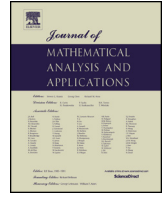




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Logarithmic Bloch spaces in the polydisc, endpoint results for Hankel operators and pointwise multipliers



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ABSTRACT

We define two notions of Logarithmic Bloch space in the polydisc for which we provide equivalent definitions in terms of symbols of bounded Hankel operators. We also provide a full characterization of the pointwise multipliers between two different Bloch spaces of the unit polydisc.

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1. Introduction

We recall that the polydisc of \mathbb{C}^n is the set $\mathbb{D}^n = \mathbb{D} \times \cdots \times \mathbb{D}$ (n times). We denote by $\mathcal{H}(\mathbb{D}^n)$ the space of all analytic functions on \mathbb{D}^n .

For $0 < p < \infty$, the Bergman space $A^p(\mathbb{D}^n)$ consists of analytic functions f in the polydisc \mathbb{D}^n such that

$$\|f\|_p^p := \int_{\mathbb{D}^n} |f(z)|^p d\nu(z) < \infty. \tag{1}$$

Here $d\nu(z) = d\nu_1(z_1) \cdots d\nu_n(z_n)$ for $z = (z_1, \dots, z_n)$, where $d\nu_j$ is the normalized Lebesgue measure on the unit disk \mathbb{D} of the complex plane \mathbb{C} . That is $A^p(\mathbb{D}^n)$ is the subspace of the Lebesgue space $L^p = L^p(\mathbb{D}^n, d\nu)$ consisting of analytic functions. In particular the space $A^2(\mathbb{D}^n)$ is a reproducing kernel Hilbert space, that is any $f \in A^2(\mathbb{D}^n)$ admits the representation

$$f(z) = P(f)(z) = \langle f, B(\cdot, z) \rangle = \int_{\mathbb{D}^n} f(w) \overline{B(w, z)} d\nu(z), \text{ for any } z \in \mathbb{D}^n, \tag{2}$$

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where the (weighted Bergman) kernel $B(\cdot, \cdot)$ is given by

$$B(z, w) := \frac{1}{\prod_{j=1}^n (1 - w_j \bar{z}_j)^2}. \quad (3)$$

The operator P is in fact the orthogonal projection from the Hilbert space $L^2(\mathbb{D}^n)$ onto its closed subspace $A^2(\mathbb{D}^n)$ and it is called the Bergman projection.

The space $H^\infty(\mathbb{D}^n)$ is the set of all bounded analytic functions in \mathbb{D}^n , and for $f \in H^\infty(\mathbb{D}^n)$, we write

$$\|f\|_\infty := \sup_{z \in \mathbb{D}^n} |f(z)| < \infty.$$

Let $f \in \mathcal{H}(\mathbb{D}^n)$. For $j \in \{1, \dots, n\}$, we put $R_j f(z) = z_j \frac{\partial f}{\partial z_j}(z)$, and we consider the operator D_j defined by

$$D_j f(z) = 2f(z) + z_j \frac{\partial f}{\partial z_j}(z) = (2I + R_j)(f),$$

where I stands for the identity operator. We put $D = D_1 \dots D_n$.

The product Bloch space of the polydisc \mathbb{D}^n is denoted $\mathcal{B}(\mathbb{D}^n)$ and consists of all analytic functions f such that

$$\|f\|_{\mathcal{B}} := |f(0)| + \sup_{z \in \mathbb{D}^n} \left[\prod_{j=1}^n (1 - |z_j|^2) \right] |Df(z)| < \infty. \quad (4)$$

The product Bloch space can be identified with the dual space of the Bergman space $A^1(\mathbb{D}^n)$ (see [5]).

We recall that given $b \in \mathcal{H}^\infty(\mathbb{D}^n)$, the (small) Hankel operator with symbol b , h_b is the operator defined for $f \in A^2(\mathbb{D}^n)$ by

$$h_b(f) := P(b\bar{f}).$$

The following equivalent characterizations of the Bloch space were obtained in [5,13]. For the corresponding result in the one parameter case, we refer the reader to [7,14].

Lemma 1.1. *Let f be holomorphic in \mathbb{D}^n . Then the following assertions are equivalent.*

- (i) $b \in \mathcal{B}(\mathbb{D}^n)$.
- (ii) There exists a function $g \in L^\infty(\mathbb{D}^n)$ such that $b = Pg$. Moreover, $\|b\|_{\mathcal{B}} \simeq \|g\|_\infty$.
- (iii) b belongs to the topological dual space $(A^1(\mathbb{D}^n))^*$ of $A^1(\mathbb{D}^n)$ under the duality pairing

$$\langle f, b \rangle := \lim_{r \rightarrow 1} \int_{\mathbb{D}^n} f(z) \overline{b(rz)} d\nu(z), \quad g \in A^1(\mathbb{D}^n), b \in \mathcal{B}(\mathbb{D}^n). \quad (5)$$

- (iv) For $1 < p < \infty$, the Hankel operator h_b extends into a bounded operator on $A^p(\mathbb{D}^n)$.

The next space is called by several authors the Bloch space but here we name it the pointwise Bloch space to avoid any confusion with the Bloch space defined above. A function f analytic in \mathbb{D}^n belongs to the pointwise Bloch space $\mathbb{B}(\mathbb{D}^n)$ if

$$\|f\|_{\mathbb{B}} := |f(0)| + \sup_{z \in \mathbb{D}^n} \sum_{j=1}^n (1 - |z_j|^2) \left| \frac{\partial f}{\partial z_j}(z) \right| < \infty. \tag{6}$$

The pointwise Bloch space can be obtained using the definition of a Bloch space in several complex variables by R. Timoney [11]. We recall that the definition by R. Timoney was given in the more general setting of bounded homogeneous domains in \mathbb{C}^n .

If in (6), we let $n = 1$, we recover, the definition of the one parameter Bloch space $\mathcal{B}(\mathbb{D})$. Note that a function is in the pointwise Bloch space if and only if it is a one parameter Bloch function in each variable (see [11]). This justifies our choice of the name for this space.

It was proved in [3, Theorem C] that $\mathbb{B}(\mathbb{D}^n)$ coincides with the space of holomorphic functions f such that the big Hankel operator associated to the symbol \bar{f} is bounded on $A^2(\mathbb{D}^n)$.

Let us introduce some other analytic function spaces in \mathbb{D}^n . First the logarithmic Bloch space $\mathcal{B}_L(\mathbb{D}^n)$ which consists of analytic functions f in \mathbb{D}^n such that

$$\|f\|_{\mathcal{B}_L} := |f(0)| + \sup_{z \in \mathbb{D}^n} \left[\prod_{j=1}^n (1 - |z_j|^2) \log \frac{4}{1 - |z_j|^2} \right] |Df(z)| < \infty. \tag{7}$$

The above notion extends the notion of logarithmic Bloch space of the unit disk.

When it comes to the pointwise Bloch space, we also have a notion of logarithmic Bloch space, $\mathbb{B}_L(\mathbb{D}^n)$. An analytic function f belongs to $\mathbb{B}_L(\mathbb{D}^n)$ if

$$\|f\|_{\mathbb{B}_L} := |f(0)| + \sup_{z \in \mathbb{D}^n} \left[\prod_{j=1}^n (1 - |z_j|^2) \right] \left[\sum_{j=1}^n \log \frac{4}{1 - |z_j|^2} \right] |Df(z)| < \infty. \tag{8}$$

If we take $n = 1$ in (7) or (8), we obtain the definition of the logarithmic Bloch space of the unit disk $\mathcal{B}_L(\mathbb{D})$.

Our last space is the space $\mathcal{B}_{LL}(\mathbb{D}^n)$ defined as the space of all analytic functions f in \mathbb{D}^n for which there is a constant $C > 0$ such that for any $K = \{k_1, \dots, k_l\} \subseteq \{1, \dots, n\}$,

$$|f(0)| + \sup_{z \in \mathbb{D}^n} \left[\prod_{j \in K} (1 - |z_j|^2) \log \frac{4}{1 - |z_j|^2} \right] |D_{k_1} \dots D_{k_l} f(z)| \leq C.$$

The smallest constant in the above definition is denoted $\|f\|_{\mathcal{B}_{LL}}$.

Note that a function f belongs to $\mathcal{B}_{LL}(\mathbb{D}^n)$ if and only if $f \in \mathcal{B}_L(\mathbb{D}^n)$ and for any $w \in \mathbb{D}^k$ fixed ($1 \leq k < n$), the function $f(\cdot, w)$ which is a function of $n - k$ variables, is uniformly in $\mathcal{B}_L(\mathbb{D}^{n-k})$.

Let us observe that $\mathcal{B}_{LL}(\mathbb{D}^n)$ is a strict subspace of $\mathcal{B}_L(\mathbb{D}^n)$. Indeed, consider the function

$$f : (z_1, \dots, z_n) \mapsto \prod_{j=1}^n \log \log \left(\frac{16}{1 - z_j} \right).$$

Then $f \in \mathcal{B}_L(\mathbb{D}^n)$, but $f \notin \mathcal{B}_{LL}(\mathbb{D}^n)$.

