$s_n(x)$ is the Sheffer polynomial for $(\beta_n, p_n(x))$ if and only if we have*

$$s_n(x) = \alpha_n p_0 + \binom{n}{2}\alpha_{n-2}p_2(x) + \dots + \alpha_0 p_n(x), n = 0, 1 \dots$$

with $\alpha_0 = \frac{1}{\beta_0}$, $\qquad \alpha_i = -\frac{1}{\beta_0}\sum_{k=0}^{i-1}\binom{i}{k}\beta_{i-k}\alpha_k, \quad i = 1, 2, \dots, n$.

Theorem 3.4.26. (Multiplication theorem)([6])$s_n(x), n = 0, 1, \dots$ *is the Sheffer sequence for $(\beta_n, p_n(x))$ if and only if* $s_n(mx) = \sum_{i=0}^{n}\binom{n}{i}s_{n-i}(x)p_i((m-1)x)$, $n = 0, 1, \dots,\ m = 1, 2, \dots$.

Proof. To proof this, it suffices to put $y = x(m-1)$ in (3.12). □

Corollary 3.4.27. ([6]) *If $s_n, n = 0, 1, \dots,$ is the Sheffer sequence for $(\beta_n, p_n(x))$, then*

$$s_n(x) = \sum_{k=0}^{n} s_{n,k}x^k = \sum_{k=0}^{n}\left(\sum_{i=k}^{n}\binom{n}{i}\alpha_{n-i}a_{i,k}\right)x^k, n = 0, 1, \dots$$

with $a_{n,k}$ defined by (3.13) *and α_n given in Theorem* (3.4.25).

Application of the Wronskian to Sheffer Sequences

Let $(s_i(x))$, $i = 0, 1, 2, \dots, k$ be a set of k continuous functions each of which is at least $k-$ times differentiable.

Definition 3.4.28. The *Wronskian matrix of n^{th} order of* $(s_i(x))$ $(i = 0, 1, \ldots, k)$ is an $(n+1)$ by k matrix defined by

$$\Phi[s_i(x)] = \begin{bmatrix} s_0 & s_1 & \ldots & s_k \\ s_0^{(1)} & s_1^{(1)} & \ldots & s_k^{(1)} \\ \vdots & & \ddots & \vdots \\ s_0^{(n+1)} & s_1^{(n+1)} & \ldots & s_k^{(n+1)} \end{bmatrix},$$

where $s_k^{(n)} := \frac{d^n}{dx^n} s_k(x)$.

Definition 3.4.29. The *Wronskian* is defined to be the determinant of the Wronskian matrix,

$$W_n(x) = det\ \Phi[s_k(x)].$$

Definition 3.4.30. Let $s_n(x)$ be the Sheffer polynomial for $(\beta_n, p_n(x))$, the vector of functions $\bar{S_n}(x) = [s_0(x), s_1(x), \ldots, s_n(x)]^T$ is called the *Sheffer vector* for $(\beta_n, p_n(x))$.

Corollary 3.4.31. *Let $s_n(x)$ be the Sheffer polynomial for $(\beta_n, p_n(x))$, then*

$$\bar{S_n}(x) = \det \begin{bmatrix} Dp_1 & Dp_2 & \ldots & Dp_{n+1} \\ \frac{D^2 p_1}{2} & \frac{D^2 p_2}{2} & \ldots & \frac{D^2 p_{n+1}}{2} \\ \vdots & & \ddots & \vdots \\ \frac{D^{n+1} p_1}{n+1} & \frac{D^{n+1} p_2}{n+1} & \ldots & \frac{D^{n+1} p_{n+1}}{n+1} \end{bmatrix}_{z=0}$$

Proof. As observed in equation (3.2), according to [25]; $\bar{S_n}(x) = W_n \left[\frac{1}{g(\bar{f}(z))} e^{x\bar{f}(z)}\right]_{z=0}$. Since from proposition (3.4.21), $s_n(x) = \frac{1}{n+1} Dp_{n+1}(x)$, it follows that;

$$\begin{aligned} \bar{S_n}(x) &= W_n \left[\frac{1}{n+1} Dp_{n+1}(x)\right]_{z=0} \\ &= \det \begin{bmatrix} Dp_1 & Dp_2 & \ldots & Dp_{n+1} \\ \frac{D^2 p_1}{2} & \frac{D^2 p_2}{2} & \ldots & \frac{D^2 p_{n+1}}{2} \\ \vdots & & \ddots & \vdots \\ \frac{D^{n+1} p_1}{n+1} & \frac{D^{n+1} p_2}{n+1} & \ldots & \frac{D^{n+1} p_{n+1}}{n+1} \end{bmatrix}_{z=0} . \end{aligned}$$

□

3.5 The Sheffer Group

In this section, we shall show that the set of all Sheffer-type polynomial sequences

$$\mathcal{S} = \left\{p_n(x) = [t^n]A(t)e^{xg(t)}\right\}$$

with an operation, "umbral composition" as used by S. Roman forms a group. This "umbral composition" can also be interpreted as polynomial sequence multiplication. To describe the umbral calculus, take any two Sheffer-type polynomial sequences

$$\left\{p_n(x) = \sum_{k=0}^{n} p_{n,k}x^k\right\} \text{ and } \left\{q_n(x) = \sum_{k=0}^{n} q_{n,k}x^k\right\} \text{ say,}$$

and define the operation

$$\otimes : \mathcal{S} \times \mathcal{S} \longrightarrow \mathcal{S} \text{ as follows}$$

$$\left\{p_n(x)\right\} \otimes \left\{q_n(x)\right\} = \sum_{k=0}^{n} (r_{n,k})\, x^k,$$

$$\text{where } r_{n,k} = \sum_{l=k}^{n} l!p_{n,l}q_{l,k}. \text{ Now call the R.H.S } \quad \{r_n(x)\}$$

$$i.e. \left\{r_n(x) = \sum_{k=0}^{n} r_{n,k}x^k\right\}.$$

Theorem 3.5.1. ([9])
The set of all Sheffer-type polynomial sequences $\mathcal{S}$ with the operation $\otimes$ forms a group called the Sheffer group, denoted by $(\{p_n(x), \otimes)$. The identity of the group is $\left\{\frac{x^n}{n!}\right\}$. The inverse of $\{p_n(x)\} \in \mathcal{S}$, denoted by $\{p_n(x)\}^{-1}$ is the Sheffer-type polynomial sequence generated by $\frac{1}{A(\bar{g}(t))}exp(x\bar{g}(t))$, where $\bar{g}$ is the compositional inverse of g. i.e $\quad (g \circ \bar{g})(t) = (\bar{g} \circ g)(t) = t$.

