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HELLY-TYPE THEOREMS ABOUT SETS

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Suppose that G is a graph. A 1-factor is a set of edges of G such that every vertex of G meets exactly one of its edges. Suppose that we have a set Y of 1-factors of G such that any two 1-factors of Y have an edge in common. We investigate the following questions:

(1) How large may Y be?

(2) When is there necessarily an edge contained in all the members of Y?

We answer these questions in the case that G is the complete graph on $2n$ vertices K_{2n} or the complete bipartite graph $K_{n,n}$. In the next section we study the first question; the third section is devoted to the second. In the final section we show that $B^2(K_{n,n}) \approx K_{n,n}$ and $B^2(K_{2n}) \approx K_{2n}$. We end with some unsolved problems. In the remainder of this section we identify the 1-factors of the two graphs, state our results, and recall the definitions of the space of colorings of G, $B(G)$. (See [4].

Let the vertices of K_{2n} be $1, 2, 3 \cdots 2n$. Then a 1-factor is a collection of n pairs $(a_1, a_2) \cdots (a_{2n-1} a_{2n})$ such that every value from 1 to $2n$ occurs exactly once. We call such a collection a *pairing*, and say that it is composed of *n pairs.*

$K_{n,n}$ has two sets of n vertices each, which we shall each number from 1 to n. Every point in one of the sets is joined to every point in the other set. Fixing one of the two sets, a 1-factor may be described by specifying the vertices of one set that the canonically ordered vertices of the other set connect to. This will be a permutation of the elements of the point set.

Two 1-factors of K_{2n} have an edge in common iff the pairings have a pair in common. Two 1-factors of $K_{n,n}$ have an edge in common iff the two permutations have an element in the same position. We can now state our results:

Theorem 1. *Let Y be a set of pairings of K_{2n}. If any two pairings of Y have a pair in common, then Y has at most F_{n-1} members, with* $F_{n-1} = (2n-3)(2n-5) \cdots 5 \cdot 3 \cdot 1$.

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Theorem 2[1]. *Let Y be a set of permutations of $K_{n,n}$. If any two permutations intersect, then Y has at most $(n-1)!$ members.*

Remark. The numbers F_{n-1} and $(n-1)!$ are simply the number of members that Y would have if Y consisted of all the 1-factors containing a fixed edge. Our next two results show that there is really only one way that maximal Y can occur.

Theorem 3. *Let Y be a set of F_{n-1} pairings of K_{2n} such that any two pairings of Y have a pair in common. Then there is a fixed pair contained in every member of Y.*

Theorem 4 *Let Y be a set of $(n-1)!$ permutations of $K_{n,n}$ such that any two intersect. Then they all intersect.*

A collection of 1-factors of a graph G such that every edge of G is in exactly one 1-factor is called a coloring of G. We make a simplicial complex $B(G)$ out of the 1-factors of G as follows: the vertices of $B(G)$ are all the 1-factors of G. The top dimensional simplices of $B(G)$ are all the colorings of G. That is, a set of 1-factors is a top simplex iff the set of 1-factors is a coloring.

2. The bounds on the number of intersecting 1-factors

If G is either K_{2n} or $K_{n,n}$ it is clear that there are automorphisms of G which carry any one 1-factor onto any other one. This means that $B(G)$ is point homogeneous: there are automorphisms of $B(G)$ which carry any one point onto any other. From these facts and the following Lemma we derive Theorems 1 and 2.

Lemma 1. *Let K be an n-complex which is point homogeneous. If Y is a set of vertices of K such that there is at most one vertex of Y in any top dimensional simplex of K, then Y has at most $g/(n+1)$ members, where g is the number of vertices of K.*

Proof. Since G is point homogeneous, every vertex of Y lies in the same number m of top simplices as any other. Counting the pairs (P, Q) where P is a vertex of Y, Q a top simplex of G and P in Q, we get $|Y|\, m \leq g'$ where g' is the number of top simplices of G. From the relation $gm = g'(n+1)$ we get the desired conclusion. □

Note. This can be false if K is not point homogeneous.

[1]Also see Deza [3].

Proofs of Theorems 1 *and* 2. If two members of Y were in the same top simplex, then they would not intersect. Therefore Y satisfies the conditions of Lemma 1. The number of 1-factors of K_{2n} is easily seen to be F_n, and a top simplex has $2n-1$ vertices in $B(G)$. The bound given by Lemma 1 is $F_n/(2n-1)=F_{n-1}$.

The number of 1-factors of $K_{n,n}$ is $n!$, and the bound given by the lemma is $(n!)/(n)=(n-1)!$.

Remark. From the proof of Lemma 1 we see that if $\|Y\|=g/(n+1)$ then every top simplex has exactly one member of Y.

3. Proofs of the Helly Type theorems

Our approach to these two theorems is first to find two 1-factors which differ in exactly two edges, one of them being in Y and the other not. We then show that one of these two edges lies in every 1-factor of Y.