In this note, for each of the above logarithmic Bloch spaces, we provide an equivalent definition in terms of a set of symbols of bounded Hankel operators. We also characterize the multiplier algebra of the product Bloch space, and the bounded pointwise multipliers from the pointwise Bloch space to the product Bloch space.

In the next section, we state our endpoints results on Hankel operators, and pointwise multipliers between Bloch spaces in the polydisc. Some useful results and observations on the spaces in consideration are provided

in Section 3. The results on Hankel operators are proved in Section 4 while in Section 5, we prove our results on pointwise multipliers.

Finally, all over the text, C will be a constant not necessarily the same at each occurrence. We will also use the notation C_k to express the fact that the constant depends on the underlined parameter. Given two positive quantities A and B , the notation $A \lesssim B$ means that $A \leq CB$ for some positive constant C . When $A \lesssim B$ and $B \lesssim A$, we write $A \simeq B$.

2. Statement of the results

Our first result is the following (see [4] for the case of the unit ball).

Theorem 2.1. *Let $b \in \mathcal{H}(\mathbb{D}^n)$. Then the Hankel operator h_b extends as a bounded operator on $A^1(\mathbb{D}^n)$ if and only if $b \in \mathcal{B}_L(\mathbb{D}^n)$.*

We also obtain that $\mathbb{B}_L(\mathbb{D}^n)$ is the exact range of symbols of bounded Hankel operators from $\mathbb{B}(\mathbb{D}^n)$ to $\mathcal{B}(\mathbb{D}^n)$.

Theorem 2.2. *Let $b \in \mathcal{H}(\mathbb{D}^n)$. Then the Hankel operator h_b extends as a bounded operator from $\mathbb{B}(\mathbb{D}^n)$ to $\mathcal{B}(\mathbb{D}^n)$ if and only if $b \in \mathbb{B}_L(\mathbb{D}^n)$.*

Given two Banach spaces X and Y of analytic functions in \mathbb{D}^n , the set of pointwise multipliers from X to Y is defined by

$$\mathcal{M}(X, Y) := \{g \in \mathcal{H}(\mathbb{D}^n) : fg \in Y, \text{ for any } f \in X\}.$$

When $X = Y$, we just write $\mathcal{M}(X)$ for $\mathcal{M}(X, X)$. The norm of the multiplication operator M_ϕ by ϕ from X to Y is denoted $\|M_\phi\|_{X \rightarrow Y}$ or $\|\phi\|_{X \rightarrow Y}$.

The multiplier algebra $\mathcal{M}(\mathbb{B}(\mathbb{D}^n))$ of $\mathbb{B}(\mathbb{D}^n)$ was characterized by R. F. Allen and F. Colonna in [1]. They proved that the only members of $\mathcal{M}(\mathbb{B}(\mathbb{D}^n))$ are constants.

Our first main result on pointwise multipliers is the following.

Theorem 2.3. *Let $\phi \in H^\infty(\mathbb{D}^n)$. Then ϕ is a multiplier from $\mathbb{B}(\mathbb{D}^n)$ to $\mathcal{B}(\mathbb{D}^n)$ if and only if $\phi \in \mathbb{B}_L(\mathbb{D}^n)$. Moreover,*

$$\|\phi\|_{\mathbb{B} \rightarrow \mathcal{B}} \lesssim \|\phi\|_\infty + \|\phi\|_{\mathbb{B}_L}.$$

Here is our characterization of the pointwise multipliers of $\mathcal{B}(\mathbb{D}^n)$.

Theorem 2.4. *Let $\phi \in \mathcal{H}(\mathbb{D}^n)$. Then ϕ is a multiplier of $\mathcal{B}(\mathbb{D}^n)$ if and only if*

$$\phi \in H^\infty(\mathbb{D}^n) \cap \mathcal{B}_{LL}(\mathbb{D}^n).$$

Moreover,

$$\|\phi\|_{\mathcal{B} \rightarrow \mathcal{B}} \simeq \|\phi\|_\infty + \|\phi\|_{\mathcal{B}_{LL}}.$$

In the unit disk \mathbb{D} of the complex plane \mathbb{C} , Theorem 2.4 was proved by J. Arazy in [2]. For the unit ball of \mathbb{C}^n , we refer the reader to [12].

Remark 2.5. The last two results are stated differently. This comes from the fact that we do not know whether or not $\mathcal{M}(\mathbb{B}(\mathbb{D}^n), \mathcal{B}(\mathbb{D}^n)) \subset H^\infty(\mathbb{D}^n)$. In fact, more generally, when X and Y two Banach spaces of analytic functions with $Y \not\subseteq X$, it is not always the case that $\mathcal{M}(X, Y) \subset H^\infty(\mathbb{D}^n)$. The question of finding appropriate conditions on the spaces X and Y so that every element of $\mathcal{M}(X, Y)$ is a bounded function was recently considered in the case of the unit disk \mathbb{D} by D. Girela and N. Merchán in [6], but our case is far from the scope of their work.

3. Some useful results

3.1. A useful estimate

The following useful result is extracted from [10, Proposition 1.4.10].

Lemma 3.1. For $z \in \mathbb{D}$, c real, and $t > -1$, define

$$J_{c,t}(z) := \int_{\mathbb{D}} \frac{(1 - |w|)^t}{|1 - \bar{w}z|^{2+c+t}} d\nu(w).$$

Then

- (i) when $c < 0$, $J_{c,t}$ is bounded on \mathbb{D} ;
- (ii) when $c > 0$, $J_{c,t}(z) \simeq (1 - |z|^2)^{-c}$;
- (iii) $J_{0,t}(z) \simeq \log\left(\frac{1}{1 - |z|^2}\right)$.

3.2. The product Bloch space and its logarithmic counterpart

For $K = (k_1, \dots, k_n)$, $k_j \in \mathbb{N}$, we define on $\mathcal{H}(\mathbb{D}^n)$ the operator D^K given by

$$D^K = D_1^{k_1} \dots D_n^{k_n},$$

where $D_j^{k_j} f(z) = [(1 + k_j)I + R_j] \dots [2I + R_j]$.

As in the one parameter setting (see [4]), we obtain equivalent norms on $\mathcal{B}(\mathbb{D}^n)$ by using higher order derivatives in each variable. More precisely, for a vector $K = (k_1, \dots, k_n) \in \mathbb{N}^n$, we have

$$\|f\|_{\mathcal{B}} \simeq |f(0)| + \sup_{z \in \mathbb{D}^n} \left[\prod_{j=1}^n (1 - |z_j|^2)^{k_j} \right] |D^K f(z)| \tag{9}$$

Also,

$$\|f\|_{\mathcal{B}_L} \simeq |f(0)| + \sup_{z \in \mathbb{D}^n} \left[\prod_{j=1}^n (1 - |z_j|^2)^{k_j} \log \frac{4}{1 - |z_j|^2} \right] |D^K f(z)| \tag{10}$$

(see also [8,9]).

Let us observe the following formula that can be proved as in the one parameter situation.

Lemma 3.2. Let f and g be two analytic polynomials in \mathbb{D}^n . Then for any $K = (k_1, \dots, k_n) \in \mathbb{N}_0^n$, there exists a constant $C = C_{K,n}$ such that the following formula holds

$$\int_{\mathbb{D}^n} f(z) \overline{g(z)} d\nu(z) = C \int_{\mathbb{D}^n} f(z) \left[\prod_{j=1}^n (1 - |z_j|^2)^{k_j} \right] \overline{D^K g(z)} d\nu(z). \quad (11)$$

The following pointwise estimate of functions in $\mathcal{B}(\mathbb{D}^n)$ will be useful, it is obtained from the representation of Bloch functions in Lemma 1.1.

Lemma 3.3. *There exists a constant $C > 0$ such that for any $f \in \mathcal{B}(\mathbb{D}^n)$ and any $z \in \mathbb{D}^n$,*

$$|f(z)| \leq C \left(\prod_{j=1}^n \log \frac{4}{1 - |z_j|^2} \right) \|f\|_{\mathcal{B}}. \quad (12)$$

3.3. The pointwise Bloch space

Let us start by considering some observations.

Lemma 3.4. *The following assertions hold*

(1) *The function $f(z) = \sum_{j=1}^n f_j(z_j)$, with $f_j \in \mathcal{B}(\mathbb{D})$ belongs to $\mathbb{B}(\mathbb{D}^n)$ and*

$$\|f\|_{\mathbb{B}(\mathbb{D}^n)} \leq \sum_{j=1}^n \|f_j\|_{\mathcal{B}(\mathbb{D})}.$$

(2) *There is a constant $C > 0$ such that for any $f \in \mathbb{B}(\mathbb{D}^n)$ and any $z \in \mathbb{D}^n$,*

$$|f(z)| \leq C \left[\sum_{j=1}^n \log \frac{4}{1 - |z_j|^2} \right] \|f\|_{\mathbb{B}(\mathbb{D}^n)} \quad (13)$$

and this is sharp.