Proof. Clearly the set $\mathcal{S}$ is non-empty, since it contains the identity polynomial sequence $\left\{\frac{t^n}{n!}\right\}$. By the above $\mathcal{S}$ is closed under $\otimes$.
It is also clear that

$$\Big\{p_n(x)\Big\} \otimes \left\{\frac{x^n}{n!}\right\} = \left\{\frac{x^n}{n!}\right\} \otimes \Big\{p_n(x)\Big\}.$$

And so for any $\Big\{p_n(x)\Big\}$, $\Big\{q_n(x)\Big\}$ and $\Big\{r_n(x)\Big\}$, we have

$$\begin{aligned}
\{p_n(x)\} \otimes (\{q_n(x)\} \otimes \{r_n(x)\}) &= \left\{\sum_{k=0}^{n}\left(\sum_{u=k}^{n}\sum_{l=u}^{n} l!u!p_{n,l}q_{l,u}r_{u,k}\right)x^k\right\} \\
&= \left\{\sum_{k=0}^{n}\sum_{u=k}^{n}\left(\sum_{l=u}^{n} l!u!p_{n,l}q_{l,u}r_{u,k}\right)x^k\right\} \\
&= (\{p_n(x)\} \otimes \{q_n(x)\}) \otimes \{r_n(x)\}.
\end{aligned}$$

Hence, the set of all Sheffer-type polynomial sequences forms a group. □

The set of Sheffer polynomial sequences has two subgroups, the Appell subgroup and the Associated subgroup. Let $(g(x), f(x))$ be a Riordan pair of GFs. A sequence $(q_n(x))$ is said to be an Appell sequence for $g(x)$ if and only if $(q_n(x))$ is a Sheffer polynomial sequence for $(g(x), x)$. Meaning, $f(x) = x$. Then the pair $((q_n(x), \otimes)$ is the Appell subgroup of the Sheffer group. Similarly, if $g(x) = 1$, then the sequence $(p_n(x))$ generated by the Riordan pair $(1, f(z))$ is the associated polynomial sequence for $f(z)$. And the pair $((p_n(x), \otimes)$ is the associated subgroup of the Sheffer group. Furthermore, the Appell subgroup is an abelian subgroup whereas the associated subgroup is not abelian. Hence, the Sheffer group is a semi direct product of the Appell and associated subgroups.

3.6 Connecting Riordan Arrays, Sheffer Polynomials and Stirling numbers

We shall in this section draw the connection between Riordan arrays, Stirling numbers, and Sheffer polynomials. It shall also be established that there exists an isomorphism between the group of Riordan arrays and the group of Sheffer polynomials.

Definition 3.6.1. Let A(t) $\in \mathcal{F}_o$ and $g(t) \in \mathcal{F}_1$, and also let

$$\frac{1}{k!}A(t)(g(t))^k = \sum_{n\geq k} \sigma(n,k)\frac{t^n}{n!}.$$

Then $\sigma(n,k)$ is called the *generalized Stirling number* with respect to $A(t)$ and $g(t)$.

Now; from definitions, it follows that

$$p_n(x) = [t^n]A(t)e^{xg(t)} = [t^n]\sum_{k\geq 0}\frac{1}{k!}A(t)(g(t))^k x^k$$

$$= \sum_{k=0} d_{n,k}\frac{x^k}{k!} = \frac{1}{n!}\sum_{k=0}^{n}\sigma(n,k)x^k,$$

where we use $d_{n,k} = \sigma(n,k) = 0$ for all $k > n$.

We observe that by multiplying through the k^{th} column of the Riordan array by the constant $\frac{1}{k!}$, the rows of the matrix yield the coefficient of the Sheffer-type polynomial sequences.

Take for instance the Riordan array of the signed Pascal triangle. We know that

$$\left(\frac{1}{1-t}, \frac{-t}{1-t}\right) = \frac{1}{1-t}\left(\frac{-t}{1-t}\right)^k = \sum_{k=0}(-1)^k\binom{n}{k}t^k.$$

From this, we obtain by the constant multiplier $\frac{1}{k!}$ the coefficients of the Laguerre polynomial sequence

$$\left\{ L_n(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{x^k}{k!} \right\}_{n \in \mathbb{N}}.$$

Deduction

From corollary (3.4.27), we have that $s_n(x) = \sum_{k=0}^{n} \left(\sum_{i=k}^{n} \binom{n}{i} \alpha_{n-i}\, a_{i,k} \right) x^k = \frac{1}{n!} \sum_{k=0}^{n} \sigma(n,k) x^k$ where $a_{i,k} = \frac{1}{k} \sum_{i=1}^{i-k+1} \binom{i}{j} a_{j,1} a_{i-1,k-1}, (k = 2, \ldots, n)$ and $\alpha_{n-1} = -\frac{1}{\beta_0} \sum_{k=0}^{n-2} \binom{n-1}{k} \beta_{n-(1+k)} \alpha_k,\ (n = 2, 3, \ldots, n)$.

The generalized Stirling number can be deduced $\sigma(n,k)$ as follows;

$$\sigma(n,k) = n! \sum_{i=0}^{n} \binom{n}{i} \alpha_{n-1} a_{i,k} \tag{3.14}$$

Theorem 3.6.2. ([9])
The Sheffer group and the Riordan group are isomorphic.

