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Lie Groups, Lie Algebras and some applications in Physics

By

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DECLARATION

This thesis was written in the Department of Mathematics, University of Ghana, Legon from August 2018 to July 2019 in partial fulfilment of the requirements for the award of Master of Philosophy degree in Mathematics under the supervision of Dr. Ralph Twum and Prof. Kinvi Kangni.

I hereby declare that except where due acknowledgement is made, this work has never been presented wholly or in part for the award of a degree at the University of Ghana or any other University.

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ABSTRACT

Given a Lie algebra $\mathfrak{g}$ and its complexification $\mathfrak{g}_{\mathbb{C}}$, the representations of $\mathfrak{g}_{\mathbb{C}}$ are isomorphic to those of $\mathfrak{g}$. Moreover, if $\mathfrak{g}$ is the corresponding Lie algebra of a connected and simply connected Lie group G then the representations of the Lie group in question are isomorphic to those of $\mathfrak{g}_{\mathbb{C}}$. This thesis explains the basic concepts of Lie groups and Lie algebras. Further, the basic representation theory of Lie groups and Lie algebras, particularly those of semisimple Lie algebras is discussed. In addition, an exposition of a method of constructing induced representations, with the particular case of the Poincaré group and an application in Physics is given. Finally, some physical applications of Lie groups and Lie algebras are outlined and discussed.

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DEDICATION

To my late father, Mr. Daniel Dzikpor.

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To the One who makes my life meaningful and beautiful, to the One who doubles as my father and my Lord I say thank You for all that You are and have been to me. Indeed God's grace is sufficient for me.

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Chapter 1

Introduction

Of a truth, the emergence of Lie groups and Lie algebras in times past and their development in current years has been a great feat to Mathematics and Physics. Although Lie groups may possess a global structure with nonlinear features, Lie algebras are linear in nature and thus are often much more accessible or rather easier to work with. Again, Lie groups can be viewed as global objects and Lie algebras as local objects [1]. Consequently, a number of features of Lie groups are realized by the properties of the Lie algebras that correspond to them [2]. For example, given two connected and simply connected Lie groups G_1, G_2, and their corresponding Lie algebras $\mathfrak{g}_1, \mathfrak{g}_2$, any homomorphism $\phi : G_1 \to G_2$ is isomorphic to the corresponding homomorphism $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$ [3].

In addition, representations can be attributed to homomorphisms in themselves. The Lie theory is characterized by a variety of beautiful and interesting theories. One of such is the theory of representations of Lie groups and Lie algebras. Generally, a number of Lie group and Lie algebra applications are centered on their representations. Given a Lie algebra $\mathfrak{g}$ and its complexification $\mathfrak{g}_{\mathfrak{C}}$, the representations of $\mathfrak{g}_{\mathfrak{C}}$ are isomorphic to those of $\mathfrak{g}$ [4]. Moreover, if $\mathfrak{g}$ is the corresponding Lie algebra of a connected and simply connected Lie group G then the representations of the Lie group in question are isomorphic to those of $\mathfrak{g}_{\mathbb{C}}$. Thus, the representations of Lie algebras are of great relevance in the study of Lie group representations.

Lie groups arise most often in physics as symmetry groups of dynamical systems, with such symmetries being associated with laws of conservation [5]. Though Lie groups

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found their way into Physics even before the development of the quantum theory and were relevant for the description of some homogeneous symmetric spaces being used in geometric theories of gravitation, they were virtually "forced" into physics by the development of the modern quantum theory in 1925 – 1926 [6]. Over the years, Lie groups and Lie algebras have been widely used in Spectroscopy, Nuclear Physics, Particle Physics, Gauge theories and in the structure of space time. This work will attempt to mention a few applications with regards to these areas after touching on basic concepts in Lie groups, Lie algebras and their representations with some focus on the induced representations of the Poincaré group. For more details on induced representations of locally compact groups the reader may refer to [7, 8]

In a bid to gain a better understanding and have a substantial knowledge of the theory of Lie groups and Lie algebras, particularly how it is applied to various fields I was motivated to carry out this work.

1.1 Organization of Studies

This work is structured as follows:

- Chapter 1 gives a brief introduction of how Lie groups are linked to Lie algebras and their relevance in Physics.
- Chapter 2 discusses basic concepts in Lie groups and that of Lie algebras. Among these concepts are some definitions, theorems and examples regarding Lie groups as well as Lie algebras, Linear Lie groups and the concept of nilpotency, solvability and semisimplicity of Lie algebras.
- Chapter 3 focuses on representation theory, specifically the representations of Lie algebras that are semisimple and explains a method of constructing induced representations of topological locally compact group.
- Chapter 4 outlines the application of induced representations of the Poincaré group in Physics and explains how Lie groups as well as Lie algebras are applied in physics.

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Chapter 2

Basic Concepts

This chapter serves as an opening into elementary concepts in Lie groups and Lie algebras.

2.1 Lie algebras

For a general understanding of Lie algebras and Linear Lie groups we make reference to [9, 3]

Definition 2.1.1. A Lie algebra $\mathfrak{g}$ is a vector space over a field $\mathbb{K}$($\mathbb{R}$ or $\mathbb{C}$) endowed with a bilinear map (Lie bracket)

$$
\begin{aligned}
[\ ,\ ] :& \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} \\
& (x, y) \to [x, y] \quad x, y \in \mathfrak{g}
\end{aligned}
$$

which satisfies the following conditions:

(i) $[x, x] = 0, \forall x \in \mathfrak{g}$

(ii) $\big[x, [y, z]\big] + \big[y, [z, x]\big] + \big[z, [x, y]\big] = 0, \forall x, y, z \in \mathfrak{g}$ (Jacobi Identity)

$\mathfrak{g}$ is bilinear $\Leftrightarrow$ for all $k_1, k_2 \in \mathbb{K}$ and $x, y, z \in \mathfrak{g}$

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(a) $[k_1x + k_2y, z] = k_1[x, z] + k_2[y, z]$

(b) $[x, k_1y + k_2z] = k_1[x, y] + k_2[x, z]$

The set of linearly independent vectors that spans a finite dimensional Lie algebra $\mathfrak{g}$ is termed *basis*.

Definition 2.1.2. Let $\{X_1, \cdots, X_n\}$ be the basis of the Lie algebra $\mathfrak{g}$. For each X_i, X_j in the basis we have

$$[X_i, X_j] = \sum_{k=1}^{n} c_{ij}^k X_k.$$

where the c_{ij}^k are referred to as *structure constants* of $\mathfrak{g}$.

With regards to the structure constants, the two conditions in Definition 2.1.1 can be written as:

(i) $c_{ii}^k = 0$

(ii) for any m, $\sum_{l=1}^{n}(c_{ij}^l c_{lk}^m + c_{jk}^l c_{li}^m + c_{ki}^l c_{lj}^m) = 0 \quad \forall i, j, k = 1, \cdots n.$

Corollary 2.1.3. *The first condition in Definition 2.1.1 connotes anticommutativity. This means, $[x, y] = -[y, x]$ for all x, y in $\mathfrak{g}$. Consequently, this anticommutativity ensures that $\mathfrak{g}$ is abelian if and only if $[x, y] = 0 \;\; \forall\, x, y \in \mathfrak{g}$.*

Proof. From the definition of a Lie algebra we know that,

$$\begin{aligned} [x + y, x + y] &= 0 \\ \Rightarrow [x, x] + [x, y] + [y, x] + [y, y] &= 0 \;\; \text{(Bilinearity)} \\ [x, y] + [y, x] &= 0 \qquad (2.1) \\ [x, y] &= -[y, x] \end{aligned}$$

Suppose $\mathfrak{g}$ is abelian then,

$$[x, y] = [y, x] \qquad (2.2)$$

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From Equations 2.1 and 2.2,

$$[x, y] + [x, y] = 2[x, y] = 0 \Rightarrow [x, y] = 0$$

Conversely, suppose $[x, y] = 0$ then

$$-[y, x] = 0 \Rightarrow [y, x] = [x, y] = 0$$

□

We give two examples of Lie algebras.

Example 2.1.4. Consider the space of $n \times n$ matrices $M(n, \mathbb{R})$. Let $S, T \in M(n, \mathbb{R})$ with $[S, T] = ST - TS$. $M(n, \mathbb{R})$ equipped with this bracket is a Lie algebra. This Lie algebra is denoted by $\mathfrak{gl}(n, \mathbb{R})$.

Example 2.1.5. Consider the vector space $\mathbb{R}^3$. For vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ define $[\mathbf{a}, \mathbf{b}] = \mathbf{a} \times \mathbf{b}$ such that,

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix} \forall \mathbf{a}, \mathbf{b} \in \mathbb{R}^3.$$

$\mathbb{R}^3$ endowed with the Lie bracket as defined in this example is a Lie algebra. To show this, let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$.

(i) Now, $\forall a \in \mathbb{R}^3$

$$\mathbf{a} \times \mathbf{a} = \begin{pmatrix} a_2 a_3 - a_3 a_2 \\ a_3 a_1 - a_1 a_3 \\ a_1 a_2 - a_2 a_1 \end{pmatrix} = \mathbf{0}$$

since $a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{R}$ and multiplication in $\mathbb{R}$ is commutative.

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(ii) Also we know from vectors that,

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a}.\mathbf{c})\mathbf{b} - (\mathbf{a}.\mathbf{b})\mathbf{c}$$
$$\mathbf{b} \times (\mathbf{c} \times \mathbf{a}) = (\mathbf{b}.\mathbf{a})\mathbf{c} - (\mathbf{b}.\mathbf{c})\mathbf{a}$$
$$\mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = (\mathbf{c}.\mathbf{b})\mathbf{a} - (\mathbf{c}.\mathbf{a})\mathbf{b}$$

The dot product is commutative hence

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}$$

We consider another example in terms of structure constants.

Example 2.1.6. Consider a basis of su(2) given by the matrices,

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

We calculate the structure constants below considering $[\sigma_i, \sigma_j] = c_{ij}^k \sigma_k$. Now,

$$[\sigma_i, \sigma_i] = \sigma_i\sigma_i - \sigma_i\sigma_i = 0 \implies c_{ii}^k = 0.$$

Also,

$$[\sigma_1, \sigma_2] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} - \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2i & 0 \\ 0 & -i \end{bmatrix} = 2i\sigma_3 \implies c_{12}^3 = 2i$$
$$[\sigma_1, \sigma_3] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} = -2i\sigma_2 \implies c_{13}^2 = -2i$$
$$[\sigma_2, \sigma_3] = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2i \\ 2i & 0 \end{bmatrix} = 2i\sigma_1 \implies c_{23}^1 = 2i$$

From these results,

$$c_{12}^3 c_{33}^m + c_{23}^1 c_{11}^m + c_{31}^2 c_{22}^m = 2i(0) + 2i(0) - 2i(0) = 0$$

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for any m.

Proposition 2.1.7. *Let V be a finite dimensional vector space. Consider the space $End(V)$ of endomorphisms of V, with Lie bracket defined as follows:*

$$\forall f, g \in End(V), \quad [f, g] = f \circ g - g \circ f.$$

Then $End(V)$ equipped with the Lie bracket $[\cdot, \cdot]$ is a Lie algebra.

Proof. Let $f, g, h \in \mathrm{End}(V)$.

(i) $[f, f] = f \circ f - f \circ f = 0$.

(ii) $\big[f, [g, h]\big] = f \circ g \circ h - f \circ h \circ g - g \circ h \circ f + h \circ g \circ f$
$\big[g, [h, f]\big] = g \circ h \circ f - g \circ f \circ h - h \circ f \circ g + f \circ h \circ g$
$\big[h, [f, g]\big] = h \circ f \circ g - h \circ g \circ f - f \circ g \circ h + g \circ f \circ h$

Hence $\big[f, [g, h]\big] + \big[g, [h, f]\big] + \big[h, [f, g]\big] = 0$

□

Definition 2.1.8. (Lie subalgebra) Let $\mathfrak{g}$ be a Lie algebra. $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$ if $\mathfrak{h}$ is a vector subspace of $\mathfrak{g}$ closed under the Lie bracket that is $\forall\, a, b \in \mathfrak{h}, [a, b] \in \mathfrak{h}$. Furthermore, $\mathfrak{I}$ is an ideal of $\mathfrak{g}$ if $\mathfrak{I}$ is a vector subspace of $\mathfrak{g}$ and $[a, b] \in \mathfrak{I}, \forall a \in \mathfrak{I}, b \in \mathfrak{g}$.

Definition 2.1.9. (Derivation) Let $\mathfrak{g}$ be a Lie algebra. A derivation is an endomorphism $D : \mathfrak{g} \to \mathfrak{g}$ such that $D[X, Y] = [DX, Y] + [X, DY], \forall\, X, Y \in \mathfrak{g}$. Denote the vector space of derivations of $\mathfrak{g}$ by $\mathfrak{D}er(\mathfrak{g})$. For $D_1, D_2 \in \mathfrak{D}er(\mathfrak{g})$, define

$$[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1.$$

We state and prove a corollary that illustrates an example of a Lie subalgebra.

Corollary 2.1.10. *The space of derivations of $\mathfrak{g}$ ($\mathfrak{D}er(\mathfrak{g})$) is a Lie subalgebra of $\mathfrak{gl}(\mathfrak{g})$, where $\mathfrak{gl}(\mathfrak{g})$ is the Lie algebra of endomorphisms of $\mathfrak{g}$.*

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Proof. We will show that $\mathfrak{Der}(\mathfrak{g})$ satisfies the following conditions.

(i) $\alpha D_1 + \beta D_2 \in \mathfrak{Der}(\mathfrak{g}) \ \forall D_1, D_2 \in \mathfrak{Der}(\mathfrak{g}), \alpha, \beta \in \mathbb{R}$

(ii) $[D_1, D_2] \ \forall D_1, D_2 \in \mathfrak{Der}(\mathfrak{g})$.

For (i),

$$\begin{aligned}
\alpha D_1[X,Y] + \beta D_2[X,Y] &= \alpha\left([D_1X,Y] + [X,D_1Y]\right) + \beta\left([D_2X,Y] + [X,D_2Y]\right) \\
&= [\alpha D_1X,Y] + [X,\alpha D_1Y] + [\beta D_2X,Y] + [X,\beta D_2Y] \\
&= [\alpha D_1X,Y] + [\beta D_2X,Y] + [X,\alpha D_1Y] + [X,\beta D_2Y] \\
&= [(\alpha D_1 + \beta D_2)X,Y] + [X,(\alpha D_1 + \beta D_2)Y] \\
&= (\alpha D_1 + \beta D_2)[X,Y]
\end{aligned}$$

Again, for (ii)

$$\begin{aligned}
[D_1,D_2][X,Y] &= D_1 \circ D_2[X,Y] - D_2 \circ D_1[X,Y] \\
&= D_1\left(D_2[X,Y]\right) - D_2\left(D_1[X,Y]\right) \\
&= D_1\left([D_2X,Y] + [X,D_2Y]\right) - D_2\left([D_1X,Y] + [X,D_1Y]\right) \\
&= [D_1D_2X,Y] + [D_2X,D_1Y] + [D_1X,D_2Y] + [X,D_1D_2Y] \\
&\quad - [D_2D_1X,Y] - [D_1X,D_2Y] - [D_2X,D_1Y] - [X,D_2D_1Y] \\
&= [D_1D_2X,Y] - [X,D_2D_1Y] + [X,D_1D_2Y] - [D_2D_1X,Y] \\
&= [[D_1,D_2]X,Y] + [X,[D_1,D_2]Y]
\end{aligned}$$

□

Definition 2.1.11. (Lie algebra homomorphism) A map $\Psi : \mathfrak{g}_1 \to \mathfrak{g}_2$ where $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are Lie algebras, is a Lie algebra homomorphism if $\forall X, Y \in \mathfrak{g}$,

(i) $\Psi(\beta X + \gamma Y) = \beta\Psi(X) + \gamma\Psi(Y), \forall\ \beta, \gamma \in \mathbb{R}$

(ii) $\Psi[X,Y] = [\Psi(X), \Psi(Y)]$

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Proposition 2.1.12. *The adjoint map* $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ *defined by* $\mathrm{ad}X(Y) = [X, Y]$ *is a homomorphism of Lie algebras.*

Proof. Let $X, Y, Z \in \mathfrak{g}$. We want to show that,

(i) $\mathrm{ad}(\alpha X + \beta Y) = \alpha \mathrm{ad}X + \beta \mathrm{ad}Y$

(ii) $\mathrm{ad}[X, Y] = [\mathrm{ad}X, \mathrm{ad}Y]$

With respect to (i),

$$\begin{aligned} \mathrm{ad}(\alpha X + \beta Y)(Z) &= [\alpha x + \beta Y, Z] \\ &= [\alpha X, Z] + [\beta Y, Z] \\ &= \alpha[X, Z] + \beta[Y, Z] \\ &= (\alpha \mathrm{ad}X + \beta \mathrm{ad}Y)Z \\ &= \alpha \mathrm{ad}X(Z) + \beta \mathrm{ad}Y(Z) \end{aligned}$$

Also, for (ii) by the definition of ad,

$$\mathrm{ad}[X, Y](Z) = \big[[X, Y], Z\big]$$

Using the Jacobi Identity,

$$\begin{aligned} \big[[X, Y], Z\big] &= \big[X, [Y, Z]\big] - \big[Y, [X, Z]\big] \\ \therefore\ \mathrm{ad}[X, Y](Z) &= \mathrm{ad}X[Y, Z] - \mathrm{ad}Y[X, Z] \\ &= \mathrm{ad}X \circ \mathrm{ad}Y(Z) - \mathrm{ad}Y \circ \mathrm{ad}X(Z) \\ &= \big(\mathrm{ad}X\mathrm{ad}Y - \mathrm{ad}Y\mathrm{ad}X\big)(Z) \\ &= [\mathrm{ad}X, \mathrm{ad}Y](Z) \end{aligned}$$

□

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Corollary 2.1.13. *The adjoint of the Lie algebra* $\mathfrak{g}$ *is an ideal of the space of derivations of* $\mathfrak{g}$.