Proof. Assertion (1) is direct from the definition of $\mathbb{B}(\mathbb{D}^n)$. Let us prove (2). We have for any $z = (z_1, \dots, z_n) \in \mathbb{D}^n$,

$$\begin{aligned} f(z) - f(0) &= \int_0^1 \frac{d}{ds} [f(sz)] ds \\ &= \sum_{j=1}^n \int_0^1 z_j \frac{\partial f}{\partial z_j}(sz) ds. \end{aligned}$$

It easily follows using the definition of $\mathbb{B}(\mathbb{D}^n)$ that

$$\begin{aligned} |f(z)| &\leq |f(0)| + \sum_{j=1}^n \int_0^1 |z_j| \left| \frac{\partial f}{\partial z_j}(sz) \right| ds \\ &\leq |f(0)| + \|f\|_{\mathbb{B}(\mathbb{D}^n)} \sum_{j=1}^n \int_0^1 \frac{|z_j|}{1 - s^2 |z_j|^2} ds \end{aligned}$$

$$\leq C \left(\sum_{j=1}^n \log \frac{4}{1 - |z_j|^2} \right) \|f\|_{\mathbb{B}(\mathbb{D}^n)}.$$

Sharpness follows by testing (13) with the function $f(z) = f_a(z) = \sum_{j=1}^n \log(1 - \bar{a}_j z_j)$. \square

Remark 3.5. Let us observe that $\mathbb{B}(\mathbb{D}^n)$ is a strict subspace of $\mathcal{B}(\mathbb{D}^n)$. To see this, one only needs to observe that for $f_j \in \mathcal{B}(\mathbb{D})$, $j = 1, \dots, n$, the tensor product

$$(f_1 \otimes f_2 \otimes \dots \otimes f_n)(z_1, \dots, z_n) = \prod_{j=1}^n f_j(z_j)$$

belongs to $\mathcal{B}(\mathbb{D}^n)$ with

$$\|f_1 \otimes f_2 \otimes \dots \otimes f_n\|_{\mathcal{B}(\mathbb{D}^n)} \leq \prod_{j=1}^n \|f_j\|_{\mathcal{B}(\mathbb{D})}$$

while $f_1 \otimes f_2 \otimes \dots \otimes f_n$ belongs to $\mathbb{B}(\mathbb{D}^n)$ only if each f_j belongs to $H^\infty(\mathbb{D})$, $j = 1, 2, \dots, n$.

A nontrivial example is the function $f(z_1, \dots, z_n) = \prod_{j=1}^n \log(1 - \bar{a}_j z_j)$ which belongs to $\mathcal{B}(\mathbb{D}^n) \setminus \mathbb{B}(\mathbb{D}^n)$.

To $\vec{j} = (j_1, \dots, j_l)$, $1 \leq l \leq n$, we associate the set $J = \{j_1, \dots, j_l\} \subseteq \{1, \dots, n\}$. We denote by $D_{\vec{j}}$ the differential operator defined by

$$D_{\vec{j}}f(z) = D_{j_1} \dots D_{j_l}f(z).$$

We observe the following.

Lemma 3.6. Let $\vec{j} = (j_1, \dots, j_l)$, $1 \leq l \leq n$, $J = \{j_1, \dots, j_l\} \subseteq \{1, \dots, n\}$ be given. Then for any $f \in \mathbb{B}(\mathbb{D}^n)$,

$$\sup_{z \in \mathbb{D}^n} \left[\prod_{k \in J} (1 - |z_k|^2) \right] |D_{\vec{j}}f(z)| \lesssim \|f\|_{\mathbb{B}}. \tag{14}$$

Proof. We can suppose without loss of generality that $\vec{j} = (1, \dots, l)$, $1 \leq l \leq n$, so that $J = \{1, \dots, l\}$. For $z = (z_1, \dots, z_n)$, we set $w = (z_{l+1}, \dots, z_n) \in \mathbb{D}^{n-l}$. We observe that for $w \in \mathbb{D}^{n-l}$ fixed, the function $g = f(\cdot, w)$ is uniformly in $\mathbb{B}(\mathbb{D}^l)$ whenever $f \in \mathbb{B}(\mathbb{D}^n)$ with $\|g\|_{\mathbb{B}(\mathbb{D}^l)} \leq \|f\|_{\mathbb{B}(\mathbb{D}^n)}$. Hence, $g = f(\cdot, w)$ is uniformly in $\mathcal{B}(\mathbb{D}^l)$ with

$$\sup_{a \in \mathbb{D}^l} \left[\prod_{k=1}^l (1 - |a_k|^2) \right] |D_{\vec{j}}f(a, w)| \leq \|g\|_{\mathbb{B}(\mathbb{D}^l)} \leq \|f\|_{\mathbb{B}(\mathbb{D}^n)}.$$

This proves that (14) holds. \square

Let us observe that as for the space $\mathcal{B}_L(\mathbb{D}^n)$, we have the following equivalent definition of the logarithmic Bloch-type space $\mathbb{B}_L(\mathbb{D}^n)$.

Lemma 3.7. For any function f in the space $\mathbb{B}_L(\mathbb{D}^n)$, we have

$$\|f\|_{\mathbb{B}_L} \simeq |f(0)| + \sup_{z \in \mathbb{D}^n} \left[\prod_{j=1}^n (1 - |z_j|^2)^{k_j} \right] \left[\sum_{j=1}^n \log \frac{4}{1 - |z_j|^2} \right] |D^K f(z)| < \infty,$$

$$K = (k_1, \dots, k_n) \in \mathbb{N}^n.$$

4. Endpoint results for Hankel operators

In this section, we prove our results on the boundedness of the Hankel operators.

Proof of Theorem 2.1. We start by the easy part which is the sufficiency. What we would like to prove is that under the condition that $b \in \mathcal{B}_L(\mathbb{D}^n)$, there is a positive constant C such that for all $f \in A^1(\mathbb{D}^n)$, we have

$$\|h_b(f)\|_1 \leq C\|f\|_1.$$

To prove this, we observe with the help of Lemma 3.2 that we have in particular that

$$\begin{aligned} h_b(f)(z) &= \int_{\mathbb{D}^n} \frac{b(w)\overline{f(w)}}{\prod_{j=1}^n (1 - \overline{w_j}z_j)^2} d\nu(w) \\ &= C_{n,K} \int_{\mathbb{D}^n} \frac{\left[\prod_{j=1}^n (1 - |w_j|^2)^{k_j}\right] D^K b(w)\overline{f(w)}}{\prod_{j=1}^n (1 - \overline{w_j}z_j)^2} d\nu(w), \end{aligned}$$

for any $K = (k_1, \dots, k_n) \in \mathbb{N}^n$. Put

$$(1 - \|w\|^2)^K = \left[\prod_{j=1}^n (1 - |w_j|^2)^{k_j} \right].$$

It follows using the above observation and Lemma 3.1 that

$$\begin{aligned} \|h_b(f)\|_1 &= \int_{\mathbb{D}^n} |h_b(f)(z)| d\nu(z) \\ &\leq C_{n,k} \int_{\mathbb{D}^n} (1 - \|w\|^2)^K |D^K b(w)| |f(w)| \left(\int_{\mathbb{D}^n} \frac{d\nu(z)}{\prod_{j=1}^n |1 - \overline{w_j}z_j|^2} \right) d\nu(w) \\ &\leq C_{n,k} \int_{\mathbb{D}^n} (1 - \|w\|^2)^K \left(\prod_{j=1}^n \log \frac{4}{1 - |w_j|^2} \right) |D^K b(w)| |f(w)| d\nu(w) \\ &\leq C_{n,k} \|b\|_{\mathcal{B}_L} \int_{\mathbb{D}^n} |f(w)| d\nu(w) \\ &= C_{n,k} \|b\|_{\mathcal{B}_L} \|f\|_1. \end{aligned}$$

From the duality (A^1, \mathcal{B}) , we have that the converse is equivalent to saying that if b is such that there exists a constant $C > 0$ so that for any $f \in A^1(\mathbb{D}^n)$ and any $g \in \mathcal{B}(\mathbb{D}^n)$,

$$|\langle b, fg \rangle| \leq C\|f\|_1 \|g\|_{\mathcal{B}}, \quad (15)$$

then $b \in \mathcal{B}_L(\mathbb{D}^n)$.

We will need the following lemma.

Lemma 4.1. Let $w_j, z_j \in \mathbb{D}$, $j = 1, \dots, n$ be given. The following formula holds.

$$\prod_{j=1}^n \log(1 - \overline{w_j}z_j) = \sum_{L \subseteq \{1, \dots, n\}} (-1)^{|L|+1} \left(\prod_{l \in L} \log \frac{1 - \overline{w_l}z_l}{1 - |w_l|^2} \right) \left(\prod_{l \in {}^cL} \log(1 - \overline{w_l}z_l) \right) + \prod_{j=1}^n \log(1 - |w_j|^2) \quad (16)$$

$[L]$ being the cardinality of the set L and cL its complementary in $\{1, \dots, n\}$.