Let $(\mathcal{R}, *)$ and $(\mathcal{S}, \otimes)$ be the Riordan group and Sheffer group respectively. Define the map

$$\theta : \mathcal{R} \longrightarrow \mathcal{S}$$

$$(A(t), g(t)) \mapsto \{p_n(x)\}$$

by

$$\theta((A(t), g(t))) := [t^n] A(t) e^{xg(t)}.$$

Suppose $\theta((A(t), g(t))) = \{p_n(x)\}$ and $\theta((B(t), f(t))) = \{q_n(x)\}$,
then $\{p_n(x)\} \otimes \{q_n(x)\} = [\theta(A(t), g(t))] \otimes [\theta(B(t), f(t))] = \theta\left[(A(t), g(t)) \ * \ (B(t), f(t))\right]$
$= \theta(A(t) \cdot f(g(t)), B(g(t))) = \{r_n(x)\} \in \mathcal{S}$.

Also, θ maps the identity of $\mathcal{R}$ to the identity of $\mathcal{S}$

since

$$\begin{aligned}\theta((1,t)) : &= e^{xt}\\ &= \sum \frac{x^n}{n!}t^n\\ &= \sum p_n(x)t^n\end{aligned}$$

for $\left\{p_n(x) = \left\{\frac{x^n}{n!}\right\}\right\}_{n\in\mathbb{N}}$ the identity of $(\mathcal{S},\otimes)$.

Hence θ is bijective and thus an isomorphism.

Corollary 3.6.3. *Let $g(t) \in \mathcal{F}_o$ and $f(t) \in \mathcal{F}_1$. Then every proper Riordan array $(g(t), f(t))$ has a Stirling number associated to it.*

To justify this, we apply FTRA. Thus, we take the product of the matrix $(g(t), f(t))$ with a column vector with G.F; $A(t) = e^t$.

Then

$$\begin{aligned}(g(t), f(t)) \cdot A(t) &= g(t)exp(f(t))\\ &= \sum \frac{g(t)(f(t))^k}{k!}\\ &= \sum_{n\geq k} \sigma(n,k)\frac{t^n}{n!}.\end{aligned}$$

3.7 Construction of Inverses of Sheffer Polynomials

Introduction

This section is motivated by definitions (3.1.1) and (3.1.2). Since both describe a Sheffer polynomial sequence, we seek to see how they are related. We find that applying definitions (3.1.1) and (3.1.2) together on any given Riordan array we obtain a polynomial sequence and its inverse.

In this section, we construct the inverse polynomial sequences for some Sheffer polynomials. They include, the Touchard, Toscano, Charlier, and Poisson - Charlier polynomial sequences.Observe that definition (3.1.2) is essential same as the map θ defined by [9].

3.7.1 Examples

Touchard Polynomial Sequence

Applying definition (3.1.1) or the map θ to the Riordan array $(1, e^t - 1)$ yields the Touchard polynomial sequence. Thus,

$$\theta\left((1, e^t - 1)\right) = e^{x(e^t-1)} = e^{xG_t} = \sum T_n(x)\frac{t^n}{n!} \quad \text{and}$$

$$\sum T_n(x)\frac{t^n}{n!} = \sum G_n\frac{t^n}{n!} = \sum_{n\geq 0}\left(\sum_{r=0}^{n}\left\{\begin{array}{c} n \\ k \end{array}\right\}x^k\right)\frac{t^n}{n!}.$$

The GF, say $f(t) = e^t - 1$ has a compositional inverse function given by $\bar{f}(t) = \ln(1+t)$. Therefore by definition (3.1.2), we obtain the inverse polynomial

$$\begin{aligned}\sum_{n\geq 0}\{T_n(x)\}^{-1}\frac{t^n}{n!} = e^{x\ln(1+t)} &= (1+t)^x \\ &= \sum_{n=0}^{x} x(x-1)...(x-n+1)\frac{t^n}{n!} \\ &= \sum_{n\geq 0}\left(\sum_{r=0}^{n} s(n,r)x^r\right)\frac{t^n}{n!}.\end{aligned}$$

Hence, we have that the Touchard polynomial sequence $T_n(x) = \sum_{r=0}^{n}\left\{\begin{array}{c} n \\ r \end{array}\right\}(x)_r$ has the inverse $\{T_n(x)\}^{-1} = \sum_{r=0}^{n} s(n,r)x^r$. From this, we easily deduce the Stirling transform. Formally,

Definition 3.7.2. The *Touchard polynomial* is a polynomial sequence of the binomial type given by the Stirling transform $T_n(x) = \sum_{k=0}^{n} \left\{ \begin{array}{c} n \\ k \end{array} \right\} x^k,\ n \geq 0$ with $T_o(x) = 1$.

Toscano Polynomial Sequence

Another polynomial of interest here is the Toscano polynomial. Given the Riordan array $(e^{\lambda t}, 1 - e^t), \lambda \neq 0, \lambda \in \mathbb{K}$, we obtain the polynomial

$$\theta((e^{\lambda t}, 1 - e^t)) = \sum_{n \geq 0} (Tos)_n^{\lambda}(x) \frac{t^n}{n!}$$

called the Toscano polynomial. So we have

$$\begin{aligned} \sum (Tos)_n^{\lambda}(x) \frac{t^n}{n!} &= e^{\lambda t} e^{x(1-e^t)} \\ &= e^{\lambda t} e^{-xG_t} \end{aligned}$$

for the parameter $\lambda \in \mathbb{R}$, where we write

$$e^{\lambda t} = \sum_{n=0}^{\infty} \frac{\lambda^n t^n}{n!} \quad \text{and}$$

$$e^{-xG_t} = \sum_{m=0}^{\infty} (-1)^m x^m G_m \frac{t^m}{m!}.$$

Therefore,

$$\begin{aligned} \sum_{n \geq 0} (Tos)_n(x) \frac{t^n}{n!} &= \left(\sum_{n=0}^{\infty} \frac{\lambda^n t^n}{n!} \right) \left(\sum_{m=0}^{\infty} (-1)^m x^m G_m \frac{t^m}{m!} \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{r=0}^{n} \frac{(\lambda t)^r}{r!} (-1)^{n-r} (xG)^{n-r} \frac{t^{n-r}}{(n-r)!} \right) \end{aligned}$$

$$\begin{aligned} &= \sum_{n=0}^{\infty}\left(\sum_{r=0}^{n}\binom{n}{r}\lambda^{r}(-1)^{n-r}(xG)^{n-r}\right)\frac{t^{n}}{n!} \\ &= \sum_{n=0}^{\infty}\left((\lambda-xG)^{n}\right)\frac{t^{n}}{n!}. \end{aligned}$$

Hence $(Tos)_n^{(\lambda)}(x) = (\lambda - xG)^n$ is a GF for the Toscano polynomial sequence.