Proof. We have already established in the previous proposition that $\text{ad}(\alpha X + \beta Y) = \alpha\text{ad}X + \beta\text{ad}Y \ \forall \ \alpha, \beta \in \mathbb{R}$ and $X, Y \in G$. Thus, $\alpha\text{ad}X + \beta\text{ad}Y \in \text{ad}(\mathfrak{g})$. We now show that $\forall X \in \text{ad}(\mathfrak{g}),\ D \in \mathfrak{Der}(\mathfrak{g})$ and $[\text{ad}X, D] \in \text{ad}(\mathfrak{g})$

We have,

$$\begin{aligned}
[adX, D](Y) &= (\text{ad}X \circ D - D \circ \text{ad}X)(Y) \\
&= \text{ad}X(D(Y)) - D(\text{ad}X(Y)) \\
&= [X, DY] - D[X, Y] \\
&= [X, DY] - [DX, Y] - [X, DY] \\
&= -[DX, Y] \\
&= -\text{ad}DX(Y) \\
&= \text{ad}(-DX)(Y)
\end{aligned}$$

So, $[\text{ad}X, D] = \text{ad}(-DX) \in \text{ad}(\mathfrak{g})$ and hence $\text{ad}(\mathfrak{g})$ is an ideal of $\mathfrak{Der}(\mathfrak{g})$. □

2.2 Lie algebra of a Lie group

Given a Lie group, it is important to know its Lie algebra because of the relationship between the representation of the Lie group and its corresponding Lie algebra. Here, we give the definition of Lie groups via smooth manifolds, define Linear Lie groups, define the Lie algebra for a certain class of Lie groups and compute the Lie algebras for these Lie groups.

For preliminaries on manifolds and Lie groups we make reference to [10].

Definition 2.2.1. (*Manifold*)[11] A topological manifold is a Hausdorff topological space M such that for any $x \in M$ there exists some open neighbourhood U of x which is homeomorphic to $\mathbb{R}^n$ for some integer $n \geq 0$. In other words, a topological manifold

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is a locally Euclidean space. In addition, if φ is the homeomorphism in question, then $(U,\ \varphi : U \to \mathbb{R}^n)$ is referred to as a chart. A collection of these charts that cover M is an atlas. As such if the atlas is not contained in a larger atlas then it is a maximal atlas. Furthermore, M is a smooth manifold if M has a maximal atlas.

Definition 2.2.2. (*Tangent space*) Let us suppose M is a manifold contained in some ambient space $\mathbb{R}^n$ where M is a smooth manifold and $x \in M$. The tangent space of M at the point x is given by,

$$T_x(M) = \{v \in \mathbb{R}^n | v = \phi'(0)\}$$

where $\phi : \mathbb{R} \to M$ is a smooth curve that passes through x and $\phi(0) = x$.

Next, we define a Lie group and its subgroup.

Definition 2.2.3. (Lie Group) A smooth manifold G with a group structure such that the map,

$$\begin{aligned} \alpha : G \times G &\to G \\ (g, h) &\to gh^{-1} \quad g, h \in G \end{aligned}$$

is smooth is called a Lie group. Moreover, if $H \subset G$ is a closed subgroup of G and H is a submanifold of G, then H is a Lie subgroup of G.

Some examples of Lie groups (linear Lie groups) have been stated earlier. We outline some other examples of Lie groups.

Example 2.2.4. $(\mathbb{R}^n, +)$ and $(\mathbb{R}^+ \backslash \{0\}, \times)$ are Lie groups.

Example 2.2.5. The quotient group $(\mathbb{R}^n / \mathbb{Z}^n, +)$ is a Lie group.

Example 2.2.6. The group of $n \times n$ invertible matrices whose entries are in $\mathbb{K}$, $\mathrm{GL}(n, \mathbb{K})$ is a Lie group. $\mathrm{M}(n, \mathbb{K}) \cong \mathbb{K}^{n^2}$ is a differentiable manifold. $\mathrm{GL}(n, \mathbb{K}) = f^{-1}(\mathbb{R} \backslash \{0\})$, where $f : \mathrm{M}(n, \mathbb{K}) \mapsto \mathbb{R}$ that is $f(A) = \det(A)$, for $A \in \mathrm{M}(n, \mathbb{K})$. $\mathrm{GL}(n, \mathbb{K})$ is an open submanifold of $\mathrm{M}(n, \mathbb{K})$ and its multiplication and inverse maps are polynomials hence smooth. Thus it is a Lie group.

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Example 2.2.7. The unit circle S^1 is a Lie group. $S^1 \subset \mathrm{GL}(1, \mathbb{C})$ is a subgroup and a smoothly imbedded submanifold of $\mathrm{GL}(1, \mathbb{C})$, thus it is a Lie group. Again, we know, $S^1 = \{x \in \mathbb{R} | (x_1)^2 + (x_2)^2 = 1\}$. We consider the following coordinate charts. These are,

$$V_1\{x \in S^1 | x_2 > 0\}, \phi_{V_1}(x) = x_1$$
$$V_2\{x \in S^1 | x_2 < 0\}, \phi_{V_2}(x) = x_1$$
$$W_1\{x \in S^1 | x_1 > 0\}, \phi_{W_1}(x) = x_2$$
$$W_2\{x \in S^1 | x_1 < 0\}, \phi_{W_2}(x) = x_2$$

These charts cover S^1. Also, with respect to $V_1 \cap W_2$ we notice that

$$x_1 = \sqrt{1 - (x_2)^2} > 0, \quad x_2 = \sqrt{1 - (x_1)^2} > 0.$$

These are C^∞ functions, so (V_1, ϕ_{V_1}) and (W_2, ϕ_{W_2}) are C^∞ compatible. These charts are enough to make S^1 a one dimensional smooth manifold. The group S^1 has its multiplicative and inverse operations smooth as well. Thus it is a Lie group.

Example 2.2.8. The torus $T^2 \cong S^1 \times S^1$ is a Lie group. If G and H are Lie groups, then $G \times H$ is a Lie group under the usual Cartesian group operations and the smooth product structure [12]. Thus T^2 is a Lie group.

Example 2.2.9. The Heisenberg group is a Lie group.

In addition, we define a Linear Lie group and state some examples.For information regarding Linear Lie groups we make use of the reference Hall [3].

2.2.10 Linear Lie groups

Definition 2.2.11. A closed subgroup of $\mathrm{GL}(n, \mathbb{K})$ is called a linear Lie group.

We state examples of linear Lie groups.

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Example 2.2.12. The special linear group, $\mathrm{SL}(n, \mathbb{K}) := \{B \in \mathrm{GL}(n, \mathbb{K}) : \det B = 1\}$, is a closed subgroup of $\mathrm{GL}(n, \mathbb{K})$ because its determinant function is continuous and $\mathrm{SL}(n, \mathbb{K}) = \det^{-1}(1)$, which is closed in $\mathrm{GL}(n, \mathbb{K})$.

Example 2.2.13. The orthogonal group, $\mathrm{O}(n) := \{B \in \mathrm{GL}(n, \mathbb{R}) : B^T B = BB^T = I\}$ and the special orthogonal group, $\mathrm{SO}(n) := \{B \in \mathrm{GL}(n, \mathbb{R}) : B^T B = BB^T = I, \det B = 1\}$ are linear Lie groups. $\mathrm{O}(n)$ is closed in $\mathrm{GL}(n, \mathbb{R})$. It is the pre-image of the map $f(B) = BB^T - I$. In addition, $\det(\mathrm{O}(n)) = \pm 1$. Thus $\mathrm{O}(n)$ has two connected components and $\mathrm{SO}(n)$ is the connected component of the identity.

Example 2.2.14. The unitary group, $\mathrm{U(n)} := \{B \in \mathrm{GL}(n, \mathbb{C}) : B^* B = BB^* = I\}$ and the special unitary group, $\mathrm{SU}(n) := \{B \in \mathrm{GL}(n, \mathbb{C}) : B^* B = BB^* = I, \det B = 1\}$, where B^* is the conjugate transpose of B are linear Lie groups. U(n) and SU(n) can be defined analogously to O(n) and SO(n).

Example 2.2.15. The symplectic group denoted by, $\mathrm{Sp}(2n, \mathbb{K}) := \{B \in \mathrm{GL}(2n, \mathbb{K}) : B^T JB = J\}$, where $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ is a closed subgroup of $\mathrm{GL}(2n, \mathbb{K})$ thus it is a linear Lie group.

We give the general definition of a representation of a linear Lie group G on a finite dimensional vector space V.

Definition 2.2.16. A representation, Π of a linear Lie group G on a finite dimensional vector space V is a homomorphism of G into $\mathrm{GL}(V)$. This map $\Pi : G \to \mathrm{GL}(V)$ satisfies,

(a) $\Pi_{xy} = \Pi_x \Pi_y, \qquad \forall x, y \in G$

(b) $\Pi_e = Id_V$

where $\Pi_x := \Pi(x) \in \mathrm{GL}(V)$.

The adjoint representation of a Lie group G is the homomorphism $\mathrm{Ad} : G \mapsto \mathrm{GL}(\mathfrak{g})$. We would discuss a little bit more about representations in the next chapter. Here, we provide the adjoint representation of the Lie group, SU(2).

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Example 2.2.17. (The adjoint representation of SU(2)) Earlier we defined the Lie group, SU(n) in Example 2.2.14. Similarly,

$$\mathrm{SU}(2) = \{Y \in \mathrm{GL}(2, \mathbb{C}) : Y^*Y = YY^* = I \ , \mathrm{det}Y = 1\}$$

First, we find matrices that take the above form. Now, for $R \in \mathrm{SU}(2)$ such that,

$$R = \begin{pmatrix} m & n \\ r & s \end{pmatrix} \text{ and } \begin{pmatrix} m & n \\ r & s \end{pmatrix} \begin{pmatrix} \bar{m} & \bar{r} \\ \bar{n} & \bar{s} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, ms - nr = 1 \ \forall \ m, n, r, s \in \mathbb{C}$$

The following equations are derived.

$$|m|^2 + |n|^2 = 1 \tag{2.3}$$

$$|r|^2 + |s|^2 = 1 \tag{2.4}$$

$$m\bar{r} + n\bar{s} = 0 \tag{2.5}$$

$$r\bar{m} + s\bar{n} = 0 \tag{2.6}$$

in addition to

$$ms - nr = 1. \tag{2.7}$$

From Equation 2.5 and 2.6,

$$m\bar{r} = -n\bar{s} \text{ and } r\bar{m} = -s\bar{n}$$

Multiplying these equations yield,

$$|m|^2|r^2| = |s|^2|n^2|$$

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Also,

$$\begin{aligned}|m|^2|r^2| &= |s|^2(1-|m|^2)\\ |m|^2|r^2| + |s|^2|m|^2 &= |s|^2\\ |m|^2(|r^2| + |s|^2) &= |s|^2\\ |m|^2 &= |s|^2\\ \Rightarrow m\bar{m} &= s\bar{s} \qquad (2.8)\end{aligned}$$

In the same vein,

$$r\bar{r} = n\bar{n} \qquad (2.9)$$

This gives us the following possibilities

$$m = \pm s, m = \pm \bar{s}, \bar{m} = \pm s, \bar{m} = \pm \bar{s}$$

and

$$r = \pm n, r = \pm \bar{n}, \bar{r} = \pm n, \pm \bar{r} = \pm \bar{n}$$

From Equations 2.8 and 2.9,

$$m(\bar{m}) - n(-\bar{n}) = |m|^2 + |n|^2 = 1$$

So, the only possibilities are

$$s = \bar{m}, r = -\bar{n}$$

Thus, elements in the special unitary group SU(2) are of the form,

$$\begin{pmatrix} m & n \\ -\bar{n} & \bar{m} \end{pmatrix} \text{ where } m, n \in \mathbb{C} \text{ with } |m|^2 + |n|^2 = 1$$

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Suppose $m = a + ib$ and $n = c + id$, an element in SU(2) is given by,

$$\begin{pmatrix} a+ib & c+id \\ -c+id & a-ib \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + c\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

Consequently, we can write down the following basis for SU(2):

$$X_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \qquad X_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad X_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

Consider a matrix $R \in \mathrm{SU}(2)$, then

$$R = r_1X_1 + r_2X_2 + r_3X_3, \qquad r_1, r_2, r_3 \in \mathbb{C}.$$

Now,

$$[X_1, X_2] = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} 0 & 2i \\ 2i & 0 \end{pmatrix} = 2X_3$$
$$[X_2, X_3] = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix} = 2X_1$$
$$[X_1, X_3] = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} = 2X_2$$

Thus, from the above results *the adjoint representation* of R in accordance with the ordered basis $\{X_1, X_2, X_3\}$ is given by,

$$\mathrm{ad}(R) = \begin{pmatrix} 0 & -2r_3 & -2r_2 \\ 2r_3 & 0 & -2r_1 \\ 2r_2 & 2r_1 & 0 \end{pmatrix}.$$

Definition 2.2.18. (One-parameter subgroup) A homomorphism $\psi : \mathbb{R} \to \mathrm{GL}(n, \mathbb{K})$

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satisfying the condition:

$$\forall s, t \in \mathbb{R}, \psi(s+t) = \psi(s)\psi(t) \text{ and } \psi(0) = I$$

is a one-parameter subgroup.