Proof. One easily checks that for $n = 2$, we have

$$\begin{aligned} \prod_{j=1}^2 \log(1 - \overline{w_j}z_j) &= \log \frac{1 - \overline{w_1}z_1}{1 - |w_1|^2} \log(1 - \overline{w_2}z_2) + \log \frac{1 - \overline{w_2}z_2}{1 - |w_2|^2} \log(1 - \overline{w_1}z_1) \\ &\quad - \log \frac{1 - \overline{w_1}z_1}{1 - |w_1|^2} \log \frac{1 - \overline{w_2}z_2}{1 - |w_2|^2} + \log(1 - |w_1|^2) \log(1 - |w_2|^2). \end{aligned}$$

Next we suppose that (16) holds for $n \geq 2$ and prove that it then also holds for $n + 1$. We first write

$$\prod_{j=1}^{n+1} \log(1 - \overline{w_j}z_j) = \left(\prod_{j=1}^n \log(1 - \overline{w_j}z_j) \right) \log(1 - \overline{w_{n+1}}z_{n+1}). \quad (17)$$

Applying our hypothesis on the first factor on the right hand side of (17), we obtain

$$\prod_{j=1}^{n+1} \log(1 - \overline{w_j}z_j) = T_1 + \left(\prod_{j=1}^n \log(1 - |w_j|^2) \right) \log(1 - \overline{w_{n+1}}z_{n+1}), \quad (18)$$

where

$$T_1 = \left(\sum_{L \subseteq \{1, \dots, n\}} (-1)^{|L|+1} \left(\prod_{l \in L} \log \frac{1 - \overline{w_l}z_l}{1 - |w_l|^2} \right) \left(\prod_{l \in {}^cL} \log(1 - \overline{w_l}z_l) \right) \right) \log(1 - \overline{w_{n+1}}z_{n+1})$$

Before going ahead, let us deal with the second term in the sum on the right hand side of (18). We clearly have that

$$\begin{aligned} M &:= \left(\prod_{j=1}^n \log(1 - |w_j|^2) \right) \log(1 - \overline{w_{n+1}}z_{n+1}) \\ &= \left(\prod_{j=1}^n \log(1 - |w_j|^2) \right) \left(\log \left(\frac{1 - \overline{w_{n+1}}z_{n+1}}{1 - |w_{n+1}|^2} \right) + \log(1 - |w_{n+1}|^2) \right) \\ &= \left(\prod_{j=1}^n \log(1 - |w_j|^2) \right) \log \left(\frac{1 - \overline{w_{n+1}}z_{n+1}}{1 - |w_{n+1}|^2} \right) + \prod_{j=1}^{n+1} \log(1 - |w_j|^2). \end{aligned}$$

Hence applying our hypothesis, we obtain

$$M = T_2 + T_3 + \prod_{j=1}^{n+1} \log(1 - |w_j|^2),$$

where

$$T_2 = \left(\prod_{j=1}^n \log(1 - \overline{w_j} z_j) \right) \log\left(\frac{1 - \overline{w_{n+1}} z_{n+1}}{1 - |w_{n+1}|^2}\right)$$

and

$$T_3 = \left(\sum_{L \subseteq \{1, \dots, n\}} (-1)^{|L|} \left(\prod_{l \in L} \log \frac{1 - \overline{w_l} z_l}{1 - |w_l|^2} \right) \left(\prod_{l \in cL} \log(1 - \overline{w_l} z_l) \right) \right) \log\left(\frac{1 - \overline{w_{n+1}} z_{n+1}}{1 - |w_{n+1}|^2}\right)$$

Taking this into (18), we obtain

$$\begin{aligned} \prod_{j=1}^{n+1} \log(1 - \overline{w_j} z_j) &= T_1 + T_2 + T_3 + \prod_{j=1}^{n+1} \log(1 - |w_j|^2) \\ &= \sum_{L \subseteq \{1, \dots, n+1\}} (-1)^{|L|+1} \left(\prod_{l \in L} \log \frac{1 - \overline{w_l} z_l}{1 - |w_l|^2} \right) \left(\prod_{l \in cL} \log(1 - \overline{w_l} z_l) \right) \\ &\quad + \prod_{j=1}^{n+1} \log(1 - |w_j|^2). \end{aligned}$$

The proof of the lemma is complete. \square

Coming back to the proof of the necessity part of the theorem, we test (15) with

$$f(z) = f_w(z) = \prod_{j=1}^n \frac{(1 - |w_j|^2)^{k_j}}{(1 - \overline{w_j} z_j)^{2+k_j}} \quad k_j \in \mathbb{N},$$

and

$$g(z) = g_w(z) = \prod_{j=1}^n \log(1 - \overline{w_j} z_j).$$

Clearly f and g are uniformly in $A^1(\mathbb{D}^n)$ and $\mathcal{B}(\mathbb{D}^n)$ respectively.

Next, we take f_w and the expansion of g_w obtained in Lemma 4.1 into (15) to obtain using Lemma 1.1 (iii), that

$$C \geq |\langle b, fg \rangle| = \left| \left(\prod_{j=1}^n \log(1 - |w_j|^2) \right) \lim_{r \rightarrow \mathbf{1}} \int_{\mathbb{D}^n} \prod_{j=1}^n \frac{(1 - |w_j|^2)^{k_j}}{(1 - \overline{w_j} z_j)^{2+k_j}} \overline{b(rz)} d\nu(z) + T \right| \quad (19)$$

where writing

$$\begin{aligned} G_w^L(z) &= \left(\prod_{l \in L} \log \frac{1 - \overline{w_l} z_l}{1 - |w_l|^2} \right) \left(\prod_{l \in cL} \log(1 - \overline{w_l} z_l) \right), \\ T &= \lim_{r \rightarrow \mathbf{1}} \int_{\mathbb{D}^n} \left(\sum_{L \subseteq \{1, \dots, n\}} (-1)^{|L|+1} G_w^L(z) \right) \left(\prod_{j=1}^n \frac{(1 - |w_j|^2)^{k_j}}{(1 - \overline{w_j} z_j)^{2+k_j}} \right) \overline{b(rz)} d\nu(z) \\ &= \sum_{L \subseteq \{1, \dots, n\}} (-1)^{|L|+1} \lim_{r \rightarrow \mathbf{1}} \int_{\mathbb{D}^n} f_w^L(z) g_w^L(z) \overline{b(rz)} d\nu(z) \end{aligned}$$

where

$$\begin{aligned} f_w^L(z) &:= \left(\prod_{j=1}^n \frac{(1 - |w_j|^2)^{k_j}}{(1 - \overline{w_j}z_j)^{2+k_j}} \right) \left(\prod_{l \in L} \log \frac{1 - \overline{w_l}z_l}{1 - |w_l|^2} \right) \\ &= \left(\prod_{j \in cL} \frac{(1 - |w_j|^2)^{k_j}}{(1 - \overline{w_j}z_j)^{2+k_j}} \right) \left(\prod_{l \in L} \frac{(1 - |w_l|^2)^{k_l}}{(1 - \overline{w_l}z_l)^{2+k_l}} \log \frac{1 - \overline{w_l}z_l}{1 - |w_l|^2} \right) \end{aligned}$$

and

$$g_w^L(z) := \prod_{l \in cL} \log(1 - \overline{w_l}z_l).$$

Clearly, g_w^L is uniformly in $\mathcal{B}(\mathbb{D}^n)$. Observing that in the unit disk of \mathbb{C} , the function $\frac{(1 - |w_j|^2)^{k_j}}{(1 - \overline{w_j}z_j)^{2+k_j}} \log \frac{1 - \overline{w_j}z_j}{1 - |w_j|^2}$ is uniformly in $A^1(\mathbb{D})$ (see [4]), we conclude that f_w^L is also uniformly in $A^1(\mathbb{D}^n)$. Hence, applying (15) to f_w^L and g_w^L , we obtain that

$$|T| \leq \sum_{L \subseteq \{1, \dots, n\}} |\langle b, f_w^L g_w^L \rangle| \leq C.$$

We deduce from (19) using $b = Pb$, that

$$\left(\prod_{j=1}^n (1 - |w_j|^2)^{k_j} |\log(1 - |w_j|^2)| \right) \left| \lim_{r \rightarrow 1} \int_{\mathbb{D}^n} \frac{\overline{b(rz)}}{\prod_{j=1}^n (1 - \overline{w_j}z_j)^{2+k_j}} d\nu(z) \right| \leq C.$$

That is

$$\left(\prod_{j=1}^n (1 - |w_j|^2)^{k_j} |\log(1 - |w_j|^2)| \right) |D^K b(w)| \leq C < \infty, \text{ for any } w \in \mathbb{D}^n.$$

By (10), we conclude that $b \in \mathcal{B}_L(\mathbb{D}^n)$. The proof is complete. \square

Let us now prove Theorem 2.2 that provides an equivalent definition of $\mathbb{B}_L(\mathbb{D})$ in terms of symbols of bounded Hankel operators.