Let $f(t) = 1 - e^t$, then we get $\bar{f}(t) = \ln(1-t)$. Then the inverse of the Toscano polynomial is given by

$$\begin{aligned} \sum_{n=0}^{\infty}\{(Tos)_n^{(\lambda)}(x)\}^{-1}\frac{t^n}{n!} &= (1-t)^{x-\lambda} \\ &= \sum_{n=0}^{x-\lambda}\binom{x-\lambda}{n}(-t)^{x-\lambda-n} \\ &= \sum_{n=0}^{x-\lambda}(-1)^n(x-\lambda)(x-\lambda-1)...(x-\lambda-n+1)\frac{t^n}{n!} \\ &= \sum_{n=0}^{x-\lambda}(-1)^n\sum_{r=0}^{n}s(n,r)(x-\lambda)^r\frac{t^n}{n!}. \end{aligned}$$

Hence, the Toscano polynomial sequence $(Tos)_n^{\lambda}(x) = (\lambda - xG)^n$ has inverse

$$\{(Tos)_n^{\lambda}(x)\}^{-1} = (-1)^n\sum_{r=0}^{n}s(n,r)(x-\lambda)^r.$$

Charlier Polynomial Sequence

The Charlier polynomial $C_n^{(\alpha)}(x)$ is generated by the Riordan array $(e^{(-\alpha t)}, \ln(1+t))$ with $\alpha \neq 0$. By definition (3.1.1), we have

$$\sum_{n=0}^{\infty}C_n^{(-\alpha)}(x)\frac{t^n}{n!} = e^{-\alpha t}(1+t)^x$$

$$\begin{aligned}
\sum_{n=0}^{\infty} C_n^{(-\alpha)}(x)\frac{t^n}{n!} &= e^{-\alpha t}\sum_{m=0}^{x} x(x-1)\dots(x-m+1)\frac{t^m}{m!} \\
&= \sum_{n=0}^{\infty}(-\alpha)^n\frac{t^n}{n!}.\sum_{m=0}^{x}\Big(\sum_{r=0}^{m} s(m,r)x^r\Big)\frac{t^m}{m!} \\
&= \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n}\frac{(-\alpha t)^k}{k!}.\sum_{r=0}^{m} s(m,r)x^{n-k}\frac{t^{n-k}}{(n-k)!}\Big) \\
&= \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n}(-\alpha)^k.\sum_{r=0}^{m}\binom{n}{r}s(m,r)x^{n-k}\Big)\frac{t^n}{n!}.
\end{aligned}$$

Now, for the inverse sequence, we put $f(t) = \ln(1+t)$ then $\bar{f}(t) = e^t - 1$. By definition (3.1.2), we have

$$\begin{aligned}
\sum_{n=0}^{\infty}\{C_n^{(\alpha)}(x)\}^{-1}\frac{t^n}{n!} &= e^{(\alpha+x)G_t} \\
&= \sum_{n=0}^{\infty}(\alpha+x)^n G_n\frac{t^n}{n!}.
\end{aligned}$$

Hence, the Charlier polynomial sequence
$C_n^{(\alpha)}(x) = \sum_{k=0}^{n}(-\alpha)^k.\sum_{r=0}^{m}\binom{n}{r}s(m,r)x^{n-k}$ has the inverse sequence

$$\{C_n^{(\alpha)}(x)\}^{-1} = (\alpha+x)^n G_n, \quad \alpha \neq 0, \quad \alpha, x \in \mathbb{K}.$$

We observe that the Riordan array for $\{C_n^{(\alpha)}(x)\}^{-1}$ is given by $(e^{\alpha G_t}, G_t)$. And that the Sheffer polynomial sequence $\{C_n^{(\alpha)}(x)\}^{-1}$ is an Appell sequence for $e^{\alpha G_t}$, since,

$$\sum_{n\geq 0}\{C_n^{(\alpha)}(x)\}^{-1}\frac{t^n}{n!} = \frac{\exp(xG_t)}{e^{-\alpha G_t}}.$$

Poisson - Charlier Polynomial Sequence

The Poisson - Charlier polynomial sequence $(PC)_n(x)$ is generated by the Riordan pair $(e^t, \ln(1+t))$ so that;

$$\begin{aligned}\sum_{n=0}^{\infty}(PC)_n(x)\frac{t^n}{n!} &= e^t.(1+t)^x \\ &= e^t.\sum_{r=0}^{x}x(x-1)\ldots(x-r+1)\frac{t^r}{r!} \\ &= \sum_{n=0}^{\infty}\frac{t^n}{n!}.\sum_{r=0}^{x}\Big(\sum_{i=0}^{r}s(r,i)x^i\Big)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{r=0}^{n}\Big(\sum_{i=0}^{r}\binom{n}{i}s(r,i)\Big)x^r\right)\frac{t^n}{n!}.\end{aligned}$$

Now, take $g(t) = e^t$ and $f(t) = \ln(1+t)$ so that $\bar{f}(t) = e^t - 1$. Then the Sheffer polynomial generated by the Riordan pair $(g(t), f(t))^{-1}$ is

$$\frac{1}{g\Big(\bar{f}(t)\Big)}e^{x\bar{f}(t)} = \sum_{n=0}^{\infty}\Big\{\Big(PC\Big)_n(x)\Big\}^{-1}\frac{t^n}{n!}.$$