Example 2.2.19. Given the sets:

$$A = \{a_t, t \in \mathbb{R}\}, \qquad N = \{n_\zeta, \zeta \in \mathbb{R}\}$$

where,

$$a_t = \begin{pmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{2}} \end{pmatrix} \qquad n_\zeta = \begin{pmatrix} 1 & \zeta \\ 0 & 1 \end{pmatrix}$$

a_t and n_ζ are one parameter subgroups of $\mathrm{SL}(2, \mathbb{R})$

In fact,

$$\begin{aligned} a_{t_1+t_2} &= \begin{pmatrix} e^{\frac{t_1+t_2}{2}} & 0 \\ 0 & e^{-\frac{t_1+t_2}{2}} \end{pmatrix} \\ &= \begin{pmatrix} e^{\frac{t_1}{2}} e^{\frac{t_2}{2}} & 0 \\ 0 & e^{-\frac{t_1}{2}} e^{-\frac{t_1}{2}} \end{pmatrix} \\ &= \begin{pmatrix} e^{\frac{t_1}{2}} & 0 \\ 0 & e^{-\frac{t_1}{2}} \end{pmatrix} \begin{pmatrix} e^{\frac{t_2}{2}} & 0 \\ 0 & e^{-\frac{t_2}{2}} \end{pmatrix} \text{ By properties of diagonal matrices} \\ &= a_{t_1} a_{t_2} \end{aligned}$$

and

$$\begin{aligned} a_0 &= \begin{pmatrix} e^0 & 0 \\ 0 & e^0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

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Therefore,

$$a_{t_1+t_2} = a_{t_1} a_{t_2} \quad \forall t_1, t_2 \in \mathbb{R} \text{ and } a_0 = I.$$

Moreover,

$$\begin{aligned} n_{\zeta_1+\zeta_2} &= \begin{pmatrix} 1 & \zeta_1 + \zeta_2 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & \zeta_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \zeta_2 \\ 0 & 1 \end{pmatrix} \\ &= n_{\zeta_1} n_{\zeta_2} \end{aligned}$$

And,

$$n_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Therefore,

$$n_{\zeta_1+\zeta_2} = n_{\zeta_1} n_{\zeta_2} \quad \forall \zeta_1, \zeta_2 \in \mathbb{R} \text{ and } n_0 = I.$$

Proposition 2.2.20. *[9] Every one-parameter subgroup ψ of $\mathrm{GL}(n, \mathbb{K})$ is of the form $\psi(t) = \exp(tB)$ where $B = \psi'(0) \in \mathrm{GL}(n, \mathbb{K})$ and $t \in \mathbb{R}$.*

Proof. From the usual definition of the derivative, we have

$$\begin{aligned} \psi'(t) &= \lim_{h \to 0} \frac{\psi(t+h) - \psi(t)}{h}, \quad h \in \mathbb{R} \\ &= \lim_{h \to 0} \frac{\psi(t)\psi(h) - \psi(t)}{h} \\ &= \psi(t) \lim_{h \to 0} \frac{\psi(h) - \psi(0)}{h} \\ &= \psi(t)\psi'(0) \end{aligned}$$

So,

$$\psi'(t) = \psi(t)\psi'(0)$$

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After integrating with respect to t we have,

$$\psi(t) = K \exp t\psi'(0)$$

Since $\psi(0) = I$, $K = 1$. Therefore,

$$\psi(t) = \exp t\psi'(0) = \exp tB$$

where $B = \psi'(0)$. $\square$

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2.2.21 Lie algebras of Linear Lie groups

Definition 2.2.22. The Lie algebra of a linear Lie group, G is given by:

$$\mathfrak{g} = \{B \in \mathfrak{gl}(n, \mathbb{R}) | \exp tB \in G, \forall t \in \mathbb{R}\}$$

We compute the Lie algebra of the linear Lie groups discussed earlier. To do this, we use the following well known result, $\det(\exp tA) = \exp[t \ \mathrm{tr}(A)]$, where $A \in \mathrm{GL}(n, \mathbb{K})$.

Example 2.2.23. We first consider the general linear group. We know that,

$$\mathrm{GL}(n, \mathbb{R}) = \{B \in \mathrm{M}(n, \mathbb{R}) | \det(B) \neq 0\}$$

Its Lie algebra is given by,

$$\mathfrak{gl}(n, \mathbb{R}) = \{B \in \mathrm{M}(n, \mathbb{R}) | \exp tB \in \mathrm{GL}(n, \mathbb{R})\}$$

This implies that,

$$\det(\exp tB) = \exp[t \ \mathrm{tr}(B)] \neq 0$$

But we know that,

$$\exp[t \ \mathrm{tr}(B)] > 0$$

Thus $\det(\exp tB) > 0$ and hence,

$$\mathfrak{gl}(n, \mathbb{R}) = \mathrm{M}(n, \mathbb{R}).$$

Example 2.2.24. For the special linear group, $\mathrm{SL}(n, \mathbb{R}) = \{B \in \mathrm{GL}(n, \mathbb{R}) : \det B = 1\}$, its Lie algebra is given by,

$$\mathfrak{sl}(n, \mathbb{R}) = \{B \in \mathrm{M}(n, \mathbb{R}) | \exp tB \in \mathrm{SL}(n, \mathbb{R}), \forall t \in \mathbb{R}\}$$

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So,

$$\det(\exp tB) = 1, \forall t \in \mathbb{R}$$
$$\exp[t \ \mathrm{tr}(B)] = 1$$
$$\mathrm{tr}(B) = 0.$$

This gives us the space of traceless matrices, that is,

$$\mathfrak{sl}(n, \mathbb{R}) = \{B \in M(n, \mathbb{R}) | \ \mathrm{tr}(B) = 0\}$$

Example 2.2.25. We consider the orthogonal and the special orthogonal groups. $O(n) = \{B \in \mathrm{GL}(n, \mathbb{R}) : B^T B = BB^T = I\}$ and $SO(n) = \{B \in \mathrm{GL}(n, \mathbb{R}) : B^T B = BB^T = I, \ \det B = 1\}$. The lie algebra of O($n$) is given by,

$$\mathfrak{o}(n) = \{B \in \mathrm{M}(n, \mathbb{R}) | \exp tB \in O(n)\}$$

Computing this, we have

$$\exp tB(\exp tB)^T = I$$
$$\exp tB \exp tB^T = I$$
$$\exp t(B + B^T) = I$$

Differentiating at $t = 0$ we have,

$$B + B^T = \mathbf{0}$$
$$B^T = -B$$

This yields the space of skew symmetric $n \times n$ matrices given by,

$$\mathfrak{o}(n) = \{B \in \mathrm{M}(n, \mathbb{R}) | B^T = -B\}$$

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In the case of $\mathfrak{so}(n)$, the trace of the matrix also vanishes in addition to the above skew symmetric nature but skew symmetric matrices have a zero trace so $\mathfrak{so}(n) = \mathfrak{o}(n)$.

Example 2.2.26. We compute the Lie algebra of the unitary and special unitary groups. Now,

$$\mathrm{U}(n) = \{B \in \mathrm{GL}(n, \mathbb{C}) | B^*B = BB^* = I\}$$

So,

$$\mathfrak{u}(n) = \{A \in \mathfrak{gl}(n, \mathbb{C}) | \exp tB \in U(n)\}$$

Computing this we have,

$$\begin{aligned} \exp tB(\exp tB)^* &= I \\ \exp tB \exp tB^* &= I \\ \exp t(B + B^*) &= I \end{aligned}$$

Differentiating at $t = 0$ we have,

$$\begin{aligned} B + B^* &= \mathbf{0} \\ B^* &= -B \end{aligned}$$

Therefore,

$$\mathfrak{u}(n) = \{B \in \mathfrak{gl}(n, \mathbb{C}) | B^* = -B\}$$

In addition to $\mathfrak{u}(n)$ the Lie algebra, $\mathfrak{su}(n)$ has the condition $tr(B) = 0$. Hence,

$$\mathfrak{su}(n) = \{B \in \mathfrak{gl}(n, \mathbb{C}) | B^* = -B \text{ and } \mathrm{tr}(B) = 0\}$$

Example 2.2.27. Consider the symplectic group, $\mathrm{Sp}(2n, \mathbb{R}) = \{B \in \mathrm{GL}(2n, \mathbb{R}) : B^T JB = J\}$, where $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ Then,

$$\mathfrak{sp}(2n, \mathbb{R}) = \{B \in \mathfrak{gl}(2n, \mathbb{R}) | \exp tB \in \mathrm{Sp}(2n, \mathbb{R})\}$$

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Computing this we have,

$$(\exp tB)^T J(\exp tB) = J$$
$$\exp tB^T J = J\exp(-tB)$$

Differentiating at $t = 0$ we have

$$B^T J = -JB \Rightarrow B^T J + JB = 0$$

Therefore, the Lie algebra of the symplectic group is given by,

$$\mathfrak{sp}(2n, \mathbb{R}) = \{B \in \mathfrak{gl}(2n, \mathbb{R}) | B^T J + JB = 0\}$$

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2.3 Nilpotent, Solvable and Semisimple Lie Algebras

Here, we define nilpotent, solvable and semisimple Lie algebras, other concepts relating to them and some examples as well. We begin by defining the centre of a Lie algebra.

Definition 2.3.1. (Centre of a Lie algebra) [13] The centre $z(\mathfrak{g})$ of a Lie algebra, $\mathfrak{g}$ is the set,

$$z(\mathfrak{g}) = \{X \in \mathfrak{g}, [X, Y] = 0 \ \forall Y \in \mathfrak{g}\}.$$

A Lie algebra is abelian if and only if $z(\mathfrak{g}) = \mathfrak{g}$.

Definition 2.3.2. (Nilpotent Lie algebra) Let $\mathfrak{g}$ be a Lie algebra. Define the descending central series $(\mathcal{C}^n\mathfrak{g})_{n \in \mathbb{N} \cup \{0\}}$ such that

$$\mathcal{C}^0 \supset \mathcal{C}^1 \supset \cdots \supset \mathcal{C}^{n-1}\mathfrak{g} \supset \mathcal{C}^n\mathfrak{g} \supset \mathcal{C}^{n+1}\mathfrak{g} \supset \cdots$$

where $\mathcal{C}^0\mathfrak{g} := \mathfrak{g}$ and $\mathcal{C}^n\mathfrak{g} := [\mathfrak{g}, \mathcal{C}^{n-1}\mathfrak{g}]$. If there exist an $n \in \mathbb{N}$ such that $\mathcal{C}^n\mathfrak{g} = \{0\}$ we say $\mathfrak{g}$ is nilpotent.

The smallest n such that $\mathcal{C}^n\mathfrak{g} = \{0\}$ is called the step of the nilpotent Lie algebra $\mathfrak{g}$.

Example 2.3.3. Abelian Lie algebras are nilpotent of step 1. For any abelian Lie algebra $\mathfrak{g}$, $[X, Y] = 0, \forall X, Y \in \mathfrak{g}$. This means that,

$$[\mathfrak{g}, \mathfrak{g}] = \{[X, Y], \forall X, Y \in \mathfrak{g}\} = \{0\}$$

Example 2.3.4. The Heisenberg Lie algebra H_3 is nilpotent. Consider the following basis

$$C_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } C_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

of the Lie algebra H_3. We see that $C_3 \in z(H_3)$ and $[C_1, C_2] = C_3$ is nilpotent. In

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addition,

$$\mathcal{C}^1 H_3 = [H_3, \mathcal{C}^0 H_3] = [H_3, H_3] = \{[C, E], C, E \in H_3\}$$
$$\mathcal{C}^1 H_3 = \{[\alpha_1 C_1 + \alpha_2 C_2 + \alpha_3 C_3,\ \beta_1 E_1 + \beta_2 E_2 + \beta_3 E_3], \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3 \in \mathbb{R}\}$$
$$= \{(\alpha_1 \beta_2 - \alpha_2 \beta_1) C_3 = \gamma C_3, \gamma \in \mathbb{R}\}$$

Also,

$$\mathcal{C}^2 H_3 = [H_3, \mathcal{C}^1 H_3] = \{[\alpha_1 C_1 + \alpha_2 C_2 + \alpha_3 C_3, \gamma C_3], \alpha_1, \alpha_2, \alpha_3, \gamma \in \mathbb{R}\} = \{0\}$$

$\therefore H_3$ is nilpotent.

Example 2.3.5. The space of strictly upper triangular matrices, $T_o(n, \mathbb{R})$ is nilpotent of step n.

Example 2.3.6. The 4 dimensional Lie algebra l with basis Q_1, Q_2, Q_3, Q_4 such that $[Q_1, Q_2] = Q_3, [Q_2, Q_3] = Q_4$ and all other products equal to 0 is a nilpotent Lie algebra.

Remark 2.3.7. If φ is a homomorphism of Lie algebras then $\varphi(\mathcal{C}^n(\mathfrak{g})) = \mathcal{C}^n \varphi(\mathfrak{g})$.

We state a corollary that is a consequence of the **Engel's theorem** [9].

Corollary 2.3.8. *Let $\mathfrak{g}$ be a Lie algebra. $\mathfrak{g}$ is nilpotent $\Leftrightarrow \forall X \in \mathfrak{g}, \mathrm{ad}X$ is nilpotent.*

Proof. Suppose $\mathfrak{g}$ is nilpotent, then there exist $n \in \mathbb{N}$ such that $\mathcal{C}^n \mathfrak{g} = [\mathfrak{g}, \mathcal{C}^{n-1}\mathfrak{g}] = \{0\}$.

So,

$$\forall X \in \mathfrak{g} \quad [X, \mathcal{C}^{n-1}\mathfrak{g}] = 0$$

Therefore,

$$\mathrm{ad}X(\mathcal{C}^{n-1}\mathfrak{g}) = 0$$

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Consequently,

$$(\text{ad}X)^2(\mathcal{C}^{n-2}\mathfrak{g}) = 0$$
$$\vdots$$
$$(\text{ad}X)^n(\mathcal{C}^0\mathfrak{g}) = (\text{ad}X)^n(\mathfrak{g}) = 0.$$

This shows that,

$$(\text{ad}X)^n = 0.$$

Thus ad$X$ is nilpotent for all $X \in \mathfrak{g}$. Now, suppose ad$X$ is nilpotent $\forall X \in \mathfrak{g}$. Then there exist $n \in \mathbb{N}$ such that $\mathcal{C}^n(\text{ad}\mathfrak{g}) = \{0\}$. Since ad is a homomorphism,

$$\mathcal{C}^n(\text{ad}\mathfrak{g}) = [\text{ad}\mathfrak{g}, \mathcal{C}^{n-1}(\text{ad}\mathfrak{g})] = [\text{ad}\mathfrak{g}, \text{ad}(\mathcal{C}^{n-1}(\mathfrak{g}))] = \text{ad}(\mathcal{C}^n\mathfrak{g}) = \{0\}$$

Thus,

$$\mathcal{C}^n\mathfrak{g} \subset \ker(\text{ad}) = z(\mathfrak{g})$$

So,

$$\mathcal{C}^n\mathfrak{g} \subset z(\mathfrak{g}).$$

Now,

$$\mathcal{C}^n\mathfrak{g} \subset [\mathfrak{g}, z(\mathfrak{g})] = \{0\}.$$

Hence $\mathfrak{g}$ is nilpotent. □

Definition 2.3.9. Let $\mathfrak{g}$ be a Lie algebra. Define the upper central series $(\mathcal{D}^n\mathfrak{g})_{n\in\mathbb{N}\cup\{0\}}$ such that,

$$\mathcal{D}^0\mathfrak{g} \supset \mathcal{D}^1\mathfrak{g} \cdots \supset \mathcal{D}^{n-1}\mathfrak{g} \supset \mathcal{D}^n\mathfrak{g} \supset \mathcal{D}^{n+1}\mathfrak{g} \supset \cdots$$

where,

$$\mathcal{D}^0\mathfrak{g} := \mathfrak{g} \text{ and } \mathcal{D}^n\mathfrak{g} := [\mathcal{D}^{n-1}\mathfrak{g}, \mathcal{D}^{n-1}\mathfrak{g}].$$

If there exist an $n \in \mathbb{N}$ such that $\mathcal{D}^n\mathfrak{g} = \{0\}$ we say $\mathfrak{g}$ is solvable.

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Corollary 2.3.10. *Every nilpotent Lie algebra is solvable.*

Proof. We prove by induction the statement $\forall n \in \mathbb{N} \cup \{0\}, \mathcal{D}^n\mathfrak{g} \subset \mathcal{C}^n\mathfrak{g}$. .

For $n = 0$,

$$\mathcal{D}^0\mathfrak{g} = \mathcal{C}^0\mathfrak{g},$$

which is clear from the definition of $\mathcal{D}^0\mathfrak{g}$ and $\mathcal{C}^0\mathfrak{g}$. Also, for $n = 1$,

$$\mathcal{D}^1\mathfrak{g} = [\mathcal{D}^0\mathfrak{g}, \mathcal{D}^0\mathfrak{g}] = [\mathfrak{g}, \mathfrak{g}] = \mathcal{C}^1\mathfrak{g}.$$

Therefore,

$$\mathcal{D}^1\mathfrak{g} \subset \mathcal{C}^1\mathfrak{g}$$

.