We begin with the proof of Theorem 3. We claim that we may assume that (a) = (12)(34)(56) · · · is in Y and (b) = (13)(24)(56) · · · is not in Y. If this were not the case, then we could start with any member of Y and change exactly two of the pairs and the resulting pairing would still be in Y. This operation would generate all pairings. After renumbering we get (a) and (b). We shall first show that with this assumption every pairing of Y contains either (12) or (34).

A coloring consists of $2n-1$ pairings such that every pair occurs in exactly one pairing. We claim that if F is a coloring that contains (b), then the member of Y in F contains (12) or (34). If x is the pairing in Y and F, then it shares a pair with (a) and none with (b). This is possible iff x contains (12) or (34) or both.

Consider the set W (resp. W') of couples (F, p) where F is a coloring containing (b), p is a pairing contained in F which contains (12) and not (34) (resp. (34) and not (12)). A pair *occurs* in (F, p) iff the pair lies in p. Let S be the set of pairs other than those containing 1 or 2 or those occurring in (a). We claim that each of the pairs in S occurs equally often in W.

To see this consider the group of all permutations that fix (a) and (b). It is clear that all pairs (i, j) with both i and j greater than 4 occur equally often. Next. any pair of the form $(3, i)$ for i greater than 4 occurs as often as any other pair of that form. Similarly for pairs of the form $(4, i)$ for $i>4$. Since each couple of W contains exactly one of either of the latter sets of pairs, we see that these latter pairs occur in a proportion of $1/(2n-4)$ of the couples. There are $2n(2n-1)/2-(n-2)-(4(2n-4))-6=(2n-4)(n-3)$ pairs of S not containing 1, 2, 3, 4. There are $n-3$ positions in each couple in W, so these too occur in a proportion of $1/(2n-4)$ of the couples.

Let us now take a pairing x in Y which does not contain (12) or (34). Since x intersects (a), it contains, say, (56). Consequently, x contains at least 3 pairs not in S: $(1, i)$, $(2, j)$ and (56). Each of the remaining $n-3$ pairs of x occurs in $1/(2n-4)$

of the W couples. Similarly, each of the $n-3$ pairs other than $(3, k)$, $(4, n)$ and (56) occur in $1/(2n-4)$ of the couples. Since $(n-3)/(2n-4)$ is less than 1/2, less than half of the colorings which occur in W have a pairing containing (12) which meets x. Similarily, less than half have a pairing with (34) meeting x. Hence there is a coloring containing (b) which does not meet x. This shows that every member of Y has either (12) or (34) in it.

In order to complete this argument it is necessary to verify that W and W' are non empty. That is, there is some coloring containing (b) that does not have (12) and (34) in the same pairing. Consider the cycle of $2n-1$ integers $3, 2, 5, 7, 9, 11, \ldots 2n-3, 2n-1, 2n, 2n-2, \ldots 8, 6, 4$. For any member i of this cycle we get a pairing by taking the pair $(1, i)$ along with all pairs (j, k) such that j and k are the same distance $(< n)$ from i on the cycle. These $2n-1$ pairings give the desired coloring.

Let there be a pairing x of the form (12)(3a)(4b) $\cdots$ in Y. Let X (resp. X') be all pairings of Y which have (12) and not (34) (resp. (34) and not (12)). x has only $n-3$ pairs to intersect members of X', so it meets at most F_{n-3} pairings in X'. If there is a pairing of Y of the form (34)(1a)(2b) $\cdots$ then $|X| \leq F_{n-3}$. If there is no such pairing then all pairings of Y have (12). The number of pairings with (12)(34) is F_{n-2}, and Y has exactly F_{n-1} members. Consequently, we have the contradiction $/Y/ = F_{n-1} < F_{n-2} + 2F_{n-3}$. □

Proof of Theorem 4. This proof makes use of the fact that it is easy to construct colorings containing a given permutation. Consider the permutations

$$a, b, \ldots, m \qquad b, c, \ldots, m, a$$
$$c, d, \ldots, m, a, b \cdots m, a, b, \ldots, n.$$

This collection forms a coloring, and is uniquely determined by any one of the permutations. We call this collection the *cycle* determined by one of the members.

We are going to show that Y has a collection of $n-1$ permutations of the form $1P_1, 1P_2, \ldots, 1P_{n-1}$ where each P_i is a permutation of $2, 3 \ldots, n$ and no two P_i have a point in common. Given this, let D be an arbitrary permutation in Y. In any given place, D can intersect only one of the $1P_j$'s. If D does not begin with 1, then there are $n-2$ places for it to intersect the $1P_j$'s. Since there are $n-1$ permutations, D must have a 1 in the first place.

Now we construct the desired permutations. If D is any permutation in Y we can perform a transposition on two of its elements. Since Y is not all permutations, there must be some permutation in Y with the permutation resulting from some transposition not in Y. We may assume that these two permutations are as (a) and (b) below. Some one of the permutations of the cycle of (b) must be in Y, since there is exactly one member of Y in every cycle. This follows from Theorem 1. There are only two possibilities for the permutation, and by symmetry we may take it to be (c). Now consider the permutation in (d). It has no intersection with (c). There are two possible cyclings of it which might be in Y, (e) and (f). Now (e) has no intersection with (c) so (f) must be in Y. Now consider the permutations of the form $1, k, k+1, \ldots, n, 2, 3 \ldots, k-2, k-1$. There are only two cyclings of these which might be in Y and one misses (c) and the other misses (f). These are the desired permutations $1P_j$.