Thus, we have

$$\begin{aligned}\sum_{n=0}^{\infty}\Big\{\Big(PC\Big)_n(x)\Big\}^{-1}\frac{t^n}{n!} &= e^{(x-1)G_t} \\ &= \sum_{n=0}^{\infty}(x-1)^nG_n\frac{t^n}{n!},\end{aligned}$$

which is just the case $\alpha = -1$ in the Charlier polynomial.
Here too, the inverse is a Sheffer sequence for the Riordan pair $(\frac{1}{e^{G_t}}, G_t)$ is an Appell sequence for $\frac{1}{e^{G_t}}$ since

$$\sum_{n=0}^{\infty}\Big\{\Big(PC\Big)_n(x)\Big\}^{-1}\frac{t^n}{n!} = \frac{\exp(xG_t)}{e^{G_t}}.$$

Therefore we obtain that the Poisson - Charlier polynomial sequence

$$(PC)_n = \sum_{r=0}^{n} \left(\sum_{k=0}^{r} \binom{n}{k} s(r,k) \right) x^r$$

has the inverse sequence

$$\left\{ \Big(PC\Big)_n (x) \right\}^{-1} = (x-1)^n G_n.$$

These constructions can be arrived at by Theorem (2.1) in He, Hsu and Shiue's paper [9],which is stated here as Theorem (3.5.1). Until after we have made our comparison, their work was unknown to us.

Chapter 4

Touchard Polynomial

4.1 Introduction

We have seen that the Riordan array $(A(t), g(t)) = (1, e^t - 1)$ where $A(t) = 1$ and $g(t) = e^t - 1$ under the map θ yields the Touchard polynomial.(Also known as the exponential polynomial). Thus

$$\theta(A(t), g(t)) = e^{x(e^t-1)}.$$

Hence in exponential terms the G.F of the Touchard polynomial is

$$\sum_{n\geq 0} T_n(x)\frac{t^t}{n!} = e^{xG_t},$$

which, making use of the substitution $e^t - 1 = G_t$, we write as

$$\sum_{n\geq 0} T_n(x)\frac{t^n}{n!} = e^{xG_t} = \sum_{r=0}^{\infty} x^r G_r \frac{t^r}{r!}.$$

Hence $T_n(x) = x^n G_n$ for all $n \in \mathbb{N}$. For $x = 1$, we have that $T_n(1) = G_n$, which implies the value of the n^{th} Touchard polynomial at $x = 1$ is the n^{th} Bell number.

It then follows from ([24]) that *For* $n \in \mathbb{N}$, $T_{n+1} = (T(1)+1)^n$ *and* $T_{n+1}(1) = \sum \binom{n}{r} T_r(1)$.

Remark 4.1.1. a . The Riordan array $(1, e^t - 1)$ which generates the Touchard polynomial is an associate Riordan array, and hence belongs to the associated subgroup of the Riordan group.

b . The Touchard (polynomial) sequence (which is a Sheffer sequence) is the associated sequence for $\ln(1+t)$, since $\bar{g}(t)$ of $g(t) = e^t - 1$ is $\bar{g}(t) = \ln(1+t)$.

c . Under the case $x = 1$, the Touchard polynomial yields the Stirling numbers of the second kind.

In [4] , Chrysaphinou showed how the Bell polynomial, the partial Bell polynomial and the Rook polynomial are related to the Touchard polynomial. According to Riordan[[13], p171], the Rook polynomial $R_{n,k}$ is related to the Laguerre polynomial $L_n^{(\lambda)}$ as follows; $R_{n,n+\lambda}(x) = n!x^n L_n^{(\lambda)}(-x^{-1})$ for any integral λ. And since;

Theorem 4.1.2. ([4] , Theorem 2.2) *The Touchard polynomials are related with the Rook polynomials as follows;* $T_n(x) := T_{n,k}(1, 2x, 0, \ldots; 1, 0, \ldots) = \binom{n}{k} R_{n-k,k}(x)$,

we obtain a relation between the Touchard polynomial and the Laguerre polynomial as follows;

$$T_{n,k}(1, 2x, 0, \ldots; 1, 0, \ldots) = \frac{n!}{k!} x^{n-k} L_{n-k}^{(2k-n)}(-x^{-1}).$$

Using the recurrence relation given as (1.1.10) for G_n, $n \in \mathbb{N}$, we derive a corresponding recurrence relation for the $T_n(1)$.

The Touchard polynomial $T_n, n \in \mathbb{N}$ forms a basis for the linear space of all polynomials. To show this, we use the definition of $T_n(x)$ in Remark (4.1.1)(c) above and the Stirling transform, and express the basis $\{T_n\}_{n\in\mathbb{N}}$ in terms of the standard basis $\{x^n\}_{n\in\mathbb{N}}$. In other words, we solve for $(x)_k$ in

$$T_n(x) = \sum_{k=0}^{n} \left\{ \begin{array}{c} n \\ k \end{array} \right\} (x)_k$$

to get

$$\begin{aligned} (x)_n &= \textstyle\sum_{k=0}^{n} (-1)^{(n-k)} \left[\begin{array}{c} n \\ k \end{array} \right] T_k \\ &= \textstyle\sum_{k=0}^{n} s(n,k) T_k. \end{aligned}$$

Hence we can express the standard basis $\{1, x, x^2, \ldots\}$ in terms of the Touchard polynomial $\{T_0, T_1, T_2, \ldots\}$ as follows

$$\begin{array}{rcc} 1 &=& T_0 \\ x &=& T_1 \\ x^2 &=& -T_1 + T_2 \\ x^3 &=& 2T_1 - 3T_2 + T_3 \\ x^4 &=& -6T_1 + 11T_2 - 6T_3 + T_4 \\ \vdots && \end{array} = \begin{bmatrix} 1 &&&& \\ 0 & 1 &&& \\ 0 & -1 & 1 && \\ 0 & 2 & -3 & 1 & \\ 0 & -6 & 11 & -6 & 1 \\ \vdots &&&& \end{bmatrix} \begin{bmatrix} T_0 \\ T_1 \\ T_2 \\ \vdots \\ T_n \\ \vdots \end{bmatrix},$$

where the infinite lower triangular matrix is the Riordan matrix of the Stirling numbers of the first kind.