Suppose $\mathcal{D}^n\mathfrak{g} \subset \mathcal{C}^n\mathfrak{g}$ is true for $n = k$. Then,

$$\mathcal{D}^{k+1}\mathfrak{g} = [\mathcal{D}^k\mathfrak{g}, \mathcal{D}^k\mathfrak{g}] \subset [\mathfrak{g}, \mathcal{C}^k\mathfrak{g}] = \mathcal{C}^{k+1}\mathfrak{g}$$

Thus we have proved that if $\mathfrak{g}$ is nilpotent then there exists an $n \in \mathbb{N}$ such that $\mathcal{D}^n\mathfrak{g} \subset \mathcal{C}^n\mathfrak{g}$, so $\mathfrak{g}$ is solvable. □

Consequently, every nilpotent Lie algebra is solvable but the converse is not true. We state examples of solvable Lie algebras.

Example 2.3.11. The 2 dimensional non- abelian Lie algebra given by $[Y_1, Y_2] = Y_1$ is solvable. Moreover, this Lie algebra is not nilpotent. Now $\mathcal{D}^0\mathfrak{g} = \mathfrak{g}$ and,

$$\begin{aligned} \mathcal{D}^1\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] &= \{[\alpha_1 Y_1 + \alpha_2 Y_2, \beta_1 Y_1 + \beta_2 Y_2,], \alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}\} \\ &= \{(\alpha_1\beta_2 - \alpha_2\beta_1)Y_1 = \gamma Y_1, \gamma \in \mathbb{R}\} \end{aligned}$$

Also,

$$\mathcal{D}^1\mathfrak{g} = [\mathcal{D}^0\mathfrak{g}, \mathcal{D}^0\mathfrak{g}] = \{[\gamma Y_1, \gamma Y_1],\ \gamma \in \mathbb{R}\} = \{0\}$$

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But,

$$\mathcal{C}^1\mathfrak{g} = \{\Gamma Y_1, \Gamma \in \mathbb{R}\}$$
$$\mathcal{C}^2\mathfrak{g} = \{\Gamma' Y_1, \Gamma' \in \mathbb{R}\}$$
$$\mathcal{C}^2\mathfrak{g} = \{\Gamma'' Y_1, \Gamma'' \in \mathbb{R}\}$$
$$\vdots$$

$$\mathcal{C}^n\mathfrak{g} = \{\rho Y_1, \rho \in \mathbb{R}\}, \forall n \in \mathbb{N}$$

Remark 2.3.12. If φ is a Lie algebra homomorphism then $\varphi(\mathcal{D}^n(\mathfrak{g})) = \mathcal{D}^n\varphi(\mathfrak{g})$.

Example 2.3.13. Every abelian Lie algebra is solvable.

We state a corollary on solvable Lie algebras. In addition, we state and proof a proposition using results from the corollary.

Corollary 2.3.14. *[9] Let $\mathfrak{g}$ and $\mathfrak{g}'$ be two Lie algebras and I an ideal of $\mathfrak{g}$.*

(i) If $\mathfrak{g}$ is isomorphic to $\mathfrak{g}'$ then $\mathfrak{g}$ is solvable if and only if $\mathfrak{g}'$ is solvable.

(ii) If $\mathfrak{g}$ is solvable then $\mathfrak{g}/I$ is solvable.

(iii) If I and $\mathfrak{g}/I$ are solvable then $\mathfrak{g}$ is solvable.

Proposition 2.3.15. *The sum of 2 solvable ideals is solvable.*

Proof. Let I and J be two solvable ideals of a Lie algebra $\mathfrak{g}$. There exist a homomorphism,

$$\phi : J \to (I + J)/I.$$

Here, $\ker\phi = I \cap J$. Thus, from the second Isomorphism theorem for Lie algebras we have, $(I + J)/I \cong J/I \cap J$. Now, if a Lie algebra $\mathfrak{g}$ is solvable then given an ideal I, $\mathfrak{g}/I$ is solvable. Thus if J is solvable then $J/I \cap J$ is solvable. Also, if $\mathfrak{g} \cong \mathfrak{g}'$ then $\mathfrak{g}$ is solvable if and only if $\mathfrak{g}'$ is solvable. Thus $(I + J)/I$ is solvable. Again, if I and $\mathfrak{g}/I$

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are solvable then $\mathfrak{g}$ is solvable. Consequently, since $(I+J)/I$ and I are solvable, so $I+J$ is solvable.

□

We define the radical of a Lie algebra and the Killing form.

Definition 2.3.16. (Radical) The largest solvable ideal of a Lie algebra $\mathfrak{g}$ is the radical of $\mathfrak{g}$ denoted by $\mathrm{Rad}(\mathfrak{g})$.

Definition 2.3.17. (Killing form) Let ρ be a representation of a Lie algebra $\mathfrak{g}$ on a finite dimensional vector space V. Then we have a symmetric bilinear form

$$B_\rho(X,Y) = \mathrm{tr}(\rho(X)\rho(Y)), \qquad X, Y \in \mathfrak{g}.$$

If $\rho = \mathrm{ad}$ then

$$B(X,Y) = \mathrm{tr}(\mathrm{ad}X\mathrm{ad}Y),$$

is the Killing form. Thus the Killing form is the symmetric bilinear form associated with the adjoint representation.

We now define simple and semisimple Lie algebras.

Definition 2.3.18. A Lie algebra, $\mathfrak{g}$ is *simple* if it is non-abelian and has no non-trivial ideal (that is, $\{0\}$ and $\mathfrak{g}$ itself are the only ideals of $\mathfrak{g}$). $\mathfrak{g}$ is semisimple if the only abelian ideal is $\{0\}$. Furthermore, a Lie algebra which is isomorphic to a direct sum of simple lie algebras is semisimple.

Theorem 2.3.19. *The 3 properties below are equivalent.*

(i) $\mathfrak{g}$ *is semisimple*

(ii) $Rad(\mathfrak{g}) = \{0\}$

(iii) The Killing form on $\mathfrak{g}$ *is non degenerate.*

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Proof. We prove (i)⇒(ii) by contraposition. Suppose $\mathrm{Rad}(\mathfrak{g}) \neq \{0\}$. Let $S = \mathrm{Rad}(\mathfrak{g})$. Since $\mathrm{Rad}(\mathfrak{g})$ is the maximal solvable ideal, then $\mathcal{D}^k S = [\mathcal{D}^{k-1}S, \mathcal{D}^{k-1}S] = \{0\}$. So $\mathcal{D}^{k-1}S \neq \{0\}$ is an abelian ideal and hence $\mathfrak{g}$ is not semisimple.

We prove (ii)⇒(iii) Let,

$$S = \mathfrak{g}^{\perp} = \{X \in \mathfrak{g} | \forall Y \in \mathfrak{g}, B(X,Y) = 0\}$$

Since S is an ideal and the restriction of B to S vanishes identically, S is solvable. Hence $S = \{0\}$.

Finally, for (iii)⇒(i), Suppose the Killing form of $\mathfrak{g}$ is non-degenerate. Let S be an abelian ideal in $\mathfrak{g}$. For $X \in S$ and $Y, Z \in \mathfrak{g}$,

$$\mathrm{ad}X\mathrm{ad}Y(Z) = [X,[Y,Z]] \in S, \text{ since } [Y,Z] \in \mathfrak{g} \text{ and } S \text{ is an ideal in } \mathfrak{g}\ .$$

Thus, $\mathrm{ad}X\mathrm{ad}Y$ maps $\mathfrak{g}$ into S. Also,$(\mathrm{ad}X\mathrm{ad}Y)^2$ maps $\mathfrak{g}$ into $[S,S] = \{0\}$ Hence, $\mathrm{ad}X\mathrm{ad}Y$ is nilpotent and

$$B(X,Y) = \mathrm{tr}(\mathrm{ad}X\mathrm{ad}Y) = 0.$$

Again, the Killing form is non-degenerate so $S = 0$. This means that we cannot have any non-trivial abelian ideal and hence $\mathfrak{g}$ is semisimple.

□

We provide 2 examples: $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{so}(3)$ and show that they are semisimple Lie algebras.

Example 2.3.20. $\mathfrak{sl}(2,\mathbb{R})$ is semisimple. We will use the following basis of $\mathfrak{sl}(2,\mathbb{R})$.

$$X_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad X_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad X_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

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Let κ denote the Killing form. We want to show that,

$$det(\kappa(X_i, X_j)) \neq 0, \quad 1 \leq i, j \leq 3$$

Now we compute the matrices for $\text{ad}X_1, \text{ad}X_2$ and $\text{ad}X_3$. For $\text{ad}X_1$ we have,

$$\text{ad}X_1(X_1) = [X_1, X_1] = 0$$

$$\begin{aligned}
\text{ad}X_1(X_2) = [X_1, X_2] &= X_1X_2 - X_2X_1 \\
&= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
&= X_3
\end{aligned}$$

Also,

$$\begin{aligned}
\text{ad}X_1(X_3) = [X_1, X_3] &= X_1X_3 - X_3X_1 \\
&= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
&= -2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
&= -2X_1
\end{aligned}$$

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Putting all results together, we have,

$$\mathrm{ad}X_1 = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

We compute $\mathrm{ad}X_2$. Now,

$$\mathrm{ad}X_2(X_1) = [X_2, X_1] = -[X_1, X_2] = -X_3$$

Also,

$$\mathrm{ad}X_2(X_2) = [X_2, X_2] = 0$$

And,

$$\begin{aligned} \mathrm{ad}X_2(X_3) &= [X_2, X_3] = X_2X_3 - X_3X_2 \\ &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \\ &= 2 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ &= 2X_2 \end{aligned}$$

Putting all results together, we have,

$$\mathrm{ad}X_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ -1 & 0 & 0 \end{pmatrix}$$

. We compute $\mathrm{ad}X_3$. Now,

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$$\text{ad}X_3(X_1) = [X_3, X_1] = -[X_1, X_3] = 2X_1$$

$$\text{ad}X_3(X_2) = [X_3, X_2] = -[X_2, X_3] = -2X_2$$

And,

$$\text{ad}X_3(X_3) = [X_3, X_3] = 0$$

Putting all results together, we have,

$$\text{ad}X_3 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So,

$$\text{ad}X_1 = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad \text{ad}X_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ -1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \text{ad}X_3 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We seek to find the matrix of the killing form. Now,

$$\text{ad}X_1\text{ad}X_1 = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ with trace } = 0 \Rightarrow \kappa(X_1, X_1) = 0$$

$$\text{ad}X_1\text{ad}X_2 = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \text{ with trace } = 4 \Rightarrow \kappa(X_1, X_2) = 4$$

$$\text{ad}X_1\text{ad}X_3 = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -2 & 0 \end{pmatrix} \text{ with trace } = 0 \Rightarrow \kappa(X_1, X_3) = 0$$

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$$\text{ad}X_2\text{ad}X_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ with trace } = 0 \Rightarrow \kappa(X_2, X_2) = 0$$

$$\text{ad}X_2\text{ad}X_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix} \text{ with trace } = 0 \Rightarrow \kappa(X_2, X_3) = 0$$

$$\text{ad}X_2\text{ad}X_3 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ with trace } = 8 \Rightarrow \kappa(X_3, X_3) = 8$$

By the symmetric property of the trace of a matrix,

$$\kappa(X_2, X_1) = \kappa(X_1, X_2), \quad \kappa(X_3, X_2) = \kappa(X_2, X_3) \text{ and } \kappa(X_3, X_1) = \kappa(X_1, X_3)$$

Putting all results together the matrix of the killing form is given by:

$$\begin{pmatrix} 0 & 4 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

Now the determinant of the above matrix is,

$$-4 \begin{vmatrix} 4 & 0 \\ 0 & 8 \end{vmatrix} = -4(32) = -128 \neq 0$$

Since

$$det(\kappa(X_i, X_j)) \neq 0, \forall i \geq 1, j \leq 3$$

we conclude that the Killing form on $\mathfrak{sl}(2, \mathbb{R})$ is non-degenerate, hence $\mathfrak{sl}(2, \mathbb{R})$ is a semi simple Lie algebra.

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Example 2.3.21. Consider the Lie algebra $\mathfrak{so}(3)$ with basis,

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad A_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.10}$$

and commutation relations

$$[A_1, A_2] = -A_3, \qquad [A_2, A_3] = -A_1, \qquad [A_3, A_1] = -A_2.$$

We show that $\mathfrak{so}(3)$ is semisimple. Using the commutation relations we find the adjoint representation with respect to the basis. These are,

$$\mathrm{ad}A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \mathrm{ad}A_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \text{ and } \quad \mathrm{ad}A_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Also,

$$\mathrm{ad}A_1\mathrm{ad}A_1 = \mathrm{diag}(0, -1, -1), \quad \mathrm{ad}A_2\mathrm{ad}A_2 = \mathrm{diag}(-1, 0, -1) \text{ and } \quad \mathrm{ad}A_3\mathrm{ad}A_3 = \mathrm{diag}(-1, -1, 0)$$

In addition,

$$\mathrm{ad}A_1\mathrm{ad}A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \mathrm{ad}A_2\mathrm{ad}A_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad \mathrm{ad}A_1\mathrm{ad}A_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

with trace,

$$\mathrm{tr}(\mathrm{ad}A_1\mathrm{ad}A_1) = \mathrm{tr}(\mathrm{ad}A_2\mathrm{ad}A_2) = \mathrm{tr}(\mathrm{ad}A_3\mathrm{ad}A_3) = -2$$
$$\mathrm{tr}(\mathrm{ad}A_1\mathrm{ad}A_2) = \mathrm{tr}(\mathrm{ad}A_2\mathrm{ad}A_3) = \mathrm{tr}(\mathrm{ad}A_1\mathrm{ad}A_3) = 0$$

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Thus the matrix of the killing form is diag$(-2,-2,-2)$ and the determinant

$$-2\begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix} = -8 \neq 0$$

Hence $\mathfrak{so}(3)$ is semisimple.

More generally, a number of classical Lie algebras are semisimple [2]. The special unitary algebra $\mathfrak{su}(n)$ with n not less than 2 and the special orthogonal algebra $\mathfrak{so}(n)$ with n not less than 3 are two of such. Others include,

(i) $\mathfrak{sl}(n,\mathbb{K}), n \geq 2, \quad \mathbb{K} = \mathbb{R}$ or $\mathbb{C}$

(ii) $\mathfrak{sp}(n,\mathbb{K}), n \geq 1, \quad \mathbb{K} = \mathbb{R}$ or $\mathbb{C}$

To conclude the chapter, we define the Casimir Operator.

Definition 2.3.22. (Casimir Operator of a Representation) [9] Let $\{Y_1, \cdots, Y_n\}$ be a basis of $\mathfrak{g}$ and

$$g_{ij} = \beta(Y_i, Y_j), \quad (g^{ij}) = (g_{ij})^{-1}$$

In addition, let π be a representation of $\mathfrak{g}$ on a vector space V. The Casimir operator Ω_π of the representation π is defined by,

$$\Omega_\pi = \sum_{ij=1}^{n} g^{ij}\pi(Y_i)\pi(Y_j)$$

In particular, if β is positive definite and $\{Y_1, \cdots, Y_n\}$ is an orthonormal basis for the Euclidean inner product defined by β then

$$\beta(Y_i, Y_j) = \delta_{ij} \quad \text{and} \quad \Omega_\pi = \sum_{i=1}^{n} \pi(Y_i)^2$$

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Chapter 3

Representation Theory

Here, we discuss the basic representation theory of Lie groups and Lie algebras and explain a method for constructing induced representations of topological locally compact groups, particularly Lie groups.

3.1 Representations of Lie algebras and Lie groups

First, we define representation of a Lie group and a Lie algebra.

Definition 3.1.1. (Representations of a Lie group) Given a Lie group G and a vector space V, a representation Π of G is a Lie group homomorphism,

$$\Pi : G \to \mathrm{GL}(V)$$

i.e it is a map which satisfies the following conditions:

(1) $\forall g_1, g_2 \in G, \quad \Pi(g_1 g_2) = \Pi(g_1)\Pi(g_2)$

(2) $\Pi(e) = I$ where e and I are the identity elements of G and $\mathrm{GL}(V)$ respectively.

Remark 3.1.2. The vector space V may be a real (resp. complex), in which case we obtain a real (resp. complex) representation of V. V may also be finite or infinite

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dimensional. Usually when V is not finite dimensional, we consider V as a Hilbert space and Π as a map from G into the space of bounded operators in V.