Proof of Theorem 2.2. Let us start by the sufficiency. Assume $b \in \mathbb{B}_L(\mathbb{D}^n)$. Then for any $f \in \mathbb{B}(\mathbb{D}^n)$ and any $g \in A^1(\mathbb{D}^n)$, and for $K = (k_1, \dots, k_n) \in \mathbb{N}^n$, using Lemma 3.2, (13) and Lemma 3.7, we obtain

$$\begin{aligned} |\langle h_b(f), g \rangle| &= |\langle b, fg \rangle| \\ &\leq C \limsup_{r \rightarrow 1} \int_{\mathbb{D}^n} (1 - \|w\|^2)^K |D^K b(w)| |f(rw)| |g(rw)| d\nu(w) \\ &\leq C \|f\|_{\mathbb{B}(\mathbb{D}^n)} \limsup_{r \rightarrow 1} \int_{\mathbb{D}^n} (1 - \|w\|^2)^K \left(\sum_{j=1}^n \log \frac{4}{1 - |w_j|^2} \right) |D^K b(w)| |g(rw)| d\nu(w) \\ &\leq C \|f\|_{\mathbb{B}(\mathbb{D}^n)} \|b\|_{\mathbb{B}_L(\mathbb{D}^n)} \int_{\mathbb{D}^n} |g(w)| d\nu(w) \\ &= C \|f\|_{\mathbb{B}(\mathbb{D}^n)} \|b\|_{\mathbb{B}_L(\mathbb{D}^n)} \|g\|_1. \end{aligned}$$

Thus

$$\|h_b(f)\|_{\mathcal{B}(\mathbb{D}^n)} = \sup_{g \in A^1(\mathbb{D}^n), \|g\|_1 \leq 1} |\langle h_b(f), g \rangle| \leq C \|f\|_{\mathbb{B}(\mathbb{D}^n)} \|b\|_{\mathbb{B}_L(\mathbb{D}^n)}.$$

That is h_b is bounded from $\mathbb{B}(\mathbb{D}^n)$ to $\mathcal{B}(\mathbb{D}^n)$ for any $b \in \mathbb{B}_L(\mathbb{D}^n)$.

For the converse, we have to prove that if b is such that there exists a constant $C > 0$ so that for any $f \in \mathbb{B}(\mathbb{D}^n)$, and any $g \in A^1(\mathbb{D}^n)$,

$$|\langle b, fg \rangle| \leq C \|f\|_{\mathbb{B}(\mathbb{D}^n)} \|g\|_1, \quad (20)$$

then $b \in \mathbb{B}_L(\mathbb{D}^n)$. For this, we test (20) with

$$f_a(z) = \sum_{j=1}^n \log \left(\frac{4}{1 - \overline{a_j} z_j} \right), \quad a = (a_1, a_2, \dots, a_n) \in \mathbb{D}^n, a_j \neq 0, j = 1, 2, \dots, n,$$

and

$$g_a(z) = \prod_{j=1}^n \frac{(1 - |a_j|^2)^{k_j}}{(1 - \overline{a_j} z_j)^{2+k_j}} \quad k_j \in \mathbb{N}.$$

We obtain at the first step using that both f_a and g_a are uniformly in $\mathbb{B}(\mathbb{D}^n)$ and $A^1(\mathbb{D}^n)$ respectively,

$$C \geq |\langle b, fg \rangle| = \left| \sum_{l=1}^n \lim_{r \rightarrow 1} \int_{\mathbb{D}^n} \log \left(\frac{4}{1 - \overline{a_l} z_l} \right) \prod_{j=1}^n \frac{(1 - |a_j|^2)^{k_j}}{(1 - \overline{a_j} z_j)^{2+k_j}} \overline{b(rz)} d\nu(z) \right|. \quad (21)$$

Next we observe that

$$\begin{aligned} & \int_{\mathbb{D}^n} \log \left(\frac{4}{1 - \overline{a_l} z_l} \right) \prod_{j=1}^n \frac{(1 - |a_j|^2)^{k_j}}{(1 - \overline{a_j} z_j)^{2+k_j}} \overline{b(rz)} d\nu(z) \\ &= \int_{\mathbb{D}^n} \log \left(\frac{1 - |a_l|^2}{1 - \overline{a_l} z_l} \right) \prod_{j=1}^n \frac{(1 - |a_j|^2)^{k_j}}{(1 - \overline{a_j} z_j)^{2+k_j}} \overline{b(rz)} d\nu(z) \\ & \quad + \int_{\mathbb{D}^n} \log \left(\frac{4}{1 - |a_j|^2} \right) \prod_{j=1}^n \frac{(1 - |a_j|^2)^{k_j}}{(1 - \overline{a_j} z_j)^{2+k_j}} \overline{b(rz)} d\nu(z) \end{aligned}$$

and observing that as $\log \left(\frac{1 - |a_l|^2}{1 - \overline{a_l} z_l} \right) \in \mathbb{B}(\mathbb{D}^n)$ and $\frac{(1 - |a_j|^2)^{k_j}}{(1 - \overline{a_j} z_j)^{2+k_j}} \in A^1(\mathbb{D})$ both uniformly, we have by (20) that

$$\left| \lim_{r \rightarrow 1} \int_{\mathbb{D}^n} \log \left(\frac{1 - |a_l|^2}{1 - \overline{a_l} z_l} \right) \prod_{j=1}^n \frac{(1 - |a_j|^2)^{k_j}}{(1 - \overline{a_j} z_j)^{2+k_j}} \overline{b(rz)} d\nu(z) \right| \leq C,$$

$l = 1, \dots, n$. It follows from the latter and (21) that

$$\begin{aligned} S &:= \left(\sum_{j=1}^n \left| \log \left(\frac{4}{1 - |a_j|^2} \right) \right| \right) \left(\prod_{j=1}^n (1 - |a_j|^2)^{k_j} \right) \left| \lim_{r \rightarrow 1} \int_{\mathbb{D}^n} \left(\prod_{j=1}^n \frac{1}{(1 - \overline{a_j} z_j)^{2+k_j}} \right) \overline{b(rz)} d\nu(z) \right| \\ &\leq C < \infty. \end{aligned}$$

That is for any $K = (k_1, \dots, k_n) \in \mathbb{N}^n$,

$$\sup_{z \in \mathbb{D}^n} \left(\sum_{j=1}^n \left(\log \frac{4}{1 - |z_j|^2} \right) \right) \left(\prod_{j=1}^n (1 - |z_j|^2)^{k_j} \right) |D^K b(z)| < \infty.$$

Thus by Lemma 3.7, $b \in \mathbb{B}_L(\mathbb{D}^n)$. The proof is complete. \square

5. Pointwise multipliers of Bloch spaces

We prove Theorem 2.3 and Theorem 2.4 in this section. We start with the following observation.

Lemma 5.1. *Let $\phi \in \mathcal{H}(\mathbb{D}^n)$. If $\phi \in \mathcal{M}(\mathbb{B}(\mathbb{D}^n), \mathcal{B}(\mathbb{D}^n))$ (resp. $\phi \in \mathcal{M}(\mathcal{B}(\mathbb{D}^n))$), then the multiplication operator M_ϕ is bounded from $\mathbb{B}(\mathbb{D}^n)$ to $\mathcal{B}(\mathbb{D}^n)$ (resp. from $\mathcal{B}(\mathbb{D}^n)$ to itself).*

Proof. We apply the closed graph theorem. We need to show the following implication. Assume that $\{f_p\} \rightarrow f$ in $\mathbb{B}(\mathbb{D}^n)$ (resp. $\{f_p\} \rightarrow g$ in $\mathcal{B}(\mathbb{D}^n)$), and $\{\phi f_p\} \rightarrow g$ in $\mathbb{B}(\mathbb{D}^n)$; then we have the equality $\phi f = g$. Indeed, since the Banach space $\mathbb{B}(\mathbb{D}^n)$ is continuously embedded in the Banach space $\mathcal{B}(\mathbb{D}^n)$, which is continuously embedded in the Bergman space $A^2(\mathbb{D}^n)$, the first assumption implies that $\{f_p\} \rightarrow f$ uniformly on compact sets of \mathbb{D}^n and this implies that $\{\phi f_p\} \rightarrow \phi f$ uniformly on compact sets of \mathbb{D}^n . Comparing with the second assumption, in view of the unicity of the limit, we conclude that $\phi f = g$. \square

Proof of Theorem 2.3. We have from Lemma 5.1 that if $\phi \in \mathcal{M}(\mathbb{B}(\mathbb{D}^n), \mathcal{B}(\mathbb{D}^n))$, then the multiplication operator M_ϕ is bounded from $\mathbb{B}(\mathbb{D}^n)$ to $\mathcal{B}(\mathbb{D}^n)$.

We would like to prove that $\phi \in H^\infty(\mathbb{D}^n)$ is so that there is a constant $C > 0$ such that for any $f \in \mathbb{B}(\mathbb{D}^n)$ and any $z = (z_1, \dots, z_n) \in \mathbb{D}^n$,

$$\left[\prod_{j=1}^n (1 - |z_j|^2) \right] |D(\phi f)(z)| < C$$

if and only if $\phi \in \mathbb{B}_L(\mathbb{D}^n)$.