Corollary 4.1.3. *For any $n \in \mathbb{N}$ and p prime,*
$T_{n+p} \equiv \sum_{k=0}^{n+p} s(n+p,k)\, T_k \Big\{ \Big(T_n(1) + T_{n+1}(1) \Big) mod\, p \Big\}$.

Proof. We have seen that $T_n(x) = x^n G_n$ and also from Lemma (1.1.10) that $G_{n+p} \equiv (G_n + G_{n+1})\, mod\, p$ for p prime. Then, it follows that $T_{n+p}(x) = x^{n+p} G_{n+p} \equiv x^{n+p}\{G_n + G_{n+1}\} mod\, p$.
By making the substitution $x^{n+p} = \sum_{k=0}^{n+p} s(n+p,k) T_k$ and $G_n = T_n(1)$, the result follows; $T_{n+p} \equiv \sum_{k=0}^{n+p} s(n+p,k) T_k \Big\{ \Big(T_n(1) + T_{n+1}(1) \Big) mod\, p \Big\}$ which is an analogue of (1.1.10) for $n \in \mathbb{N}$ and p prime. $\square$

For $n = 0$ and $x = 0$, we obtain the already known result

$$T_p(1) \equiv 2 \, mod \, p.$$

Thus, for

$$n = 0, \quad T_p(x) \equiv \sum_{k=0}^{p} s(p,k) T_k \Big(2 \, mod \, p \Big),$$

where

$$\sum_{k=0}^{p} s(p,k) T_k = 1 \quad \text{for} \quad x = 1.$$

4.2 Congruence

4.2.1 Introduction

Given any two polynomials

$$f(x) = \sum_n a_n x^n \quad \text{and} \quad g(x) = \sum_n b_n x^n$$

with integer coefficients, $a_n, b_n \in \mathbb{Z}$. Whenever $a_n \equiv b_n \, mod \, q$ for all k, then

$$f(x) \equiv g(x) \, mod \, q \tag{4.1}$$

and we say "f congruent g modulo q".

Theorem 4.2.2. (Lagrange [5]) *For each prime p, we have in the sense of the relation* (4.1) $(x)_p := x(x-1)\dots(x-p+1) \equiv (x^p - x) \, mod \, p$.

In other words, the Stirling numbers of the first kind satisfy $s(p,k) \equiv 0 \, mod \, p$, except $s(p,p) = 1$ and $s(p,1) = (p-1)! \equiv -1 \, mod \, p$.

Consequently, we obtain upon putting $x = a$ in Theorem (4.2.2);

Corollary 4.2.3. (Fermat theorem, [5]) *For all integers $a \geq 0$, and each prime number p, $a^p \equiv a \, mod \, p$.*

Theorem 4.2.4. ([5]) *For each prime number p, the Stirling numbers of the second kind satisfy* $\left\{ \begin{matrix} p \\ k \end{matrix} \right\} \equiv 0 \, mod \, p,$ *except* $\left\{ \begin{matrix} p \\ 1 \end{matrix} \right\} = \left\{ \begin{matrix} p \\ p \end{matrix} \right\} = 1$.

4.3 Another Analogue of Lemma (1.1.10)

In definition (3.7.2), the Touchard polynomial sequence was given by the Stirling transform as $T_n(x) = \sum_{k=1}^{n} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} x^k$. We have already seen that when $x = 1$, $T_n(1) = G_n$, from which we proceeded to obtain corollary (4.1.3). Thus, from Lemma (1.1.10) we have; $\sum_{k=1}^{n+p} \left\{ \begin{matrix} n+p \\ k \end{matrix} \right\} \equiv \left(\sum_{k=1}^{n} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} + \sum_{k=1}^{n+1} \left\{ \begin{matrix} n+1 \\ k \end{matrix} \right\} \right)$ *mod p.* Thus,

$$\begin{aligned} \sum_{k=1}^{n+p} \left\{ \begin{matrix} n+p \\ k \end{matrix} \right\} &\equiv \left(\sum_{k=1}^{n} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} + \sum_{k=1}^{n+1} \left\{ \begin{matrix} n+1 \\ k \end{matrix} \right\} \right) mod \, p \\ &\equiv \left(\sum_{k=1}^{n+1} \left(\left\{ \begin{matrix} n \\ k \end{matrix} \right\} + \left\{ \begin{matrix} n+1 \\ k \end{matrix} \right\} \right) \right) mod \, p, \quad \text{since} \left\{ \begin{matrix} n \\ n+1 \end{matrix} \right\} = 0 \\ &\equiv \left(\sum_{k=1}^{n+p} \left(\left\{ \begin{matrix} n \\ k \end{matrix} \right\} + \left\{ \begin{matrix} n+1 \\ k \end{matrix} \right\} \right) \right) mod \, p, \end{aligned}$$

since all the extra terms on the right are zero for $p > 1$.

This says p divides $\sum_{k=1}^{n+p} \left\{ \begin{matrix} n+p \\ k \end{matrix} \right\} - \sum_{k=1}^{n+p} \left(\left\{ \begin{matrix} n \\ k \end{matrix} \right\} + \left\{ \begin{matrix} n+1 \\ k \end{matrix} \right\} \right)$ i.e. $p | \sum_{k=1}^{n+p} \left(\left\{ \begin{matrix} n+p \\ k \end{matrix} \right\} - \left\{ \begin{matrix} n \\ k \end{matrix} \right\} + \left\{ \begin{matrix} n+1 \\ k \end{matrix} \right\} \right)$. Although this does not mean that $p \,|\, \left(\left\{ \begin{matrix} n+p \\ k \end{matrix} \right\} - \left\{ \begin{matrix} n \\ k \end{matrix} \right\} + \left\{ \begin{matrix} n+1 \\ k \end{matrix} \right\} \right)$, it is however valid to ask for which values of n, p, and k (if any) does the relation

$$\left\{ \begin{matrix} n+p \\ k \end{matrix} \right\} \equiv \left(\left\{ \begin{matrix} n \\ k \end{matrix} \right\} + \left\{ \begin{matrix} n+1 \\ k \end{matrix} \right\} \right) mod \, p. \tag{4.2}$$

holds.