Definition 3.1.3. (Representations of a Lie algebra) Let $\mathfrak{g}$ be a Lie algebra and V a vector space. A representation of $\mathfrak{g}$ on V is a Lie algebra homomorphism $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$. Hence $\forall X, Y \in \mathfrak{g},\ \pi[X, Y] = [\pi(X), \pi(Y)] = \pi(X)\pi(Y) - \pi(Y)\pi(X)$.

We say that a representation is *faithful* if it is injective. In addition, from *Ado's theorem* every finite dimensional Lie algebra admits a faithful finite dimensional representation. More specifically, semisimple Lie algebras have a faithful adjoint representation. To explain this we state the fact that the center of a semisimple Lie algebra is trivial and recall that $\ker(\text{ad}) \subset z(\mathfrak{g})$. We state some examples of representations of Lie groups and Lie algebras.

Example 3.1.4. (The Adjoint representation) Given a Lie group G and its Lie algebra $\mathfrak{g}$ the map $\text{Ad} : G \to \text{Aut}(\mathfrak{g})$ defined by $\text{Ad}(g)X = gXg^{-1}$ for all $X \in G$ is the adjoint representation of the Lie group G. In addition, the adjoint map defined in Proposition 2.1.12 is the adjoint representation of the Lie algebra $\mathfrak{g}$.

More precisely, given $\text{Ad} : G \to \text{Aut}(\mathfrak{g})$

$$\text{ad} = (D(\text{Ad}))_e$$

that is,

$$\forall X \in \mathfrak{g},\ \text{ad}X = \frac{d}{dt}\text{Ad}(\exp tX)\Big|_{t=0}.$$

The reason being that given any $Y \in G$ we have,

$$\begin{aligned}
\frac{d}{dt}\text{Ad}(\exp tX)(Y)\Big|_{t=0} &= X\exp(tX)Y\exp(-tX) - \exp(tX)YX\exp(-tX)\Big|_{t=0} \\
&= XY - YX \\
&= [X, Y] \\
&= \text{ad}X(Y)
\end{aligned}$$

Thus the tangent space at the identity of the Adjoint map is,

$$T_e(\text{Ad}) : T_eG \to T_I(\text{Aut}\mathfrak{g}).$$

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This results in the map,

$$ad : \mathfrak{g} \to \mathfrak{D}er(\mathfrak{g})$$

since,

$$T_eG = \mathfrak{g} \text{ and } T_I(\text{Aut}\mathfrak{g}) = \text{Lie}(\text{Aut}(\mathfrak{g}) = \mathfrak{D}er(\mathfrak{g}).$$

We can express this relationship using the commutative diagram below.

$$\begin{array}{ccc} G & \xrightarrow{\text{Ad}} & \text{Aut}(\mathfrak{g}) \\ {\scriptstyle T_e}\downarrow & & \downarrow{\scriptstyle T_I} \\ \mathfrak{g} & \xrightarrow{\text{ad}} & \mathfrak{D}\text{er}(\mathfrak{g}) \end{array}$$

Example 3.1.5. Let's consider a linear Lie group G. The map $G \mapsto \text{GL}(n, \mathbb{C})$ defined by $\Pi(T) = T$ for all $T \in G$ is the standard representation of G. Also, if $\mathfrak{g}$ is a linear Lie algebra then the representation π defined by $\pi(X) = X$ for $X \in \mathfrak{g}$ is the standard representation of $\mathfrak{g}$.

Again, when we take the direct sum and tensor product of existing representations we are able to form new representations. Next, we define the direct sum of two or more representations, the tensor product of representations and the dual representation.

Definition 3.1.6. (Direct sum and Tensor product of representations)
Let $\Pi_1, \Pi_2, \cdots \Pi_m$ (resp. $\pi_1, \pi_2, \cdots \pi_m$) be representations of G(resp. $\mathfrak{g}$) acting on vector spaces $V_1, V_2, \cdots, V_m$ where $V_i,\ i = 1, \cdots, m$ is invariant by $\Pi_i(\text{resp.}\pi_i),\ i = 1, \cdots m$.

(i) The direct sum, $\Pi_1 \oplus \Pi_2 \oplus \cdots \oplus \Pi_m$ (resp. $\pi_1 \oplus \pi_2 \oplus \cdots \oplus \pi_m$) acting on $V_1 \oplus V_2 \oplus \cdots \oplus V_m$ is a representation.

(ii) The tensor product $\Pi_1 \otimes \Pi_2$ of $G_1 \times G_2$ acting on $V_1 \otimes V_2$ is defined as,

$$(\Pi_1 \otimes \Pi_2)(X) = \Pi_1(X) \otimes \Pi_2(X), \quad X \in G.$$

Taking the differential at $t = 0$ we find the corresponding Lie algebra represen-

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tation. That is,

$$\begin{aligned}
\pi_1 \otimes \pi_2(B) &= d\Pi_1 \otimes d\Pi_2(B) \\
&= \frac{d}{dt}\left(\Pi_1(\exp tB) \otimes \Pi_2(\exp tB)\right)\Big|_{t=0} \\
&= d\Pi_1(B) \otimes I + I \otimes d\Pi_2(B).
\end{aligned}$$

for all $B \in \mathfrak{g}$.

Definition 3.1.7. (Dual representation) Let V and V^* be a finite dimensional vector space and the dual space of V respectively. Suppose G (resp. $\mathfrak{g}$) is a Linear Lie group (resp. Lie algebra) and $\Pi : G \to \mathrm{GL}(V)$ (resp. $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$) is a representation of G (resp. $\mathfrak{g}$) acting on V then $\Pi^* : G \to \mathrm{GL}(V^*)$ (resp. $\pi^* : G \to \mathfrak{gl}(V^*)$) defined by $\Pi^*(g) = [\Pi(g^{-1})]^{\mathrm{tr}}$ (resp. $\pi^*(x) = -\pi(X)^{\mathrm{tr}}$) for $g \in G, x \in \mathfrak{g}$ is a representation of G (resp. $\mathfrak{g}$) acting on V^*.

Further, we define the equivalence of representations, discuss the concept of irreducibilty of representations, state some examples of irreducible representations and then define a unitary representation.

Definition 3.1.8. (Equivalence of representations)[14] Let ρ, ρ' be two representations of G or $\mathfrak{g}$. We say that ρ and ρ' are equivalent, if there is an isomorphism $L : V \to V'$ such that $\rho'(X)(Lv) = L(\rho(X)v)$ for every $X \in G$ or $\mathfrak{g}, v \in V$. L is called an intertwining operator.

Definition 3.1.9. (Irreducible representation)[2] Let ρ be a representation of G or $\mathfrak{g}$ on a vector space V. If there are no nontrivial subspaces $W \subset V$ for every $X \in G$ or $\mathfrak{g}$ such that $\rho(X)W \subset W$ (invariant), then we say ρ is irreducible. Otherwise ρ is reducible. Moreover, if a representation is isomorphic to a direct sum of irreducible representations then it is completely reducible.

Example 3.1.10. Let G be a Linear Lie group. The trivial representation $\Pi : G \to GL(1, \mathbb{C})$ (resp. $\pi : \mathfrak{g} \to \mathfrak{gl}(1, \mathbb{C})$) of G (resp. $\mathfrak{g}$) defined by, $\Pi(B) = I$ (resp. $(\pi(X) = 0)$) is irreducible. This is because $\mathbb{C}$ has no nontrivial subspaces.

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Example 3.1.11. The representation $\rho : SL(n, \mathbb{C}) \to \mathbb{C}^n$ is irreducible.

Remark 3.1.12. For a connected Lie group, every representation of its Lie algebra is derived from a representation of the Lie group. In this case, given that Π is a representation of a Lie group, when we find the derivative of Π and evaluate it at the identity, the result is the corresponding Lie algebra representation. This means that if Π is a representation of the Lie group G then,

$$\forall X \in \mathfrak{g}, \; d\Pi(X) = \frac{d}{dt}\Pi(\exp tX)\bigg|_{t=0}$$

where $d\Pi$ is the corresponding Lie algebra representation. Moreover,

(1) $d\Pi$ is irreducible $\Leftrightarrow \Pi$ is irreducible.

(2) $\Pi \sim \Pi' \Leftrightarrow d\Pi \sim d\Pi'$, $\quad$ ($\sim$ connotes equivalence)

Example 3.1.13. Let's take a look at the Heisenberg Lie group,

$$H = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}, \quad (a, b, c) \in \mathbb{R}^3 \right\}.$$

Its Lie algebra is,

$$\text{Lie}(H) = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}, \quad a, b, c \in \mathbb{R}, \forall g \in H \right\}$$

with basis,

$$E = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad G = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

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and V is the space of square integrable functions on $\mathbb{R}$. Given a representation,

$$\Pi_g f(x) = e^{-i\hbar b} e^{-i\hbar cx} f(x-a),$$

where $\hbar \in \mathbb{R}\backslash\{0\}$, we can find $d(\Pi)$ in relation to each basis E, F and G.

Now,

$$E^2 = F^2 = G^2 = 0$$

Also,

$$\exp tX = \sum_{n=0}^{\infty} \frac{t^n X^n}{n!}$$

Then,

$$\exp tE = I + tE = \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\exp tF = I + tF = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$$

$$\exp tG = I + tG = \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Consequently,

$$d\Pi(E) = \frac{d}{dt}\Pi(\exp tE)\Big|_{t=0} = \frac{d}{dt}(f(x-t))\Big|_{t=0} = -f^{'}(x)$$

$$d\Pi(F) = \frac{d}{dt}\Pi(\exp tE)\Big|_{t=0} = \frac{d}{dt}(e^{i\hbar tx} f(x))\Big|_{t=0} = i\hbar x e^{i\hbar tx} f(x)\Big|_{t=0} = i\hbar x f(x)$$

$$d\Pi(G) = \frac{d}{dt}\Pi(\exp tE)\Big|_{t=0} = \frac{d}{dt}(e^{-i\hbar t} f(x))\Big|_{t=0} = -i\hbar e^{i\hbar t} f(x)\Big|_{t=0} = -i\hbar f(x)$$

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Lemma 3.1.14. *(Schur's Lemma)[15] Let $\rho : G \to GL(V_1)$ and $\rho' : G \to GL(V_2)$ be two irreducible representations of a Lie group G over complex vector spaces and let $L : V_1 \to V_2$ be an intertwining map. Then,*

(i) If ρ and ρ' are not isomorphic, we have $L = 0$

(ii) If $V_1 = V_2$ and $\rho = \rho'$, L is a homothety, that is $L = \lambda I, \lambda \in \mathbb{C}$.

Proof. (a) First, we show that ker$L$ and ImL are invariant subspaces of V_1 and V_2 respectively. Now $\ker L = \{u \in V_1 | L(u) = 0\} \subseteq V_1$. Let $s \in G$

$$\forall u \in \ker L, \quad (L\rho_s)(u) = (\rho'_s L)(u) = \rho'_s L(u) = \rho'_s(0) = 0.$$

Therefore, $\rho_s u \in \ker L$ so ker$L$ is invariant.
Similarly, $\mathrm{Im}L = \{v \in V_2 | v = L(u),\ u \in V_1\}$ and

$$\rho'_s v = \rho'_s L(u) = L(\rho_s u) = L(w) \in V' = \mathrm{Im}L,\ \text{where } w = \rho_s u$$

Therefore, ImL is an invariant subspace of V_2. Now, since ρ_1 and ρ_2 are irreducible, the only invariant subspaces of V_1 and V_2 are $0, V_1$ or V_2. Hence $\ker L = 0$ or $\ker L = V_1$ and $\mathrm{Im}L = 0$ or $\mathrm{Im}L = V_2$. We look out for possibilities that may arise.
What happens when $\ker L = 0$? From the first isomorphism theorem ImL is isomorphic to $V_2/\ker L$. In this case, the only possibility for the image of L is V_2. Hence L is an isomorphism.
Again, by the first isomorphism theorem when $\ker L = V_1$ the only possibility for the image of L is 0. Therefore L is a zero map.

(b) From (ii) $L : V \to V$ and $\rho_s \circ L = L \circ \rho_s, \forall s \in G$. Let λ be an eigenvalue of L. Then $L - \lambda I$ is not invertible so it is not an isomorphism. Hence $L - \lambda I = 0 \Rightarrow L = \lambda I$. So L is a homothety.

□

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Remark 3.1.15. Lemma 3.1.14 works for Lie algebras.

A consequence of Schur's Lemma with respect to the Casimir operator defined in 2.3.22 is that, if ρ is a $\mathbb{C}$ linear irreducible representation of a Lie algebra $\mathfrak{g}$ on a finite dimensional complex vector space V, then $\exists k_\rho \in \mathbb{C}$ such that, $\Omega_\rho = -k_\rho I$.

Definition 3.1.16. (Unitary representation)[9] Let $\mathbf{H}$ be a Hilbert space and let G be a Lie group. A representation Π of G on $\mathbf{H}$ is said to be unitary if for every $g \in G$, $\Pi(g)$ is a unitary operator. That is, $\forall g \in G, \forall v \in \mathbf{H}, ||g \cdot v|| = ||v||$ where $g \cdot v = \Pi(g)v$ and $|| \cdot ||$ is the norm induced from the inner product on $\mathbf{H}$.

Some definitions state that $\langle g \cdot v_1,\ g \cdot v_2 \rangle = \langle v_1,\ v_2 \rangle, \quad v_1, v_2 \in \mathbf{H}$

Example 3.1.17. The standard representation of $SU(n)$ on $\mathbb{C}^n$ is unitary.

Remark 3.1.18. More generally, any representation of a compact group is unitarizable. That is, for any compact group we can construct a unitary representation Π of the compact group on a Hilbert space H.

3.1.19 Representations of Semisimple Lie algebras

Recall that a completely reducible representation is isomorphic to a direct sum of irreducible representations. This property of complete reducibilty enables us to study representations by making use of irreducible representations which are much easier to work with. We state a theorem regarding semisimple Lie algebras and complete reducibility.

Theorem 3.1.20. *[2] Every finite dimensional representation of a semisimple Lie algebra $\mathfrak{g}$ is completely reducible.*

Proof. Though there is a purely algebraic proof to the above theorem we will make use of the famous unitary trick by Hermann Weyl as seen in [4, 16].

Let $\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C}$ be a semisimple Lie algebra. In this case, $\mathfrak{g}_0$ is the real form of $\mathfrak{g}$ and its simply connected form is a compact Lie group G. When we restrict a given representation of $\mathfrak{g}_\mathbb{C}$ to $\mathfrak{g}$ we can take the expoent of that representation to

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get a representation of G which satisfies the property of complete reducibility as such we can deduce from this the complete reducibility of the original representation (representation of $\mathfrak{g}$). $\square$

The Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ is very useful in the study of other semisimple Lie algebras. It is a well known fact that embedded in every semisimple Lie algebra are copies of $\mathfrak{sl}(2, \mathbb{C})$. Consequently, when we study the representations of $\mathfrak{sl}(2, \mathbb{C})$ we are able to understand the representations of semisimple Lie algebras much better. Recall that, elements of $\mathfrak{sl}(2, \mathbb{C})$ are traceless two by two matrices. We write the following basis of $\mathfrak{sl}(2, \mathbb{C})$ and its commutation relations. They are

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$[H, X] = HX - XH = 2X \qquad [H, Y] = HY - YH = -2Y \qquad [X, Y] = XY - YX = H$$

Let's take V as an irreducible finite dimensional representation of $\mathfrak{sl}(2, \mathbb{C})$. H is a diagonal matrix. As such we observe that the action of H on V is diagonalizable. Thus we can split V into eigenspaces V_α where

$$V_\alpha = \{v \in V | Hv = \alpha v, \alpha \in \mathbb{C}\}.$$

Therefore we have

$$V = \bigoplus_{\alpha \in \mathbb{C}} V_\alpha.$$

Moreover, the action of H on Xv and Yv implies that $X : V_\alpha \to V_{\alpha+2}$ and $Y : V_\alpha \to V_{\alpha-2}$ since

$$HXv = XHv + [H, X]v = X(\alpha v) + 2Xv = (\alpha + 2)Xv$$

and

$$HYv = YHv + [H, Y]v = Y(\alpha v) - 2Yv = (\alpha - 2)Yv$$

For any semisimple Lie algebra we can have a collection of elements which behave like H, that is they are diagonalizable. This set of elements commute, and are *simultane-*

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ously diagonalizable. These elements, called semisimple elements, form a subalgebra of G which we call a Cartan subalgebra.