We observe the following.

$$D(\phi f) = \phi Df + f D\phi + \sum_{\emptyset \neq J \subset \{1, \dots, n\}, K = cJ} D_{\vec{j}} \phi D_{\vec{k}} f \tag{22}$$

where $\vec{j} = (j_1, \dots, j_l)$ is the vector associated to the set $J = \{j_1, \dots, j_l\}$ and the same for \vec{k} and the set K the complement of J in $\{1, \dots, n\}$. We also observe that if $J \cap K = \emptyset$, $J \cup K = \{1, \dots, n\}$ with $J, K \neq \{1, \dots, n\}$, we have using (14) and the continuous embedding $H^\infty(\mathbb{D}^n) \hookrightarrow \mathbb{B}(\mathbb{D}^n)$, that

$$\begin{aligned} M &:= \left[\prod_{j=1}^n (1 - |z_j|^2) \right] |D_{\vec{j}} \phi(z) D_{\vec{k}} f(z)| \\ &\leq \left(\sup_{z \in \mathbb{D}^n} \left[\prod_{j \in J} (1 - |z_j|^2) \right] |D_{\vec{j}} \phi(z)| \right) \left(\sup_{z \in \mathbb{D}^n} \left[\prod_{k \in K} (1 - |z_k|^2) \right] |D_{\vec{k}} f(z)| \right) \\ &\leq \|\phi\|_{\mathbb{B}} \|f\|_{\mathbb{B}}. \end{aligned}$$

Remark that if $\phi \in H^\infty(\mathbb{D}^n)$, then for any $z \in \mathbb{D}^n$,

$$\left[\prod_{j=1}^n (1 - |z_j|^2) \right] |\phi(z)Df(z)| \leq \|\phi\|_\infty \|f\|_{\mathbb{B}(\mathbb{D}^n)}.$$

All the above observations amount to saying that a bounded function ϕ is a multiplier from $\mathbb{B}(\mathbb{D}^n)$ to $\mathcal{B}(\mathbb{D}^n)$ if and only if there is a constant $C > 0$ such that for any $f \in \mathbb{B}(\mathbb{D}^n)$ and any $z \in \mathbb{D}^n$,

$$\left[\prod_{j=1}^n (1 - |z_j|^2) \right] |f(z)D\phi(z)| \leq C. \quad (23)$$

Let us suppose that $\phi \in \mathbb{B}_L(\mathbb{D}^n)$ and prove that in this case, (23) holds. Using the pointwise estimate of functions in $\mathbb{B}(\mathbb{D}^n)$ given by Lemma 3.4, we obtain

$$\begin{aligned} \left[\prod_{j=1}^n (1 - |z_j|^2) \right] |f(z)D\phi(z)| &\leq \|f\|_{\mathbb{B}} \left[\prod_{j=1}^n (1 - |z_j|^2) \right] \left[\sum_{j=1}^n \log \frac{4}{1 - |z_j|^2} \right] |D\phi(z)| \\ &\leq \|f\|_{\mathbb{B}} \|\phi\|_{\mathbb{B}_L}. \end{aligned}$$

Conversely, if (23) holds, then testing with the function $f(z) = f_a(z) = \sum_{j=1}^n \log \left(\frac{4}{1 - a_j z_j} \right)$ with $a \in \mathbb{D}^n$ fixed, $a_j \neq 0$, $j = 1, 2, \dots, n$ (note that by Lemma 3.4, $f \in \mathbb{B}(\mathbb{D}^n)$), we obtain

$$\left[\prod_{j=1}^n (1 - |z_j|^2) \right] \left| \sum_{j=1}^n \log \left(\frac{4}{1 - a_j z_j} \right) \right| |D\phi(z)| \leq C.$$

Taking $z_j = a_j$, $j = 1, \dots, n$ in the latter inequality, we obtain that for any $z \in \mathbb{D}^n$,

$$\left[\prod_{j=1}^n (1 - |z_j|^2) \right] \left[\sum_{j=1}^n \log \frac{4}{1 - |z_j|^2} \right] |D\phi(z)| < C$$

which proves that $\phi \in \mathbb{B}_L(\mathbb{D}^n)$ \square

Let us now characterize the multiplier algebra of the product Bloch space. We start with the following elementary result.

Lemma 5.2. *Let $\vec{j} = (j_1, \dots, j_l)$, $1 \leq l \leq n$, and the associated set $J = \{j_1, \dots, j_l\} \subseteq \{1, \dots, n\}$. Then there is a constant $C > 0$ such that for any $f \in \mathcal{B}(\mathbb{D}^n)$ and any $z = (z_1, \dots, z_n) \in \mathbb{D}^n$,*

$$\left[\prod_{j=1}^n (1 - |z_j|^2) \right] |D_{\vec{j}}f(z)| \leq C \left[\prod_{j \notin J} (1 - |z_j|^2) \log \frac{4}{1 - |z_j|^2} \right] \|f\|_{\mathcal{B}}.$$

Proof. The proof uses the representation formula of $f \in \mathcal{B}(\mathbb{D}^n)$ in assertion (ii) of Lemma 1.1. Indeed, by this assertion, $b = Pg$ with $g \in L^\infty(\mathbb{D}^n)$. Using this and Lemma 3.1, we obtain

$$|D_{\vec{j}}f(z)| = \left| \int_{\mathbb{D}^n} \left[\prod_{j \in J} \frac{4}{(1 - z_j \bar{w}_j)^3} \right] \left[\prod_{j \notin J} \frac{1}{(1 - z_j \bar{w}_j)^2} \right] g(w) d\nu(w) \right|$$

$$\begin{aligned} &\leq C \|g\|_\infty \int_{\mathbb{D}^n} \left| \left[\prod_{j \in J} \frac{1}{(1 - z_j \bar{w}_j)^3} \right] \right| \left| \left[\prod_{j \notin J} \frac{1}{(1 - z_j \bar{w}_j)^2} \right] \right| d\nu(w) \\ &\leq C \left[\prod_{j \in J} (1 - |z_j|^2)^{-1} \right] \left[\prod_{j \notin J} \log \frac{4}{(1 - |z_j|^2)} \right] \|f\|_{\mathcal{B}}. \end{aligned}$$

The proof is complete. \square

As a consequence of the above lemma, we have the following result.

Proposition 5.3. *Let \vec{j} and \vec{k} be two vectors such that their respective associated sets J and K are complementary in $\{1, \dots, n\}$, with none of them empty. Suppose that $f \in \mathcal{B}(\mathbb{D}^n)$ and $\phi \in \mathcal{B}_{LL}(\mathbb{D}^n)$. Then for any $z \in \mathbb{D}^n$,*

$$\left[\prod_{j=1}^n (1 - |z_j|^2) \right] |D_{\vec{j}}f(z)| |D_{\vec{k}}\phi(z)| \lesssim \|f\|_{\mathcal{B}} \|\phi\|_{\mathcal{B}_{LL}}.$$

Proof of Theorem 2.4. We note again with Lemma 5.1 that if $\phi \in \mathcal{M}(\mathcal{B}(\mathbb{D}^n))$, then the multiplication operator M_ϕ is bounded from $\mathcal{B}(\mathbb{D}^n)$ to itself.

Let us start by proving the sufficiency. Let $\phi \in H^\infty(\mathbb{D}^n) \cap \mathcal{B}_{LL}(\mathbb{D}^n)$. We would like to prove that for any $f \in \mathcal{B}(\mathbb{D}^n)$, the analytic function ϕf belongs to $\mathcal{B}(\mathbb{D}^n)$. From the formula (22), one sees that we only have to estimate the following three terms.

$$\begin{aligned} M_1 &:= \left[\prod_{j=1}^n (1 - |z_j|^2) \right] |\phi(z)Df(z)|; \\ M_2 &:= \left[\prod_{j=1}^n (1 - |z_j|^2) \right] |D_{\vec{j}}\phi(z)D_{\vec{k}}f(z)|, \quad \emptyset \neq J, K \subset \{1, \dots, n\} = J \cup K, \quad J \cap K = \emptyset; \\ M_3 &:= \left[\prod_{j=1}^n (1 - |z_j|^2) \right] |f(z)D\phi(z)|. \end{aligned}$$

From that $\phi \in H^\infty(\mathbb{D}^n)$, we obtain

$$M_1 := \left[\prod_{j=1}^n (1 - |z_j|^2) \right] |\phi(z)Df(z)| \leq \|\phi\|_\infty \|f\|_{\mathcal{B}}.$$

The estimate of the term M_2 is given by Proposition 5.3. For the last term, we use the pointwise estimate of $f \in \mathcal{B}(\mathbb{D}^n)$ to get

$$\begin{aligned} M_3 &:= \left[\prod_{j=1}^n (1 - |z_j|^2) \right] |f(z)D\phi(z)| \\ &\leq C \|f\|_{\mathcal{B}} \left[\prod_{j=1}^n (1 - |z_j|^2) \log \frac{4}{1 - |z_j|^2} \right] |D\phi(z)| \\ &\leq C \|f\|_{\mathcal{B}} \|\phi\|_{\mathcal{B}_L}. \end{aligned}$$

We conclude that for $\phi \in H^\infty(\mathbb{D}^n) \cap \mathcal{B}_{LL}(\mathbb{D}^n)$,

$$\|\phi f\|_{\mathcal{B}} \leq C(\|\phi\|_\infty + \|\phi\|_{\mathcal{B}_{LL}}) \|f\|_{\mathcal{B}}.$$

We now prove the necessity part in Theorem 2.4. We suppose that ϕ is such that for any $f \in \mathcal{B}(\mathbb{D}^n)$, the function ϕf belongs to $\mathcal{B}(\mathbb{D}^n)$. That is there is a constant $C > 0$ such that for any $z \in \mathbb{D}^n$,

$$(1 - \|z\|^2) |D(\phi f)(z)| = (1 - \|z\|^2) \left| \sum_{J \subseteq \{1, \dots, n\}, K=cJ} D_{\bar{j}} \phi(z) D_{\bar{k}} f(z) \right| \leq C, \quad (24)$$

with $(1 - \|z\|^2) = (1 - |z_1|^2) \dots (1 - |z_n|^2)$ and for $J = \emptyset$, $D_{\bar{j}} f = f$.