First observation; for all $n \in \mathbb{N}, p$ prime, and $k > n+p$; $\left\{ \begin{array}{c} n+p \\ k \end{array} \right\} = 0$. Since $0 \equiv 0 \,(mod\, p)$, the relation (4.2) holds.

Second observation; for all $n \geq 1$ and for all prime p, the relation (4.2) always holds for $k = 0$. Since $\left\{ \begin{array}{c} n+p \\ 0 \end{array} \right\} = 0$ for all $n \geq 1,$ and prime p.

i.e. $\left\{ \begin{array}{c} n+p \\ 0 \end{array} \right\} \equiv \left(\left\{ \begin{array}{c} n \\ 0 \end{array} \right\} + \left\{ \begin{array}{c} n+1 \\ 0 \end{array} \right\} \right) mod\, p.$

To proceed from the above observations, we want to find a k satisfying $0 < k \leq n+p$, for which the relation holds. However, the R.H.S of (4.2) motivates that whenever $n+1 < k$, our first observation (also) accounts for the values of k that satisfy (4.2). Hence, we restrict our discussion here to $0 < k \leq n+1$. We do so for the first four primes.

To start with, take $p = 2$

then for $n = 0$ we have

$$\left\{ \begin{array}{c} 2 \\ k \end{array} \right\} \equiv \left(\left\{ \begin{array}{c} 0 \\ k \end{array} \right\} + \left\{ \begin{array}{c} 1 \\ k \end{array} \right\} \right) mod\, 2$$

which holds when $k = 1$.

For $n = 1$, $\left\{ \begin{array}{c} 3 \\ k \end{array} \right\} \equiv \left(\left\{ \begin{array}{c} 1 \\ k \end{array} \right\} + \left\{ \begin{array}{c} 2 \\ k \end{array} \right\} \right) mod\, 2$ is satisfied by $k = 2$.

Continuing, we consider when $n = 2$. We have that

$$\left\{ \begin{array}{c} 4 \\ k \end{array} \right\} \equiv \left(\left\{ \begin{array}{c} 2 \\ k \end{array} \right\} + \left\{ \begin{array}{c} 3 \\ k \end{array} \right\} \right) mod\, 2$$

fails (even) for $0 < k \leq 4$.

We consider only one more case here. Thus, when $n = 3$.

This gives $\left\{ \begin{array}{c} 5 \\ k \end{array} \right\} \equiv \left(\left\{ \begin{array}{c} 3 \\ k \end{array} \right\} + \left\{ \begin{array}{c} 4 \\ k \end{array} \right\} \right) mod\, 2$ which holds for $k = 3$.

Next, we take $p = 3$ and seek for values $k \in \mathbb{N}$ for which

$$\left\{ \begin{array}{c} n+3 \\ k \end{array} \right\} \equiv \left(\left\{ \begin{array}{c} n \\ k \end{array} \right\} + \left\{ \begin{array}{c} n+1 \\ k \end{array} \right\} \right) mod\, 3$$

is satisfied.

For $n = 0$, $\left\{ \begin{array}{c} 3 \\ k \end{array} \right\} \equiv \left(\left\{ \begin{array}{c} 0 \\ k \end{array} \right\} + \left\{ \begin{array}{c} 1 \\ k \end{array} \right\} \right) mod\, 3$ holds for $k = 1$.

For $n = 1$, $\left\{ \begin{array}{c} 4 \\ k \end{array} \right\} \equiv \left(\left\{ \begin{array}{c} 1 \\ k \end{array} \right\} + \left\{ \begin{array}{c} 2 \\ k \end{array} \right\} \right) mod\, 3$ holds for $k = 2$.

For $n = 2$, $\left\{ \begin{array}{c} 5 \\ k \end{array} \right\} \equiv \left(\left\{ \begin{array}{c} 2 \\ k \end{array} \right\} + \left\{ \begin{array}{c} 3 \\ k \end{array} \right\} \right) mod\, 3$ holds for $k = 3$.

For $n = 3$, $\left\{ \begin{array}{c} 6 \\ k \end{array} \right\} \equiv \left(\left\{ \begin{array}{c} 3 \\ k \end{array} \right\} + \left\{ \begin{array}{c} 4 \\ k \end{array} \right\} \right) mod\, 3$ holds for $k = 2$.

See, this is the (special) case $n = p = 3$.

For $n = 4$, $\left\{ \begin{array}{c} 7 \\ k \end{array} \right\} \equiv \left(\left\{ \begin{array}{c} 4 \\ k \end{array} \right\} + \left\{ \begin{array}{c} 5 \\ k \end{array} \right\} \right) mod\, 3$ holds for $k = 3, 4$.

And lastly for this case we consider when $n = 5$,

$\left\{ \begin{array}{c} 8 \\ k \end{array} \right\} \equiv \left(\left\{ \begin{array}{c} 5 \\ k \end{array} \right\} + \left\{ \begin{array}{c} 6 \\ k \end{array} \right\} \right) mod\, 3$ holds for $k = 2, 4$.

Next, we take $p = 5$ and list the values $k \in \mathbb{N}$ for $n \in \mathbb{N}$ for which

$$\left\{ \begin{array}{c} n+5 \\ k \end{array} \right\} \equiv \left(\left\{ \begin{array}{c} n \\ k \end{array} \right\} + \left\{ \begin{array}{c} n+1 \\ k \end{array} \right\} \right) mod\, 5$$

is satisfied.