Definition 3.1.21. (Cartan Subalgebra) [3] A maximal commutative subalgebra of diagonalizable elements of a semisimple Lie algebra $\mathfrak{g}$ is called a Cartan subalgebra of $\mathfrak{g}$. In addition, consider $\mathfrak{h}$ as a Cartan subalgebra and the set A as a set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Then the Lie algebra decomposes as

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in A} \mathfrak{g}_\alpha \text{ (Cartan Decomposition)}$$

with $\mathfrak{g}_\alpha$ being the root space which corresponds to α.

Example 3.1.22. In the case of $\mathfrak{sl}(2, \mathbb{C}), mH, m \in \mathbb{C}$ is a Cartan subalgebra since mH consists of diagonal matrices in $\mathfrak{sl}(2, \mathbb{C})$ and diagonal matrices commute. Also, every semisimple Lie algebra possesses a Cartan subalgebra [3]. Here, the Cartan decomposition for every element $M \in \mathfrak{sl}(2, \mathbb{C})$ is given by,

$$M = \beta_1 H + \beta_2 X + \beta_3 Y, \text{ where } \beta_1, \beta_2, \beta_3 \in \mathbb{C}$$

Let's take $\mathfrak{h}$ as a Cartan subalgebra of $\mathfrak{g}$. We may use $\mathfrak{h}$ to decompose $\mathfrak{g}$ by means of the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$, $\text{ad} : \mathfrak{h} \to \text{gl}(\mathfrak{g})$. Define

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} | [H, X] = \alpha(H)X, \; H \in \mathfrak{h}\}.$$

Since elements of $\mathfrak{h}$ are simultaneously diagonalizable, $\alpha : \mathfrak{h} \to \mathbb{C}$ is well-defined, and is called a generalized eigenvalue of $\mathfrak{h}$. Generalized eigenvalues corresponding to the adjoint action are known as roots of $\mathfrak{g}$. Therefore $\alpha \in \mathfrak{h}^*$ is a root of $\mathfrak{g}$ provided that

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g} | [H, X] = \alpha(H)X\} \neq 0.$$

We see that,

$$\mathfrak{g}_0 = \{X \in \mathfrak{g} | [H, X] = 0, \forall H \in \mathfrak{h}\}.$$

Thus, $\mathfrak{g}_0$ is the centralizer of $\mathfrak{h}$. If $\mathfrak{g}$ is semisimple, it can be shown that H is its own centralizer. Moreover, if $\mathfrak{g}$ is finite dimensional, then the set of roots is finite. We can summarize the above information in the following definition.

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Definition 3.1.23. [17] Let $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of the semisimple Lie algebra $\mathfrak{g}$ and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. Then,

(i) A weight of the representation π is a linear functional $\alpha \in \mathfrak{h}^*$ such that there exist a vector $v \in V\backslash\{0\}$ which satisfies the relation $\pi(x)v = \alpha(x)v$ for $x \in \mathfrak{h}$.

(ii) The set of all elements v in V that satisfy the above relation is called the weight space associated to α and usually denoted by V_α. The multiplicity of α in π is the dimension of V_α.

(iii) The **weights** of the adjoint representation of $\mathfrak{g}$ that are **non-zero** are referred to as the **roots** of $\mathfrak{g}$. In addition, the **weight space** associated to a **root** is known as the **root space** of α.

(iv) A highest weight of π is that weight which corresponds to a vector $v \in V\backslash\{0\}$ contained in a certain weight space of V such that,

$$g_\alpha . v = 0$$

is true for all positive roots α of g. This vector is none other than the highest weight vector.

Example 3.1.24. 2 and -2 are the roots of $\mathfrak{sl}(2,\mathbb{C})$. This is so, because $[H, X] = 2X$ and $[H, Y] = -2Y$.

Remark 3.1.25. The irreducible representations of $\mathfrak{sl}(2,\mathbb{C})$ could be applied to the representations of (connected/ simply connected) Lie groups whose Lie algebras have $\mathfrak{sl}(2,\mathbb{C})$ as their complexification.

Example 3.1.26. We consider the 8 dimensional semisimple Lie algebra $\mathfrak{sl}(3,\mathbb{C})$. We write the following basis of the aforementioned Lie algebra.

$$X_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad X_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad X_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

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$$Y_1 = X_1^T \qquad Y_2 = X_2^T \qquad Y_3 = X_3^T \qquad H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

In this case, the Cartan subalgebra is,

$$\mathfrak{h} = \{m_1 H_1 + m_2 H_2, \text{`} \quad m_1, m_2 \in \mathbb{C}\}$$

For an elementary matrix E_{ij} and a diagonal matrix $H \in \mathfrak{h}$,

$$[H, E_{ij}] = (a_i - a_j)E_{ij}, \quad a_i : H \to a_i H.$$

Thus the one dimensional subspace $\mathbb{C}E_{ij}$ is a simultaneous eigenspace for all elements H in the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{sl}(3, \mathbb{C})$ with eigenvalues given by the weight $\gamma_i - \gamma_j \in \mathfrak{h}^*$ Also,

$$[H_i, E_j] = \alpha_j(H_i)E_j, 1 \leq i, j \leq 3$$

$$[H_i, F_j] = -\alpha_j(H_i)E_j, 1 \leq i, j \leq 3$$

The linear functionals $\alpha_j, -\alpha_j, 1 \leq j \leq 3$ are the roots of $sl(3, \mathbb{C})$. With respect to the weight, $\gamma_i - \gamma_j \in \mathfrak{h}^*$,

$$\alpha_1 = \gamma_1 - \gamma_2, \quad \alpha_2 = \gamma_2 - \gamma_3, \quad \alpha_3 = \gamma_1 - \gamma_3$$

Thus the roots of $\mathfrak{sl}(3, \mathbb{C})$ are,

$$\{\gamma_1 - \gamma_2,\ \gamma_2 - \gamma_3,\ \gamma_1 - \gamma_3,\ \gamma_2 - \gamma_1,\ \gamma_3 - \gamma_2,\ \gamma_3 - \gamma_1\}$$

Remark 3.1.27. When we analyze the roots and weights of a semisimple Lie algebra, find the interaction between their root and weight spaces, the multiplicities of their weights and the highest weight we can then classify the irreducible representations of the semisimple Lie algebra.

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3.2 Induced representations

Given a "canonical" representation of a closed subgroup K of any topological locally compact group G, we can construct a representation of G with the known representation of K. Representations of a group that are constructed from a known representation of its subgroup are called *induced representations*. We note that finite dimensional Lie groups are locally compact. This is because they are locally homeomorphic to the euclidean space which is locally compact. Moreover, all Lie groups are topological in nature. In this section, we define and apply one of the methods for constructing induced representations of topological locally compact groups. We will also state some examples of induced representations for some Lie groups and take a look at the induced representations of the semidirect product of two topological and locally compact groups.

The results in this section are adapted from Barut[16]. Before we proceed to define the concept of induced representations we would recall some key terms needed in the definition.

(1) A topological group G is a group, a topological space and whose group operations xy and x^{-1} are continuous for all $x, y \in G$. Further, G is a locally compact group if the topology on it is Hausdorff and locally compact.

(2) Let G be a group. X is referred to as a homogeneous space if for every pair of points x_1, x_2 in X there exists an element $g \in G$ such that $x_2 = gx_1$. This means that G acts transitively on X.

(3) We say that a measure $d\mu(x)$ on a homogeneous space X is quasi-invariant if $d\mu(xg)$ is equivalent to $d\mu(x)$ for every $g \in G$.

(4) Given a quasi-invariant measure $d\mu(x)$, there exist a function $\Gamma(x)$ such that $d\mu(xg) = \Gamma(x)d\mu(x)$, with $\Gamma(x) \geq 0$. $\Gamma(x) = \dfrac{d\mu(xg)}{d\mu(x)}$ is called the Radon-Nikodym derivative.

Definition 3.2.1. [16]Let G be a topological locally compact group, K a closed subgroup of G, $L : k \to L_k$ a unitary representation of K on a Hilbert space H and μ a quasi-invariant measure in the homogeneous space $X = K\backslash G = \{Kg, g \in G\}$ of

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the right K - cosets. Consider the set H^L of all functions u with domain in G and range H satisfying the following conditions:

(i) $g \mapsto \langle u(g), v \rangle$ is measurable for all $v \in H$.

(ii) $u(kg) = L_k u(g)$ for all $k \in K$ and all $g \in G$.

(iii) $\int_X ||u(g)||^2 d\mu(\dot{g}) < \infty, \dot{g} = Kg$ where $||u(g)||$ is the norm in the space H.

These functions H^L form a Hilbert space with respect to the scalar product

$$(u_1, u_2)_{H^L} = \int \langle u_1(g), u_2(g) \rangle_H d\mu(g)$$

The map $g_0 \to U^L_{g_0}$ defined by,

$$U^L_{g_0} u(g) = \left(\frac{d\mu(\dot{g}g_0)}{d\mu(\dot{g})} \right)^{\frac{1}{2}} u(gg_0) \tag{3.1}$$

is a representation of G in H^L where $\frac{d\mu(\dot{g}g_0)}{d\mu(\dot{g})}$ is the Radon-Nikodym derivative of the quasi-invariant measure $d\mu$ in X.

In addition, the Mackey Decomposition theorem is relevant in the concept of induced representations of a topological group G. It states that, for a locally compact group G, there exists a Borel set S in G such that every element $g \in G$ can be written as

$$g = ks, k \in K, \ s \in S.$$

Remark 3.2.2. $H^L \cong L^2(X, \mu, H)$. Given that $u(g) \in L^2(X, \mu, H)$, this isomorphism is represented by $u(g) = L_{k_g} \tilde{u}(\dot{g})$ where $\dot{g} = Kg$. Here, k_g is the factor of g in the Mackey decomposition $g = k_g s_g$.

For more details on Remark 3.2.2 check Barut [16]. Alternatively, we can define the induced representation with respect to the Mackey decomposition. We denote the element of G that represents the coset Kg by s_g and the factor of the Mackey

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decomposition in Remark 3.2.2 of the element $s_g g_0$ by $k_{s_g g_0}$. We note that $k_{s_g g_0} \in K$. Also,

$$Kg = Kk_g s_g = Ks_g$$

Moreso, Definition 3.2.1 can be re-defined in a different way when we consider an operator $g \to B_g$ from G into the set of unitary operators in H which satisfy a number of properties.

Definition 3.2.3. [16] Let $g \to B_g$ be an operator function from G into the set of unitary operators in H which satisfies the following properties.

(i) $B_{kg} = L_k B_g, \forall k \in K$ and all $g \in G$

(ii) For each pair of $u, v \in H$ the function $g \to \langle B_g u, v \rangle$ is dg measurable.

Also, Let K be a closed subgroup of G and let $k \to L_k$ be a unitary representation of K in H. The map $g_0 \to Ug_0^L$ given by

$$U_{g_0}^L u(\dot{g}) = \left(\frac{d\mu(\dot{g}g_0)}{d\mu(\dot{g})} \right)^{\frac{1}{2}} B_g^{-1} B_{gg_0} u(\dot{g}g_0) \tag{3.2}$$

provides a unitary representation of G in $L^2(X, \mu, H)$. Further, if we substitute L_{kg} for B_g and x for $\dot{g}$ then we would have,

$$U_{g_0}^L u(x) = \left(\frac{d\mu(xg_0)}{d\mu(x)} \right)^{\frac{1}{2}} L_{k_{s_g g_0}} u(xg_0) \tag{3.3}$$

Equations 3.1 and 3.3 provide a way of constructing induced representations for G given representations of K, a closed subgroup of G. We consider some examples.

Example 3.2.4. Let $G = \mathrm{SL}(2, \mathbb{R})$. $\mathrm{SL}(2, \mathbb{R})$ is a locally compact Lie group though a non compact group as such it is a topological locally compact group. Recall that for all $g \in G$,

$$g = \begin{pmatrix} a & z \\ y & w \end{pmatrix}, \text{ such that } aw - zy = 1$$

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Consider the subgroup K of G consistsing of upper triangular matrices $K \in G$ which can be expressed in the form,

$$k = \begin{pmatrix} \alpha & \beta \\ 0 & \frac{1}{\alpha} \end{pmatrix}, \ \alpha \neq 0.$$

We consider one dimensional unitary representations of the subgroup K, of the form

$$L_k \equiv L_{(\beta,\alpha)} = |\alpha|^{i\sigma} \left(\frac{\alpha}{|\alpha|}\right)^{\varepsilon}, \quad \sigma \in \mathbb{R}, \varepsilon = 0, \text{or } 1 \tag{3.4}$$

We notice that the carrier space H of L is $\mathbb{C}^1$, so the carrier space of U^L is $H^L = L^2(X, \mu)$ where X, μ is as defined in Definition 3.2.1.

Using the Mackey decomposition $g = k_g s_g$ we seek to find the realization of the homogeneous space X and how G acts on X.

Now, with regards to matrices $k \in K$, we can write k_g in two ways, that is,

(a) $k_g = \begin{pmatrix} \frac{1}{w} & z \\ 0 & w \end{pmatrix}$

(b) $k_g = \begin{pmatrix} z & -a \\ 0 & \frac{1}{z} \end{pmatrix}$

Then given $g = \begin{pmatrix} a & z \\ y & w \end{pmatrix}$, we have

(a)

$$g = \begin{pmatrix} \frac{1}{w} & z \\ 0 & w \end{pmatrix} \begin{pmatrix} p_1 & q_1 \\ r_1 & m_1 \end{pmatrix} = \begin{pmatrix} \frac{p_1}{w} + zr_1 & \frac{q_1}{w} + zm_1 \\ wr_1 & wm_1 \end{pmatrix} \tag{3.5}$$

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(b)

$$g = \begin{pmatrix} z & -a \\ 0 & \frac{1}{z} \end{pmatrix} \begin{pmatrix} p_2 & q_2 \\ r_2 & m_2 \end{pmatrix} = \begin{pmatrix} zp_2 - ar_2 & zq_2 - am_2 \\ \frac{r_2}{z} & \frac{m_2}{z} \end{pmatrix} \tag{3.6}$$

From Equation 3.5,

$$r_1 = \frac{y}{w}, \ wm_1 = w \Rightarrow m_1 = 1, \ \ p_1 = aw - zy = 1 \text{ and } q_1 = zw(1 - m_1) \Rightarrow q_1 = 0$$

Also from Equation 3.6 we obtain,

$$p_2 = m_2 = 0, \ \ q_2 = 1 \text{ and } r_2 = -1$$

Therefore all elements

$$g = \begin{pmatrix} a & z \\ y & w \end{pmatrix} \in G$$

with respect to k_g could be written in the form

$$g = \begin{pmatrix} \frac{1}{w} & z \\ 0 & w \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{y}{w} & 1 \end{pmatrix}$$

and

$$g = \begin{pmatrix} z & -a \\ 0 & \frac{1}{z} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

This implies that for every $k_g \in K$ we have s_g such that

$$s_g = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \text{ or } s_g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \equiv S_0, \ \ x \in \mathbb{R}.$$

We choose

$$s_g = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$$

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in this case since KS_0 has the measure zero in $K\backslash G = X$. Then,

$$\forall x \in X, x \equiv \dot{g} = K_g = Kk_g s_g = Ks_g.$$

Hence there is a one to one correspondence between the points in X and the points in $\mathbb{R}$. To find the action of $g_0 \in G$ in X on s_g we have,

$$\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \begin{pmatrix} a_0 & z_0 \\ y_0 & w_0 \end{pmatrix} = \begin{pmatrix} a_0 & z_0 \\ a_0 x + y_0 & z_0 x + w_0 \end{pmatrix} = \begin{pmatrix} (z_0 x + w_0)^{-1} & z_0 \\ 0 & z_0 x + w_0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \dfrac{a_0 x + y_0}{z_0 x + w_0} & 1 \end{pmatrix}$$

This action is given by,

$$x \to x g_0 = \frac{a_0 x + y_0}{z_0 x + w_0}$$

Finding the quasi invariant measure $d\mu(x)$ on X we have

$$d(xg_0) = (z_0 x + w_0)^{-2} dx$$

Thus, the Radon-Nikodym derivative is given by,

$$\frac{d(xg_0)}{dx} = (z_0 x + w_0)^{-2}$$

Then,

$$k_{s_g g_0} = \begin{pmatrix} (z_0 x + w_0)^{-1} & z_0 \\ 0 & z_0 x + w_0 \end{pmatrix}$$

Also, from Equation 3.4,

$$L_{k_{s_g g_0}} = |z_0 x + w_0|^{-i\sigma} \left(\frac{z_0 x + w_0}{|z_0 x + w_0|} \right)^{\varepsilon}.$$

Hence, the induced representations of $\mathrm{SL}(2, \mathbb{R})$ here are of the form

$$U_g^L u(x) = |z_0 x + w_0|^{-i\sigma - 1} \left(\frac{z_0 x + w_0}{|z_0 x + w_0|} \right)^{\varepsilon} u \left(\frac{a_0 x + y_0}{z_0 x + w_0} \right)$$

Remark 3.2.5. These induced representations are irreducible except for cases where

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$\varepsilon = 1$ and $\sigma = 0$

3.2.6 Induced Representations of Semi-direct Products

In this subsection, we provide a method for constructing induced representations of semi-direct products.