We first prove that this necessarily provides that ϕ is bounded. Clearly, if $\phi \in \mathcal{M}(\mathcal{B}(\mathbb{D}^n))$, then the pointwise estimate of functions in $\mathcal{B}(\mathbb{D}^n)$ gives that there exists a constant $C > 0$ such that for any $f \in \mathcal{B}(\mathbb{D}^n)$ and $z \in \mathbb{D}^n$,

$$|f(z)\phi(z)| \leq C \left(\prod_{j=1}^n \log \frac{4}{1 - |z_j|^2} \right) \|f\phi\|_{\mathcal{B}} \leq C \|\phi\|_{\mathcal{B}(\mathbb{D}^n) \rightarrow \mathcal{B}(\mathbb{D}^n)} \left(\prod_{j=1}^n \log \frac{4}{1 - |z_j|^2} \right) \|f\|_{\mathcal{B}}. \quad (25)$$

We test (25) with the function

$$f(z) = f_a(z) = \prod_{j=1}^n \log \left(\frac{4}{1 - \overline{a_j} z_j} \right),$$

$a = (a_1, \dots, a_n)$ given in \mathbb{D}^n with $a_j \neq 0$ for each j . We obtain that for any $z \in \mathbb{D}^n$,

$$\left| \left(\prod_{j=1}^n \log \left(\frac{4}{1 - \overline{a_j} z_j} \right) \right) \phi(z) \right| \leq C \|\phi\|_{\mathcal{B}(\mathbb{D}^n) \rightarrow \mathcal{B}(\mathbb{D}^n)} \left(\prod_{j=1}^n \log \frac{4}{1 - |z_j|^2} \right) \|f\|_{\mathcal{B}}.$$

Taking in particular $z_j = a_j$ ($j = 1, \dots, n$) in the above inequality, we obtain that for any $z \in \mathbb{D}^n$,

$$|\phi(z)| \leq C \|\phi\|_{\mathcal{B}(\mathbb{D}^n) \rightarrow \mathcal{B}(\mathbb{D}^n)} < \infty,$$

that is $\phi \in H^\infty(\mathbb{D}^n)$.

We next prove that $\phi \in \mathcal{B}_{LL}(\mathbb{D}^n)$. For this, we first observe the following fact.

Lemma 5.4. *If $\phi \in \mathcal{H}(\mathbb{D}^n)$ is a multiplier of $\mathcal{B}(\mathbb{D}^n)$, then for any fixed $a_1 \in \mathbb{D}$, the function $\phi(a_1, \cdot)$ is a multiplier of $\mathcal{B}(\mathbb{D}^{n-1})$. Moreover,*

$$\|\phi(a_1, \cdot)\|_{\mathcal{B}(\mathbb{D}^{n-1}) \rightarrow \mathcal{B}(\mathbb{D}^{n-1})} \lesssim \|\phi\|_{\mathcal{B}(\mathbb{D}^n) \rightarrow \mathcal{B}(\mathbb{D}^n)}.$$

Here the constant in \lesssim does not depend on a_1 .

Proof. We first prove that for any $w_n \in \mathbb{D}$ fixed, for any $b \in \mathcal{B}(\mathbb{D}^n)$, the function $b(\cdot, w_n)$ which is a function of $n - 1$ variables, is in $\mathcal{B}(\mathbb{D}^{n-1})$ with

$$\|b(\cdot, w_n)\|_{\mathcal{B}(\mathbb{D}^{n-1})} \lesssim \log \frac{4}{1 - |w_n|^2} \|b\|_{\mathcal{B}(\mathbb{D}^n)}. \quad (26)$$

Let $z = (z_1, \dots, z_{n-1}) \in \mathbb{D}^{n-1}$. From the integral representation of elements of $\mathcal{B}(\mathbb{D}^n)$, we have that for some $g \in L^\infty(\mathbb{D}^n)$,

$$b(z, w_n) = \int_{\mathbb{D}^n} \frac{g(\xi) d\nu(\xi)}{(1 - z_1 \bar{\xi}_1)^2 (1 - z_2 \bar{\xi}_2)^2 \dots (1 - z_{n-1} \bar{\xi}_{n-1})^2 (1 - w_n \bar{\xi}_n)^2},$$

hence

$$\begin{aligned} & \left(\prod_{j=1}^{n-1} (1 - |z_j|^2) \right) |D_1 \dots D_{n-1} b(z, w_n)| \\ &= \left(\prod_{j=1}^{n-1} (1 - |z_j|^2) \right) \left| \int_{\mathbb{D}^n} \frac{2^n g(\xi) d\nu(\xi)}{(1 - z_1 \bar{\xi}_1)^3 (1 - z_2 \bar{\xi}_2)^3 \dots (1 - z_{n-1} \bar{\xi}_{n-1})^3 (1 - w_n \bar{\xi}_n)^2} \right| \\ &\leq 2^n \left(\prod_{j=1}^{n-1} (1 - |z_j|^2) \right) \int_{\mathbb{D}^n} \frac{|g(\xi)| d\nu(\xi)}{|1 - z_1 \bar{\xi}_1|^3 |1 - z_2 \bar{\xi}_2|^3 \dots |1 - z_{n-1} \bar{\xi}_{n-1}|^3 |1 - w_n \bar{\xi}_n|^2} \\ &\lesssim \left(\prod_{j=1}^{n-1} (1 - |z_j|^2) \right) \|g\|_{L^\infty(\mathbb{D}^n)} \int_{\mathbb{D}^n} \frac{d\nu(\xi)}{|1 - z_1 \bar{\xi}_1|^3 |1 - z_2 \bar{\xi}_2|^3 \dots |1 - z_{n-1} \bar{\xi}_{n-1}|^3 |1 - w_n \bar{\xi}_n|^2} \\ &\lesssim \|b\|_{\mathcal{B}(\mathbb{D}^n)} \log \frac{4}{1 - |w_n|^2}. \end{aligned}$$

Now let ϕ be a multiplier of $\mathcal{B}(\mathbb{D}^n)$. Then from (26) we obtain that for any $b \in \mathcal{B}(\mathbb{D}^n)$ and any $w_n \in \mathbb{D}$ fixed, for every $z = (z_1, \dots, z_{n-1}) \in \mathbb{D}^{n-1}$,

$$\left(\prod_{j=1}^{n-1} (1 - |z_j|^2) \right) |D_1 \dots D_{n-1} (\phi b)(z, w_n)| \lesssim \left(\log \frac{4}{1 - |w_n|^2} \right) \|\phi b\|_{\mathcal{B}(\mathbb{D}^n)}$$

and so

$$\left(\prod_{j=1}^{n-1} (1 - |z_j|^2) \right) |D_1 \dots D_{n-1} (\phi b)(z, w_n)| \lesssim \left(\log \frac{4}{1 - |w_n|^2} \right) \|\phi\|_{\mathcal{B}(\mathbb{D}^n) \rightarrow \mathcal{B}(\mathbb{D}^n)} \|b\|_{\mathcal{B}(\mathbb{D}^n)} \tag{27}$$

Let us take in (27), $b(z, \xi) = g(z) \log \left(\frac{4}{1 - \xi \bar{w}_n} \right)$, $g \in \mathcal{B}(\mathbb{D}^{n-1})$, $z \in \mathbb{D}^{n-1}$ and $\xi \in \mathbb{D}$. We obtain

$$\begin{aligned} S &:= \left(\prod_{j=1}^{n-1} (1 - |z_j|^2) \right) \left| \log \left(\frac{4}{1 - \xi \bar{w}_n} \right) \right| |D_1 \dots D_{n-1} (\phi g)(z, \xi)| \\ &= \left(\prod_{j=1}^{n-1} (1 - |z_j|^2) \right) |D_1 \dots D_{n-1} (\phi b)(z, \xi)| \\ &\lesssim \log \frac{4}{1 - |\xi|^2} \|\phi\|_{\mathcal{B}(\mathbb{D}^n) \rightarrow \mathcal{B}(\mathbb{D}^n)} \|g\|_{\mathcal{B}(\mathbb{D}^{n-1})}. \end{aligned}$$

Taking $\xi = w_n$ in the above inequalities, we obtain that for any $g \in \mathcal{B}(\mathbb{D}^{n-1})$, and any $z \in \mathbb{D}^{n-1}$,

$$\left(\prod_{j=1}^{n-1} (1 - |z_j|^2) \right) |D_1 \dots D_{n-1}(\phi g)| \lesssim \|\phi\|_{\mathcal{B}(\mathbb{D}^n) \rightarrow \mathcal{B}(\mathbb{D}^n)} \|g\|_{\mathcal{B}(\mathbb{D}^{n-1})}.$$

Thus for any $w_n \in \mathbb{D}$ fixed, $\phi(\cdot, w_n)$ is a multiplier of $\mathcal{B}(\mathbb{D}^{n-1})$. The proof of the lemma is complete. \square

We next proceed by induction on the number of parameters $n \geq 2$ to prove that if ϕ is a multiplier of $\mathcal{B}(\mathbb{D}^n)$, then $\phi \in \mathcal{B}_{LL}(\mathbb{D}^n)$. We start with the case $n = 2$. Let ϕ be a multiplier of $\mathcal{B}(\mathbb{D}^2)$. Then there exists a constant $C > 0$ such that for any $f \in \mathcal{B}(\mathbb{D}^2)$ and any $z = (z_1, z_2)$,

$$(1 - |z_1|^2)(1 - |z_2|^2) |D(\phi f)(z)| \leq C \|f\|_{\mathcal{B}(\mathbb{D}^2)}. \quad (28)$$