For $n = 0,\ k = 1$

$n = 1,\ k = 2$

$n = 2,\ k = 3$
$n = 3,\ k = 4$
$n = 4,\ k = 5$
$n = 5,\ k = 2, 3$ since $\exists p < 5$ when $n = p$.
$n = 6,\ k = 3, 4, 5, 6$.

Lastly, we consider the case where the prime $p = 7$. For $n \in \mathbb{N}$, here are the $k \in \mathbb{N}$ which satisfy the relation

$$\left\{ \begin{matrix} n+7 \\ k \end{matrix} \right\} \equiv \left(\left\{ \begin{matrix} n \\ k \end{matrix} \right\} + \left\{ \begin{matrix} n+1 \\ k \end{matrix} \right\} \right) mod\, 7.$$

When $n = 0, \quad k = 1$
$n = 1, \quad k = 2$
$n = 2, \quad k = 3$
$n = 3, \quad k = 4$
$n = 4, \quad k = 2, 5$ (a case worth considering)
$n = 5, \quad k = 6$
$n = 6, \quad k = 7$
$n = 7, \quad k = 2, 3, 4, 5, 6$.

This leads to the following conjecture.

Conjecture 4.3.1. *For any prime p, $\left\{ \begin{matrix} n+p \\ k \end{matrix} \right\} \equiv \left(\left\{ \begin{matrix} n \\ k \end{matrix} \right\} + \left\{ \begin{matrix} n+1 \\ k \end{matrix} \right\} \right) mod\, p$ holds for all $n \in \mathbb{N}$ with;*

a. $k > n + p$,

b. $k = 0$.

And the congruence relation also holds for

i. $k = n + 1$ *and* $n < p$,

ii. *for all k such that $2 \leq k < p$ whenever $n = p$.*

Finally, we observe that we can replace $\left\{ \begin{array}{c} n+1 \\ k \end{array} \right\}$ by $\left\{ \begin{array}{c} n \\ k-1 \end{array} \right\} + k \left\{ \begin{array}{c} n \\ k \end{array} \right\}$ using the recurrence of Lemma (2.2.2) so that

$$\begin{aligned} \left\{ \begin{array}{c} n+p \\ k \end{array} \right\} &\equiv \left(\left\{ \begin{array}{c} n \\ k \end{array} \right\} + \left\{ \begin{array}{c} n+1 \\ k \end{array} \right\} \right) mod\, p \\ \Leftrightarrow \left\{ \begin{array}{c} n+p \\ k \end{array} \right\} &\equiv \left((1+k) \left\{ \begin{array}{c} n \\ k \end{array} \right\} + \left\{ \begin{array}{c} n \\ k-1 \end{array} \right\} \right) mod\, p. \end{aligned}$$

4.4 Conclusion

We have seen that the Riordan arrays, the Stirling numbers and the Sheffer polynomial sequences are related. The Sheffer polynomial sequences as an array of generating functions give rise to Riordan arrays. Also, we've seen that for any proper Riordan array $(g(z), f(z))$, there is a Stirling number associated to it. These connections enabled us to use the inverses of Riordan arrays to construct the corresponding polynomial inverses of their respective Sheffer polynomials.

There are many problems in several branches of mathematics, particularly in applied mathematics (in the area of approximation theory), engineering, theoretical physics and chemistry that give rise to Sheffer polynomial sequences. We based our study of this kind of sequences both on formal power series and operators.

Several works [2], [6] and [15] have showed the Sheffer sequence related to a sequence $p_n(x)$ of binomial - type if it satisfies the functional equation
$s_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} s_k(x) p_{n-k}(y) = \sum_{k=0}^{n} \binom{n}{k} p_k(y) s_{n-k}(x)$. It was suggested that if $Q \in \mathcal{L}_s$ has $p_n(x)$ as its basis sequence, then definition (3.4.16) allows us to introduce the operator Q into the above to obtain $s_n(x+y) = \sum_{k \geq 0} \frac{1}{(n-k)!} Q^k s_n(x) p_k(y)$.

In [2] Theorem 2.10, a necessary and sufficient condition is given for $Q \in \mathcal{L}_s$ to be a delta operator. Following from that, the result we obtained in proposition (3.4.21) suggests a kind of generalised form of this theorem. We notice that it relates the delta operator Q, the basic sequence $p_n(x)$ of Q, the Sheffer sequence relative to Q and the derivative operator D in a single equation: $(Q^k s_n)(x) = ((D^k \circ S^{k-1}) p_n)(x)$.

Consequently, $(Qs_n)(x) = (Dp_n)(x)$ whenever $k = 1$. From this we wrote $s_n(x) = \frac{1}{n+1}Dp_{n+1}(x)$ and applied the idea of Wronskian to obtain the Sheffer vector $\bar{S}(x)$.

Motivated by Lemma (1.1.10) and the fact that $G_n = T_n(1) = \sum_{k=1}^{n}\left\{\begin{matrix} n \\ k \end{matrix}\right\}$, we arrived at the result $T_{n+p} \equiv \sum_{k=0}^{n+p} s(n+p,k)\, T_k\Big\{\Big(T_n(1) + T_{n+1}(1)\Big)(modp)\Big\}$ where p is a prime for any $n \in \mathbb{N}$. Finally, we deduced the congruence relation $\sum_{k=1}^{n+p}\left\{\begin{matrix} n+p \\ k \end{matrix}\right\} \equiv \left(\sum_{k=1}^{n+p}\left(\left\{\begin{matrix} n \\ k \end{matrix}\right\} + \left\{\begin{matrix} n+1 \\ k \end{matrix}\right\}\right)\right) mod\, p$ and made a conjecture concerning the values of n, p, and k for which the congruence $\left\{\begin{matrix} n+p \\ k \end{matrix}\right\} \equiv \left(\left\{\begin{matrix} n \\ k \end{matrix}\right\} + \left\{\begin{matrix} n+1 \\ k \end{matrix}\right\}\right) mod\, p$ is true.

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