Let N be a locally compact abelian group and S a locally compact group. Also let G be a semi-direct product of these two groups such that G in itself is a locally compact group. The composition law of G in this case is given by

$$(n_1, s_1)(n_2, s_2) = (n_1 + s_1 n_2, s_1 s_2) \qquad \text{for } n_1, n_2 \in N \text{ and } s_1, s_2 \in S.$$

Let us consider the following notation:

(a) $\hat{N}$ denotes the dual space of N.

(b) $\hat{O}$ represents an orbit.

(c) $S_{\hat{o}} = \{s \in S | \hat{n}_o \cdot s = \hat{n}_o\}$, where $\hat{n}_o$ is that element of $\hat{O}$ contained in $N \rtimes S_{\hat{o}}$.

(d) Given a Hilbert space H, we denote an irreducible unitary representation of $S_{\hat{o}}$ on H by L and a representation of $N \rtimes S_{\hat{o}}$ by $\hat{L}$.

(e) The quasi-invariant measure in $\hat{O}$ is denoted by μ.

Let W be a unitary representation of G that is irreducible. Then we can have an orbit $\hat{O}$ in the dual space of the abelian group N that corresponds to W.

Remark 3.2.7. An irreducible representation of a semi-direct product is unitarily equivalent to a representation of the group obtained from the semi-direct product induced by a representation of the subgroup of the semi-direct product.

We state a theorem without proof. For proof see [16].

Theorem 3.2.8. *[16] Let $G = N \rtimes S$, $\hat{n}L$ be an irreducible unitary representation of the subgroup $N \rtimes S_{\hat{o}}$ and $\hat{O}$ be an orbit in $\hat{N}$. Then,*

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(1) We can have a corresponding induced representation $U^{\hat{n}L}$ with respect to every orbit and the related subgroup $N \rtimes S_{\hat{o}}$.

(2) The representation $U^{\hat{n}L}$ realized in $H^{\hat{n}L} = L^2(\hat{O}, \mu, H)$. is given by,

$$U^{\hat{n}L}_{(n,s)}u(\hat{n}) = \langle n, \hat{n} \rangle_S U^L_s u(\hat{n}) \tag{3.7}$$

where ${}_SU^L$ is a representation of S given by Equation 3.3. This representation is induced by the representation of the subgroup $S_{\hat{o}}$ which is a subset of S.

Example 3.2.9. Let $G = \mathbb{R}^3 \rtimes SO(3)$. So, $N = \mathbb{R}^3, \quad S = SO(3)$. Now, $\hat{N} \cong \mathbb{R}^3$ since $\mathbb{R}^3$ is a non compact. Also, we can have a unitary character of the form,

$$\langle n, \hat{n} \rangle = \exp i(n_1\hat{n}_1 + n_2\hat{n}_2 + n_3\hat{n}_3), \quad \text{for } n = (n_1, n_2, N_3) \in N \text{ and } \hat{n} = (\hat{n}_1, \hat{n}_2, \hat{n}_3).$$

In addition the orbit is given by,

$$\hat{O} = \{\hat{n}_o s \; | \hat{n}_o \in \hat{N}, s \in SO(3)\}.$$

This particular orbit corresponds to a character $\hat{n}_o$. With respect to each character in $\hat{N}$ there is an orbit. Thus we have a collection of them. These represent spheres with center $(0, 0, 0)$ and radius $r \geq 0$. Hence, with regards to Remark 3.2.7 and Theorem 3.2.8, each irreducible unitary representations of $\mathbb{R}^3 \rtimes SO(3)$ is induced by an irreducible representation of the subgroup $N \rtimes S_{\hat{o}}$ corresponding to the orbits $r = 0, r > 0$. We consider these two cases.

Case 1 (Radius equals 0)
Here we have, $\hat{n}_o = (0, 0, 0)$. In this case, $S_{\hat{o}}$ is nothing but the group $SO(3)$. Then the irreducible induced representations of $\mathbb{R}^3 \rtimes SO(3)$ associated with this orbit are finite dimensional irreducible unitary reps of $SO(3)$ that are in $\mathbb{R}^3 \rtimes SO(3)$.

Case 2 (Radius greater than 0)
Here we have, $\hat{n}_o = (0, 0, r)$. In this case $S_{\hat{o}}$ is isomorphic to the space of 3×3 matrices

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of the special orthogonal group.

As such, the irreducible representations of $S_{\hat{o}}$ are of the form,

$$L : \phi \to \exp(il\phi), \quad \phi \in [0, 2\pi], \; l = 0, \pm 1, \pm 2, \cdots$$

Also,

$$H = \mathbb{C} \text{ and } \hat{O} = S_{\hat{o}} \backslash S = SO(2) \backslash SO(3) \cong S^2$$

The measure μ on $\hat{O}$ is the usual quasi-invariant measure associated with rotations on the sphere S^2. The induced irreducible representation is given by,

$$U^{\hat{n}L}_{(n,s)} u(\hat{n}) = \exp[i(n_1\hat{n}_1 + n_2\hat{n}_2 + n_3\hat{n}_3)]_S U^L_s u(\hat{n}), \quad n \in N, s \in S, \hat{n} \in S^2$$

where ${}_S U^L$ is the induced representation of SO(3). Here the representations L of SO(2) are used to obtain the induced representations.

In the following chapter, we consider an application of these results and some known applications of Lie groups and Lie algebras in Physics.

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Chapter 4

Applications

In this chapter we discuss how the induced representations of the Poincaré group are applied in Physics and other applications of Lie groups and Lie algebras in Physics.

4.1 An application of Induced Representations of the Poincaré group

4.1.1 The Lorentz group

Definition 4.1.2. Consider the Minkowski space $\big(M, \langle , \rangle\big)$ where for all $v = \big\{v_u\big\}_{u=0}^{3}$ and $w = \big\{w_u\big\}_{u=0}^{3}$,

$$\langle v, w\rangle = v_0 w_0 - v_1 w_1 - v_2 w_2 - v_3 w_3$$

The Lorentz group comprises a collection of linear transformations of M that preserves the Minkowski metric. This is given by,

$$L = \{\Lambda \in \mathrm{GL}(4, \mathbb{R}) | \Lambda^T g \Lambda = g\}, \qquad \text{where } g = \mathrm{diag}(1, -1, -1, -1).$$

The Lorentz group has 4 components. These are, $L_{+}^{\downarrow}, L_{-}^{\downarrow}, L_{+}^{\uparrow}$ and $L_{-}^{\uparrow}$. Among these

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components, only $L_+^\uparrow$ is a subgroup of L. It is called the proper orthochronous or restricted Lorentz group [18]. It is actually an invariant subgroup of L since for any 2 elements $\Lambda \in L$ and $A \in L_+^\uparrow$, $\Lambda A \Lambda^{-1} \in L_+^\uparrow$.

Proposition 4.1.3. *The universal covering group of* $L_+^\uparrow$ *is* $\mathrm{SL}(2, \mathbb{C})$.

Proof. Let F represent the space of Hermitian matrices,

$$F = \{E \in \mathrm{M}(2, \mathbb{C}) : E = E^*\}$$

where E^* denotes the complex conjugate transpose of E. The Minkowski space can be seen as the space of 2×2 Hermitian matrices, since for an element (v_0, v_1, v_2, v_3) in the Minkowski space M, the corresponding element in F is represented by,

$$E = \begin{pmatrix} v_0 + v_3 & v_1 - iv_2 \\ v_1 + iv_2 & v_0 - v_3 \end{pmatrix}, \quad \text{with det } E = {v_0}^2 - {v_1}^2 - {v_2}^2 - {v_3}^2.$$

This means that any vector in M could be associated to a 2×2 Hermitian matrix. That is,

$$(v_0, v_1, v_2, v_3) \mapsto \begin{pmatrix} v_0 + v_3 & v_1 - iv_2 \\ v_1 + iv_2 & v_0 - v_3 \end{pmatrix}$$

The map $M \mapsto F$ is linear and injective. Under this map the standard basis $\{(1, 0, 0, 0)\ (0, 1, 0, 0),\ (0, 0, 1, 0),\ (0, 0, 0, 1)\}$ have respective images,

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \text{ and } \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Also, σ_1, σ_2 and σ_3 are called Pauli matrices. This suggests that a point (v_0, v_1, v_2, v_3) in M corresponds to $v_0\sigma_0 + v_1\sigma_1 + v_2\sigma_2 + v_3\sigma_3$ in F. Conversely, any 2×2 Hermitian matrix could be associated to an element in the Minkowski space. For two matrices, A and C we can define an inner product,

$$\left\langle A, C \right\rangle = \frac{1}{2}\mathrm{tr}(AC).$$

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From this inner product we note that the basis are orthonormal. As such we can have an inverse map, $F \mapsto M$ so that for $E \in F$,

$$E \mapsto \Big(\Big\langle E, \sigma_0 \Big\rangle, \Big\langle E, \sigma_1 \Big\rangle, \Big\langle E, \sigma_2 \Big\rangle, \Big\langle E, \sigma_3 \Big\rangle\Big) \in M.$$

Now, we consider $\lambda \in \mathrm{SL}(2, \mathbb{C})$ and let $\mathrm{SL}(2, \mathbb{C})$ act on F such that,

$$E \mapsto \lambda E \lambda^* \in F$$

Thus there is a homomorphism $\lambda \mapsto L_\lambda$ from $\mathrm{SL}(2, \mathbb{C})$ to a subgroup of the Lorentz group L. Moreover, $\mathrm{SL}(2, \mathbb{C})$ is a simply connected manifold and the map $\lambda \mapsto L_\lambda$ is continuous thus we have the homomorphism,

$$\varphi : \mathrm{SL}(2, \mathbb{C}) \to L_+^\uparrow$$

since $L_+^\uparrow$ is the only proper connected subgroup of L. Now,

$$\ker(\varphi) = \{\lambda \in \mathrm{SL}(2, \mathbb{C}) |\ \lambda E \lambda^* = E,\ E \in F\}.$$

In particular, for $E = I$, we notice that,

$$\lambda \lambda^* = I \implies \lambda = \lambda^{-1}.$$

Thus,

$$E\lambda = \lambda E \implies \lambda = aI, a \in \mathbb{C}.$$

Again, since $\lambda \in \mathrm{SL}(2, \mathbb{C})$ its determinant is 1. As a result, we can only have $\lambda = \pm I$. Hence, $\ker(\varphi) = \{\pm I\}$. Thus, we have the isomorphism

$$\mathrm{SL}(2, \mathbb{C}) \Big/ \{\pm I\} \cong L_+^\uparrow.$$

This informs us that $\mathrm{SL}(2, \mathbb{C})$ is a double cover of $L_+^\uparrow$. Moreover, $\mathrm{SL}(2, \mathbb{C})$ is the

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universal cover of the proper orthochronous Lorentz group $L_+^\uparrow$ since it is simply connected. □

4.1.4 The Poincaré group

The Poincaré group is a 10 dimensional non abelian non compact Lie group of the Minkowski space isometries.

Definition 4.1.5. The Poincaré group is the semi direct product of the translations of M and the Lorentz group. We denote it by denoted by $\Pi = \Pi^4 \rtimes L$.
Thus for any element (n, L) in the Poincaré group the composition law is given by,

$$(n_1, L_1)(n_2, L_2) = (n_1 + L_1 n_2, L_1 L_2), \quad n_1, n_2 \in \Pi^4 \text{ and } L_1, L_2 \in L.$$

The Poincaré group has four components which correspond to the components of the Lorentz group. These are, $\Pi_+^\downarrow, \Pi_-^\downarrow, \Pi_+^\uparrow$ and $\Pi_-^\uparrow$. We would consider $\Pi_+^\uparrow = \Pi^4 \rtimes L_+^\uparrow$ as it is relevant to the first application we would outline here. In addition, the semi direct product $\Pi^4 \rtimes \mathrm{SL}(2, \mathbb{C})$ is the universal covering group of the Poincaré group. Its composition law is defined by ,

$$(n_1, \Lambda_1)(n_1, \Lambda_2) = (n_1 + L_{\Lambda_1} n_2, \Lambda_1 \Lambda_2)$$

4.1.6 Induced Representations of the Poincaré Group

The concepts in this subsection are adapted from [16]. It is quite easier to work with the universal covering group of a group than the group itself. Moreover the representations of the universal covering group of a Lie group are isomorphic to those of the group itself. Given $\widetilde{\Pi} = \Pi^4 \rtimes \mathrm{SL}(2, \mathbb{C})$, let $N = \Pi^4$ (Translation group). We note the following,

(i) For every $\hat{n} = (\hat{n}_0, \hat{n}_1, \hat{n}_2, \hat{n}_3) \in \hat{N}$ there is a corresponding character given by,

$$\langle n, \hat{n} \rangle = \exp[i(n_0\hat{n}_0 - n_1\hat{n}_1 - n_2\hat{n}_2 - n_3\hat{n}_3)]$$

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(ii) SL$(2,\mathbb{C})$ acts on $\hat{N}$. This is shown as follows.,

$$\langle L_\Lambda n, \hat{n}\rangle = \langle n, {L_\Lambda}^{-1}\hat{n}\rangle, \text{ where } \Lambda \in \mathrm{SL}(2,\mathbb{C}) \text{ and } L_\Lambda^{-1} \in L_+^\uparrow.$$

(iii) The orbit associated with the character $\hat{n}_0$ is given by,

$$\hat{O} = \{L_\Lambda \hat{n}_0 |\ \hat{n}_0 \in N, \forall L_\Lambda \in L_+^\uparrow\}.$$

Thus, every orbit is contained in any of the hyperboloids,

$$\hat{n}_0^2 - \hat{n}_1^2 - \hat{n}_2^2 - \hat{n}_3^2 = m^2, \quad m^2 \in \mathbb{R}$$

(iv) All cases, $m^2 = 0$, $m^2 > 0$ and $m^2 < 0$ yields six types of orbits. For the purpose of our work we consider one of them which we represent by,

$$\hat{O}_m^+ : \hat{n}_0^2 - \hat{n}_1^2 - \hat{n}_2^2 - \hat{n}_3^2 = m^2, \qquad m > 0,\ \hat{n}_0 > 0.$$

Here, the stability subgroup $S_{\hat{O}_m^+}$ of the point $\hat{n}_0$ is SU(2).

(v) SU(2) has irreducible representations $L^j \equiv D^j$ of dimension $2j+1$, $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots$

(vi) The representations D^j of SU(2) depend on the parameter j, that is the *spin*. The representations of the Poincaré group induced by the representations D^j which we denote by $U^{m,+;\,j}$ depends on the parameters m and j. These represent *mass* and *spin* respectively.

In addition, to find the representations $U^{m,+;\,j}$ of the Poincaré group the following information is needed.

(a) The stability group of the orbit, $\hat{O}_m^+$ is the group $K = \Pi^4 \rtimes \mathrm{SU}(2)$, where K is a subgroup of $G = \widetilde{\Pi} = \Pi^4 \rtimes \mathrm{SL}(2,\mathbb{C})$

(b) The irreducible unitary representations of $\Pi^4 \rtimes \mathrm{SU}(2)$ are take the form,

$$L_k^j = L_{(a,r)}^j = \exp(i\mathring{p}a)D^j(r), \quad a \in \Pi^4, r \in \mathrm{SU}(2).$$

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where $\mathring{p} = (m, 0, 0, 0)$ and $D^j(r)$ is an irreducible unitary representation of SU(2).