(a)	1	2	3	4	$\cdots$		$n-1$	n
(b)	2	1	3	4	$\cdots$		$n-1$	n
(c)	1	3	4	5	$\cdots$		n	2
(d)	3	4	5	6	$\cdots$	1	2	n
(e)	2	n	3	4	$\cdots$		$n-1$	1
(f)	1	2	n	3	$\cdots$		$n-2$	$n-1$ □

4. Relations with coloring theory

In order to study the iterates of the coloring functor B, we must have it operating on a complex through coloring points, not lines. We achieve this by taking the "line complex" of our graphs . Let G be a graph. The *Line complex of* G, written $L(G)$, has for its vertices the edges of G. The top simplices of $L(G)$ correspond to the vertices of G: a top simplex being all the edges containing a point. The space of colorings $B(K)$ of an n-complex K is again an n-complex. (See [4, Ch. 8]). We are going to show

Theorem 5. *If* $G = L(K_{n,n})$, *then* $G = B^2(G)$.

Before we can prove this, we need an extension lemma:

Lemma 3. *Let x and y be two 1-factors of $K_{n,n}$ which have no edge in common. Then there is a coloring of $K_{n,n}$ which contains x and y.*

Proof. If we consider the 1-factors as permutations, then x and y form a partial Latin Square. Now it is the case that every partial Latin Square extends to a Latin Square (Ryser [6]). Moreover, a coloring of $K_{n,n}$ corresponds to a Latin Square. □

Proof of Theorem 5. There is a map $\phi: G \to B^2(G)$ given by $\phi(p) =$ all vertices of $B(G)$ containing p. We first show that ϕ is 1–1. This is obvious, for given any two edges of $K_{n,n}$, we can certainly find a 1-factor which contains one of the edges and not the other.

Now suppose that we have a vertex Y of $B^2(G)$. This is a set of 1-factors such that there is exactly one in every coloring of G. By Lemma 3, any two of these must intersect. Theorem 2 shows that Y must contain exactly $(n-1)!$ members.

By Theorem 4 all these members have an edge in common. Since Y contains $(n-1)!$ members, and each one contains a fixed point p, Y must be the collection of all the 1-factors containing p. Thus Y is of the form $\phi(p)$. Since ϕ is 1–1 and onto, it is an isomorphism. □

Theorem 6. *If $G = L(K_{2n})$, $n > 2$, then $G \approx B^2 G$.*

This follows from

Lemma 4. *If X and Y are two 1-factors of K_{2n} which have no edge in common, then there is a coloring which contains X and Y.*

Proof (due to A. Kotzig). Let F consist of all edges in X or Y. We can partition the vertices of K_{2n} into two classes V_0 and V_1 in such a way that every edge of F joins one vertex of V_0 with one vertex of V_1. We have $|V_0| = |V_1| = n$.

Denote by G_0 (resp. G_1) the complete subgraph of K_{2n} containing all the vertices of V_0 (resp. V_1). Denote by H the subgraph of K_{2n} whose vertices are those of K_{2n}, and contains all edges joining a vertex of V_0 with one of V_1. Then clearly $G_0 \approx G_1 \approx K_n$, and $H \approx K_{n,n}$, with $F \subset H$.

If n is even, then we can find an edge coloring of $K_n \approx G_0 \approx G_1$, and by Lemma 3, F extends to a coloring of H. Combining these gives a coloring of K_{2n} extending F.

If n is odd, let L be the set of edges in H not in F. Choose a 1-factor L' of L. Consider the graph with vertices those of K_{2n}, and edges those in G_0, G_1, or L'. Now Kotzig [5] showed that this graph has a coloring. Combining the coloring of $H-L$ with that of $G_0 \cup G_1 \cup L$ gives a coloring of K_{2n} extending F. □

Remarks. In case $n=2$, $B(G) = K_3$, so ϕ is not 1–1. Bruch [1] studied some of the structure of $B(G)$, for $n = 2, 3, 4$. His calculations can be used to show that $B(L(K_6)) = L(K_6)$. This paper originated from trying to prove Theorem 6.

Here is another interpretation of Theorem 5. Let us define the Latin Square Complex LSC. The vertices of LSC are all potential rows of a Latin Square. The top simplices of LSC are all the rows which make up a Latin Square. In the notation of Theorem 5, LSC is B(G). Hence LSC $= B^2$(LSC).

The following generalization of Theorem 6 seems very difficult:

Define a complex $S(n, K)$ as follows. Take nK elements and let the vertices of $S(n, K)$ be all the K-subsets of these nK elements. n K-subsets form a top dimensional simplex of $S(n, K)$ iff they are disjoint. Prove

$$B^2(S(n, K)) \approx S(n, K).$$

It is a theorem of Barlotti that $B(S(n, K))$ is non empty (Cameron [2, Ch. 1]).

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