Recall that in this case, $D = D_1 D_2$ and

$$D(\phi f)(z) = f(z)D\phi(z) + \phi(z)Df(z) + D_1\phi(z)D_2f(z) + D_1f(z)D_2\phi(z).$$

But by Lemma 5.4 and the characterization of multipliers of $\mathcal{B}(\mathbb{D})$ (see [2]), $\phi(\cdot, z_2)$ and $\phi(z_1, \cdot)$ are uniformly in $\mathcal{B}_L(\mathbb{D})$, that is there is a constant $C > 0$ such that for any $z = (z_1, z_2) \in \mathbb{D}^2$,

$$(1 - |z_1|^2) \left(\log \frac{4}{1 - |z_1|^2} \right) |D_1\phi(z)| \leq C \quad (29)$$

and

$$(1 - |z_2|^2) \left(\log \frac{4}{1 - |z_2|^2} \right) |D_2\phi(z)| \leq C. \quad (30)$$

Hence for any $z = (z_1, z_2) \in \mathbb{D}^2$, we obtain using Lemma 5.2,

$$(1 - |z_1|^2)(1 - |z_2|^2) |D_1\phi(z)| |D_2f(z)| \lesssim (1 - |z_1|^2) \left(\log \frac{4}{1 - |z_1|^2} \right) |D_1\phi(z)| \|f\|_{\mathcal{B}(\mathbb{D}^2)},$$

and consequently, using (29), we obtain

$$(1 - |z_1|^2)(1 - |z_2|^2) |D_1\phi(z)| |D_2f(z)| \leq C \|f\|_{\mathcal{B}(\mathbb{D}^2)}. \quad (31)$$

In the same way, using (30), we obtain for any $z = (z_1, z_2) \in \mathbb{D}^2$,

$$(1 - |z_1|^2)(1 - |z_2|^2) |D_2\phi(z)| |D_1f(z)| \leq C \|f\|_{\mathcal{B}(\mathbb{D}^2)}. \quad (32)$$

Also, note that as $\phi \in H^\infty(\mathbb{D}^2)$, we have that for any $z = (z_1, z_2) \in \mathbb{D}^2$,

$$(1 - |z_1|^2)(1 - |z_2|^2) |\phi(z)| |Df(z)| \leq \|\phi\|_\infty \|f\|_{\mathcal{B}(\mathbb{D}^2)}. \quad (33)$$

From (28), (31), (32) and (33), we deduce that there exists a constant $C > 0$ such that for any $f \in \mathcal{B}(\mathbb{D}^2)$ and any $z = (z_1, z_2) \in \mathbb{D}^2$,

$$(1 - |z_1|^2)(1 - |z_2|^2) |f(z)| |D\phi(z)| \leq C \|f\|_{\mathcal{B}(\mathbb{D}^2)}. \quad (34)$$

For $a = (a_1, a_2) \in \mathbb{D}^2$ given, $a_j \neq 0$, $j = 1, 2$, we test (34) with

$$f(z) = f_a(z) = \log \left(\frac{4}{1 - z_1 \bar{a}_1} \right) \log \left(\frac{4}{1 - z_2 \bar{a}_2} \right)$$

which is uniformly in $\mathcal{B}(\mathbb{D}^2)$ and obtain for any $z = (z_1, z_2) \in \mathbb{D}^2$,

$$(1 - |z_1|^2)(1 - |z_2|^2) \left| \log \left(\frac{4}{1 - z_2 \bar{a}_2} \right) \right| \left| \log \left(\frac{4}{1 - z_1 \bar{a}_1} \right) \right| |D\phi(z)| \leq C \|f\|_{\mathcal{B}(\mathbb{D}^2)}. \tag{35}$$

Taking in particular $z_1 = a_1$ and $z_2 = a_2$ in (35), we conclude that there is a constant $C > 0$ such that for any $z = (z_1, z_2) \in \mathbb{D}^2$,

$$(1 - |z_1|^2)(1 - |z_2|^2) \left(\log \frac{4}{1 - |z_1|^2} \right) \left(\log \frac{4}{1 - |z_2|^2} \right) |D\phi(z)| \leq C,$$

that is $\phi \in \mathcal{B}_L(\mathbb{D}^2)$. Combining this with (29) and (30), we conclude that $\phi \in \mathcal{B}_{LL}(\mathbb{D}^2)$. This completes the proof for the case $n = 2$.

Now for $n > 2$, we suppose that ϕ is a multiplier of $\mathcal{B}(\mathbb{D}^n)$ implies that $\phi \in \mathcal{B}_{LL}(\mathbb{D}^n)$. We prove that this implies that if ϕ is a multiplier of $\mathcal{B}(\mathbb{D}^{n+1})$, then $\phi \in \mathcal{B}_{LL}(\mathbb{D}^{n+1})$.

Let ϕ be a multiplier of $\mathcal{B}(\mathbb{D}^{n+1})$. Then by Lemma 5.4, for any $w_{n+1} \in \mathbb{D}$ fixed, $\phi(\cdot, w_{n+1})$ is a multiplier of $\mathcal{B}(\mathbb{D}^n)$ with uniformly bounded multiplier norm. Hence by our hypothesis, $\phi(\cdot, w_{n+1}) \in \mathcal{B}_{LL}(\mathbb{D}^n)$ uniformly. It follows in particular that there is a constant $C > 0$ such that for any $\vec{j} = (j_1, \dots, j_l)$, $l \leq n$, with associated set $J = \{j_1, \dots, j_l\} \subset \{1, 2, \dots, n + 1\}$, and any $z = (z_1, \dots, z_{n+1}) \in \mathbb{D}^{n+1}$,

$$\left(\prod_{j \in J} (1 - |z_j|^2) \log \frac{4}{1 - |z_j|^2} \right) |D_{\vec{j}}\phi(z)| \leq C. \tag{36}$$

For the vector \vec{k} whose associated set K is the complement of J in $\{1, 2, \dots, n + 1\}$, we obtain from (36) and Lemma 5.2 that for any $f \in \mathcal{B}(\mathbb{D}^{n+1})$, and any $z = (z_1, \dots, z_{n+1}) \in \mathbb{D}^{n+1}$,

$$\left(\prod_{j=1}^{n+1} (1 - |z_j|^2) \right) |D_{\vec{j}}\phi(z)| |D_{\vec{k}}f(z)| \leq C \|f\|_{\mathcal{B}(\mathbb{D}^{n+1})} \left(\prod_{j \in J} (1 - |z_j|^2) \log \frac{4}{1 - |z_j|^2} \right) |D_{\vec{j}}\phi(z)|.$$

Hence applying (36) to the above, we obtain

$$\left(\prod_{j=1}^{n+1} (1 - |z_j|^2) \right) |D_{\vec{j}}\phi(z)| |D_{\vec{k}}f(z)| \leq C \|f\|_{\mathcal{B}(\mathbb{D}^{n+1})}. \tag{37}$$

Also we have since $\phi \in H^\infty(\mathbb{D}^{n+1})$, that for any $z = (z_1, \dots, z_{n+1}) \in \mathbb{D}^{n+1}$,

$$\left(\prod_{j=1}^{n+1} (1 - |z_j|^2) \right) |\phi(z)| |Df(z)| \leq \|\phi\|_{H^\infty} \|f\|_{\mathcal{B}(\mathbb{D}^{n+1})}. \tag{38}$$

From (37), (38) and the fact that we have a constant $C > 0$ such that for any $f \in \mathcal{B}(\mathbb{D}^{n+1})$ and any $z = (z_1, \dots, z_{n+1}) \in \mathbb{D}^{n+1}$,

$$\left(\prod_{j=1}^{n+1} (1 - |z_j|^2) \right) |D(\phi f)(z)| \leq C \|f\|_{\mathcal{B}(\mathbb{D}^{n+1})}, \tag{39}$$

we obtain that there exists a constant $C > 0$ such that for any $f \in \mathcal{B}(\mathbb{D}^{n+1})$ and for any $z = (z_1, \dots, z_{n+1}) \in \mathbb{D}^{n+1}$,

$$\left(\prod_{j=1}^{n+1} (1 - |z_j|^2) \right) |f(z)| |D\phi(z)| \leq C \|f\|_{\mathcal{B}(\mathbb{D}^{n+1})}. \quad (40)$$

Testing (40) with a product of logarithmic functions as in (34), we obtain that there exists a constant $C > 0$ such that for any $z = (z_1, \dots, z_{n+1}) \in \mathbb{D}^{n+1}$,

$$\left(\prod_{j=1}^{n+1} (1 - |z_j|^2) \left(\log \frac{4}{1 - |z_j|^2} \right) \right) |D\phi(z)| \leq C.$$

That is $\phi \in \mathcal{B}_L(\mathbb{D}^{n+1})$. The latter and (36) allow us to conclude that $\phi \in \mathcal{B}_{LL}(\mathbb{D}^{n+1})$. The proof is complete. \square

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