(c) Let $g \in \widetilde{\Pi}$. With respect to the Mackey decomposition and $p = L_{\Lambda_p}\mathring{p}$, we have,

$$L^{-1}_{k_{g_o^{-1}s_g}} = \exp(ipa)D^j(r\Lambda_0)$$

where

$$r\Lambda_0 = {\Lambda_p}^{-1}\Lambda_0\Lambda_{L^{-1}_{\Lambda_0}p}, \qquad \Lambda_p = \begin{pmatrix} f & z \\ 0 & f^{-1} \end{pmatrix}, f \in \mathbb{R}\backslash\{0\}, z \in \mathbb{C}$$

So,

$$U^{m,+;\,j}_{(a,\Lambda)}u(p) = \exp(ipa)D^j(r\Lambda)u(L^{-1}_{\Lambda}p), \tag{4.1}$$

Then for, $a \in \Pi^4,\ \Lambda \in \mathrm{SL}(2, \mathbb{C})$ the action of $U^{m,+;\,j}$ on the carrier space $H^{m,+;\,j}$ of a *particle* $[m,\ j]$ is given by,

$$\left(U^{m,+;\,j}_{(a,\Lambda)}u\right)_n(p) = \exp(ipa)D^j_{nn'}(r_\Lambda)u_{n'}(L^{-1}_{\Lambda}p) \tag{4.2}$$

where $u_n(p)$ is a vector valued function on the positive mass hyperboloid,

$$\hat{n}_0^2 - \hat{n}_1^2 - \hat{n}_2^2 - \hat{n}_3^2 = m^2.$$

Remark 4.1.7. The set of functions $\{u_n(p)\}_{n=-j}^{j}$ could be identified with the *wave functions* of a free *physical system* with *mass* m and *spin* j.

The *wave function* $\psi(p)$ corresponding to a *massive particle* with a *spin* j transforms into,

$$U^{m,+;\,j}_{(a,\Lambda)}\psi(p) = \exp(ipa)D^{(j,o)}(\Lambda)\psi(L^{-1}_{\Lambda}p), \qquad D^{(j,0)} \in \mathrm{SL}(2, \mathbb{C}) \tag{4.3}$$

and hence, it satisfies the trivial wave equation,

$$(p^2 - m^2)\psi(p) = 0$$

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In the rest system $\mathring{p} = (m, 0, 0, 0)$ with $(0, r) \in SU(2)$, Equation 4.2 becomes

$$\left(U^{m,+;\ j}_{(0,r)} u\right)_n (\mathring{p}) = D^j_{nn'}(r) u_{n'}(\mathring{p})$$

This is exactly the property of a *free physical system* with *total spin* j, where the number, $n = -j, -j+1, \cdots, j-1, j$ in the rest system represents the *projection of the spin*. To bring the chapter to a close we outline and explain other applications of Lie groups and Lie algebras.

4.2 Some Applications of Lie Groups and Lie Algebras

4.2.1 $\mathbf{SU(2), SO(3), SO(4)}$ and the Hydrogen Atom

This application is obtained from[1]. Consider a number of observers who are interested in investigating or studying the *hydrogen atom*. The result of the experiment does not depend on the angle of observation since the idea is to get equal measurements from different angles. Thus, the experiment is invariant under rotations, so we can say that the symmetry group of the hydrogen atom contains the group SO(3). In other words, when we view the hydrogen atom as a *lone electron* we can say that the rotation group SO(3) is the *physical symmetry* group of the hydrogen atom.

Moreover, this observation gives as a representation of $\rho : \mathrm{SO}(3) \to L^2(\mathbb{R}^3)$,that is,

$$g \cdot f \begin{pmatrix} x \\ y \\ z \end{pmatrix} = f\left(g^{-1}\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right), \qquad \text{for } g \in SO(3),\ \ f \in L^2(\mathbb{R}^3).$$

When we see the *hydrogen atom* as a *stable nucleus* with one particle moving around it, then the space of states is $L^2(\mathbb{R}^3)$. Consequently, this representation explains the natural way for rotations of the sphere to move around vectors in the space of the *states* of the *hydrogen atom*. SU(2) is the double cover of SO(3) hence any representation of SO(3) is also a representation of SU(2).
As seen in [1], Fock showed that " the *Schrödinger equation* for the *hydrogen atom* in *momentum space* was identical to the integral equation for the *spherical harmonics*

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of four-dimensional *potential theory* and as such the transformation group of the *hydrogen atom* is the four-dimensional rotation group, SO(4)."

Thus the Lie groups SU(2), SO(3) and SO(4) are needful in the understanding of the *spatial symmetry* of the *hydrogen atom*.

4.2.2 $\mathfrak{u}(n)$ and Harmonic Oscillators

This application is obtained from [19]. Representations of $\mathfrak{u}(n)$ describe some *particles* in Physics. The totally symmetric and antisymmetric representations of $\mathfrak{u}(n)$ describe *bosons* and *fermions* respectively.

Consequently, the Lie algebra $\mathfrak{u}(n)$ has been realized in functions of *boson operators* by physicists. These are very useful in applications to different kinds of problems in physics, more importantly to *oscillator problems in quantum mechanics and algebraic models of rotation-vibration spectra of molecules and nuclei.* For example, we can construct $\mathfrak{u}(1)$, with one *boson operator* b which satisfies the commutation relation,

$$[b, b^\dagger] = 1, \quad [b, b] = [b^\dagger, b^\dagger] = 0, \quad g = b^\dagger b, \quad g \in \mathfrak{u}(1).$$

Here, the *boson operators* are expressed in terms of coordinate x and *momentum*, $p_x = -i\frac{d}{dx}$ by,

$$b = \frac{1}{\sqrt{2}}(x + ip_x) \qquad b^\dagger = \frac{1}{\sqrt{2}}(x - ip_x)$$

The *"quantum mechanical" Hamiltonian* is given by,

$$H = \frac{1}{2}\left({p_x}^2 + x^2\right) = \frac{1}{2}\left(-\frac{d^2}{dx^2} + x^2\right). \tag{4.4}$$

When we write Equation 4.4 in terms of elements in $\mathfrak{u}(1)$ we have,

$$H = b^\dagger b + \frac{1}{2}.$$

Thus the Lie algebra $\mathfrak{u}(1)$ describes the *one-dimensional harmonic oscillator* to an extent. Also, $\mathfrak{u}(2)$ can be used to explain the 2 dimensional *harmonic oscillator*. It is actually called the *degeneracy algebra* of the 2 dimensional *harmonic oscillator*.

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The Lie algebra $\mathfrak{u}(3)$ is also useful in describing the *harmonic oscillator* in spherical coordinates. Again, when we construct $\mathfrak{u}(3)$ in terms of a *singlet boson* σ and a doublet τ_x, τ_y with *circular boson operators*

$$\tau_{\pm} = \frac{1}{\sqrt{2}}\left(\tau_x \pm i\tau_y\right), \quad \tau_{\pm}^{\dagger} = \frac{1}{\sqrt{2}}\left(\tau_x \mp i\tau_y\right)$$

we ascertain practical applications in the study of *vibration-rotation spectra of molecules* in 2 dimensions.

In the same vein, *fermion operators* can be used to realize the Lie algebra $u(n)$ with antisymmetric representations. These are useful in investigating the *spectroscopy of nuclei*.

4.2.3 Representations of $\mathfrak{sl}(3, \mathbb{C})$ and elementary particles

The applications in this subsection are is obtained from [5, 20]. The Lie algebra $\mathfrak{sl}(3, \mathbb{C})$ is the complexification of $\mathfrak{su}(3)$. Also,the Lie group SU(3) is connected and simply connected. Thus every map in SU(3) corresponds bijectively to those of its Lie algebra, $\mathfrak{su}(3)$. As a result, the representations of $\mathfrak{sl}(3, \mathbb{C})$ can directly be applied to SU(3).

With regards to the Cartan subalgebra,

$$\mathfrak{h} = \{\text{diag}(a_1, a_2, a_3)|\ a_1 + a_2 + a_3 = 0, \quad a_1, a_2, a_3 \in \mathbb{C}\} \subset \mathfrak{sl}(3, \mathbb{C})$$

We can have 2 sets of basis, $\{T, Z\}$ and $\{T, R\}$ of $\mathfrak{h}$ such that,

$$T = \frac{1}{2}H_1, \quad Z = \frac{1}{3}(H_1 + 2H_2) = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad R = \frac{\sqrt{3}}{2}Z = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

where H_1, H_2 is as defined in Example 3.1.26.

Generally, *elementary particles* are considered to be weight vectors of a representation. Each weight is defined by the eigenvalues of H_1 and H_2 or of T and Z or of T and R, depending on the chosen basis. The eigenvalues in this case are *quantum*

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numbers which characterize a particle. The eigenvalue (quantum number) associated with T is the *third component of the isospin*. The eigenvalue associated with Y is the *the hypercharge*. In addition, the eigenvalue corresponding to $Q = \frac{1}{2}Z + T$ is the *electric charge* where $Q = \frac{1}{2}Z + T$ is a result of the original Gell-Mann-Nishijima formula.

The standard representation of $\mathfrak{sl}(3, \mathbb{C})$ and quarks

Let's consider the 3 dimensional standard representation of $\mathfrak{sl}(3, \mathbb{C})$ acting on $\mathbb{C}^3$ in the usual way. Here we have,

$$H_1 e_1 = e_1, \quad H_1 e_2 = -e_2, \quad H_1 e_3 = 0, \quad H_2 e_1 = 0, \quad H_2 e_2 = e_2, \quad H_2 e_3 = -e_3 \tag{4.5}$$

where

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \text{and } e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{4.6}$$

From 4.5 we see that the simultaneous eigenvectors for H_1 and H_2 in the standard representation are the ususal basis elements e_1, e_2, e_3 with weights $(1, 0), (-1, 1), (0, -1)$ respectively.The weight vectors $e_1 = u, e_2 = d$ and $e_3 = s$ of this represntation are called **quarks** (u, d, s).

Moreover, when we take a look at the dual of the standard representation of $\mathfrak{sl}(3, \mathbb{C})$ we obtain the weights $(-1, 0), (1, -1), (0, 1)$. The **antiquarks** are particles corresponding to this representation. They are the weight vectors of this representation.

Other representations of $\mathfrak{sl}(3, \mathbb{C})$ and elementary particles

The *baryon* and *meson octets* transforms with respect to the 8 dimensional adjoint representation of $SU(3)$. Each vector of the adjoint representation represents a *particle*. In addition, when we take the tensor product of the standard representation and its dual decomposes into the direct sum of 2 irreducible representations, the ad-

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joint representation the trivial (identity) representation . The vectors in this tensor product space of *quarks* and *antiquarks* form an octet of composite particles that transform uniformly according to the adjoint representation and the *mesons* consists quark-antiquark pair [5]. The *baryons* are obtained in the split up of the triple tensor product of the standard representation that is, a *baryon* is composed of 3 *quarks* [20].

4.2.4 $\mathfrak{so}(3)$, the angular momentum algebra

This application is obtained from [21]. Let's take a look at the Lie algebra $\mathfrak{so}(3)$ with basis as defined in 2.10, and commutation relations

$$[A_1, A_2] = -A_3, \qquad [A_2, A_3] = -A_1, \qquad [A_3, A_1] = -A_2.$$

Its differential realization is given by,

$$\begin{aligned} P(A_1) &= y\frac{\partial}{\partial y} - z\frac{\partial}{\partial y} \\ P(A_2) &= z\frac{\partial}{\partial y} - x\frac{\partial}{\partial y} \\ P(A_3) &= x\frac{\partial}{\partial y} - y\frac{\partial}{\partial y}, \qquad (x, y, z) \in \mathbb{R}^3. \end{aligned}$$

Also,

$$[P(A_1), P(A_2)] = -P(A_3), [P(A_2), P(A_3)] = -P(A_1), [P(A_3), P(A_1)] = -P(A_2) \tag{4.7}$$

We notice that the *angular momentum* operators L_x, L_y and L_z are actually multiples of $P(A_1), P(A_2)$ and $P(A_2)$. That is,

$$\begin{aligned} L_x &= \left(\frac{\hbar}{i}\right) P(A_1) \\ L_y &= \left(\frac{\hbar}{i}\right) P(A_2) \\ L_z &= \left(\frac{\hbar}{i}\right) P(A_3) \end{aligned}$$

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As such, the Equations 4.7 imply the angular momentum commutation relations,

$$[L_x, L_y] = i\hbar L_z, \qquad [L_y, L_z] = i\hbar L_x, \qquad [L_z, L_x] = i\hbar L_y$$

Thus, there is a relationship between the *quantum theory of angular momentum* and the Lie algebra $\mathfrak{so}(3)$. Again, these operators are associated with a complexified SU(2). This is because $\mathfrak{su}(2)$ is isomorphic to $\mathfrak{so}(3)$ and SU(2) is connected as well as simply connected. SU(2) is also the universal covering group SO(3). Thus, $\mathfrak{so}(3)$ depicts the *classical* and SU(2) depicts the *angular momentum* with regards to *quantum mechanics.*

4.2.5 The Heisenberg group and Photon operators

The application in this subsection is obtained from [22]. Elements of the Heisenberg group, H_3 are not only related to the *photon creation* and *annihilation* operators $(a^\dagger, a, I)$ but also the group generated by the exponentials of the *position* and *momentum* operators (p and q) and their commutator $[p, q] = \frac{\hbar}{i}$ [22]. Moreover, the set of change of basis transformations

$$\langle p|q\rangle = \frac{1}{\sqrt{2}} e^{\frac{2\pi ipq}{h}}$$

known in quantum mechanics is a unitary representation of the Heisenberg group, H_3.

4.2.6 Lorentz group and Ray optics

This application is obtained from [23]. The Lorentz group proves to be the basic language for *classical ray optics.* This includes, *polarization optics, interferometers, one-lens optics, multi-lens optics* as well as *multilayer optics.* In *quantum optics*, the *coherent squeezed states* have been shown in [23] to be nothing but representations of the Lorentz group.

Also, it has been proved that *optical components* which includes lasers, polarizers, interferometers and multi-layers can all be formulated in terms of the Lorentz group. These make use of a representation of $SL(2, \mathbb{C})$ which is a universal covering group

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of the Lorentz group. In [23], it is shown that the two-by-two representation of the Lorentz group is the common scientific language for all aspects of *ray optics* in the aforementioned.

Apart from these, there are so many physcal applications of Lie groups and Lie algebras that have been left unsaid in this work that one could explore.

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Conclusion

Lie groups naturally play out as symmetry groups and their symmetric nature suits the needs of certain aspects of physics since such areas rely on symmetries to operate [22]. That notwithstanding, Lie algebras help us understand Lie groups better and as such both go hand in hand in physical applications. As a result based on certain properties of the Lie group, a representation of its Lie algebra is likely to be on a one to one correspondence to its representation [2].

In our work, we outlined the basic concepts of Lie groups and Lie algebras, the basic representation theory of Lie groups and Lie algebras. In addition, we outlined a method of constructing induced representations and constructed induced representations of the Poincaré group. We found out that these representations of the Poincaré group depend on the mass and spin of a particle[16]. Some set of functions $\{u_n(p)\}_{n=-j}^{j}$ were noted in Remark 4.1.7. We found out that these functions could be identified with the *wave functions* of a free *physical system* with *mass* m and *spin* j. Further, we found some other useful applications of Lie groups and Lie algebras in Physics. These included, the role of SO(3) and others in the symmetry that governs the hydrogen atom [1], the connection of $\mathfrak{u}(n)$ and harmonic oscillators [19], SU(3) and elementary particles [20], $\mathfrak{so}(3)$ as the angular momentum algebra[21], Lorentz group in ray optics [23] and other equally useful applications.

Further Work

Semisimple Lie groups and Lie algebras play an important role in non commutative harmonic analysis and its applications. Reductive Lie groups are an extension of semisimple Lie groups. We can then study the harmonic analysis of these Lie groups and explore their applications in quantum mechanics.

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